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Marvis Pararai
Indiana University of Pennsylvania

Gayan Warahena-Liyanage
Indiana University of Pennsylvania

Broderick O. Oluyede
Georgia Southern University, boluyede@georgiasouthern.edu

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A New Class of Generalized Power Lindley Distribution with Applications to Lifetime Data

Mavis Pararai[1], Gayan Warahena-Liyanage[2] and Broderick O. Oluyede[3]

Abstract

A new class of distribution called the beta-exponentiated power Lindley (BEPL) distribution is proposed. This class of distributions includes the Lindley (L), exponentiated Lindley (EL), power Lindley (PL), exponentiated power Lindley (EPL), beta-exponentiated Lindley (BEL), beta-Lindley (BL), and beta-power Lindley distributions (BPL) as special cases. Expansion of the density of BEPL distribution is obtained. Some mathematical properties of the new distribution including hazard function, reverse hazard function, moments, mean deviations, Lorenz and Bonferroni curves are presented. Entropy measures and the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters. Finally, real data examples are discussed to illustrate the usefulness and applicability of the proposed distribution.

Mathematics Subject Classification: 60E05; 62E15

[1] Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA. E-mail: pararaim@iup.edu
[2] Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA. E-mail: g.j.warahenaliyanage@iup.edu
[3] Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460,USA. E-mail: boluyede@georgiasouthern.edu

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1 Introduction

Lindley [1] developed Lindley distribution in the context of fiducial and Bayesian statistics. Properties, extensions and applications of the Lindley distribution have been studied in the context of reliability analysis by Ghitany et al. [2], Zakerzadeh and Dolati [3], and Warahena-Liyanage and Pararai [4]. Several other authors including Sankaran [5], Asgharzadeh et al. [6] and Nadarajah et al. [7] proposed and developed the mathematical properties of various generalized Lindley distributions. The probability density function (pdf) of the Lindley distribution is given by

$$f(y;\beta)=\frac{\beta^2}{\beta+1}(1+y)e^{-\beta y},\quad y>0,\beta>0.$$

The power Lindley (PL) distribution proposed by Ghitany et al. [8] is an extension of the Lindley (L) distribution. Using the transformation $X=Y^{\frac{1}{\alpha}}$, Ghitany et al. [8] derived and studied the power Lindley (PL) distribution with the probability density function (pdf) given by

$$f(x;\alpha,\beta)=\frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha},\quad x>0,\alpha>0,\beta>0.$$

The cumulative distribution function (cdf) of the power Lindley distribution is

$$F(x)=1-S(x)=1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}$$

for $x>0,\alpha,\beta>0$. Warahena-Liyanage and Pararai [4] studied the properties of the exponentiated Power Lindley (EPL) distribution. The EPL cdf and pdf are given by

$$G_{EPL}(x;\alpha,\beta,\omega)=\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^\omega \tag{1.1}$$

and

$$g_{EPL}(x;\alpha,\beta,\omega) = \frac{\alpha\beta^2\omega}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1} \quad (1.2)$$

respectively, for $x > 0, \alpha > 0, \beta > 0, \omega > 0$. The hazard rate function of the EPL distribution is given by

$$h_{G_{EPL}}(x;\alpha,\beta,\omega) = \frac{g(x;\alpha,\beta,\omega)}{\overline{G}(x;\alpha,\beta,\omega)} = \frac{\frac{\alpha\beta^2\omega}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1}}{1-\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega}}.$$

The r^{th} moment of the EPL distribution is given by

$$E(X^r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^{j-k-r\alpha^{-1}+1}\Gamma(k+r\alpha^{-1}+1)}{(\beta+1)^{i+1}(i+1)^{(k+r\alpha^{-1}+1)}}.$$

The purpose of this paper is to develop a five-parameter alternative to several lifetime distributions including the gamma, Weibull, exponentiated Weibull, exponentiated Lindley, lognormal, beta Weibull geometric (BWG) [9], and beta Weibull Poisson (BWP) [10] distributions. In this context, we propose and develop the statistical properties of the beta exponentiated power Lindley (BEPL) distribution and show that it is a competitive model for reliability analysis. Our aim in this paper is to discuss some important statistical properties of the BEPL distribution. This discussion includes the shapes of the density, hazard rate and reverse hazard rate functions, moments, moment generating function and parameter estimation by using the method of maximum likelihood. Finally, applications of the model to real data sets in order to illustrate the applicability and usefulness of the BEPL distribution are presented.

This paper is organized as follows. In section 2, the model, sub-models and some of its statistical properties including shapes and behavior of the hazard function are presented. Moments, conditional moments, reliability and related measures are given in section 3. Mean deviations, Bonferroni and Lorenz curves are presented in section 4. Section 5 contains distribution of order statistics and measures of uncertainty. In section 6, we present the maximum likelihood method for estimating the parameters of the distribution. Applications are given in section 7 followed by concluding remarks.

2 The Model, Sub-models and Some Properties

In this section, we present the BEPL distribution and derive some properties of this class of distributions including the cdf, pdf, expansion of the density, hazard and reverse hazard functions, shape and sub-models. Let $G(x)$ denote the cdf of a continuous random variable X and define a general class of distributions by

$$F(x) = \frac{B_{G(x)}(a,b)}{B(a,b)}, \tag{2.1}$$

where $B_{G(x)}(a,b) = \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt$ and $1/B(a,b) = \Gamma(a+b)/\Gamma(a)\Gamma(b)$. The class of generalized distributions above was motivated by the work of Eugene et al. [11]. They proposed and studied the beta-normal distribution. Some beta-generalized distributions discussed in the literature include work by Jones [12], Bidram et al. [9]. Nadarajah and Kotz [13], Nadarajah and Gupta [14], Nadarajah and Kotz [15], Barreto-Souza et al. [16] proposed the beta-Gumbel, beta-Frechet, beta-exponential (BE), beta-exponentiated exponential (BEE) distributions, respectively. Gusmao et al. [17] presented results on the generalized inverse Weibull distribution. Pescim et al. [18] proposed and studied the beta-generalized half-normal distribution which contains some important distributions such as the half-normal and generalized half normal (Cooray and Ananda [19]) as special cases. Furthermore, Cordeiro et al. [20] presented the generalized Rayleigh distribution and Carrasco et al. [21] studied the generalized modified Weibull distribution with applications to lifetime data. More recently, Oluyede and Yang [22] studied the beta generalized Lindley distribution with applications.

By considering $G(x)$ as the cdf of EPL distribution we obtain the beta-exponentiated power Lindley (BEPL) distribution with a broad class of distributions that may be applicable in a wide range of day to day situations including applications in medicine, reliability and ecology. The cdf and pdf of the five-parameter BEPL distribution are given by

$$\begin{aligned} F_{BEPL}(x;\alpha,\beta,\omega,a,b) &= \frac{1}{B(a,b)}\int_0^{G_{EPL}(x;\alpha,\beta,\omega)} t^{a-1}(1-t)^{b-1}dt \\ &= \frac{B_{G(x)}(a,b)}{B(a,b)}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} f_{BEPL}(x;\alpha,\beta,\omega,a,b) &= \frac{1}{B(a,b)}\left[G_{EPL}(x)\right]^{a-1}\left[1-G_{EPL}(x)\right]^{b-1} g_{EPL}(x), \\ &= \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \\ &\times \left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega a-1} \\ &\times \left\{1-\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega}\right\}^{b-1}, \end{aligned} \tag{2.3}$$

respectively, for $x > 0, \alpha > 0, \beta > 0, \omega > 0, a > 0, b > 0$. Plots of the pdf of BEPL distribution for several combinations of values of α, β, ω, a and b are given in Figure 2.1. The plots indicate that the BEPL pdf can be decreasing or right skewed. The BEPL distribution has a positive asymmetry.

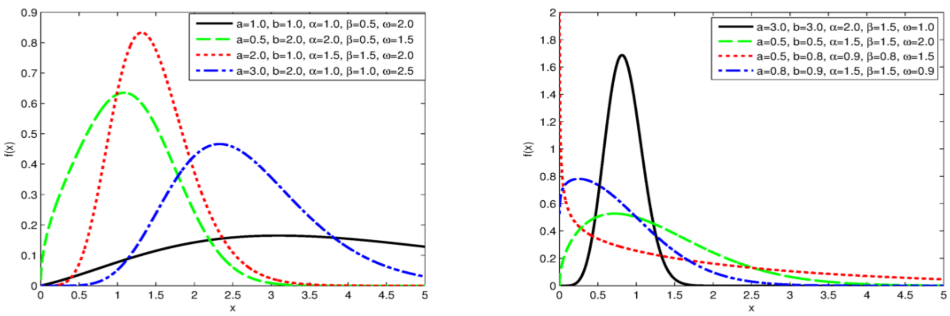

Figure 2.1: Plots of the PDF for different values of α, β, ω, a and b

2.1 Expansion of density

The expansion of the pdf of BEPL distribution is presented in this section.

For $b > 0$ a real non-integer, we use the series representation

$$(1 - G_{EPL}(x))^{b-1} = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \left[G_{EPL}(x)\right]^i, \tag{2.4}$$

where

$$G_{EPL}(x; \alpha, \beta, \omega) = \left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^\omega.$$

If a is an integer, from Equation (2.3) and the above expansion (2.4), we can rewrite the density of the BEPL distribution as

$$\begin{aligned} f_{BEPL}(x; \alpha, \beta, \omega, a, b) &= \frac{g_{EPL}(x)}{B(a,b)} \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \left[G_{EPL}(x)\right]^{a+i-1} \qquad (2.5) \\ &= \frac{\alpha\beta^2\omega}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^{\omega-1} \\ &\times \sum_{i=0}^{\infty} \frac{(-1)^i \binom{b-1}{i}}{B(a,b)} \left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^{\omega(a+i-1)} \\ &= \frac{\alpha\beta^2\omega}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \\ &\times \sum_{i=0}^{\infty} l_i \left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^{\omega(a+i)-1}, \qquad (2.6) \end{aligned}$$

where the coefficients l_i are

$$l_i = l_i(a,b) = \frac{(-1)^i \binom{b-1}{i}}{B(a,b)}$$

and $\sum_{i=0}^{\infty} l_i = 1$, for $x > 0, \alpha > 0, \beta > 0, \omega > 0, a > 0, b > 0$.

