Interpolation Theorems for Self-Adjoint Operators

Shijun Zheng
Georgia Southern University, szheng@georgiasouthern.edu

Follow this and additional works at: https://digitalcommons.georgiasouthern.edu/math-sci-facpubs

Part of the Education Commons, and the Mathematics Commons

Recommended Citation
https://digitalcommons.georgiasouthern.edu/math-sci-facpubs/260

This article is brought to you for free and open access by the Department of Mathematical Sciences at Georgia Southern Commons. It has been accepted for inclusion in Department of Mathematical Sciences Faculty Publications by an authorized administrator of Georgia Southern Commons. For more information, please contact digitalcommons@georgiasouthern.edu.
INTERPOLATION THEOREMS FOR SELF-ADJOINT OPERATORS

SHIJUN ZHENG

Abstract. We prove a complex and a real interpolation theorems on Besov spaces and Triebel-Lizorkin spaces associated with a self-adjoint operator \( L \), without assuming the gradient estimate for its spectral kernel. The result applies to the cases where \( L \) is a uniformly elliptic operator or a Schrödinger operator with electromagnetic potential.

1. Introduction and main result

Interpolation of function spaces has played an important role in classical Fourier analysis and PDEs \([1, 6, 12, 15, 18]\). Let \( L \) be a selfadjoint operator in \( L^2(\mathbb{R}^n) \). Then, for a Borel measurable function \( \phi: \mathbb{R} \rightarrow \mathbb{C} \), we define \( \phi(L) \) using functional calculus. In \([15, 11, 2, 17]\) several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with Schrödinger operators. In this note we present an interpolation result on these spaces for \( L \).

Let \( \{ \varphi_j \}_{j=0}^{\infty} \subset C_{0}^{\infty}(\mathbb{R}) \) be a dyadic system satisfying (i) \( \text{supp } \varphi_0 \subset \{ x : |x| \leq 1 \} \), \( \text{supp } \varphi_j \subset \{ x : 2^{j-2} \leq |x| \leq 2^j \}, j \geq 1 \), (ii) \( |\varphi_j^{(k)}(x)| \leq c_k 2^{-kj} \) for all \( j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), (iii) \( \sum_{j=0}^{\infty} |\varphi_j(x)| \approx 1, \ \forall x \). Let \( \alpha \in \mathbb{R}, 1 \leq p, q \leq \infty \). The inhomogeneous Besov space associated with \( L \), denoted by \( B^\alpha_{p,q}(L) \), is defined to be the completion of \( \mathcal{S}(\mathbb{R}^n) \), the Schwartz class, with respect to the norm

\[
\|f\|_{B^\alpha_{p,q}(L)} = \left( \sum_{j=0}^{\infty} 2^{j\alpha q} \|\varphi_j(L)f\|_{L^p}^q \right)^{1/q}.
\]

Date: February 20, 2013.


Key words and phrases. interpolation, functional calculus.
Similarly, the inhomogeneous Triebel-Lizorkin space associated with \( \mathcal{L} \), denoted by \( F^{s,q}_p(\mathcal{L}) \), is defined by the norm
\[
\|f\|_{F^{s,q}_p(\mathcal{L})} = \|\left( \sum_{j=0}^{\infty} 2^{js} |\varphi_j(\mathcal{L}) f|^q \right)^{1/q}\|_{L^p}.
\]

The following assumption on the kernel of \( \phi_j(\mathcal{L}) \) is fundamental in the study of function space theory. Let \( \phi(\mathcal{L})(x,y) \) denote the integral kernel of \( \phi(\mathcal{L}) \).

**Assumption 1.1.** Let \( \phi_j \in C_0^\infty(\mathbb{R}) \) satisfy conditions (i), (ii) above. Assume that there exist some \( \varepsilon > 0 \) and a constant \( c_n > 0 \) such that for all \( j \)
\[
|\phi_j(\mathcal{L})(x,y)| \leq c_n \frac{2^{nj/2}}{(1 + 2^{j/2}|x - y|^{n+\varepsilon})}.
\]

This is the same condition assumed in \[28, 18\] except that we drop the gradient estimate condition on the kernel. This is the case when \( \mathcal{L} \) is a Schrödinger operator \(-\Delta + V, V \geq 0\) belonging to \( L_{loc}^1(\mathbb{R}^n) \) \[14, 19\] or \( \mathcal{L} \) is a uniformly elliptic operator in \( L^2(\mathbb{R}^n) \) \[8, Theorem 3.4.10\].

In what follows, \([A,B]_\theta\) denotes the usual complex interpolation between two Banach spaces; \((A,B)_\theta,r\) the real interpolation, see Section 2. The notion \( T: X \to Y \) means that the linear operator \( T \) is bounded from \( X \) to \( Y \).

