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A Journey to Fuzzy Rings

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A JOURNEY TO FUZZY RINGS

by

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(Under the Direction of Jimmy Dillies)

ABSTRACT

Enumerative geometry is a very old branch of algebraic geometry. In this thesis, we will describe several classical problems in enumerative geometry and their solutions in order to motivate the introduction of tropical geometry. Finally, fuzzy rings, a powerful algebraic framework for tropical and algebraic geometry is introduced.

INDEX WORDS: Tropical geometry, Fuzzy rings, Enumerative geometry

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TABLE OF CONTENTS

Page

LIST OF FIGURES

Figure Page

CHAPTER 1

INTRODUCTION

Enumerative geometry is the study of certain numerical invariants of algebraic varieties, loosely it can be thought of the study of the question how many solutions do certain geometric problems have? Recently, a powerful technique emerged for tackling problems in enumerative geometry called tropical geometry. Tropical geometry is a young field without firm axiomatic foundations, but one of the most promising foundations proposed is the category of fuzzy rings. We undertake a journey to fuzzy rings by motivating the study of them via enumerative geometry, passing through tropical geometry on the way.

CHAPTER 2

RINGS, FIELDS, AND VARIETIES

2.1 Rings and Fields

Before we discuss varieties, tropical or otherwise, we need to introduce a few algebraic structures. Specifically, commutative rings and fields. A commutative ring, R, is a set together with two binary operations, $+ : R \times R \mapsto R$ and $* : R \times R \mapsto R$. And these operations obey the following axioms:

1. **Identity**: There exists two elements, 0 and 1 in S, such that $\forall x \in R$ the following holds true: $x + 0 = 0 + x = x$ and $1 * x = x * 1 = x$

2. **Commutativity**: $\forall x, y \in R$ we have $x + y = y + x$

3. **Distributivity**: $\forall x, y, z \in R$ we have the following relations: $z*(x+y) = z*x+z*y$ and $(x + y) * z = x * z + y * z$

4. **Additive Inverses**: $\forall x \in R$ we have that $\exists -x \in R$ with $x + -x = -x + x = 0$

5. **Associativity**: $\forall x, y, z \in R$ the following holds: $x * (y * z) = (x * y) * z$ and $x + (y + z) = (x + y) + z$

6. **Commutativity of multiplication**: $\forall x, y \in R$ we have $x * y = y * x$

With these restrictions in place on our binary operations, we have effectively captured the essential properties of addition and multiplication on the integers.

Example 1.1 The prototypical example is the *ring of integers* with addition and multiplication.

There are several constructions to build new rings from old ones, as well.

Definition 1.1 *A polynomial function* of degree $m \neq 0$ in n-variables over k is a function of the form:

$$p(x_1, ..., x_n) := \sum_{i_1+i_2...+i_n \leq m} a_{i_1,i_2,...,i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$$

such that $(x_1, ...x_n) \in \mathbb{A}^n(k)$ and $a_{i_1,i_2,...,i_n} \in k$ and not all $a_{i_1,i_2,...,i_n}$ with $i_1 + ...i_n = m$ equal to zero. As an example, consider $p(x) = x^2$, which is a polynomial of degree 2 over $\mathbb{A}^1(\mathbb{R}) = \mathbb{R}$ and then, one recalls that the relation $y = x^2$ which generates a particular geometric curve in $\mathbb{R}^2$ this set is in fact identical to the set where $p(x, y) = x^2 - y$ which is a polynomial of degree 2 over $\mathbb{A}^2(\mathbb{R})$, is 0. And this set, the parabola, is what's called an affine algebraic set.

Example 1.3 The set of polynomials with coefficients in R denoted $R[x]$ forms a ring, called a *polynomial ring.*

Where a ring is an abstraction of the integers, a field is an abstraction of the rational numbers.

Definition 1.2 A *field* is a ring that obeys one additional axiom,

(6.)Multiplicative inverses: $\forall x \in S$ such that $x \neq 0$ we have that $\exists x^{-1} \in S$ with $x * x^{-1} = x^{-1} * x = 1$

Along with the rational numbers, other examples of fields, typically denoted with a K, include:

Example 1.4 The real numbers under the standard addition and multiplication operation is a field, as are the complex numbers

Example 1.5 The finite fields, that is $\mathbb{Z}/p\mathbb{Z}$ where p is a prime forms a field. To see this, notice that the collection of elements of $\mathbb{Z}/p\mathbb{Z}$ coprime to p form a group under multiplication, and since every integer less than a $prime$ must be coprime with that prime, we have that this is a group under multiplication and so has multiplicative inverses.

Example 1.6 The ring of formal Laurent series, denoted $R((x))$, forms a field when $R = K$, where K is a field. Consider sums of the form

$$\sum_{n=-\infty}^{\infty} a_i * x^i,$$

with the additional constraint that only a finite number of terms of negative index may be non-zero. Then, with multiplication and addition defined analogously to how they are for polynomials with coefficients in our field K, we obtain the *field of formal Laurent series*[5].

Example 1.7 The *Puiseux series*, denoted $K\{\{x\}\}$, is the set consisting of elements of the form $\sum_{k=k_0}^{+\infty} c_k T^{\frac{k}{n}}$ where the c_k are elements of a field characteristic zero, that is, where there does not exist an n such that the following identity for n-fold sums, $x + ... + x = 0$ holds for all x. And where T is an indeterminate. Then, this set of series forms a field with operations defined analogously to those of polynomials.[5]

2.2 Affine Algebraic Sets and Ideals

We are prepared to introduce the notion of affine algebraic sets and ideals of rings! These will form the basis of our investigations into algebraic geometry. Affine algebraic sets are defined as special kinds of subsets of a structure called the *affine n-space over a field k*, denoted $\mathbb{A}^n(k)$ which is simply the n-fold cartesian product of k with itself. That is, it's the set of elements of the form $(x_1, x_2, x_3, ...x_n)$ with $x_i \in k, \forall 1 \leq i \leq n$. $\mathbb{A}^n(k)$ has the structure of a vector space over a field, with multiplication and addition defined component-wise. That is, if $x, y \in \mathbb{A}^n(k)$ then $x + y = (x_1, ..., x_n) + (y_1, ...y_n) = (x_1 + y_1, ..., x_n + y_n)$ and multiplication by scalars defined by $\alpha * (x_1, ..., x_n) = (\alpha x_1, ..., \alpha x_n)$.

Definition 1.3 An *affine algebraic set* is a subset of $\mathbb{A}^n$ denoted $V(S) \subset \mathbb{A}^n(k)$ is simply the set of common zeros of a collection polynomials, S, over $\mathbb{A}^n(k)$, that is, $V(S) := \{x \in \mathbb{A}^n(k) : \forall\, p \in S,\ p(x) = 0\}$ or equivalently $\bigcap_{p \in S}\{x \in \mathbb{A}^n(k) : p(x) = 0\}$

Definition 1.4 Let R be a ring. An ideal, $S \subset R$ is a subset of R such that $\forall x, y \in S$ we have that $x + y \in S$ and $x * y \in S$. Furthermore, if $z \in R$ and $x \in S$ then we have that $zx \in S$. We may define the ideal of an affine algebraic set. We say that an ideal is prime if $xz \in S$ then either x or z is in S. And an ideal is maximal if it is not contained in any proper ideal. Where a proper ideal means an ideal that isn't all of R or the set $\{0\}$

Example 1.7 $\mathbb{Z}/p\mathbb{Z}$, that is, the integers modulo some integer, p. Is an example of a quotient ring. More generally, let R be a ring, and $S \subset R$ be an ideal, that is, S is closed under the addition and multiplication operations of R and $RS \subseteq S$. Then the cosets $R/S := \{x + S : x \in R\}$ form a ring, with identities $1 + S$, $0 + S$ and operations $x + S + y + S = (x + y) + S$ and $(x + S)(y + S) = x * y + S$.

Definition 1.5 Let $V \subset \mathbb{A}^n(k)$ be an affine algebraic set. With S a set of polynomials in $\mathbb{A}^n(k)$ that are zero on all of V. Then, the ideal of V, denoted $I(V)$ is the ideal generated by the polynomials in S. That is, all polynomials of the form $\sum_{s(x) \in S} p_s(x) * s(x)$. Where only a finite number of the terms in this sum are non-zero. This is called the *the ideal of V*.[1]

We say that an affine algebraic set is $reducible$ if it can be written as the union of two proper subsets that are also affine algebraic sets. If not, then the set is irreducible. There is actually a really simple characterization of when an affine algebraic set is irreducible!

Theorem 1.1 An algebraic set is irreducible if and only if the ideal of that algebraic set is prime[1].

2.3 Algebraic Varieties, definitions and examples

We can now introduce the notion of an affine variety!

Definition 1.6 An *affine variety* is an irreducible affine algebraic set.