If a is real non-integer, we can expand $\left[G_{EPL}(x)\right]^{a+i-1}$ as follows:

$$\begin{aligned} \left[G_{EPL}(x)\right]^{a+i-1} &= \{1 - [1 - G_{EPL}(x)]\}^{a+i-1} \\ &= \sum_{j=0}^{\infty} \binom{a+i-1}{j} (-1)^j \left[1 - G_{EPL}(x)\right]^j, \end{aligned}$$

with

$$\left[1 - G_{EPL}(x)\right]^j = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \left[G_{EPL}(x)\right]^k,$$

so that

$$[G_{EPL}(x)]^{a+i-1} = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{a+i-1}{j}\binom{j}{k}(-1)^{j+k}\left[G_{EPL}(x)\right]^{k}. \quad (2.7)$$

From Equations (2.5) and (2.7), the BEPL density can be rearranged in the form

$$\begin{aligned} f_{BEPL}(x;\alpha,\beta,\omega,a,b) &= g_{EPL}(x)\sum_{i,j=0}^{\infty}\sum_{k=0}^{j} l_{i,j,k}\left[G_{EPL}(x)\right]^{k} \quad (2.8)\\ &= \frac{\alpha\beta^2\omega}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}})\\ &\times \sum_{i,j=0}^{\infty}\sum_{k=0}^{j} l_{i,j,k}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega(k+1)-1}, \end{aligned}$$

where the coefficients $l_{i,j,k}$ are

$$l_{i,j,k} = l_{i,j,k}(a,b) = \frac{(-1)^{i+j+k}\binom{b-1}{i}\binom{a+i-1}{j}\binom{j}{k}}{B(a,b)}$$

and $\sum_{i,j=0}^{\infty}\sum_{k=0}^{j} l_{i,j,k} = 1$, for $x>0, \alpha>0, \beta>0, \omega>0, a>0, b>0$. Hence, for any real non-integer, the BEPL density is given by three (two infinite and one finite) weighted power series sums of the baseline cdf $G_{EPL}(x)$. By changing $\sum_{j=0}^{\infty}\sum_{k=0}^{j}$ to $\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}$ in Equation (2.8), we obtain

$$\begin{aligned} f_{BEPL}(x;\alpha,\beta,\omega,a,b) &= g_{EPL}(x)\sum_{i,k=0}^{\infty} p_i\left[G_{EPL}(x)\right]^{k}\\ &= \frac{\alpha\beta^2\omega}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\\ &\times \sum_{i,k=0}^{\infty} p_i\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega(k+1)-1}, \end{aligned}$$

where the coefficient p_i is

$$p_i = p_i(a,b) = \frac{(-1)^{i}\binom{b-1}{i}q_k(a+i-1)}{B(a,b)},$$

with

$$q_k = q_k(a+i-1) = \sum_{j=k}^{\infty}\binom{a+i-1}{j}\binom{j}{k}(-1)^{j+k},$$

for $x > 0, \alpha > 0, \beta > 0, \omega > 0, a > 0, b > 0$, respectively. Note that the BEPL density is given by three infinite weighted power series sums of the baseline distribution function $G_{EPL}(x)$. When $b > 0$ is an integer, the index i in the previous series representation stops at $b - 1$.

2.2 Some sub-models of the BEPL distribution

Some sub-models of the BEPL distribution for selected values of the parameters α, β, ω, a and b are presented in this section.

(1) $a = b = 1$
When $a = b = 1$, we obtain the exponentiated power Lindley (EPL) distribution whose cdf and pdf are given in (1.1) and (1.2), (Warahena-Liyanage and Pararai [4]).

(2) $\omega = 1$
When $\omega = 1$, we obtain the beta-power Lindley (BPL) distribution. The BPL cdf is given by

$$F_{BPL}(x;\alpha,\beta,a,b) = \frac{1}{B(a,b)} \int_0^{G_{PL}(x;\alpha,\beta)} t^{a-1}(1-t)^{b-1}dt$$

for $x > 0, \alpha > 0, \beta > 0, a > 0, b > 0$. The corresponding pdf is given by

$$\begin{aligned} f_{BPL}(x;\alpha,\beta,a,b) &= \frac{\alpha\beta^2}{B(a,b)(\beta+1)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \\ &\times \left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{a-1}\left[\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{b-1} \end{aligned}$$

for $x > 0, \alpha > 0, \beta > 0, a > 0, b > 0$.

(3) $\alpha = 1$
When $\alpha = 1$, we obtain beta-exponentiated Lindley (BEL) distribution (Oluyede and Yang [22]). The BEL cdf is given by

$$F_{BEL}(x;\beta,\omega,a,b) = \frac{1}{B(a,b)} \int_0^{G_{EL}(x;\beta,\omega)} t^{a-1}(1-t)^{b-1}dt$$

for $x > 0, \beta > 0, \omega > 0, a > 0, b > 0$. The corresponding pdf is given by

$$\begin{aligned} f_{BEL}(x;\beta,\omega,a,b) &= \frac{\beta^2\omega}{B(a,b)(\beta+1)}(1+x)e^{-\beta x} \\ &\times \left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega a-1} \\ &\times \left\{1-\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega}\right\}^{b-1} \end{aligned}$$

for $x > 0, \beta > 0, \omega > 0, a > 0, b > 0$.

(4) $\omega = \alpha = 1$
When $\omega = \alpha = 1$, we obtain beta-Lindley (BL) distribution (Oluyede and Yang [22]). The BL cdf and pdf are given by

$$F_{BL}(x;\beta,a,b) = \frac{1}{B(a,b)}\int_0^{G_L(x;\beta,\omega)} t^{a-1}(1-t)^{b-1}dt$$

and

$$\begin{aligned} f_{BL}(x;\beta,a,b) &= \frac{\beta^2}{B(a,b)(\beta+1)}(1+x)e^{-\beta x} \\ &\times \left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{a-1}\left[\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{b-1}, \end{aligned}$$

respectively, for $x > 0, \beta > 0, \omega > 0, a > 0, b > 0$.

(5) $\omega = a = b = 1$
When $\omega = a = b = 1$, we obtain the power Lindley (PL) distribution (Ghitany et al. [8]). The PL cdf and pdf are respectively given by

$$F_{PL}(x;\alpha,\beta) = 1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}$$

and

$$f_{PL}(x;\alpha,\beta) = \frac{\beta^2}{(\beta+1)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}$$

for $x > 0, \alpha > 0, \beta > 0$.

(6) $\alpha = a = b = 1$

When $\alpha = a = b = 1$, we obtain exponentiated-Lindley (EL) distribution. The EL cdf is given by

$$F_{EL}(x;\beta,\omega) = \left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega}$$

for $x > 0, \beta > 0, \omega > 0$. The corresponding pdf is given by

$$f_{EL}(x;\beta,\omega) = \frac{\beta^2\omega}{(\beta+1)}(1+x)e^{-\beta x}\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega-1}$$

for $x > 0, \beta > 0, \omega > 0$.

(7) $\alpha = \omega = a = b = 1$

When $\alpha = \omega = a = b = 1$, we obtain Lindley distribution. The Lindley cdf and pdf are respectively given by

$$F_L(x;\beta) = 1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}$$

and

$$f_L(x;\beta) = \frac{\beta^2}{(\beta+1)}(1+x)e^{-\beta x}$$

for $x > 0, \beta > 0$.

(8) $\omega = a = 1$

When $\omega = a = 1$, the cdf of BEPL distribution reduces to

$$F_{BPL}(x;\alpha,\beta,b) = 1 - \left[\left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{b}$$

for $x > 0, \alpha > 0, \beta > 0, b > 0$. The corresponding pdf is

$$f_{BPL}(x;\alpha,\beta,b) = \frac{b\alpha\beta^2}{(\beta+1)}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left[\left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{b-1}$$

for $x > 0, \alpha > 0, \beta > 0, b > 0$.

(9) $\alpha = a = 1$

When $\alpha = a = 1$, the cdf of BEPL distribution reduces to

$$F_{BEL}(x;\alpha,\beta,b) = 1 - \left\{1 - \left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega}\right\}^{b}$$

for $x > 0, \beta > 0, \omega > 0, b > 0$. The corresponding pdf is given by

$$\begin{aligned} f_{BEL}(x;\beta,\omega,b) &= \frac{b\omega\beta^2}{(\beta+1)}(1+x)e^{-\beta x} \\ &\times \left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega-1} \\ &\times \left\{1-\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{\omega}\right\}^{b-1}, \end{aligned}$$

for $x > 0, \beta > 0, \omega > 0, b > 0$. This is the Kumaraswamy Lindley distribution with parameters β, ω and b.

(10) $\alpha = \omega = a = 1$
When $\alpha = \omega = a = 1$, the cdf of BEPL distribution reduces to

$$F_{BL}(x;\beta,b) = 1-\left[\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{b}$$

for $x > 0, \beta > 0, b > 0$. The corresponding pdf is given by

$$f_{BL}(x;\beta,b) = \frac{b\beta^2}{(\beta+1)}(1+x)e^{-\beta x}\left[\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]^{b-1}$$

for $x > 0, \beta > 0, b > 0$.

2.3 Hazard and Reverse Hazard Functions

The hazard and reverse hazard functions of the BEPL distribution are presented in this section. Graphs of these functions for selected values of parameters α, β, ω, a and b are also presented. The hazard and reverse hazard functions of the BEPL distribution are given respectively by

$$\begin{aligned} h_{BEPL}(x;\alpha,\beta,\omega,a,b) &= \frac{f_{BEPL}(x;\alpha,\beta,\omega,a,b)}{\overline{F}_{BEPL}(x;\alpha,\beta,\omega,a,b)} \\ &= \frac{g_{EPL}(x)\left[G_{EPL}(x)\right]^{a-1}\left[1-G_{EPL}(x)\right]^{b-1}}{B(a,b)-B_{G_{EPL}(x)}(a,b)} \end{aligned}$$

and

$$\begin{aligned} \tau_{BEPL}(x;\alpha,\beta,\omega,a,b) &= \frac{f_{BEPL}(x;\alpha,\beta,\omega,a,b)}{F_{BEPL}(x;\alpha,\beta,\omega,a,b)} \\ &= \frac{g_{EPL}(x)\left[G_{EPL}(x)\right]^{a-1}\left[1-G_{EPL}(x)\right]^{b-1}}{B_{G_{EPL}(x)}(a,b)}, \end{aligned}$$

for $x > 0, \alpha > 0, \beta > 0, \omega > 0, a > 0, b > 0$, where $G_{EPL}(x)$ and $g_{EPL}(x)$ are the cdf and pdf of the EPL distribution given by Equations (1.1) and (1.2), respectively. Plots of the hazard function for selected values of parameters α, β, ω, a and b are given in Figures 2.2 and 2.3. The graphs of the hazard function for several combinations of the parameters represent various shapes including monotonically increasing, monotonically decreasing, bathtub and upside down bathtub shapes. This attractive flexibility makes BEPL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

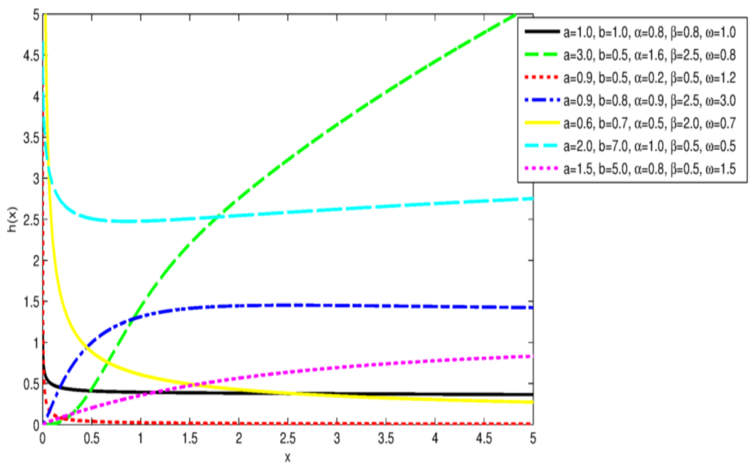

Figure 2.2: Plots of the hazard function for different values of α, β, ω, a and b