**Theorem 1.2 (complex interpolation).** Suppose that \( \mathcal{L} \) is a selfadjoint operator satisfying Assumption \[14\]. Let \( 0 < \theta < 1, s = (1 - \theta)s_0 + \theta s_1, s_0, s_1 \in \mathbb{R} \) and
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]

(a) If \( 1 < p_i < \infty, 1 < q_i < \infty, i = 0, 1, \) then
\[
[F^{s_0,q_0}_{p_0}(\mathcal{L}), F^{s_1,q_1}_{p_1}(\mathcal{L})]_\theta = F^{s,q}_{p}(\mathcal{L}).
\]

(b) If \( 1 \leq p_i \leq \infty, 1 \leq q_i \leq \infty, i = 0, 1, \) then
\[
[B^{s_0,q_0}_{p_0}(\mathcal{L}), B^{s_1,q_1}_{p_1}(\mathcal{L})]_\theta = B^{s,q}_{p}(\mathcal{L}).
\]

(c) If \( T: F^{s_0,q_0}_{p_0}(\mathcal{L}) \to F^{s_0,q_0}_{\tilde{p}_0}(\mathcal{L}) \) and \( T: F^{s_1,q_1}_{p_1}(\mathcal{L}) \to F^{s_1,q_1}_{\tilde{p}_1}(\mathcal{L}) \), then \( T: F^{s,q}_{p}(\mathcal{L}) \to F^{s,q}_{\tilde{p}}(\mathcal{L}) \), where \( \tilde{s}, \tilde{p}, \tilde{q} \) and \( \bar{s_i}, \bar{p_i}, \bar{q_i} \), satisfy the same relations as those for \( s, p, q \) and \( s_i, p_i, q_i \), \( 1 < p_i, q_i < \infty \). Similar statement holds for \( B^{s,q}_{p}(\mathcal{L}) \).

Complex interpolation method originally was due to Calderón \[4\] and Lions and Peetre \[16\]; see also \[13, 24\]. The classical interpolation theory for Besov and Triebel-Lizorkin spaces on \( \mathbb{R}^n \) has been given
systematic treatments in [20], [3], and [25, 26]. There are interesting discussions on interpolation theory in [20] and [26, 25, 22] for generalized Besov spaces associated with differential operators, which requires certain Riesz summability for $\mathcal{L}$ that seems a nontrivial condition to verify. Nevertheless, we would like to mention that the Riesz summability, the spectral multiplier theorem and the decay estimate in (1) are actually intimately related [10, 18].

The real interpolation result for $B^{\alpha,q}_p(\mathbb{R}^n)$, $F^{\alpha,q}_p(\mathbb{R}^n)$ can be found in [20] and [26, 25]. Following the proof as in the classical case, but applying the estimate in (1) instead of spectral multiplier result, we obtain

**Theorem 1.3** (real interpolation). Suppose that $\mathcal{L}$ satisfies Assumption 1.1. Let $0 < \theta < 1$, $1 \leq r \leq \infty$, $s = (1 - \theta)s_0 + \theta s_1$, $s_0 \neq s_1$.

(a) If $1 \leq p < \infty$, $1 \leq q_1, q_2 \leq \infty$, then
$$ (F^{s_0, q_0}_p(\mathcal{L}), F^{s_1, q_1}_p(\mathcal{L}))_{\theta, r} = B^{s, r}_p(\mathcal{L}). $$

(b) If $1 \leq p, q_1, q_2 \leq \infty$, then
$$ (B^{s_0, q_0}_p(\mathcal{L}), B^{s_1, q_1}_p(\mathcal{L}))_{\theta, r} = B^{s, r}_p(\mathcal{L}). $$

The homogeneous spaces $\dot{B}^{\alpha,q}_p(\mathcal{L})$ and $\dot{F}^{\alpha,q}_p(\mathcal{L})$ can be defined using $\{\varphi_j\}_{j=-\infty}^{\infty}$ in (i) to (iii), instead of $\{\varphi_j\}_{j=0}^{\infty}$. Then the analogous results of Theorem 1.2 and Theorem 1.3 hold.

2. Interpolation for $\mathcal{L}$

Theorem 1.2 and Theorem 1.3 are part of the abstract interpolation theory for $\mathcal{L}$. In this section we present the outline of their proofs. It was mentioned in [22] that the interpolation associated with $\mathcal{L}$ is a “subtle and difficult” subject, which normally relies on the very property of $\mathcal{L}$.