Example 1.8 As a trivial, and not very exciting example, the affine plane itself is an affine variety! It is the roots of the polynomial $f(x) = 0$

Example 1.9 Let k be the real numbers with affine 2-space $\mathbb{A}^2(k)$. Then a conic section is the set of zeros of the polynomial, $P(x, y) := ax^2 + bxy + cy^2 + dx + ey + f$. A conic section is not always an example of an affine variety, and in fact are possibly the prototypical examples of interesting algebraic curves that are not varieties to keep in mind. To see that the ideal generated is not always prime, consider the case of $P(x, y) := x^2 - y^2$ Then, the ideal generated by $P(x, y)$ is not prime, as it factors, $P(x, y) = (x - y)(x + y)$ and thus is the union of the affine algebraic sets defined by $(x - y)$ and $(x + y)$

We can investigate another class of varieties called projective algebraic varieties. Let us return to affine 2-space over a field k, $\mathbb{A}^2(k)$. A point in this space can be denoted with coordinates (x, y). Then we, can map to a corresponding point in $\mathbb{A}^3(k)$ by sending (x, y) to $(x, y, 1)$. The point $(x, y, 1)$ then lies on a unique line through the origin, defined by $f(t) := (xt, yt, t)$. Conversely, every line through the origin corresponds to a unique point in $\mathbb{A}^2(k)$ except for the lines $g(t) := (tx, ty, 0)$ which we say correspond to "points at infinity." We generalize this construction to affine n-space, $\mathbb{A}^n(k)$.

2.4 Projective Varieties

Definition 1.10 *Projective n-space*, denoted $\mathbb{P}^n(k)$ is then the space of lines through the origin in $\mathbb{A}^{n+1}(k)$. Each point $x := (x_1, ..., x_{n+1})$ in $\mathbb{A}^{n+1}(k)$ determines a line by $x(t) := (tx_1, ...tx_{n+1})$. We can see immediately that two non-zero points x, y determine the same line if $\exists t \in \mathbb{R}$ such that $tx = y$. So we can say that $\mathbb{P}^n(k)$ is the collection of equivalence classes of points in $\mathbb{A}^{n+1}(k)\backslash\{0\}$ under the relation $x \sim y$ iff $\exists t \in \mathbb{R}$ such that $t \neq 0$ and $tx = y$. We can then write elements of $\mathbb{P}^n(k)$ in what are called homogeneous coordinates as $[x_1, ..., x_{n+1}]$, that is, we select an element of the equivalence class in affine space as a representative. We will also call elements of the projective plane points. An immediate issue is that the i-th coordinate is only actually defined relative to the other terms in the (n+1)-tuple. Specifically, the ratios $\frac{x_j}{x_i}$ are always well-defined if x_i is non-zero. We can define another kind of coordinates on elements of $\mathbb{P}^n(k)$.

Definition 1.11 $U_i := \{x \in \mathbb{P}^n(k) : x_i \neq 0\}$. Then, if $p \in U_i$ we define the non-homogeneous coordinates of p with respect to U_i to be $[x_1, ...x_{i-1}, 1, x_{i+1}, ...x_{n+1}]$. Which defines each element of U_i uniquely, as each p can be viewed as a set of points of the form $(tx_1, ...tx_{n+1})$ and there is only one element of $\mathbb{R}$ sending x_i to 1 under multiplication. Thus, we have a bijection between elements of U_i and elements of $\mathbb{A}^n(k)$, $\phi_i : U_i \mapsto \mathbb{A}^n(k)$,

$\phi_i(p) := (p_1, ...p_{i-1}, p_{i+1}, ..., p_n)$

We finally have the machinery required to define projective algebraic sets!

Definition 1.12 Let $k[X_1, ...X_{n+1}]$ be the ring of polynomials on $\mathbb{A}^{n+1}(k)$ with terms corresponding to homogeneous coordinates in $\mathbb{P}^n(k)$. Then, $P(x) \in k[X_1, ...X_{n+1}]$ has a zero at point $p \in \mathbb{P}^n(k)$ if $P(p) = 0$ for any representative of p in homogeneous coordinates. Then, a projective algebraic set is simply the set of common zeros in $\mathbb{P}^n(k)$ of a collection of polynomials on $\mathbb{A}^{n+1}(k)$, and a projective algebraic variety is an irreducible projective algebraic set.

Example 1.10 An example of a collection of varieties that will be explored later are the *cubic surfaces*. These are the surfaces defined by the zeros of homogeneous polynomials of degree 3 in four variables, that is, polynomials of the form: $\sum_{X_i+Y_i+Z_i+W_i=3} a_i X^{x_i} Y^{y_i} Z^{z_i} W^{w_i}$. Not all cubic surfaces are irreducible. In fact, in projective algebraic sets, we have a nice irreducibility criterion, by a version of the nullstellensatz, a projective hypersurface is irreducible if and only if the polynomial defining it is irreducible! Thus, if the cubic polynomial defining our surface is reducible, our surface must be also. Given that we are considering cubic polynomials in projective 4-space, any homogeneous polynomial of degree three must then be a hypersurface. An example, then, of a reducible cubic is $P(X, Y, Z, W) := X(YZ - W^2)$ as this is a homogeneous polynomial of degree 3, which factors into homogeneous polynomials of degree 2 and 1.

Another important class of varieties are the *toric varieties.* This is a very large class of varieties that have a very nice combinatorial description. We will describe a specific kind of toric variety here, that is, the toric variety of a cone. We will need several definitions first.

Definition 1.13 *A cone*, C, in a vector space, V, over an ordered field, F, is a subset of V such that if $a \in F$ and $c \in C$ with $a \geq 0$ then $ac \in C$. We say that C is a convex cone if $\forall x, y \in C$ if $\alpha, \beta \geq 0$ then $\alpha x + \beta y \in C$. This cone is called polyhedral if there is a finite set of vectors $v_1, ..., v_k \in C$ such that if $w \in C$ then we have $w = \sum_{i=1}^{k} \lambda_i v_i$ for some set of constants λ_i in $\mathbb{R}$. A polyhedral cone is rational if we can insist that $\lambda_i \in \mathbb{Z}$

Definition 1.14 If V is a finite dimensional vector space over the reals, then V has an inner-product and we may define the *dual cone* of C, D, as $\{w \in V : \forall v \in C, \langle w, v \rangle \geq 0\}$. Then, we say that a cone is strongly convex if $dim(D) = dim(C)$ where the dimension of a subset of V is taken to be the dimension of the smallest subspace containing that subset.

Definition 1.15 The *affine toric variety* of a strongly convex rational polyhedral cone is the spectrum of the semi-group algebra formed by the ”basis vectors” of the cone.

Example 1.11 The affine variety $V(x^3 - y^2)$ in $\mathbb{C}^2$ is a toric variety. Specifically, we have that $V(x^3 - y^2) = Spec(\mathbb{C}[\mathbb{N}\mathcal{A}])$ where $\mathcal{A} = \{2, 3\}$[3].

CHAPTER 3

ENUMERATIVE PROBLEMS IN ALGEBRAIC GEOMETRY

3.1 Bézout's Theorem

The simplest and most well known enumerative result in algebraic geometry is Bézout's Theorem for projective plane curves. It is a fundamental result in algebraic geometry. But we'll need some definitions before introducting it. For further reading, we refer the reader to [1]

Definition 2.1 We denote by $\mathcal{O}_p(\mathbb{A}^2)$ the set of *rational functions* defined at $p \in \mathbb{A}^2$, where a rational function defined at p is a function $F(X) = \frac{P(X)}{Q(X)}$ where $P(X)$ and $Q(X)$ are polynomials such that $Q(p) \neq 0$. The rational functions defined at a point form a ring.

Definition 2.2 Let f, g be affine plane curves. That is, an equivalence class of polynomials under the relation defined on homogeneous polynomials, also called **forms** below. Then, if f and g intersect at a point $p \in \mathbb{A}^2(k)$, we define the *intersection number*, I, of f and g at p by the formula $I(p, fg) := dim_k(\mathcal{O}_p(\mathbb{A}^2)/(f, g))$.

Definition 2.3 Let $F, G \in \mathbb{C}[X, Y, Z]$ be homogeneous polynomials, then we construct an equivalence relation by $F \sim G$ if $\exists \lambda \in \mathbb{C}\backslash\{0\} : \lambda F = G$. A *projective plane curve* is then an equivalence class of homogeneous polynomials on $\mathbb{C}[X, Y, Z]$.

Bézout's Theorem Let X and Y be two plane projective curves in $\mathbb{C}[X, Y, Z]$ with no components(irreducible factors) in common. Then we have
$\sum_p (I, F \cap G) = mn$ where m and n are the degrees of F and G.