2.4 Monotonicity Properties

The monotonicity properties of the BEPL distribution are discussed in this section. Let

$$V(x) = G_{PL}(x;\alpha,\beta) = 1 - \left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right) e^{-\beta x^{\alpha}}.$$

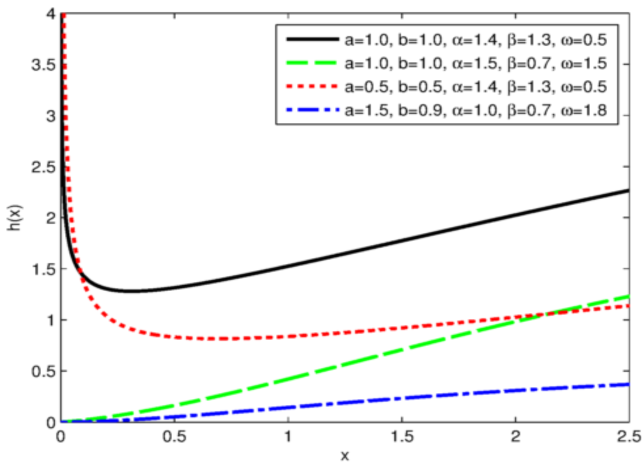

Figure 2.3: Plots of the hazard function for different values of α, β, ω, a and b

From Equation (2.3) we can rewrite the BEPL pdf as

$$f_{BEPL}(x;\alpha,\beta,\omega,a,b) = \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \times [V(x)]^{\omega a-1}[1-V^\omega(x)]^{b-1}$$

for $x > 0, \alpha > 0, \beta > 0, \omega > 0, a > 0, b > 0$. It follows that

$$\log f_{BEPL}(x) = \log\left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right) + \log(1+x^\alpha) + (\alpha-1)\log(x) - \beta x^\alpha + (\omega a-1)\log V(x) + (b-1)\log\left[1-V^\alpha(x)\right], \tag{2.9}$$

and

$$\frac{d\log f_{BEPL}(x)}{dx} = \frac{\alpha x^{\alpha-1}}{1+x^\alpha} + \frac{\alpha-1}{x} - \alpha\beta x^{\alpha-1} + \frac{(\omega a-1)(1-V^\omega(x)) - \omega(b-1)V^\omega(x)}{V(x)\left[1-V^\omega(x)\right]}V'(x). \tag{2.10}$$

Substituting $V'(x) = dV(x)/dx = (\alpha\beta^2/(\beta+1))(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}$ into Equation (2.10), we have

$$\frac{d\log f_{BEPL}(x)}{dx} = \frac{\alpha x^{\alpha-1}}{1+x^\alpha} + \frac{\alpha-1}{x} - \alpha\beta x^{\alpha-1} + \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} \times \left\{\frac{(\omega a-1)(1-V^\omega(x)) - \omega(b-1)V^\omega(x)}{V(x)\left[1-V^\omega(x)\right]}\right\}.$$

Since $\alpha > 0, \beta > 0, \omega > 0, a > 0$ and $b > 0$, we have

$$V'(x) = \frac{dV(x)}{dx} = \frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha} > 0, \forall x > 0. \tag{2.11}$$

If $x \longrightarrow 0$, then

$$V(x) = 1 - \left(1 + \frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha} \longrightarrow 0.$$

If $x \longrightarrow \infty$, then

$$V(x) = 1 - \left(1 + \frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha} \longrightarrow 1.$$

Thus $V(x)$ is monotonically increasing from 0 to 1. Note that, since $0 < V(x) < 1$,
$0 < V^\omega(x) < 1, \forall\omega > 0, 0 < 1 - V^\omega(x) < 1, \forall\omega > 0$ and $V'(x) > 0$, we have $V'(x)/V(x)[1 - V^\omega(x)] > 0$.
If $\alpha \leqslant 1/2, \omega a < 1$ and $b > 1$. we obtain

$$\begin{aligned}\frac{d\log f_{BEPL}(x)}{dx} &= \frac{\alpha x^{\alpha-1}}{1+x^\alpha} + \frac{\alpha-1}{x} - \alpha\beta x^{\alpha-1} \\ &+ \frac{(\omega a-1)(1-V^\omega(x)) - \omega(b-1)V^\omega(x)}{V(x)\left[1-V^\omega(x)\right]}V'(x) < 0\end{aligned} \tag{2.12}$$

since $[\alpha x^{\alpha-1}/(1+x^\alpha)] + [(\alpha-1)/x] = [(2\alpha-1)x^\alpha + (\alpha-1)]/x(1+x^\alpha) < 0$, $(\omega a-1)(1-V^\omega(x)) - \omega(b-1)V^\omega(x) < 0$ and $V'(x)/V(x)[1-V^\omega(x)] > 0$.
In this case, $f_{BEPL}(x;\alpha,\beta,\omega,a,b)$ is monotonically decreasing for all x.
If $\alpha > 1/2$, $f_{BEPL}(x;\alpha,\beta,\omega,a,b)$ could attain a maximum, a minimum or a point of inflection according to whether

$$\frac{d^2\log f_{BEPL}(x)}{dx^2} < 0, \quad \frac{d^2\log f_{BEPL}(x)}{dx^2} > 0 \quad \text{or} \quad \frac{d^2\log f_{BEPL}(x)}{dx^2} = 0.$$

3 Moments, Conditional Moments and Reliability

In this section, moments, conditional moments and reliability and related measures including coefficients of variation, skewness and kurtosis of the BEPL distribution are presented. A table of values for mean, variance, coefficient of skewness (CS) and coefficient of kurtosis (CK) is also presented.

3.1 Moments

The r^{th} moment of the BEPL distribution, denoted by μ_r' is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f_{_{BEPL}}(x)dx \quad \text{for } r = 0, 1, 2, \ldots \quad .$$

In order to find the moments of the BEPL distribution, consider the following lemma.

Lemma 3.1. *Let*

$$L_1(\alpha, \beta, m, r) = \int_0^{\infty} (1 + x^{\alpha})x^{\alpha+r-1} \left[1 - \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) e^{-\beta x^{\alpha}}\right]^{m-1} e^{-\beta x^{\alpha}} dx,$$

then

$$L_1(\alpha, \beta, m, r) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{l=0}^{k+1} \binom{m-1}{j} \binom{j}{k} \binom{k+1}{l} \frac{(-1)^j \beta^k \Gamma(l + r\alpha^{-1} + 1)}{\alpha(\beta + 1)^j \left[\beta(j + 1)\right]^{(l+r\alpha^{-1}+1)}}.$$

Proof. Using the series expansion

$$(1 - z)^{a-1} = \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i z^i, \tag{3.1}$$

where $\mid z \mid < 1$ and $b > 0$ is a real non-integer, we have

$$L_1(\alpha, \beta, m, r) = \sum_{j=0}^{\infty} \binom{m-1}{j} (-1)^j \int_0^{\infty} \left[\frac{1 + \beta(1 + x^{\alpha})}{\beta + 1}\right]^j e^{-j\beta x^{\alpha}} (1 + x^{\alpha})x^{\alpha+r-1} e^{-\beta x^{\alpha}} dx$$

$$= \sum_{j=0}^{\infty} \binom{m-1}{j} \frac{(-1)^j}{(\beta + 1)^j} \sum_{k=0}^{j} \binom{j}{k} \beta^k \int_0^{\infty} (1 + x^{\alpha})^{k+1} x^{\alpha+r-1} e^{(-j\beta x^{\alpha} - \beta x^{\alpha})} dx$$

$$= \sum_{j=0}^{\infty} \binom{m-1}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k+1} \binom{k+1}{l} \frac{(-1)^j \beta^k}{(\beta + 1)^j} \int_0^{\infty} x^{\alpha+\alpha l+r-1} e^{(-j\beta x^{\alpha} - \beta x^{\alpha})} dx.$$

Now consider,

$$\int_0^\infty x^{\alpha+\alpha l+r-1} e^{(-j\beta x^\alpha - \beta x^\alpha)} dx. \tag{3.2}$$

Let $u = \beta(j+1)x^\alpha$, then $\frac{du}{dx} = \alpha\beta(j+1)x^{\alpha-1}$ and $x = \left[\frac{u}{\beta(j+1)}\right]^{1/\alpha}$.
Consequently,

$$L_1(\alpha,\beta,m,r) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}\binom{m-1}{j}\binom{j}{k}\binom{k+1}{l}\frac{(-1)^j\beta^k\Gamma(l+r\alpha^{-1}+1)}{\alpha(\beta+1)^j\left[\beta(j+1)\right]^{(l+r\alpha^{-1}+1)}}.$$

□

Therefore, the rth moment of the BEPL distribution from equation (2.6) is given by

$$\begin{aligned} \mu'_r &= \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i \\ &\times \int_0^\infty x^r(1+x^\alpha)x^{\alpha-1}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega(a+i)-1} e^{-\beta x^\alpha}dx. \end{aligned}$$

Now, using Lemma 3.1 with $m = \omega(a+i)$, we have

$$\mu'_r = \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i L_1(\alpha,\beta,\omega(a+i),r). \tag{3.3}$$

The mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\mu = \mu'_1 = \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i L_1(\alpha,\beta,\omega(a+i),1), \tag{3.4}$$

$$\sigma^2 = \mu'_2 - \mu^2, \tag{3.5}$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2-\mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2}-1}, \tag{3.6}$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2-\mu^2)^{3/2}}, \tag{3.7}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2-\mu^2)^2}, \tag{3.8}$$

respectively.

Table 3.1 lists the first six moments of the BEPL distribution for selected values of the parameters by fixing $\alpha = 1.5, \beta = 1.0$ and $\omega = 1.5$. These values can be determined numerically using R and MATLAB. Algorithms to calculate the pdf moments, reliability, mean deviations, Rényi entropy, maximum likelihood estimators and variance-covariance matrix of the BEPL distribution are provided in the appendix.

Table 3.1: Moments of the BEPL distribution for some parameter values; $\alpha = 1.5$, $\beta = 1.0$ and $\omega = 1.5$

μ'_s	$a = 0.5, b = 1.5$	$a = 1.5, b = 1.5$	$a = 1.5, b = 2.5$	$a = 2.5, b = 1.5$
μ'_1	0.8348214	1.407729	1.129684	1.685774
μ'_2	1.035256	2.304562	1.472533	3.13659
μ'_3	1.608199	4.258207	2.149913	6.366501
μ'_4	2.928858	8.712316	3.450266	13.97437
μ'_5	6.037089	19.4759	6.007677	32.94413
μ'_6	13.77834	47.09621	11.24152	82.9509
Variance	0.33832923	0.322861063	0.19634706	0.294756021
Skewness	0.90987141	0.572390725	0.491885701	0.532003392
Kurtosis	3.760808259	3.40572536	3.236667109	3.434460209

3.2 Conditional Moments

For lifetime models, it is useful to know the conditional moments defined as $E(X^r \mid X > x)$. In order to calculate the conditional moments, we consider the following lemma:

Lemma 3.2. *Let*

$$L_2(\alpha,\beta,m,r,t) = \int_t^{\infty} (1+x^{\alpha})x^{\alpha+r-1}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{m-1} e^{-\beta x^{\alpha}}\,dx.$$

then

$$L_2(\alpha,\beta,m,r,t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}\binom{m-1}{j}\binom{j}{k}\binom{k+1}{l}\frac{(-1)^j\beta^k\Gamma(l+r\alpha^{-1}+1,\beta(j+1)t^{\alpha})}{\alpha(\beta+1)^j\left[\beta(j+1)\right]^{(l+r\alpha^{-1}+1)}},$$

where $\Gamma(a,t)=\int_t^\infty x^{a-1}s^{-x}dx$ *is the upper incomplete gamma function.*

Proof. Using the same procedure that was used in Lemma 3.1, this can be simplified into the following form.

$$L_2(\alpha,\beta,m,r,t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}\binom{m-1}{j}\binom{j}{k}\binom{k+1}{l}\frac{(-1)^j\beta^k}{(\beta+1)^j} \tag{3.9}$$
$$\times \int_t^\infty x^{\alpha+\alpha l+r-1}e^{(-j\beta x^\alpha-\beta x^\alpha)}dx. \tag{3.10}$$

Now consider, $\int_t^\infty x^{\alpha+\alpha l+r-1}e^{(-j\beta x^\alpha-\beta x^\alpha)}dx$, and let $u=\beta(j+1)x^\alpha$, then $\frac{du}{dx}=\alpha\beta(j+1)x^{\alpha-1}$ and $x=\left[\frac{u}{\beta(j+1)}\right]^{1/\alpha}$. The above integral can be rewritten by using the complementary incomplete gamma function $\Gamma(a,t)=\int_t^\infty x^{a-1}e^{-x}dx$. Consequently,

$$L_2(\alpha,\beta,m,r,t)=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k+1}\binom{m-1}{j}\binom{j}{k}\binom{k+1}{l}\frac{(-1)^j\beta^k\Gamma(l+r\alpha^{-1}+1,\beta(j+1)t^\alpha)}{\alpha(\beta+1)^j\left[\beta(j+1)\right]^{(l+r\alpha^{-1}+1)}}.$$

□

Now using Lemma 3.2, the rth conditional moment of the BEPL distribution is given by

$$\begin{aligned} E(X^r|X>x) &= \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i\frac{L_2(\alpha,\beta,\omega(a+i),r,x)}{1-F_{BEPL}(x;\alpha,\beta,\omega,a,b)} \\ &= \frac{\alpha\beta^2\omega}{(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i\frac{L_2(\alpha,\beta,\omega(a+i),r,x)}{B(a,b)-B_{G_{ELP}(x)}(a,b)}. \end{aligned}$$

The mean residual lifetime function is given by $E(X|X>x)-x$. The moment generating function (MGF) of the BEPL distribution is given by

$$M_X(t)=\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\binom{b-1}{i}(-1)^i\frac{t^n}{n!}L_1(\alpha,\beta,\omega(a+i),n). \tag{3.11}$$

3.3 Reliability

We derive the reliability R when X and Y have independent BEPL$(\alpha_1,\beta_1,\omega_1,a_1,b_1)$ and BEPL$(\alpha_2,\beta_2,\omega_2,a_2,b_2)$ distributions, respectively. Note from Equation

(2.2) that the BEPL cdf can be written as:

$$\begin{aligned} F_{BEPL}(x;\alpha,\beta,\omega,a,b) &= \frac{1}{B(a,b)}\int_0^{G_{EPL}(x;\alpha,\beta,\omega)} t^{a-1}(1-t)^{b-1}dt, \\ &= \frac{1}{B(a,b)}\sum_{j=0}^{\infty}\binom{b-1}{j}\frac{(-1)^j}{a+j}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega(a+j)}. \end{aligned} \tag{3.12}$$

Now, from Equations (3.12) and (2.3), we obtain

$$\begin{aligned} R &= P(X>Y) \\ &= \int_0^{\infty} f_X(x;\alpha_1,\beta_1,\omega_1,a_1,b_1)F_Y(x;\alpha_2,\beta_2,\omega_2,a_2,b_2)dx \\ &= \int_0^{\infty}\frac{\alpha_1\beta_1^2\omega_1}{B(a_1,b_1)(\beta_1+1)}(1+x^{\alpha_1})x^{\alpha_1-1}e^{-\beta_1x^{\alpha_1}} \\ &\times \left[1-\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)e^{-\beta_1x^{\alpha_1}}\right]^{\omega_1a_1-1} \\ &\times \left\{1-\left[1-\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)e^{-\beta_1x^{\alpha_1}}\right]^{\omega_1}\right\}^{b_1-1} \\ &\times \frac{1}{B(a_2,b_2)}\sum_{j=0}^{\infty}\binom{b_2-1}{j}\frac{(-1)^j}{a_2+j}\left[1-\left(1+\frac{\beta_2x^{\alpha_2}}{\beta_2+1}\right)e^{-\beta_2x^{\alpha_2}}\right]^{\omega_2(a_2+j)}dx. \end{aligned} \tag{3.13}$$

We apply the following series representations:

$$\begin{aligned} \left[1-\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)e^{-\beta_1x^{\alpha_1}}\right]^{\omega_1a_1-1} &= \sum_{k=0}^{\infty}\binom{\omega_1a_1-1}{k}(-1)^k\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)^k e^{-\beta_1kx^{\alpha_1}} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{\omega_1a_1-1}{k}\binom{k}{m}\frac{(-1)^k\beta_1^m x^{m\alpha_1}}{(\beta_1+1)^m}e^{-\beta_1kx^{\alpha_1}}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \left\{1-\left[1-\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)e^{-\beta_1x^{\alpha_1}}\right]^{\omega_1}\right\}^{b_1-1} &= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\omega_1l}{p}\binom{p}{n} \\ &\times \frac{(-1)^{l+p}\beta_1^n x^{n\alpha_1}}{(\beta_1+1)^n}e^{-\beta_1px^{\alpha_1}} \end{aligned} \tag{3.15}$$

and

$$\left[1-\left(1+\frac{\beta_2 x^{\alpha_2}}{\beta_2+1}\right)e^{-\beta_2 x^{\alpha_2}}\right]^{\omega_2(a_2+j)} = \sum_{q=0}^{\infty}\sum_{t=0}^{q}\binom{\omega_2(a_2+j)}{q}\binom{q}{t}\frac{(-1)^q\beta_2^t x^{t\alpha_2}}{(\beta_2)^t}e^{-\beta_2 q x^{\alpha_2}}. \tag{3.16}$$

By substituting Equations (3.14), (3.15) and (3.16) into Equation (3.13), we obtain

$$\begin{aligned}
R &= \int_0^{\infty}\frac{\alpha_1\beta_1^2\omega_1}{B(a_1,b_1)(\beta_1+1)}(1+x^{\alpha_1})x^{\alpha_1-1}e^{-\beta_1 x^{\alpha_1}} \\
&\times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{\omega_1 a_1-1}{k}\binom{k}{m}\frac{(-1)^k\beta_1^m x^{m\alpha_1}}{(\beta_1+1)^m}e^{-\beta_1 k x^{\alpha_1}} \\
&\times \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\omega_1 l}{p}\binom{p}{n}\frac{(-1)^{l+p}\beta_1^n x^{n\alpha_1}}{(\beta_1+1)^n}e^{-\beta_1 p x^{\alpha_1}} \\
&\times \frac{1}{B(a_2,b_2)}\sum_{j=0}^{\infty}\binom{b_2-1}{j}\frac{(-1)^j}{a_2+j} \\
&\times \sum_{q=0}^{\infty}\sum_{t=0}^{q}\binom{\omega_2(a_2+j)}{q}\binom{q}{t}\frac{(-1)^q\beta_2^t x^{t\alpha_2}}{(\beta_2)^t}e^{-\beta_2 q x^{\alpha_2}}dx \\
&= \frac{\alpha_1\omega_1}{B(a_1,b_1)B(a_2,b_2)}\sum_{k,l,p,j,q=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\sum_{t=0}^{q}\binom{\omega_1 a_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\omega_1 l}{p} \\
&\times \binom{p}{n}\binom{b_2-1}{j}\binom{\omega_2(a_2+j)}{q}\binom{q}{t}\frac{(-1)^{k+l+p+j+q}\beta_1^{m+n+2}\beta_2^t}{(\beta_1+1)^{m+n+1}(\beta_2+1)^t(a_2+j)} \\
&\times \int_0^{\infty}(1+x^{\alpha_1})x^{(m+n+1)\alpha_1+t\alpha_2-1}exp(-\left[\beta_1(1+p+k)x^{\alpha_1}+\beta_2 q x^{\alpha_2}\right])dx.
\end{aligned} \tag{3.17}$$

Note that,

$$\begin{aligned}
&\int_0^{\infty}(1+x^{\alpha_1})x^{(m+n+1)\alpha_1+t\alpha_2-1} \\
&\quad\times e^{-\beta_1(1+p+k)x^{\alpha_1}-\beta_2 q x^{\alpha_2}}dx = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\binom{\alpha_1}{r}(-1)^s\beta_1^s(1+p+k)^s \\
&\qquad\times \int_0^{\infty}x^{(m+n+s+1)\alpha_1+t\alpha_2+r-1}e^{-\beta_2 q x^{\alpha_2}}dx.
\end{aligned} \tag{3.18}$$

Using the definition of gamma function, we have

$$\int_0^\infty x^{(m+n+s+1)\alpha_1+t\alpha_2+r-1}e^{-\beta_2 q x^{\alpha_2}}dx = \frac{\Gamma((m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t)}{\alpha_2(\beta_2 q)^{(m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t}}. \tag{3.19}$$

Substituting Equation (3.19) into Equation (3.18), we obtain

$$\begin{aligned}\int_0^\infty &(1+x^{\alpha_1})x^{(m+n+1)\alpha_1+t\alpha_2-1}\\ \times e^{-\beta_1(1+p+k)x^{\alpha_1}-\beta_2 q x^{\alpha_2}}dx &= \sum_{s=0}^\infty\sum_{r=0}^\infty\binom{\alpha_1}{r}(-1)^s\beta_1^s(1+p+k)^s\\ &\times \frac{\Gamma((m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t)}{\alpha_2(\beta_2 q)^{(m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t}}.\end{aligned} \tag{3.20}$$