Proof Here, dim will always denote dimension as a field extension of k. We know that FG has finite cardinality, else F and G would have a component in common. Thus, we may assume without loss of generality that none of the points of intersection lay on a line at infinity, $Z = 0$. To see this, assume that $p \in F$ and p is on the line at infinity, then we can make a projective change of coordinates that maps some line not in our intersection to the line at infinity, then by injectivity, the image of any point in our intersection under this transformation will not be on the line at infinity. Let $\pi : R \mapsto \Gamma$ be the natural map onto the quotient ring. Then define $\phi : R \times R \mapsto R$ by $\phi(A, B) := AF + BG$ and $\psi : R \mapsto R \times R$ by $\psi(C) = (GC, -FC)$. Since F and G have no common factors, we can construct an exact sequence $0 \to R \xrightarrow{\psi} R \times R \xrightarrow{\phi} R \xrightarrow{\pi} \Gamma \to 0$. To see that this sequence is exact, we note that $ker(\psi)$ must be 0 since k is a field and thus an integral domain, thus, if $\phi(A, B) = 0$ then we have that $AF + BG = 0$ but this means $AF = -BG$ and since F, G have no common factors, this means that A must be of the form GC and B must be of the form $-FC$. But this is exactly $image(\psi)$. Finally, the natural projection is only 0 if and only if the $f \in (F, G)$ which means $f = AF + BG$ which is in $image(\phi)$. Hence, this sequence is exact. Then by restricting the maps in the exact sequence we obtain a second exact sequence, $0 \to R_{d-m-n} \xrightarrow{\psi} R_{d-m} \times Rd = n \xrightarrow{\phi} R_d \xrightarrow{\pi} \Gamma_d \to 0$. Finally, $dim(R_d) = \frac{(d+1)(d+2)}{2}$ so if dmn then by a result on the dimension of field extensions in exact sequences we then have that $dim(\Gamma_d) = mn$.

Consider the map, $a : \Gamma \mapsto \Gamma$ defined by $a(H) = residue(ZHmod(F, G))$ we will prove is one-to-one. First, let $J \in \mathbb{C}[X, Y, Z]$, then let $J_0 = J(X, Y, 0)$, and since F, G, and Z do not have any common zeros, we have that F_0 and G_0 are relatively prime in $\mathbb{C}[X, Y]$. Assume now that $ZH = AF + BG$, then $A_0F_0 = -G_0B_0$ and thus $\exists C \in \mathbb{C}[X, Y], A_0 = G_0C$, and $B_0 = -F_0C$. Set $A_1 = A + CG$ and $B_1 = B - CF$, then $(A_1)_0 = (B_1)_0 = 0$ therefore $\exists A', B' \in \mathbb{C}[X, Y, Z]$ with $A_1 = ZA'$ and $B_1 = ZB'$ and since $ZH = FA_1 + GB_1$ we have that $H = A'F + B'G$ and so the map a is one-to-one.

Letting $dm + n$ we choose a sequence of polynomials in R_d, $A_1, ..., A_{mn}$ such that $res((A_1)mod(F, G), ..., res((A_{mn})) \in \Gamma_d$ form a basis. And define $A_{i^*} = A_i(X, Y, 1) \in \mathbb{C}[X, Y]$ then let a_i be the residue of A_{i^*} in Γ_*. Then these are a basis for Γ_*. To see this, first notice that since a is an injective map, the restriction to Γ_d is an isomorphism onto Γ_{d+1} since injective maps between vector spaces of the same dimension must be isomorphisms. And so the residues of $Z^r A_1, ...Z^r A_{mn}$ form a basies of Γ_{d+r} if $r0$. To see spanning, let $h = res(H)mod(F_*, G_*)$ with $H \in \mathbb{C}[X, Y]$, for some $N0$ we have that $Z^N H^*$ is a form of degree $d + r$ and so $Z^N H^* = \sum_{i=1}^{mn} \lambda_i Z^r A_i + BF + CG$ for some $\lambda_i \in k$ and $B, C \in \mathbb{C}[X, Y, Z]$. Then $H = (Z^N H^*)_* = \sum \lambda_i A_{i*} + B_* F_* + C_* G_*$ and so h is in the span of the a_i.

To see independence, assume $\sum \lambda_i a_i = 0$, then $(\sum \lambda_i A_{i*}) = BF_* + CG_*$. This implies $Z^r \sum \lambda_i A_i = Z^s B^* F + Z^t C^* G$ by a property of dehomogenizations of forms. But then we have that $\sum \lambda_i (res(Z^r A_i mod(F_*, G_*)) = 0$ in Γ_d and so λ_i must all be zero, as the $res(Z^r A_i mod(F_*, G_*))$ form a basis. And so, we have shown that $dim(\Gamma_d) = mn$ and $dim(\Gamma_d) = dim(\Gamma_*$ and so Bézout's theorem has been proved. ■

Bézout's theorem is a fundamental result in enumerative geometry, and has a generalization to hypersurfaces. If we have n projective hypersurfaces in $n + 1$ variables then the number of intersection points counting multiplicity is the product of the degrees of the polynomials defining the hypersurfaces, or infinite.

3.2 27 Lines on a Smooth Cubic

It is a classical result in enumerative geometry that there are only 27 straight lines on a smooth cubic surface over the complex numbers. We will provide a lemma, and sketch an interesting proof of this result using it. The proof in its entirety is in[3].

The proof will be sketched below, and will not be transcribed in its entirety as the whole proof takes up 10 pages!

1. Show that if S is a non-singular cubic surface, that every plane in $\mathbb{P}^3$ intersects S in an irreducible cubic, a line plus a conic, or a 3 distinct lines. And if P is a point in S, at most 3 lines pass through it, and if there are 2 or 3 the most be coplanar.

2. Show that there is at least one line l on S.

3. Show that if there is a line $l \subset S$ then there exists exactly 5 pairs (l_i, l_i') meeting l, and they do so in such a way that 1)For all i, $l \cup l_i \cup l_i'$ is coplanar and 2)For $i \neq j$ we have $l_i \cup l_i' \cap l_j \cup l_j' = \emptyset$

4. Show as a corollary of the above that there exist two disjoint lines in S

5. Prove that if $l_1, ...l_4$ are disjoint lines in $\mathbb{P}^3$ then either all four lines lie on a smooth quadric and have an infinite number of transversals, or else they do not lie on any quadric and have either 1 or 2 common transversals.

6. Notice that if l and m are two distinct lines in S then m meets exactly one line in each of the five pairs meeting l,and so from this we have 17 lines defined, by noticing that there must be five pairs for m too

7. Prove that if n is any line in S other than the 17 already found, then n meets exactly 3 out of 5 lines in $l_1, ...l_5$ and given any choice of three elements from $\{1, 2, ..., 5\}$, say $\{ijk\}$ then there is a unique line meeting l_{ijk} meeting l_i, l_j, l_k.

8. Deduce that there must be 27 lines on a non-singular cubic ■

As we can see, enumerative problems can involve a substantial amount of work. Next, we will sketch and describe an important result that will hopefully impart an understanding of the power of tropical methods in enumerative geometry.

3.3 Some curve counting

Definition 2.4 Consider a complex projective plane curve, C, in $\mathbb{P}^2$, then this curve is a variety of complex dimension 1, and thus of real dimension 2. And thus is topologically equivalent to a surface. The genus of this surface is then the genus of C. Moreover, if C is a non-singular curve of a degree d, then $g(C)$, the genus of C is, $\frac{1}{2}(d-1)(d-2)$

The space of all projective curves of degree d is itself a projective space[1]. If we view our projective curve as a point of this space, then varying the coefficients of our curve can be viewed as moving it through this space of curves, and this transformation may cause it to acquire singular points. If this singular point is a node, that is, an ordinary double point, then the genus of the curve is reduced by one, and so we obtain a formula for a curve with only p nodes and no other singularities: $g(C_{sing}) = \frac{1}{2}(d-1)(d-2) - p$.

Definition 2.5 The *Gromov-Witten number*, $N_{g,d}$, is the number of irreducible curves of genus g and degree d that pass through $g+3d-1$ points in the complex projective plane. This number is well-defined, since the space of curves, formally called the moduli space of projective curves, has dimension $\frac{1}{2}(d-1)(d-2)+3d-1 = g+3d-1$[5] and since a curve with a node has codimension 1, the number $N_{g,d}$ should be finite.

Example 2.1 If a cubic polynomial is smooth, then by the genus formula it has genus 1. If it possesses a node, then it has genus 0 and thus is a rational curve. And this happens if and only if the discriminant vanishes. In this case, the discriminant is a sum of 2040 monomials! Instead, if one looks at the hessian H of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, we see that it is a polynomial of degree 3. Then, if we form a matrix whose entries are the coefficients of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial y}$, $\frac{\partial H}{\partial z}$ and then the discriminant of f is the determinant of this matrix! If our cubic is required to pass through eight points in $\mathbb{P}^2$, then we have eight linear equations in the coefficients of our cubic. Combining this with the condition $discriminant(f) = 0$ then we have a system of equations with 12 solutions and so $N_{0,3} = 12$.

Phew! Even with handwaving, that was quite the escapade. If only there was a consistently simpler way to compute these numbers? Enter, tropical geometry.

Mikhalkin's Correspondence Principle In a landmark work by Grigory Miklhakin[4], the following result was established: The number of simple tropical cuves of degree d and genus g that pass through $3 + 3d - 1$ points in $\mathbb{R}^2$, where each curve is counted with its contribution, equals the Gromov-Witten number $N_{g,d}$ of the complex projective plane, $\mathbb{P}^2$. Enter, tropical varieties. Moreover, they even derived a recursive formula for the numbers $N_{0,d}$!

3.4 Tropical Varieties

A tropical variety is an object that is much simpler than a classical algebraic variety in many ways. But their definition is more abstract and requires significantly more machinery. Despite this, there is a one-to-one correspondence between tropical curves and directed graphs with certain conditions placed upon them.

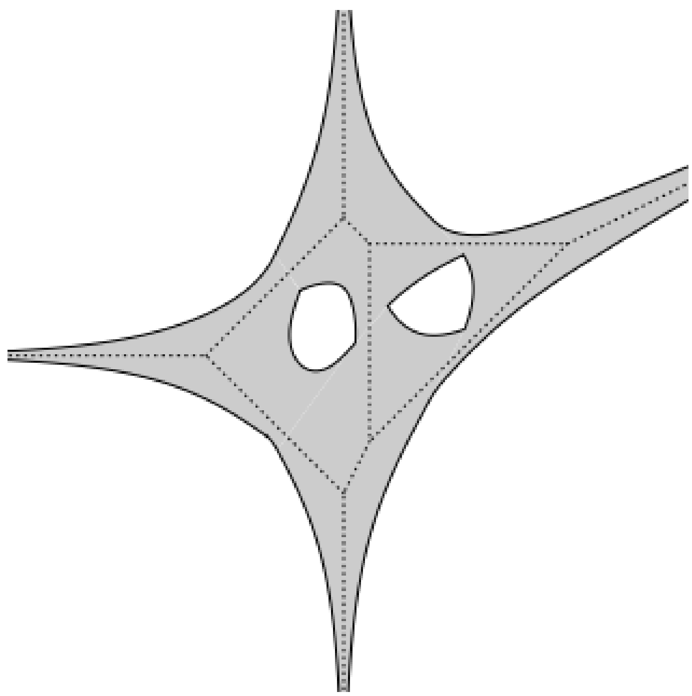

Figure 3.1: The amoeba and spine for the polynomial $1 + 5zw + w^2 + -z^3 + 3z^2w - z^2w^2$ taken from[5]

CHAPTER 4

AMOEBAS AND TROPICAL VARIETIES

4.1 Varieties from amoebas

Let I be an ideal in the ring of Laurent polynomials in n-variables as defined above. By the first section on varieties and ideals, this ideal has a variety associated to it, $V(I)$. The *amoeba* of I, $\mathcal{A}(I)$ is $\{(log(|z_1|), ..., log(|z_n|)) \in \mathbb{R}^n : \mathbf{Z} = (z_1, ..., z_n) \in V(I)\}$ An amoeba has finitely many tentacles, where tentacles are maximal(with respect to inclusion of convex sets) convex subsets with unbounded norm, and each one of these contains a ray, and more specifically, it converges to a ray in the euclidean norm. The union of these with a certain polygon embedded in the variety is then the tropicalization of the variety. An example of such a construction is given above, using the spine of the amoeba, as defined below.

The Passare Construction A more rigorous way of constructing tropical varieties is from spines, which are canonical tropical hypersurfaces inside $\mathcal{A}(f)$ where f is a polynomial defining a complex hypersurface. Let $f(z, w)$ be a polynomial in 2 variables, then its Ronkin Function is:

$$N_f(u, v) = \frac{1}{(2\pi i)^n} \int_{Log^{-1}(u,v)} log|f(z, w)| \frac{dz}{z} \wedge \frac{dw}{w}$$

This function is convex and linear on each connected component of the complement of $\mathcal{A}(f)$ Then, if $q(u, v)$ is the negated maximum of these affine-linear functions, $q(u, v)$ is what's called a tropical polynomial function. And the associated tropical curve is called the spine of the amoeba.

There are numerous ways to construct tropical varieties from amoebas, but what we want is to put tropical varieties on a firm algebraic footing that can be generalized to any variety over any field. To this end, we introduce the following definition:

Definition 3.1 Let K be a field. We say that a K has a *valuation* if there exists a map, $val : K \mapsto \mathbb{R} \cup \infty$ with the following properties:

(V1) $val(a) = \infty$ if and only if $a = 0$

(V2) $val(ab) = val(a) + val(b)$

(V3) for $a, b \in K$, $val(a + b) \geq min\{val(a), val(b)\}$

4.2 Tropical varieties, proper

Defition 3.2 Let K be a field equipped with a valuation and f a polynomial in $K[x_0, ...x_n]$. Writing f as $\sum_{\mathbf{u} \in U} c_{\mathbf{u}} x^{\mathbf{u}}$, where $U \subset \mathbb{N}^{n+1}$ is a set of finite cardinality, then the *tropicalization* of f is:

$$trop(f)(\mathbf{w}) = min\{val(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} : \mathbf{u} \in U, c_{\mathbf{u}} \neq 0\}$$

Definition 3.3 the *tropical hypersurface* associated with f as $\{\mathbf{w} \in R^n$: the minimum in $trop(f)(\mathbf{w})$ is achieved at least twice$\}$. If this is confusing, it helps to keep in mind that first, the valuation creates a set, possibly a multi-set, and then checks this for the minimum element. If the minimum element in this set appears at least twice at some point, $p \in (K^*)^n$, then $p \in V(f)$. Note, tropical varieties are considered most naturally as subvarieties of the torus $T = (K^*)^n$ since monomials are invertible on T.[5]

Definition 3.4 Let X be a variety in $(K^*)^n)$, of some ideal I. then, the tropicalization of X is $trop(X) := \bigcap_{f\in I} trop(V(f))$. That is, the intersection of the tropical hypersurfaces of the tropicalizations of polynomials in the ideal of X.

We conclude this section with some examples of tropical hypersurfaces and basic theorems.

4.3 Examples and tropical structure theorems

Example 3.1 A very simple, yet interesting tropical curve is the tropicalization of $f \in K[x^{\pm 1}, y^{\pm 1}]$ where f is defined by the equation $f = x + y + 1$. That is, f is an element of the field of bivariate Laurent polynomials. With a tropicalization defined by the valuation sending coefficients to the lowest power of t. So under the valuation we get

$$min\{val(t^0) + (w_1, w_2) \cdot \{1, 0\}, val(t^0) + (w_1, w_2) \cdot \{0, 1\}, valt^0 + (w_1, w_2) \cdot (0, 0)$$

since,

$$val(t^0) = 0$$

We have that this simplifies to

$$trop(V(f)) = min(w_1, w_2, 0).$$

As a set, this is $\{w_1 = w_2 \leq 0\} \cup \{w_1 = 0 \leq w_2\} \cup \{w_2 = 0 \leq w_1\}$

Example 3.2: For a more complex tropical curve:

Let $f = t^2x^2 + xy + (t^2 + t^3)y^2 + (1 + t)x + t^{-1}y + t^3$, then $trop(V(f)) = min(2 + 2w_1, w_1 + w_2, 2 + 2w_2, w_1, -1 + w_2, 3)$. The hypersurface is shown above:

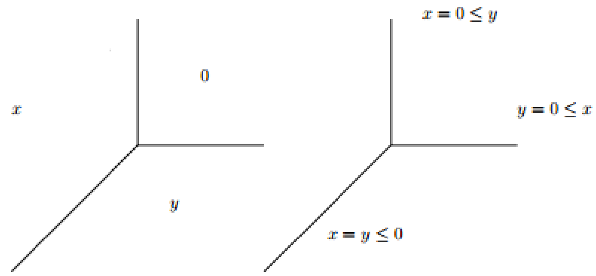

Figure 4.1: The tropicalization of $f(x, y) = x + y + 1$[5]

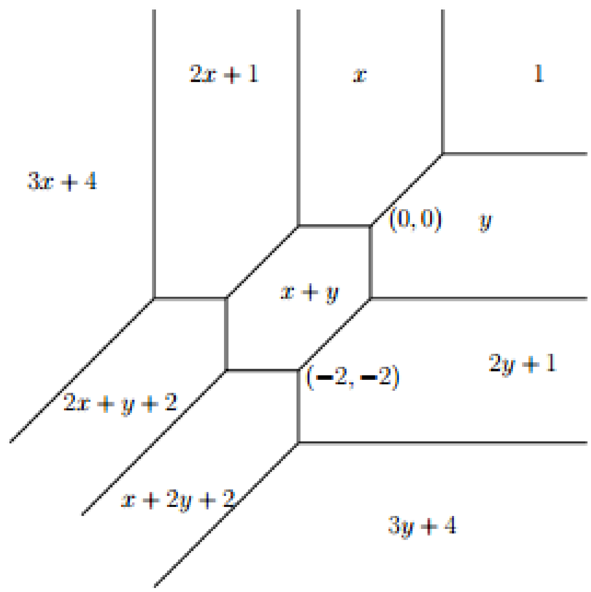

Figure 4.2: The tropicalization of $f(x, y) = f = t^2x^2 + xy + (t^2 + t^3)y^2 + (1 + t)x + t^{-1}y + t^3$[5]

To get an idea of what higher dimensional tropical varieties look like, we state a result and give an example of a higher dimensional tropical variety

Theorem 3.1 Let f be a Laurent polynomial in n-variables. The tropical hypersurface $trop(V(f))$ is the support of a a pure Γ_{val}-rational polyhedral complex of dimension $n-1$ in $\mathbb{R}^{\ltimes}$. It is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of $f = \sum_{\mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ given by the weights $val(c_{\mathbf{u}})$ on the lattice points of Newt(f).