Finally, substituting Equation (3.20) into (3.17), we obtain

$$\begin{aligned}R &= \frac{\alpha_1\omega_1}{B(a_1,b_1)B(a_2,b_2)}\sum_{k,l,p,j,q=0}^\infty\sum_{m=0}^k\sum_{n=0}^p\sum_{t=0}^q\binom{\omega_1 a_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\omega_1 l}{p}\\ &\times \binom{p}{n}\binom{b_2-1}{j}\binom{\omega_2(a_2+j)}{q}\binom{q}{t}\frac{(-1)^{k+l+p+j+q}\beta_1^{m+n+2}\beta_2^t}{(\beta_1+1)^{m+n+1}(\beta_2+1)^t(a_2+j)}\\ &\times \sum_{s=0}^\infty\sum_{r=0}^\infty\binom{\alpha_1}{r}(-1)^s\beta_1^s(1+p+k)^s\\ &\times \frac{\Gamma((m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t)}{\alpha_2(\beta_2 q)^{(m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t}}.\\ &= \frac{\alpha_1\omega_1}{B(a_1,b_1)B(a_2,b_2)}\sum_{k,l,p,j,q,s,r=0}^\infty\sum_{m=0}^k\sum_{n=0}^p\sum_{t=0}^q\binom{\omega_1 a_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\omega_1 l}{p}\\ &\times \binom{p}{n}\binom{b_2-1}{j}\binom{\omega_2(a_2+j)}{q}\binom{q}{t}\binom{\alpha_1}{r}\frac{(-1)^{k+l+p+j+q+s}\beta_1^{m+n+s+2}\beta_2^t(1+p+k)^s}{(\beta_1+1)^{m+n+1}(\beta_2+1)^t(a_2+j)}\\ &\times \frac{\Gamma((m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t)}{\alpha_2(\beta_2 q)^{(m+n+s+1)\alpha_1\alpha_2^{-1}+r\alpha_2^{-1}+t}}.\end{aligned}$$

4 Mean Deviations, Bonferroni and Lorenz Curves

In this section, we present the mean deviation about the mean, the mean deviation about the median, Bonferroni and Lorenz curves. Bonferroni and

Lorenz curves are income inequality measures that are also useful and applicable in other areas including reliability, demography, medicine and insurance. The mean deviation about the mean and mean deviation about the median are defined by

$$D(\mu) = \int_0^\infty | x - \mu | f(x)dx \quad and \quad D(M) = \int_0^\infty | x - M | f(x)dx,$$

respectively, where $\mu = E(X)$ and $M = Median(X) = F^{-1}(1/2)$ is the median of F. These measures $D(\mu)$ and $D(M)$ can be calculated using the relationships:

$$D(\mu) = 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty xf(x)dx = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx,$$

and

$$D(M) = -\mu + 2\int_M^\infty xf(x)dx = \mu - 2\int_0^M xf(x)dx.$$

Now using Lemma 3.2, we have

$$D(\mu) = 2\mu F_{BEPL}(\mu) - 2\mu + \frac{2\alpha\beta^2\omega}{B(a,b)(\beta+1)} \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i L_2(\alpha, \beta, \omega(a+i), 1, \mu)$$

and

$$D(M) = -\mu + \frac{2\alpha\beta^2\omega}{B(a,b)(\beta+1)} \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i L_2(\alpha, \beta, \omega(a+i), 1, M).$$

Lorenz and Bonferroni curves are given by

$$L(F_{BEPL}(x)) = \frac{\int_0^x xf_{BEPL}(x)dx}{E(X)}, \quad \text{and} \quad B(F_{BEPL}(x)) = \frac{L(F_{BEPL}(x))}{F_{BEPL}(x)},$$

or

$$L(p) = \frac{1}{\mu}\int_0^q xf_{BEPL}(x)dx, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q xf_{BEPL}(x)dx,$$

respectively, where $q = F_{BEPL}^{-1}(p)$. Using Lemma 3.2, we can re-write Lorenz and Bonferroni curves as

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\int_0^q xf_{BEPL}(x)dx \\ &= \frac{1}{p\mu}\left[\int_0^\infty xf_{BEPL}(x)dx - \int_q^\infty xf_{BEPL}(x)dx\right] \\ &= \frac{1}{p\mu}\left[\mu - \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)} \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i L_2(\alpha, \beta, \omega(a+i), 1, q)\right], \end{aligned}$$

and

$$\begin{aligned} L(p) &= \frac{1}{\mu}\int_0^q x f_{BEPL}(x)dx \\ &= \frac{1}{\mu}\left[\int_0^\infty x f_{BEPL}(x)dx - \int_q^\infty x f_{BEPL}(x)dx\right] \\ &= \frac{1}{\mu}\left[\mu - \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i L_2(\alpha,\beta,\omega(a+i),1,q)\right]. \end{aligned}$$

5 Order Statistics and Measures of Uncertainty

In this section, we present distribution of order statistics, Shannon entropy [23], [24], as well as Rényi entropy [25] for the BEPL distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and is a good measure of randomness or uncertainty.

5.1 Distribution of Order Statistics

Order Statistics play an important role in probability and statistics. In this section, we present the distribution of the order statistics for the BEPL distribution. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. If $X_1, X_2, \ldots, X_n$ is a random sample from BEPL distribution, it follows from Equations (2.2) and (2.3) that the pdf of the k^{th} order statistic, say $Y_k = X_{k:n}$ is given by

$$\begin{aligned} f_k(y_k) &= \frac{n!}{(k-1)!(n-k)!} f_{BEPL}(y_k)\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\left[\frac{B_{G_{EPL}(y_k;\alpha,\beta,\omega)}(a,b)}{B(a,b)}\right]^{k+l-1} \\ &\times \frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}(1+y_k^\alpha)y_k^{\alpha-1}\exp(-\beta y_k^\alpha)\left[V(y_k)\right]^{\omega a-1}\left[1-V^\omega(y_k))\right]^{b-1} \\ &= \frac{\alpha\beta^2\omega n!(1+y_k^\alpha)y_k^{\alpha-1}\exp(-\beta y_k^\alpha)}{(\beta+1)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{m=0}^{b-1}\binom{n-k}{l}\binom{b-1}{m} \\ &\times \frac{(-1)^{l+m}}{(B(a,b))^{k+l-1}}\left(B_{G_{EPL}(y_k;\alpha,\beta,\omega)}(a,b)^{k+l-1}\right)\left[V(y_k)\right]^{\omega(a+m)-1}, \end{aligned}$$

where $V(y_k) = G_{PL}(y_k;\alpha,\beta,\omega) = 1 - \left(1 + \frac{\beta y_k^\alpha}{\beta+1}\right)\exp(-\beta y_k^\alpha)$ and $G_{EPL}(y_k;\alpha,\beta,\omega) = V^\omega(y_k)$. The corresponding cdf of Y_k is

$$\begin{aligned} F_k(y_k) &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[F_{_{BEPL}}(y_k)\right]^{j+l} \\ &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[\frac{B_{G_{EPL}(y_k;\alpha,\beta,\omega)}(a,b)}{B(a,b)}\right]^{j+l} \\ &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}\frac{(-1)^l}{[B(a,b)]^{j+l}}\left[B_{G_{EPL}(y_k;\alpha,\beta,\omega)}(a,b)\right]^{j+l}. \end{aligned}$$

5.2 Shannon Entropy

Shannon entropy [23],[24] is defined by $H\left[f_{BEPL}\right] = E\left[-\log(f_{_{BEPL}}(X;\alpha,\beta,\omega,a,b))\right]$. Thus, we have

$$\begin{aligned} H\left[f_{BEPL}\right] &= \log\left[\frac{B(a,b)(\beta+1)}{\alpha\beta^2\omega}\right] - E\left[\log(1+X^\alpha)\right] \\ &- (\alpha-1)E\left[\log(X)\right] + \beta E\left[X^\alpha\right] \\ &- (\omega a-1)E\left[\log\left\{1-\left(1+\frac{\beta X^\alpha}{1+\beta}\right)e^{-\beta X^\alpha}\right\}\right] \\ &- (b-1)E\left[\log\left\{1-\left[1-\left(1+\frac{\beta X^\alpha}{1+\beta}\right)e^{-\beta X^\alpha}\right]^\omega\right\}\right]. \quad (5.1) \end{aligned}$$

Note that, for $|x|<1$, using the series representation $\log(1+x) = \sum_{q=1}^{\infty}\frac{(-1)^{q+1}x^q}{q}$, we obtain

$$E\left[\log(1+X^\alpha)\right] = -\sum_{q=1}^{\infty}\frac{(-1)^q}{q}E\left[X^{q\alpha}\right], \quad (5.2)$$

$$E\left[\log(X)\right] = -\sum_{p=1}^{\infty}\sum_{s=0}^{p}\binom{p}{s}\frac{(-1)^s}{p}E\left[X^s\right], \quad (5.3)$$

$$\begin{aligned} E\left[\log\left\{1-\left(1+\frac{\beta X^\alpha}{1+\beta}\right)e^{-\beta X^\alpha}\right\}\right] &= -\sum_{t=1}^{\infty}\sum_{u=0}^{t}\binom{t}{u}\frac{\beta^u}{t(\beta+1)^u} \\ &\times E\left[X^{u\alpha}e^{-\beta t X^\alpha}\right] \quad (5.4) \end{aligned}$$

and

$$\begin{aligned} E\left[\log\left\{1-\left[1-\left(1+\frac{\beta X^{\alpha}}{1+\beta}\right)e^{-\beta X^{\alpha}}\right]^{\omega}\right\}\right] &= \sum_{c=1}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{d}\binom{\omega c}{d}\binom{d}{e}\frac{(-1)^{d+1}\beta^{e}}{c(\beta+1)^{e}} \\ &\times\ E\left[X^{e\alpha}e^{-\beta d X^{\alpha}}\right]. \end{aligned} \quad (5.5)$$

By using the results in Lemma 3.1, we can calculate Equations (5.2), (5.3), (5.4) and (5.5).

Now, we obtain Shannon entropy for the BEPL distribution as follows:

$$\begin{aligned} H\left[f_{BEPL}\right] &= \log\left[\frac{B(a,b)(\beta+1)}{\alpha\beta^{2}\omega}\right]+\frac{\alpha\beta^{2}\omega}{B(a,b)(\beta+1)}\sum_{i=1}^{\infty}\binom{b-1}{i}(-1)^{i} \\ &\times\ \left[\sum_{q=1}^{\infty}\frac{(-1)^{q}}{q}L_{1}(\alpha,\beta,\omega(a+i),q\alpha)\right. \\ &+\ (\alpha-1)\sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\binom{p}{s}\frac{(-1)^{s}}{p}L_{1}(\alpha,\beta,\omega(a+i),s) \\ &+\ \beta L_{1}(\alpha,\beta,\omega(a+i),\alpha) \\ &+\ (\omega a-1)\sum_{t=1}^{\infty}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\binom{t}{u}\frac{\beta^{u+v}t^{v-1}(-1)^{v}}{(\beta+1)^{u+1}v!} \\ &\times\ L_{1}(\alpha,\beta,\omega(a+i),\alpha(u+v)) \\ &+\ (b-1)\sum_{c=1}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{d}\sum_{f=0}^{\infty}\binom{\omega c}{d}\binom{d}{e}\frac{\beta^{e+f}d^{f}(-1)^{d+f}}{c(\beta+1)^{e+1}f!} \\ &\times\ \left.L_{1}(\alpha,\beta,\omega(a+i),\alpha(e+f))\right]. \end{aligned} \quad (5.6)$$

5.3 Rényi Entropy

Rényi entropy [25] is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_{R}(v)=\frac{1}{1-v}\log\left(\int_{0}^{\infty}f_{BEPL}^{v}(x;\alpha,\beta,\omega,a,b)dx\right),\quad v\neq 1, v>0. \quad (5.7)$$