Example 3.3 Let $K = \mathbb{Q}$ with the $2-$adic valuation. Then

$$f = 12x^2 + 20y^2 + 8z^2 + 7xy + 22xz + 3yz + 5x + 9y + 6z + 4$$

defines in a smooth surface in T^3_K. The newton The newton polytope of this variety is exceptionally simple, being the convex hull of the points $(2,0,0), (0,2,0), (0,0,2), (0,0,0)$. Then, the 2-skeleton of $\sum_{trop(f)}$ is the tropical quadric surface pictured below:

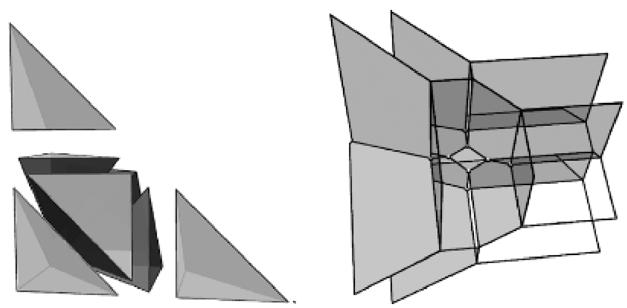

Figure 4.3: The tropicalization of f in Example 3.3[5]

Let $f \in K[x_1^{\pm}...x_n^{\pm 1}]$ then we can define

$$in_{\mathbf{w}}(f) = \sum_{\mathbf{u}:val(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=trop(f(\mathbf{w}))} \overline{t^{-val(c_{\mathbf{u}})}c_{\mathbf{u}}} x^{\mathbf{u}}$$

Kapranov's Theorem Let K be an algebraically closed field with a nontrivial valuation. Fix a Laurent polynomial $f = \sum_{\mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ in $K[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ then the following three sets coincide:

1) The tropical hypersurface $trop(V(f))$

2) The set $\{\mathbf{w} : in_{\mathbf{w}}(f)$ is not a monomial$\}$

3) The closure in $\mathbb{R}^n$ of $\{(val(y_1), ...val(y_n)) : (y_1, ..., y_n) \in V(f)\}$[5]

There are actually multiple ways of defining tropical varieties. To see this, we consider tropical curves as varieties in the min-plus algebra.

Definition 3.4 Consider the set $\mathbb{R} \cup \{\infty\}$ with the operations $\oplus, \otimes$ defined by $x \oplus y = min\{x, y\}$ and $x \otimes y = x + y$. This is a structure called a semi-ring. Furthermore, Then, a tropical polynomial is a map of the form $p(x, y) : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $p(x, y) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \otimes x^{\mathbf{u}}$ and we can define a tropical curve to be the set of points in $\mathbb{R}^2$ such that the minimum is achieved twice.

We introduce one more way of defining tropical curves using weighted graphs.

Definition 3.5 A graph, in this context is a collection of points in $\mathbb{R}^2$ called vertices, V, along with a collection of line segments connecting them, denoted E. The pair (V, E) is said to be a graph in $\mathbb{R}^2$. A graph is a *weighted graph* if there is a map from E to $\mathbb{Z}^+$.

Tropical curves in the euclidean plane can also be defined as follow, let $\Gamma \in \mathbb{R}^2$ be a graph with the following properties:

(a)Every edge is a line segment with rational slope

(b)Γ has d ends each in the directions $(-1, 0), (0, -1), (1, 1)$ where an end of weight w counts w times.

(c)The balancing condition: At every vertex v of Γ, the weighted sum of the primitive integral vectors of edges around v is zero.

Given the already presented number of working definitions of tropical varieties, and the differing construction of classical algebraic varieties, we might want a mathematical framework that encapsulates both of these structures! And in fact, this framework exists, and it is the construction of varieties over fuzzy rings.

CHAPTER 5

FUZZY RINGS

5.1 The min-plus algebra and fuzzy rings

Defintion 4.1 A *quasi-fuzzy ring*, $K = (K; +; \cdot; K_0; \epsilon$ is a set K together with two binary operations.

$$+ : K \times K \to K : (k, \lambda) \mapsto k + \lambda$$

And

$$\cdot : K \times K \to K : (k, \lambda) \mapsto k \cdot \lambda$$

Along with a distinguished element ϵ and subset $_0 \subset K$ that obey the following axioms:

(FR0) $(K, +)$ and $(K, \cdot)$ are abelian semi-groups with identities 0 and 1 respectively.

(FR1) $0 \cdot k = 0, \forall k \in K$

(FR2) $\alpha \cdot (\kappa_1, \kappa_2) = \alpha \cdot \kappa_1 + \alpha \cdot \kappa_2 \; \forall \kappa_1, \kappa_2 \in K, \alpha \in K^*$ where K^* is the group of units in K.

(FR3) $\epsilon^2 = 1$;

(FR4) $K_0 + K_0 \subset K_0$ and $K \cdot K_0 \subset K_0$, $0 \in K_0$, $1 \notin K_0$

(FR5') $1 + \epsilon \in K_0$

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ and $\lambda_1 + \kappa_1, \lambda_2 + \kappa_2 \in K_0$ implies that $\kappa_1 \cdot \kappa_2 + \epsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0$

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2 \, inK$ and $\kappa + \lambda \cdot (\kappa_1 + \kappa_2) \in K_0$ implies that $\kappa + \lambda \cdot \kappa_1 + \lambda \cdot \kappa_2 \in K_0$

We call a quasi-fuzzy ring a *fuzzy ring* if a stronger version of (FR5') holds

(FR5) $\alpha \in K^*$ implies that $1 + \alpha \in K_0$ if and only if $\alpha = \epsilon$

Intuitively, a fuzzy ring K can be thought of like a ring without additive inverses, where instead of -1 we have a special element ϵ. The reader should notice that K_0 is a set of elements that act very similar to zero. The elements of K_0 will be called *null* and the set K_0 will be called the *null-set*. These can be considered heuristically as "extra-zeros." Finally, elements that are units behave like traditional ring elements except for the lack of unique inverses. We consider some examples to elucidate the differences between fuzzy and traditional rings.

Example 4.1 Commutative unitary rings are a in a one-to-one correspondence with fuzzy rings for which $K_0 = \{0\}$ and $\epsilon = -1$

Example 4.2 More interesting is when we consider a commutative unitary ring $(R; +; \cdot)$ and an ideal $N \subset R$ then $(R; +; \cdot; -1; N)$ is a quasi-fuzzy ring if and only if N is a proper ideal. If N is not proper, we would have $1 \in N$ which would imply that 1 is in the null set. Which contradicts (FR4). If N is proper, then we get that (FR6) holds since $\kappa_1 + N \equiv -\lambda_1 + N$ and $\kappa_2 \equiv -\lambda_2 + N$ so then $\kappa_1 \cdot \kappa_2 + N \equiv \lambda_1 \cdot \lambda_2 + N$ by the definition of multiplication in factor groups, and so $\kappa_1 \cdot \kappa_2 + -1(\lambda_1 \cdot \lambda_2) \in N$ and thus (FR6) is satisfied. All other axioms are fairly easy to verify. Furthermore, $(R; +; \cdot; -1; N)$ is a fuzzy ring if and only if $1 + x \in N$ implies that $x = -1$.

To proceed with constructing tropical varieties in the context of fuzzy rings, we must introduce the notion of a linear ordering on an abelian group:

Definition 4.2 An abelian group with group operation $*$ and an ordering $(\Gamma, \leq)$ is *linearly ordered* if it is totally ordered and if $\alpha, \beta, \gamma \in \Gamma$ and $\alpha < \beta$ then $\gamma * \alpha < \gamma * \beta$.
Let $\gamma_0 \notin \Gamma$, then define $\overline{\Gamma}$ by $\Gamma \dot{\cup} \{\gamma_0\}$, and write $\gamma_0 * \alpha = \alpha * \gamma_0 = \gamma_0, \forall \alpha \in \overline{\Gamma}$. Furthermore, let us extend the ordering on Γ by writing $\gamma_0 < \alpha$ for all $\alpha \in \Gamma$. Then, we can define a

valuation on a field K in a way similar to before, we say that $v : K \to \overline{\Gamma}$ is a valuation if the following axioms hold:

(V0) $v(x) = \gamma_0 \Leftrightarrow x = 0$

(V1) $\forall x, y \in K$, then $v(x \cdot y) = v(x) * v(y)$

(V2) $\forall x, y \in K$, $v(x + y) \leq max\{v(x), v(y)\}$

The astute observer will notice that in this valuation, the element γ_0 is taking place that $-\infty$ would in a valuation that maps to the real numbers, and for this reason this element is often denoted $-\infty$.

Defintion 4.3 If $v : K \to \overline{\Gamma}$ is a valuation, then the valuation ring is defined as

$$R := \{x \in K | v(x) \leq e\}$$

where e is the identity in Γ. We claim that this is a ring.

Proof: All the ring operation axioms are inherited from the field K. All that remains is to check closure and the existence of identity elements. But, if $x, y \in R$ we have that $v(x + y) \leq max\{v(x), v(y)\} \leq e\}$ and thus $v(x + y) \in R$. Then $v(x \cdot y) = v(x) * v(y)$ but then $v(x) * v(y) \leq v(x) * e$ since the ordering is linear, applying linearity again we obtain: $v(x) * v(y) \leq e$ and thus $x \cdot y \in R$. By (V0) and (V1) we have that $0 \in R$ and that $v(1) = e$ and thus R has the identity elements. Finally, if $x \in R$ then $-x \in R$ since $v(x) * v(-1) \leq e * v(-1) = v(-1) \leq e\}$. We also note that the group of units is then simply $R^* := \{x \in K : v(x) = 1\}$ ■

Assume that K is a field with a surjective valuation $v : K \to \overline{\Gamma}$ we have that K/R^* is a fuzzy ring using the following construction:

Let $K = (K; +; \cdot; \epsilon; K_0)$ be a (quasi)-fuzzy ring, then if U is a subgroup of the group of units, we may a construct a new quotient (quasi)-fuzzy ring by

$$K/U := (P(K)^U; +; \cdot; \epsilon \cdot U; P(K)_0^U)$$

Where $P(K)^U$ is the collection of subsets, $\{F \subset K : F \neq 0, U \cdot F = F\}$ that is, the subsets invariant under multiplication by U. And similary, $P(K)_0^U$ is $\{F \in P(K)^U : F \cap K_0 \neq \emptyset\}$ and where sets in $P(K)^u$ are added and multiplied as complexes.

Note: In the rest of this paper, we let $\dot{\cup}$ be the disjoint union operation.

The fuzzy ring $L := K/R^*$ has the property that L^* is canonically isomorphic to the valuation group. Since, $K^*/R^* = L^*$ which is isomorphic to Γ under the map $\overline{v} \circ \phi$ where ϕ is the natural projection and $\overline{v}$ is the valuation applied to any representative of K^*/R^*. We also assume that the index of $[R : m] \geq 3$, where m is the maximal ideal $\{x \in K : v(x) < e\}$. Writing $\mathring{L}$ to be the smallest subset of L such that $L^* \cup \{0\} \subset \mathring{L}$ and $\mathring{L} + \mathring{L} \subset \mathring{L}$ and $\mathring{L} \cdot \mathring{L} \subset \mathring{L}$ That is, the smallest subset closed under the fuzzy ring operations that contains both the group of units and 0. Define $F := \mathring{L}$ then we have that

$$F = \{\{0\}\} \dot{\cup} \{x \cdot R^* : x \in K^*\} \dot{\cup} \{x \cdot R : x \in K^*\}$$

Since F clearly would contain $\mathring{L}$, and by definition $\mathring{L}$ must contain $\{0\}$ and $\{x \cdot R^* : x \in K^*\}$ since these are all the units in L. Finally, $\{x \cdot R : x \in K^*\}$ must be in F since because $[R : m] > 2$, where m is the maximal ideal $\{x \in K : v(x) < e\}$, we have that $x \cdot R^* + x \cdot R^* = x \cdot R$ for all $x \in K^*$. We also mention that $\{x \cdot R : x \in K^*\} = F_0$ along with the following

$$x \cdot R^* + y\dot{R}^* = y \cdot R^* \text{ whenever } v(x) < v(y)$$

$$x \cdot R + y \cdot R = y \cdot R \text{ whenever } v(x) \leq v(y)$$

$$x \cdot R + y \cdot R^* = y \cdot R^* \text{ whenever } v(x) < v(y)$$

$$x \cdot R^* + y \cdot R = y \cdot R \text{ whenever } v(x) \leq v(y)$$

Intuitively, this tells us that elements with greater valuation are absorptive under the addition operation in F. Next, define

$$\widetilde{\gamma} := \{\gamma' \in \Gamma : \gamma' \leq \gamma\} \text{ for } \gamma \in \Gamma$$

And

$$K_\Gamma := \overline{\Gamma}\dot{\cup}\{\widetilde{\gamma} : \gamma \in \Gamma\}$$

Given the way our elements act as absorptive elements under addition, it is natural to make the following identifications of cosets of the form $x \cdot R^*$ with the image $v(x) = \gamma \in \Gamma$ and the cosets $x \cdot R$ with the subsets $\overline{\gamma}$.

Remark Assume $m \in \mathbb{N}$ and $m \geq 1$, and that $\kappa_1, ...\kappa_m \in \Gamma$, then we have that $\kappa_1 + ... + \kappa_m \in \widetilde{\Gamma} := \{\overline{\gamma} : \gamma \in \Gamma\}$ if and only if the maximum is attained twice. And from here, as you might have guessed by comparing to the min-plus condition of tropical varieties, we can construct the so-called tropical fuzzy ring, the ring used to build tropical varieties as fuzzy varieties using these tools. Note, above we have used the identification established before hand to write the group operation additively.

Consider the set,

$$S_\Gamma := \{0\} \cup \{(\gamma_1, ...\gamma_k) \in \mathcal{T}(\overline{\Gamma}) : k \geq 2, \gamma_i \leq max\{\gamma_j | j \neq i\} \forall 1 \leq i \leq k\}$$

Where $\mathcal{T}(\overline{\Gamma})$ is the set of tuples of elements in $\overline{\Gamma}$. We also identify γ_0 and 0.

Definition 4.3 If K is a (quasi)-fuzzy ring, and there is a group isomorphism, $\phi : G \to K^*$ then we say that the set

$$S(\phi) := \left\{ (g_1, ..., g_k) \in \mathcal{T}(\overline{G}) : k \geq 1 \sum_{i=1}^{i=k} \phi(g_i) \in K_0 \right\}$$

Conforms to K, relative to ϕ. Furthermore, $S(\phi)$ is not the only set that conforms to K_Γ. Relevant to our focus on tropical varieties, we have that the following set conforms to K_Γ; first note that $\phi_0^{-1} : \Gamma \to \Gamma$ defined by $\phi_0 \gamma) = \gamma^1$ is an isomorphism, then we have:

$$S'_\Gamma = \{0\} \cup \{(\gamma_1, ..., \gamma_k) \in \mathcal{T}(\overline{\Gamma} : k \geq 2, \text{ if } \gamma_i > \gamma_0 \text{ there exists a} j \neq i \text{ with } \gamma_0 < \gamma_j \leq \gamma_i\}$$

By construction we have that a tuple is in S'_ϕ if and only if the minimum is achieved at least twice! Hence, this conforms to K_Γ relative to ϕ_0 as we obtain that the sum of a tuple is in $K_{\Gamma 0}$ if and only if the image of the elements of this tuple under ϕ_0 meet the criteria for being elements of $S(\phi)$

Examining the above, morally we can see that what it means for a fuzzy ring to conform to a set is to say that the elements of the group are subject to certain constraints on nullity that are imposed by the existence of the isomorphism onto the group of units. Essentially, this allows to use the existence of sums of elements in G being in K_0 to detect when group elements meet certain criteria.

Example 4.3 Consider the semi-ring $(\mathbb{R} \cup \infty, \oplus, \otimes)$ defined as in the section on the min-plus algebra. Then, letting $(R, +)$ be the group $(\Gamma, *)$, we have $\mathbb{R} \cup \infty = \overline{\Gamma}$. Our isomorphism corresponding to ϕ_0 is $\phi_0(x) = -x$, taking ∞ to $-\infty$. We obtain

$$\alpha + \alpha = \widetilde{\alpha} \text{ for } \alpha \in \Gamma$$

As opposed to $\alpha \oplus \alpha = \alpha$. Now, instead of checking whether or not the minimum occurs twice in our tropical operations, we can simply check to see that a sum of elements lands in $K_{\Gamma 0}$, and so we come to the notion of fuzzy varieties.