Rényi entropy tends to Shannon entropy as $v\rightarrow 1$. Note that by using the series expansion in Equation (3.1), and Equation (2.3), we have

$$\begin{aligned}\int_0^\infty f_{BEPL}^v(x;\alpha,\beta,\omega,a,b)dx &= \left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right)^v \sum_{i,j,p,q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{q}\binom{v}{i}\binom{\omega av-v}{j}\\ &\times \binom{j}{k}\binom{bv-v}{p}\binom{\omega p}{q}\binom{q}{r}\frac{(-1)^{j+p+q}\beta^{k+r}}{(\beta+1)^{k+r}}\\ &\times \int_0^\infty x^{\alpha(i+k+r+v)-v}e^{-\beta(v+j+q)x^\alpha}dx.\end{aligned}$$

Now let $u=\beta(v+j+q)x^\alpha$, then

$$\int_0^\infty x^{\alpha(i+k+r+v)-v}e^{-\beta(v+j+q)x^\alpha}dx = \frac{\Gamma(i+k+r+v-\frac{(v-1)}{\alpha})}{\alpha\left[\beta(v+j+q)\right]^{i+k+r+v-\frac{(v-1)}{\alpha})}}.$$

Consequently, Rényi entropy is given by

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}\log\left[\left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right)^v \sum_{i,j,p,q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{q}\binom{v}{i}\binom{\omega av-v}{j}\binom{j}{k}\right.\\ &\times \binom{bv-v}{p}\binom{\omega p}{q}\binom{q}{r}\frac{(-1)^{j+p+q}\beta^{k+r}}{(\beta+1)^{k+r}}\\ &\times \left.\frac{\Gamma(i+k+r+v-\frac{(v-1)}{\alpha})}{\alpha\left[\beta(v+j+q)\right]^{i+k+r+v-\frac{(v-1)}{\alpha}}}\right], \qquad (5.8)\end{aligned}$$

for $v\neq 1, v>0$.

5.4 s-Entropy

The s-entropy for the BEPL distribution is defined by

$$H_s\left[f_{BEPL}(X;\alpha,\beta,\omega,a,b)\right] = \begin{cases}\frac{1}{s-1}\left[1-\int_0^\infty f_{BEPL}^s(x;\alpha,\beta,\omega,a,b)dx\right] & \text{if } s\neq 1, s>0,\\ E\left[-\log f(X)\right] & \text{if } s=1.\end{cases}$$

Now, using the same procedure that was used to derive Equation (5.8), we have

$$\begin{aligned}\int_0^\infty f_{BEPL}^s(x;\alpha,\beta,\omega,a,b)dx &= \left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right)^s \sum_{i,j,p,q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{q}\binom{s}{i}\\ &\times \binom{\omega a s-s}{j}\binom{j}{k}\binom{bs-s}{p}\binom{\omega p}{q}\binom{q}{r}\\ &\times \frac{(-1)^{j+p+q}\beta^{k+r}}{(\beta+1)^{k+r}}\frac{\Gamma(i+k+r+s-\frac{(s-1)}{\alpha})}{\alpha\left[\beta(s+j+q)\right]^{i+k+r+s-\frac{(s-1)}{\alpha}}}.\end{aligned}$$

Consequently, s-entropy is given by

$$\begin{aligned}H_s\left[f_{BEPL}(X;\alpha,\beta,\omega,a,b)\right] &= \frac{1}{s-1}-\frac{1}{s-1}\left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right)^s\\ &\times \sum_{i,j,p,q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{q}\binom{s}{i}\binom{\omega a s-s}{j}\binom{j}{k}\\ &\times \binom{bs-s}{p}\binom{\omega p}{q}\binom{q}{r}\frac{(-1)^{j+p+q}\beta^{k+r}}{(\beta+1)^{k+r}}\\ &\times \frac{\Gamma(i+k+r+s-\frac{(s-1)}{\alpha})}{\alpha\left[\beta(s+j+q)\right]^{i+k+r+s-\frac{(s-1)}{\alpha}}}\end{aligned}$$

for $s \neq 1, s > 0$.

6 Maximum Likelihood Estimation

In this section, the maximum likelihood estimates of the BEPL parameters α, β, ω, a and b are presented. If $x_1, x_2, \ldots, x_n$ is a random sample from BEPL distribution, the log-likelihood function is given by

$$\begin{aligned}\log L(\alpha,\beta,\omega,a,b) &= n\log\left(\frac{\alpha\beta^2\omega}{B(a,b)(\beta+1)}\right)+\sum_{i=1}^{n}\log(1+x_i^\alpha)\\ &+ (\alpha-1)\sum_{i=1}^{n}\log(x_i)-\beta\sum_{i=1}^{n}x_i^\alpha+(\omega a-1)\sum_{i=1}^{n}\log V(x_i)\\ &+ (b-1)\sum_{i=1}^{n}\log\left[1-V^\omega(x_i)\right],\end{aligned}$$

where $V(x_i) = G_{PL}(x_i;\alpha,\beta) = 1-\left(1+\frac{\beta x_i^\alpha}{\beta+1}\right)\exp(-\beta x_i^\alpha)$. The partial derivatives of $\log L(\alpha,\beta,\omega,a,b)$ with respect to the parameters a, b, α, β and

ω are:

$$\begin{aligned}\frac{\partial \log L(\alpha,\beta,\omega,a,b)}{\partial a} &= n\left[\psi(a+b)-\psi(a)\right]+\omega\sum_{i=1}^{n}\log V(x_i),\\ \frac{\partial \log L(\alpha,\beta,\omega,a,b)}{\partial b} &= n\left[\psi(a+b)-\psi(b)\right]+\sum_{i=1}^{n}\log\left[1-V^{\omega}(x_i)\right],\\ \frac{\partial \log L(\alpha,\beta,\omega,a,b)}{\partial \alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\log(x_i)\left[\frac{x_i^{\alpha}}{1+x_i^{\alpha}}-\beta x_i^{\alpha}+1\right]\\ &- (\omega a-1)\sum_{i=1}^{n}\frac{\partial V(x_i)/\partial\alpha}{V(x_i)}\\ &+ \omega(b-1)\sum_{i=1}^{n}\frac{[V(x_i)]^{\omega-1}\,\partial V(x_i)/\partial\alpha}{1-V^{\omega}(x_i)},\end{aligned}$$

$$\begin{aligned}\frac{\partial \log L(\alpha,\beta,\omega,a,b)}{\partial \beta} &= \frac{n(\beta+2)}{\beta(\beta+1)}-\sum_{i=1}^{n}x_i^{\alpha}-(\omega a-1)\sum_{i=1}^{n}\frac{\partial V(x_i)/\partial\beta}{V(x_i)}\\ &+ \omega(b-1)\sum_{i=1}^{n}\frac{[V(x_i)]^{\omega-1}\,\partial V(x_i)/\partial\beta}{1-V^{\omega}(x_i)}\end{aligned}$$

and

$$\frac{\partial \log L(\alpha,\beta,\omega,a,b)}{\partial \omega} = \frac{n}{\omega}-(b-1)\sum_{i=1}^{n}\frac{V^{\omega}(x_i)\log V(x_i)}{1-V^{\omega}(x_i)}+a\sum_{i=1}^{n}\log V(x_i),$$

respectively, where

$$\frac{\partial V(x_i)}{\partial\alpha} = \frac{\beta^2}{\beta+1}\log(x_i)(1+x_i^{\alpha})x_i^{\alpha}\exp(-\beta x_i^{\alpha})$$

and

$$\frac{\partial V(x_i)}{\partial\beta} = \left[\left(1+\frac{\beta x_i^{\alpha}}{\beta+1}\right)-\frac{1}{(\beta+1)^2}\right]x_i^{\alpha}\exp(-\beta x_i^{\alpha}).$$

When all the parameters are unknown, numerical methods must be applied to determine the estimates of the model parameters since the system of equations is not in closed form. Therefore, the maximum likelihood estimates, $\hat{\Theta}$ of $\Theta=(\alpha,\beta,\omega,a,b)$ can be determined using an iterative method such as the Newton-Raphson procedure.

6.1 Fisher Information Matrix

In this section, we present a measure for the amount of information. This information can be used to obtain bounds on the variance of estimators and as well as approximate the sampling distribution of an estimator obtained from a large sample. Moreover, it can be used to obtain an approximate confidence interval in the case of a large sample.

Let X be a random variable with the BEPL pdf $f_{BEPL}(\cdot;\Theta)$, where $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^T = (\alpha, \beta, \omega, a, b)^T$. Then, Fisher information matrix (FIM) is the 5×5 symmetric matrix with elements:

$$\mathbf{I}_{ij}(\Theta) = E_{\Theta}\left[\frac{\partial \log(f_{BEPL}(X;\Theta))}{\partial \theta_i}\frac{\partial \log(f_{BEPL}(X;\Theta))}{\partial \theta_j}\right].$$

If the density $f_{BEPL}(\cdot;\Theta)$ has a second derivative for all i and j, then an alternative expression for $\mathbf{I}_{ij}(\Theta)$ is

$$\mathbf{I}_{ij}(\Theta) = E_{\Theta}\left[\frac{\partial^2 \log(f_{BEPL}(X;\Theta))}{\partial \theta_i \partial \theta_j}\right].$$

For the BEPL distribution, all second derivatives exist; therefore, the formula above is appropriate and most importantly significantly simplifies the computations. Elements of the FIM can be numerically obtained by MATLAB or MAPLE software. The total FIM $\mathbf{I}_n(\Theta)$ can be approximated by

$$\mathbf{J}_n(\hat{\Theta}) \approx \left[-\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j}\bigg|_{\Theta=\hat{\Theta}} |.\right]_{5\times 5} \tag{6.1}$$

For real data, the matrix given in Equation (6.1) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

6.2 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the BEPL distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\omega}, \hat{a}, \hat{b})$ be the maximum likelihood estimate of $\Theta = (\alpha, \beta, \omega, a, b)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{d} N_5(\underline{0}, \mathbf{I}^{-1}(\Theta))$, where

$\mathbf{I}(\Theta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $\mathbf{I}(\Theta)$ is replaced by the observed information matrix evaluated at $\hat{\Theta}$, that is $\mathbf{J}(\hat{\Theta})$. The multivariate normal distribution with mean vector $\underline{0} = (0,0,0,0,0)^T$ and covariance matrix $\mathbf{I^{-1}}(\Theta)$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for α, β, ω, a and b are given by

$$\hat{\alpha} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\alpha\alpha}(\hat{\Theta})}, \quad \hat{\beta} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\beta\beta}(\hat{\Theta})}, \quad \hat{\omega} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{\omega\omega}(\hat{\Theta})},$$
$$\hat{a} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{aa}(\hat{\Theta})} \text{ and } \hat{b} \pm Z_{\eta/2}\sqrt{\mathbf{I}^{-1}_{bb}(\hat{\Theta})}$$

respectively, where $\mathbf{I}^{-1}_{\alpha\alpha}(\hat{\Theta}), \mathbf{I}^{-1}_{\beta\beta}(\hat{\Theta}), \mathbf{I}^{-1}_{\omega\omega}(\hat{\Theta}), \mathbf{I}^{-1}_{aa}(\hat{\Theta})$ and $\mathbf{I}^{-1}_{bb}(\hat{\Theta})$ are diagonal elements of $\mathbf{I}_n^{-1}(\hat{\Theta}) = (n\mathbf{I}\hat{\Theta}))^{-1}$ and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the BEPL distribution with its sub-models for a given data set. For example, to test $\alpha = \omega = 1$, the LR statistic is $\omega^* = 2[\ln(L(\hat{a}, \hat{b}, \hat{\beta}, \hat{\alpha}, \hat{\omega})) - \ln(L(\tilde{a}, \tilde{b}, \tilde{\beta}, 1, 1))]$, where $\hat{a}$, $\hat{b}$, $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\omega}$ are the unrestricted estimates, and $\tilde{a}$, $\tilde{b}$, and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\delta^* > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

7 Applications

In this section, the BEPL distribution is applied to real data in order to illustrate the usefulness and applicability of the model. We fit the density functions of the beta-exponentiated power Lindley (BEPL), beta exponentiated Lindley (BEL), exponentiated power Lindley (EPL) [4], beta power Lindley (BPL), power Lindley (PL), and Lindley (L) distributions. We provide examples to illustrate the flexibility of the BEPL distribution in contrast to other models including the BEL, BPL, PL, L, beta-Weibull (BW) [26], beta-exponential (BE) [15] and Weibull (W) distributions for data modeling purposes. The pdf of the BW distribution [26] is given by

$$f_{BW}(x; \alpha, \lambda, a, b) = \frac{\alpha\lambda^{\alpha}}{B(a,b)} x^{\alpha-1}\exp(-b(\lambda x)^{\alpha})[1-\exp(-(\lambda x)^{\alpha})]^{a-1},$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$. When $\alpha = 1$, the beta exponential pdf is obtained, [15].

Estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion ($AIC = 2p - 2\log(L)$), Consistent Akaike Information Criterion ($AICC = AIC + \frac{2p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\log(n) - 2\log(L)$), where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are obtained.

The first data set represents the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Alven [27], Chhikara and Folks [28] and Dimitrakopoulou et al. [29]. It consists of the observations listed below: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

The second data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang ([30]). See the table below.

Table 7.1: Cancer Patients Data, Lee and Wang [30]

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52
4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80
25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54
3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32
10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90
2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23	5.41
7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12
1.26	2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46
4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02
2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
21.73	2.07	3.36	6.93	8.65	12.63	22.69	-	-	-	-

The third data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes (Proschan [31]). The data is presented in Table 7.2.

Table 7.2: Air conditioning system data

194	413	90	74	55	23	97	50	359	50	130	487	57
102	15	14	10	57	320	261	51	44	9	254	493	33
18	209	41	58	60	48	56	87	11	102	12	5	14
14	29	37	186	29	104	7	4	72	270	283	7	61
100	61	502	220	120	141	22	603	35	98	54	100	11
181	65	49	12	239	14	18	39	3	12	5	32	9
438	43	134	184	20	386	182	71	80	188	230	152	5
36	79	59	33	246	1	79	3	27	201	84	27	156
21	16	88	130	14	118	44	15	42	106	46	230	26
59	153	104	20	206	5	66	34	29	26	35	5	82
31	118	326	12	54	36	34	18	25	120	31	22	18
216	139	67	310	3	46	210	57	76	14	111	97	62
39	30	7	44	11	63	23	22	23	14	18	13	34
16	18	130	90	163	208	1	24	70	16	101	52	208
95	62	11	191	14	71	-	-	-	-	-	-	-

Estimates of the parameters of BEPL distribution (standard error in parentheses), Akaike Information Criterion, Consistent Akaike Information Criterion and Bayesian Information Criterion are given in Table 7.3 for the active repair time data, in Table 7.4 for the cancer patient data and in Table 7.5 for the air conditioning system data.

Table 7.3: Estimates of Models for Repair Times Data

	Estimates					Statistics			
Model	α	β	ω	a	b	$-2\log L$	AIC	AICC	BIC
BEPL$(\alpha,\beta,\omega,a,b)$	0.08792	7.5983	69.3171	41.0765	2.1953	199.3	209.3	210.8	218.5
	(0.2992)	(15.9158)	(1260.74)	(15.5515)	(6.2258)				
PL$(\alpha,\beta,1,1,1)$	0.7581	0.6757	1	1	1	210.0	214.0	214.3	217.7
	(0.07424)	(0.1016)							
L$(1,\beta,1,1,1)$	1	0.4664	1	1	1	220.0	222.0	222.1	223.8
		(0.0499)							
BL$(1,\beta,1,a,b)$	1	1.6145	1	0.9513	0.2007	212.9	218.9	219.5	224.4
		(0.03449)		(0.2505)	(0.03284)				
BW$(\alpha,\beta,-$, a, b)	0.5408	36.6023	-	41.4065	0.1263	198.0	206.0	207.0	213.3
	(0.1821)	(21.0749)		(3.4654)	(0.1214)				
W$(\alpha,\beta,-$, 1, 1)	0.8986	0.2949	-	1	1	208.9	212.9	213.2	216.6
	(0.09576)	(0.05138)							
BE$(1,\beta,-$, a, b)	1	0.01218	-	0.9322	21.2530	209.9	215.9	216.4	221.3
		(0.003109)		(0.1793)	(1.7710)				

For the repair times data set, the LR statistic for the hypothesis H_0: $PL(\alpha,\beta,1,1,1)$ against H_a: $BEPL(\alpha,\beta,\omega,a,b)$, is $\omega^* = 10.7$. The p-value is $0.01346379 < 0.05$. Therefore, there is a significant difference between PL and BEPL distributions. A LR test of H_0: $L(1,\beta,1,1,1)$ vs $H_a : BEPL(\alpha,\beta,\omega,a,b)$ shows that $\omega^* = 20.7$, and p-value$=0.00036312 < 0.001$. Therefore, there is a significant difference between L and BEPL distributions. There is also a significant difference between PL and L distributions where $\omega^* = 10.0$ with a p-value of $0.00107136 < 0.01$. Moreover, the values of the statistics AIC and AICC are smaller for the BEPL distribution and show that the BEPL distribution is a "better" fit than its sub-models for the repair times data, however a comparison of BEPL and BW distributions shows that the four parameter BW distribution is slightly better.

The asymptotic covariance matrix of MLEs for BEPL model parameters, which is the **FIM** $\mathbf{I}_n^{-1}(\hat{\Theta})$, is given by

$$\begin{pmatrix} 0.08952 & -4.5464 & -370.36 & 4.2174 & -1.6999 \\ -4.5464 & 253.31 & 19939 & -184.68 & 74.4387 \\ -370.36 & 19939 & 1589474 & -15981 & 6439.33 \\ 4.2174 & -184.68 & -15981 & 241.85 & -96.1693 \\ -1.6999 & 74.4387 & 6439.33 & -96.1693 & 38.761 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for α, β, ω, a and b are given by $0.08792 \pm 0.586432, 7.5983 \pm 31.194968, 69.3171 \pm 2471.0504, 41.0765 \pm 30.48094$ and 2.1953 ± 12.202568, respectively. Plots of the fitted densities and the histogram of the repair time data are given in Figure 7.1.

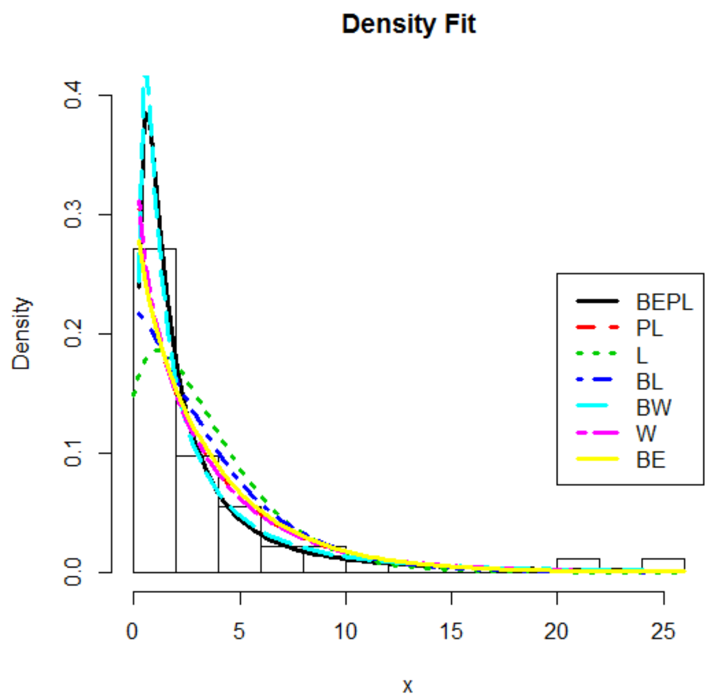

Figure 7.1: Plot of the fitted densities for the Repair Times Data

Table 7.4: Estimates of Models for Cancer Patient Data

	Estimates					Statistics			
Model	α	β	ω	a	b	$-2\log L$	AIC	AICC	BIC
BEPL$(\alpha,\beta,\omega,a,b)$	0.9049	0.3352	34.3398	0.0358	0.3598	818.8736	828.8736	829.3659	843.1337
	(0.2657)	(0.2508)	(0.0039)	(0.0199)	(0.2251)				
BPL$(\alpha,\beta,1,a,b)$	0.60245	0.8686	1	2.5744	0.7605	820.8393	828.8393	829.1645	840.2474
	(0.2299)	(0.4169)	-	(1.5238)	(1.1546)				
PL$(\alpha,\beta,1,1,1)$	0.8302	0.2943	1	1	1	826.7076	830.7076	830.8636	836.4117
	(0.0472)	(0.0370)							
L$(1,\beta,1,1,1)$	1	0.19614	1	1	1	839.0596	841.0596	841.0916	843.9118
		(0.0499)							
BW$(\alpha,\beta,-,$ a, b)	0.6689	0.3304	-	2.7257	0.8808	821.3575	829.3575	829.6827	840.7657
	(0.2368)	(0.4177)		(1.5572)	(1.3743)				
W$(\alpha,\beta,-,$ 1, 1)	1.0479	0.1046	-	1	1	828.1738	832.1738	832.2698	837.8778
	(0.0676)	(0.0093)							

For the cancer patients data, the LR statistics for the test of the hypotheses $H_0 : PL(\alpha,\beta,1,1,1)$ against $H_a : BEPL(\alpha,\beta,\omega,a,b)$ and $H_0 : L(1,\beta,1,1,1)$ against $H_a : BEPL(\alpha,\beta,\omega,a,b)$ are 7.844 ($p - value = 0.04956 < 0.05$) and

20.186 ($p - value = 0.000459 < 0.001$), respectively. Consequently, we reject the null hypothesis in favor of the BEPL distribution and conclude that the BEPL distribution is significantly better than the PL and L distributions. However, there is no significant difference between the BPL and BEPL distributions based on the LR test. Also, based on the values of the statistics: AIC, AICC and BIC, we conclude that the BPL distribution is the better fit for the cancer patient data. The BPL distribution is also slightly better that the BW distribution based on the values of these statistics. Plots of the fitted densities and the histogram for the cancer patient data are given in Figure 7.2.