5.2 Zariski Systems

In the setting of classical algebraic varieties, there are two topologies. The classical topology, which we are not concerned with here, and the Zariski topology. The Zariski topology is a unique topology with the property that the closed sets in it are exactly the algebraic subsets of the variety on which it is defined.

Definition 4.4 Let $K = (K; +; \cdot; \epsilon; K_0$ be a (quasi)-fuzzy ring, then an $ideal$ in K, I is a set in K such that the following hold:
(I1) $0 \in I$
(I2) If $x, y \in I$ then $x + y \in I$
(I3) For $a \in K$ and $x \in I$ one has $a \cdot x \in I$
Furthermore, we call an ideal $proper$ if we also have that $K_0 \subseteq I \subset K$
Then, a proper ideal is called prime if $x \cdot y \in I$ implies that $xory \in I$

We say K is a *(quasi)-fuzzy domain* if K_0 is a prime ideal. And a (quasi)-fuzzy field if $K = K_0 \dot{\cup} K^*$

Definition 4.5 Let K be a (quasi)-fuzzy domain using the standard notation, let $M \subset K$ be non-empty, and $\mathcal{F}$ be the set of maps from M to K. Then the triple, $(K, M, \mathcal{F})$ is called a *Zariski system* if the following properties hold true: (Z1) If $f, g \in \mathcal{F}$ then $f \cdot g = f(x) \cdot g(x)$ is also in $\mathcal{F}$
(Z2) If $a \in M$, we have that $\exists f_a \in M$ with $f_a(a) \notin K_0$

If $(K, M, \mathcal{F})$ is a Zariski system and $\mathcal{T} \subset \mathcal{F}$ then we define:

$$Z(\mathcal{T}) := \{a \in M : f(a) \in K_0, \forall f \in \mathcal{T}\}$$

$$Z(f) := Z(\{f\}), f \in \mathcal{T}$$

$$\mathcal{V} := \{Z(\mathcal{T}) : \mathcal{T} \subset \mathcal{F}\}$$

Proposition The Zariski system is a topology.

Proof First, note, $\mathcal{Z}\{\emptyset\} \in \mathcal{V}$ trivially. Thus, $M \in \mathcal{V}$. Moreover, $\emptyset = \mathcal{Z}(\mathcal{F})$ and thus $\emptyset \in \mathcal{V}$. Next, let $\{\mathcal{T}_i\}_{i \in I}$ be a collection of subsets of $\mathcal{F}$, then we have that $\bigcap_{i \in I} Z(\mathcal{T}_i) = Z(\bigcup_{i \in I} \mathcal{T}_i) \in \mathcal{V}$ holds by the defintion of $Z(\mathcal{T}_i)$. Finally, let $\mathcal{T} = \mathcal{T}_1 \cdot \mathcal{T}_2 := \{f_1 \cdot f_2 : f_1 \in \mathcal{T}_1, f_2 \in \mathcal{T}_2\}$. By (Z1), we have $\mathcal{T} \in \mathcal{F}$. Let $a \in \mathcal{T}_1$, when $f_1 \in \mathcal{T}$ then $f_1(a) \in K_0$ and so $f_1 \cdot f_2(a) \in K_0, \forall f_1 \in \mathcal{T}_1, f_2 \in \mathcal{T}_2\}$ therefore $Z(\mathcal{T}_1) \subset Z(\mathcal{T})$ and $Z(\mathcal{T}_2) \subset \mathcal{T}$ by similar reasoning. Let $a \in Z/Z(\mathcal{T})$ then $\exists f_1 \in \mathcal{T}_1$ with $f_1(a) \notin K_0$, so we must have that $f_1(a) \cdot f_2(a) \in K_0 \; \forall f_2 \in \mathcal{T}_2$ and since K is a (quasi)-fuzzy domain, we obtain that $f_2(a)$ must be in K_0 for every f_2 and so $a \in Z(\mathcal{T}_2)$. ■

And thus, the sets defined above using a Zariski system satisfies all the axioms required for a topology of closed sets on M.

5.3 Fuzzy polynomials and tropical varieties

5.3.1 Construction: the tropical polynomial functions

Let $L \subseteq K$ be non-empty. Then define $P_I^1(L, K) := \{f : L^I \to K |$f is a constant map or $f = P_j$ for some $j \in I\}$. For $n \geq 1$ set

$$P_I^{n+1}(L, K) := \{f + g : f, g \in P_I^n(L, K)\} \cup \{f \cdot g : f, g \in P_I^n(L, K)\}$$

Finally define $P_I(L,K) := \bigcup_{i=1}^{\infty} P_I^n$

In the case of a field, the Zariski system $(K^n, K, P_n(K,K))$ recovers the classical Zariski topology on K^n and so we recover the classical theory of algebraic varieties over a field.

Definition 4.6 Let K be a (quasi)-fuzzy domain, and L a multiplicatively closed subset, with $K^* \subseteq L$. Assume there exists some $n \in N$ with $\mathcal{F} \subset P_n(L,K)$ and $(L^n, K, \mathcal{F})$ a Zariski system. A *fuzzy variety*, or an $\mathcal{F}$-fuzzy variety if context does not make it clear, is a closed set in the Zariski system $(L^n, K, \mathcal{F})$. With this we can finally introduce the tropical Zariski topology.

5.3.2 Tropical Zariski Topology

Let $(\Gamma, \cdot, \leq)$ be a linearly ordered abelian group. We extend $\leq$ to a total ordering defined on K_Γ by writing:

$$0 \leq \gamma_1 \leq \widetilde{\gamma} \leq \gamma_2 \leq \widetilde{\gamma_2} \text{ if } \gamma_1, \gamma_2 \text{ satisfy } \gamma_1 < \gamma_2$$

Then, for $c \in K_\Gamma$ put:

$$c^* = c \text{ if } c \in \Gamma \cup \{0\} \text{ or } c^* = \gamma \text{ if } c = \widetilde{\gamma} \text{ for some } \gamma \in \Gamma$$

Denote by $\mathcal{F} = \mathcal{F}_{\Gamma,n}$ the set of functions defined by $\sum_{a \in \mathcal{A}}(c_a \cdot \prod_{i=1}^{n} x_i^{a_i})$, where $\mathcal{A}$ is a finite subset of $\mathbb{N}_0^n$, and for $a \in \mathcal{A}$ then $a = (a_1, ..., a_n)$ and $c_a \in \Gamma$

(*)*Remark* We wish to show that this is Zariski system. To do this, we make two notes. The following relations hold from direct computation:
(i)If $\gamma_1, ...\gamma_m \in \Gamma$ with $\gamma_1 \leq ... \leq \gamma_m$, then we have the absorption property:

$$\gamma_1 + ... + \gamma_m = \gamma_m \text{ if } \gamma_m > \gamma_{m-1}$$

Or:

$$\gamma_1 + ... + \gamma_m = \gamma_m \text{ if } \gamma_m = \gamma_{m-1}$$

(ii) For $\gamma_1 \in K_\Gamma$ and

$\gamma_2 \in \widetilde{\Gamma}$ with

$\gamma_1 \leq \gamma_2$ we then have that

$\gamma_1 + \gamma_2 = \gamma_2$

(iii) For $\gamma_1 \in \widetilde{\Gamma}$ and $\gamma_2 \in \Gamma$ we have $\gamma_1 \cdot \gamma_2 = \gamma_1 \cdot \widetilde{\gamma_2}$

5.3.3 The Tropical Zariski System

Assume that $P = \sum_{a \in \mathcal{A}} (c_a \cdot \prod_{i=1}^n X_i^{a_i})$ and that $P = \sum_{b \in \mathcal{B}} (d_b \cdot \prod_{i=1}^n X_i^{b_i})$. Where $\mathcal{A}$ and $\mathcal{B}$ are finite subsets of $\mathbb{N}_0^n$ with $a = (a_1, ..., a_n)$ and $c_a \in \Gamma, \forall a \in \mathcal{A}$ and that $b = (b_1, ..., b_n)$. Furthermore, for a suitable finite subset $\mathcal{S} \subset \mathbb{N}_0^n$ and $s = (s_{1],..s_n)}$ as well as $r_s \in K_\Gamma / \{0\}$ we have that $P \cdot Q = \sum_{s \in \mathcal{S}} (r_s \cdot \prod_{i=1}^n X_i^{s_i})$ that We may assume that $|\mathcal{A}|, |\mathcal{B}| \geq 2$. Suppose that $x_1, ..., x_n \in K_\Gamma$ are fixed. Then, if $\overline{P}(x_1, ...x_n) \in \Gamma \dot{\cup} \{0\}$ and $\overline{Q}(x_1, ...x_n) \in \Gamma \dot{\cup} \{0\}$, then one also has $f(x_1, ..., x_n) \in \Gamma \dot{\cup} \{0\}$ where $f = \overline{P \cdot Q} = \overline{P} \cdot \overline{Q}$ and so from the relation remark we obtain that $f(x_1, ..., x_n) = \sum_{s \in \mathcal{S}} (r_s^* \cdot \prod_{i=1}^n x_i^{s_i}) \forall x_1, ..., x_n \in K_\Gamma$. Again, from the relation remark, we have that there exists $a = (a_1, ...a_n)$ and $a' = (a'_1, ...a'_n)$ both in $\mathcal{A}$ with $a \neq a'$ and similar b and b' for $\mathcal{B}$ such that the following holds:

$f(x_1, ...x_n) = P \cdot Q(x_1, ...x_n) =$

$$= (c_a \cdot \prod_{i=1}^n x_i^{a_i}) + c_{a'} \prod_{i=1}^n x_i^{a'_i}) \cdot (d_b \cdot \prod_{i=1}^n x_i^{b_i})$$

$$= (c_a \cdot d_b \cdot \prod_{i=1}^n x_i^{a_i + b_i}) + c_{a'} \cdot d_b \cdot \prod_{i=1}^n x_i^{a'_i + b_i})$$

$$\leq \sum_{s \in \mathcal{S}} (r_s^* \cdot \prod_{i=1}^{n} x_i^{s_i})$$
$$\leq f(x_1, ...x_n)$$

Where our estimates follow from $c_a \cdot d_b \in \Gamma$ and $c_a' \cdot d_b \in \Gamma$ and the fact that $a \neq a'$.

Finally, we have now that (Z1) is a consequence of the inequalities above, since then $f(x_1, ...x_n) = \sum_{s \in \mathcal{S}} (r_s^* \cdot \prod_{i=1}^{n} x_i^{s_i})$ and then (Z2) is true because $\mathcal{F}_{\Gamma,n}$ contains the constant functions. Thus, we have that $(\Gamma^n, K_\Gamma, \mathcal{F}_{\Gamma,n})$ is a Zariski system.■

Theorem 4.1 This theorem allows us to establish the equivalence of the construction of tropical Zariski system using the max-plus or min-plus algebras. Let $f \in \mathcal{F}_{\Gamma,n}$ $f = \sum_{a \in \mathcal{A}} (c_a \cdot \prod_{i=1}^{n} x_i^{a_i})$, with $c_a \in \Gamma$ for all $(a_1, ...a_n) \in \mathcal{A}$. Then there exists $g \in \mathcal{F}_{\Gamma,n}$ written $\sum_{\overline{a} \in \overline{\mathcal{A}}} (d_{\overline{a}} \cdot \prod_{i=1}^{n} x_i^{\overline{a}_i})$, with the terms of our series defined as before $d_{\overline{a}} \in \Gamma$ etc, such that all $x_1, ...x_n \in \Gamma$ satisfy the following:

(i)The maximum(or minimum) of the values $(c_a \cdot \prod_{i=1}^{n} x_i^{a_i})$, $a \in \mathcal{A}$ occurs at least twice if and only if the minimum(or maximum) of the values $(d_{\overline{a}} \cdot \prod_{i=1}^{n} x_i^{\overline{a}_i})$, $\overline{a} \in \overline{\mathcal{A}}$ at least twice. In particular, this implies that the Zariski topology of $(\Gamma^n, K_\Gamma, \mathcal{F}_{\Gamma,n})$ is invariant under the group automorphism sending $\gamma \in \Gamma$ to γ^{-1}. ■

Definition(The tropical Zariski System):

Let $\Gamma = (\mathbb{R}, +, \leq)$, then by the above for our $f(x_1, ...x_n) \in \mathcal{F}_{\Gamma,n}$ we have that we may assume the summands of this function(under our group operation), to be of the form:

$$c_a + a_1 \cdot x_1 + ... + a_n \cdot x_n$$

Using ordinary multiplication and addition in $\mathbb{R}$. But, this coincides with tropicalization of our polynomial, that is, the minimum is achieved twice only when the above sum is an element of K_0, and so we have recovered the tropical zariski topology on the min-plus algebra!

5.4 Fuzzy Rings and Projective Planes

As an addendum, we consider another structure called hyperrings that can also be used to axiomatize tropical geometry. There is a natural correspondence between hyperrings and projective geometry. We show how the embedding of the category of hyperrings can be circumvented, and how projective planes can be generated on fuzzy rings.

Definition 4.7 A *canonical hypergroup*, is a set, H, with a binary operation, hyperaddition, $+ : H \times H \to 2^{H*}$ where 2^{H*} is the non-empty elements of the power set of H. We use the notation $A + B := \{\cup(a + b) : a \in A, b\}$. The hypergroup operation must obey the following axioms:

(H1) $x + y = y + x$ if $x, y \in H$

(H2) $\forall x, y, z \in H$ we have $x + (y + z) = (x + y) + z$

(H3) $\exists 0 \in H$ such that $x + 0 = 0 + x = x$

(H4) $\forall x \in H, \exists!\ y = -x \in H$ such that $0 \in x + y$

(H5) $x \in y + z \implies z \in y - x$

Definition 4.8 A *hyperring* is a set $(R, +, \cdot)$ with a hyperaddition, $+$ and multiplication operation, $\cdot$ satisfying the following axioms:

(HR1) $(R, +)$ is a canonical hypergroup

(HR2) $(R, \cdot)$ is a monoid with multiplicative identity 1.

(HR3) $\forall r, s, t \in R : r(s + t) = rs + tr$ and $(s + t)r = sr + tr$

(HR4) $\forall r \in R$ we have $0 \cdot r = r \cdot 0 = 0$

(HR5) $1 \neq 0$

Defintion 4.9 If $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ then a map $f : R_1 \to R_2$ is a *hyperring homorphism* if:

(1) $f(a +_1 b) \subseteq f(a) +_2 f(b), \forall a, b \in R_1$

(2) $f(a \cdot_1 b) = f(a) \cdot_2 f(b),\ \forall a, b \in R_1$

A homormorphism is strict when we have that $f(a +_1 b) = f(a) +_2 f(b)\ \forall a, b \in R_1$. And if $R_1 \subseteq R_2$ are hyperrings and the inclusion is a strict homomorphism, we say that R_2 is an extension of R_1. Also, if $(R - \{0\}, \cdot)$ is a group, then we call R a hyperfield.

Definition 4.10 We denote by $\mathbb{K}$ the object called the *Krasner hyperfield*, which is the set $(\{0, 1\}, +, \cdot)$ with additive neutral element 0, satisfying the hyper-rule: $1 + 1 = 0$ and the usual multiplication on the integers. We call extensions of this field $\mathbb{K}$-vector spaces

Theorem 4.2 There is a fully faithful functor embedding the category of hyperrings into the category of fuzzyrings. Let $(R, +\cdot)$ by a hyperring. Then the fuzzy ring $F(R)$ has the non-empty subsets of R as its elements, and multiplication is given by $A \times_F B = \{a \times b : a \in A, b \in B\}$ and addition is given by $A +_F B := \bigcup_{a \in A, b \in B}(a + b)$, with the following identifications $K := 2^{R*}$, $F(0) = \{0\}$, $F(1) = \{1\}$, $K_0 := \{T \in 2^{R*} : 0 \in T\}$[8].

Theorem 4.3 Let R be a hyperring extension of the Krasner hyperfield, $\mathbb{K}$. Then there is a unique projective geometry $\mathcal{P}$ such that the set of points of $\mathcal{P}$ is $R - \{0\}$ and the line through two distinct points of $x, y \in \mathcal{P}$, $L(x, y) = (x + y) \cup \{x, y\}$. We recall that a projective geometry is any set of elements, called points, obeying the following axioms:

(P1) Two distinct points determine a unique line

(P2) If a line, L, meets two sides of a triangle, not at their intersection, then it meets the third side

(P3)Every line contains at least three points.

These projective geometries though, obey a stronger version of (P3), called (P3') which is that every line contains at least four points[7].

Theorem 4.4 A fuzzy ring K is an embedding of a hyperring extension of the Krasner hyperfield if and only if the following conditions are met:

(i) For all $x\ inK\ x+x \in K_0$

(ii) There is a collection of *independent elements*, that is a set of elements, B, such that:

(B1) If $x, y \in B$ and $x \neq y$ then $x+y \notin K_0$

(B2) If $w, v \in K$ ∃B$_w \subset B$, $B_w \neq \emptyset$ such that $b \in B_w$ implies $b+w \in K_0$ and if $B_w = B_v$ then $v = w$.

(B3) If $x, y, z \in B$ we have distributivity, that is $z \cdot (x+y) = z \cdot x + z \cdot y$.

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