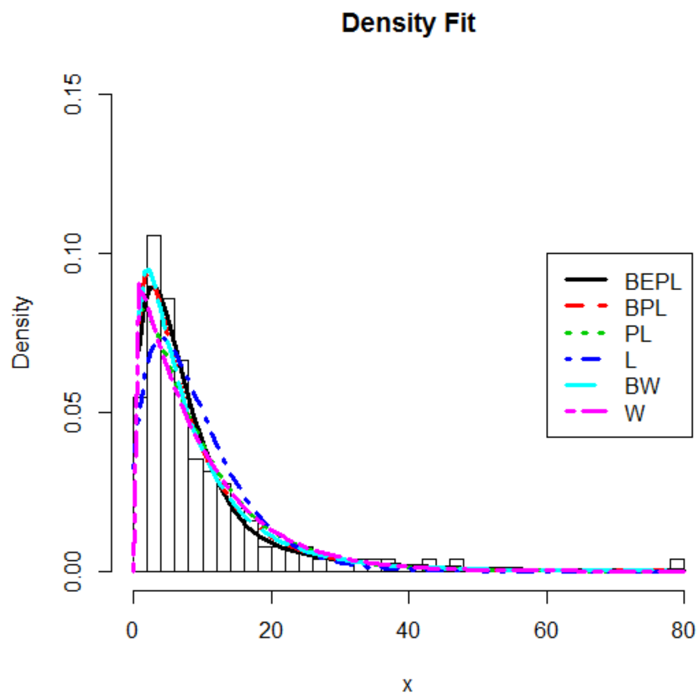

Figure 7.2: Plot of the fitted densities for the Cancer Patients Data

Table 7.5: Estimates of Models for Air Conditioning System Data

	Estimates					Statistics			
Model	α	β	ω	a	b	$-2\log L$	AIC	AICC	BIC
BEPL$(\alpha,\beta,\omega,a,b)$	0.7945	0.1509	6.7278	0.2035	0.2303	2064.1	2074.1	2074.4	2090.2
	(0.2706)	(0.2102)	(3.4546)	(0.2146)	(0.1512)				
BPL$(\alpha,\beta,1,a,b)$	0.4316	0.4867	1	3.1251	0.9630	2066.7	2074.7	2074.9	2087.6
	(0.0573)	(0.1658)		(0.4284)	(0.8737)				
BEL$(1,\beta,\omega,a,b)$	1	0.0453	7.5488	0.1048	0.2034	2064.8	2072.8	2073.0	2085.8
		(0.0194)	(3.9156)	(0.0623)	(0.0649)				
BL$(1,\beta,1,a,b)$	1	0.02343	1	0.4842	0.5378	2080.6	2086.6	2086.7	2096.3
		(0.00972)		(0.0538)	(0.2302)				
PL$(\alpha,\beta,1,1,1)$	0.6609	0.1807	1	1	1	2071.4	2075.4	2075.5	2081.9
	(0.0316)	(0.0165)							
L$(1,\beta,1,1,1)$	1	0.0215	1	1	1	2165.3	2167.3	2167.3	2170.5
		(0.00111)							
BW$(\alpha,\beta,-,$ a, b)	0.7383	0.2719	-	2.7250	0.1188	2064.6	2072.6	2078.8	2085.6
	(0.1114)	(0.7861)		(4.7308)	(0.2421)				
W$(\alpha,\beta,-,$ 1, 1)	0.9109	0.0114	-	1	1	2073.5	2077.5	2077.6	2084.0
	(0.0504)	(0.00097)							
BE$(1,\beta,-,$ a, b)	1	0.00129	-	0.9048	7.6602	2075.2	2081.2	2081.4	2090.9
		(0.000184)		(0.0864)	(0.3651)				

For the air conditioning system data, the LR statistics for the test of the hypotheses $H_0 : BL(1,\beta,1,a,b)$ against $H_a : BEPL(\alpha,\beta,\omega,a,b)$ is 16.5 ($p-value = 0.000263 < 0.001$.) Consequently, we reject the null hypothesis in favor of the BEPL distribution and conclude that the BEPL distribution is significantly better than the BL distribution. The LR test statistics for the test of the hypotheses $H_0 : BL(1,\beta,1,a,b)$ against $H_a : BEL(1,\beta,\omega,a,b)$ is 15.8 ($p-value = 0.000704 < 0.001$), so that the null hypothesis of BL model is rejected in favor of the alternative hypothesis of BEL model. The BPL distribution is also significantly better than the PL and BL models based on the LR test. However, there is no significant difference between the BPL and BEPL distributions, as well as between the BEL and BEPL distributions based on the LR test. The sub-models: BPL and BEL are better fits than the BEPL distribution for the air conditioning system data. Also, the values of the statistics: AIC, AICC and BIC, points to the BEL distribution, so we conclude that the BEL distribution is the better fit for the air conditioning system data. The BEL distribution also compares favorably with the BW distribution based on the values of these statistics. Plots of the fitted densities and the histogram for the air conditioning system data are given in Figure 7.3.

Based on the values of these statistics, we conclude that the BEPL distribution and its sub-models can provide good fits for lifetime data. In the first data set, the BEPL distribution performed better than the BL, PL, L, BE, and Weibull distributions. The four parameter BW distribution was slightly better based on the values of AIC, AICC and BIC. In the second data set, the BPL distribution performed better than the other models including the beta Weibull distribution. In the third data set, the BEL distribution as well as the BPL distribution seem to be the better fits, and the BEL distribution compares favorably with the BW distribution. The BEPL and its sub-models including the BEL and BPL distributions can provide better fits than other common lifetime models.

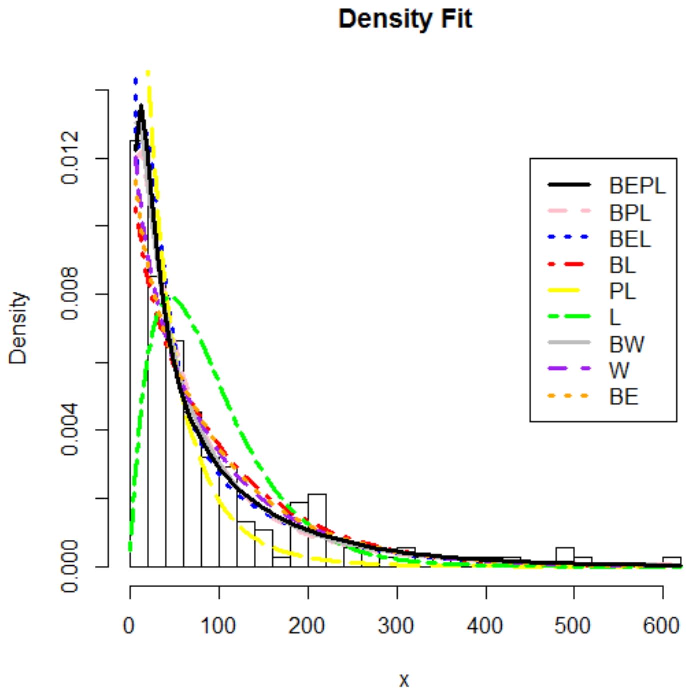

Figure 7.3: Plot of the fitted densities for the Air Conditioning System Data

8 Concluding Remarks

We have developed and presented the mathematical properties of a new class of distributions called the beta-exponentiated power Lindley (BEPL) distribution including the hazard and reverse hazard functions, monotonicity properties, moments, conditional moments, reliability, entropies, mean deviations, Lorenz and Bonferroni curves, distribution of order statistics, and maximum likelihood estimates. Applications of the proposed model to real data in order to demonstrate the usefulness of the distribution are also presented.

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Appendices

R Algorithms

```
#Define the pdf of BEPL
f1=function(x,alpha,beta,omega,a,b){y=(alpha*beta^2*omega*(1+x ^alpha)
*x ^(alpha-1)*exp(-beta*x^alpha) *(1-(1+beta*x ^alpha /(1+beta))
*exp(-beta *x ^alpha)) ^(omega*a-1)) *(1-(1-(1+beta*x ^alpha /(1+beta))
*exp(-beta *x ^alpha))^omega) ^(b-1) /(beta(a,b)*(beta+1))
return(y)
}
#Define the cdf of BEPL
F1=function(x,alpha,beta,omega,a,b){
y=pbeta((1-(1+beta*x^alpha/(1+beta))*exp(-beta*x^alpha))^omega,a,b)
return(y)
}
#Define the moments of BEPL
moment=function(alpha,beta,omega,a,b,r){
f=function(x,alpha,beta,omega,a,b,r)
{(x^r)*(f1(x,alpha,beta,omega,a,b))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,alpha=alpha,beta=beta,omega=omega,a=a,b=b,r=r)
return(y)
}
```

```
#Define the reliability of BEPL
reliability=function(alpha1,beta1,omega1,a1,b1,alpha2,
beta2,omega2,a2,b2){
f=function(x,alpha1,beta1,omega1,a1,b1,alpha2,beta2,omega2,a2,b2)
{f1(x,alpha1,beta1,omega1,a1,b1)*(F1(x,alpha2,beta2,omega2,a2,b2))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100,alpha1=alpha1,
beta1=beta1,omega1=omega1,a1=a1,b1=b1,alpha2=alpha2,
beta2=beta2,omega2=omega2,a2=a2,b2=b2)
return(y)
}
#Define Mean Deviation about the mean of BEPL
delta1=function(alpha,beta,omega,a,b){
mu=moment(alpha,beta,omega,a,b,1)$ value
f=function(x,alpha,beta,omega,a,b){(abs(x-mu)*f1(x.alpha,beta,omega,a,b)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,alpha=alpha,beta=beta,omega=omega,a=a,b=b)
return(y)
}
#Define Mean Deviation about the median of BEPL
delta2=function(alpha,beta,omega,a,b){
M=median(c(X)) #X is the data set
f=function(x,alpha,beta,omega,a,b){(abs(x-M)*f1(x.alpha,beta,omega,a,b)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,alpha=alpha,beta=beta,omega=omega,a=a,b=b)
return(y)
}
```

Define the Renyi entropy of BEPL

```
t=function(alpha,beta,omega,a,b,gamma){
f=function(x,alpha,beta,omega,a,b,gamma)
{(f1(x,alpha,beta,omega,a,b))^(gamma)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,alpha=alpha,beta=beta,omega=omega,a=a,b=b,gamma=gamma)$ value
return(y)
}
Renyi=function(alpha,beta,omega,a,b,gamma){
y=log(t(alpha,beta,omega,a,b,gamma))/(1-gamma)
return(y)
}
#Calculate the maximum likelihood estimators and
variance-covariance matrx of the BEPL
library('bbmle');
xvec<-c(X) #X is the data set
fn1<-function(alpha,beta,omega,a,b){
-sum(log(alpha*beta^2*omega/(beta(a,b)*(beta+1)))+log(1+xvec^alpha)
+(alpha-1)*log(xvec)-beta*xvec^alpha+(omega*a-1)
*log(1-(1+beta*xvec^alpha/(beta+1))
*exp(-beta*xvec^alpha))+(b-1)*log(1-(1-(1+beta*xvec^alpha/(beta+1))
*exp(-beta*xvec^alpha))^omega))
}
mle.results1<-mle2(fn1,start=list(alpha=alpha,beta=beta,
omega=omega,a=a,b=b),hessian.opt=TRUE)
summary(mle.results1)
vcov(mle.results1)
```