2.1. Complex interpolation. The proof of Theorem 1.2 is similar to that given in [25] in the Fourier case. The insight is that the three line theorem (involved in Riesz-Thorin or Calderón’s constructive proof for $L^p$ spaces) reflects the fact that the value of an analytic function in the interior of a domain is determined by its boundary values.

**Definition 2.2.** Let $(A_0, A_1)$ be an interpolation couple, i.e., $A_0, A_1$ are (complex) Banach spaces, linearly and continuously embedded in a Hausdorff space $\mathcal{H}$. The space $A_0 \cap A_1$ is endowed with the norm $\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_j}, j = 0, 1\}$. The space $A := A_0 + A_1$ is endowed with the norm $\|a\|_A = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1\}$. 
Let \( S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\} \) and \( \bar{S} \) its closure. Denote \( F \) the class of all \( A \)-valued functions \( f(z) \) on \( S \) such that \( z \mapsto f(z) \in A \) is analytic in \( S \) and continuous on \( \bar{S} \), satisfying

(i) \[ \sup_{z \in \bar{S}} \| f(z) \|_A \text{ is finite.} \]

(ii) The mapping \( t \mapsto f(j + it) \) \( \in A_j \) are continuous from \( \mathbb{R} \) to \( A_j \), \( j = 0, 1 \).

Then \( F \) is a Banach space with the norm \( \| f \|_F = \max_j \{ \sup_t \| f(j + it) \|_{A_j} \} \).

For \( 0 < \theta < 1 \) we define the interpolation space \( [A_0, A_1]_\theta \) as

\[ [A_0, A_1]_\theta := \{ a \in A : \exists f \in F \text{ with } f(\theta) = a \}. \]

Then \( [A_0, A_1]_\theta \) is a Banach space equipped with the norm \( \| a \|_\theta := \inf \{ \| f \|_F : f \in F \text{ and } f(\theta) = a \} \).

### 2.3. Outline of the proof of Theorem 1.2.

Let \( \{ \phi_j \}, \{ \psi_j \} \) satisfy the conditions in (i)-(iii) and \( \sum j \psi_j(x)\phi_j(x) = 1 \). Define the operators

- \( S : f \mapsto \{ \phi_j(\mathcal{L})f \} \), and
- \( R : g \mapsto \sum j \psi_j(\mathcal{L})g \).

The proof for part (a) follows from the commutative diagram

\[
\begin{array}{ccc}
F_{p,q}^{*} & \xrightarrow{S} & L^p(\ell^q) \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
F_{p,q}^{*} & \leftarrow R & L^p(\ell^q)
\end{array}
\]

and Lemma 2.4 and Lemma 2.5 which are interpolation results for Banach space valued \( L^p \) and \( \ell^q \) spaces [25].

**Lemma 2.4.** Let \( 0 < \theta < 1, 1 \leq p_0, p_1 < \infty \) and \( p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1} \). Let \( A_0, A_1 \) be Banach spaces. Then

\[
[L^{p_0}(A_0), L^{p_1}(A_1)]_\theta = L^p([A_0, A_1]_\theta).
\]

If \( p_1 = \infty \), then (2) holds with \( L^{p_1}(A_1) \) replaced by \( L^{\infty}_0(A_1) \), the completion of simple \( A_1 \)-valued functions with the esssup norm.

As in [25], denote \( \ell^q(A_j) \) the space of functions consisting of \( a = \{ a_j \} \), \( a_j \in A_j \) \( (A_j \text{ being Banach spaces}) \) equipped with the norm

\[
\| a \|_{\ell^q(A_j)} = \left( \sum_j \| a_j \|_{A_j}^q \right)^{1/q}.
\]
Lemma 2.5. Let $0 < \theta < 1$, $1 \leq q_0, q_1 < \infty$ and $q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$. Let $A_j$ be Banach spaces, $j \in \mathbb{N}$. Then

$$\|A_j\|_{\ell^q(B_j)} = \|\cdot\|_{\ell^q(|A_j, B_j|)}.$$  

If $q_1 = \infty$, then

$$\|A_j\|_{\ell^q(B_j)} = \|\cdot\|_{\ell^q(|A_j, B_j|)}.$$  

where $\ell^\infty(B_j) := \{\{c_j\} \in \ell^\infty(B_j) : \|c_j\|_{B_j} \to 0 \text{ as } j \to \infty\}.$

If $1 \leq q_0, q_1 < \infty$, (3) also follows from Lemma 2.4 as a special case where the underlying measure space can be taken as $(X, \mu) = \mathbb{Z}$. If $q_1 = \infty$, then the remark in [25, Subsection 1.18.1] shows that the second statement in Lemma 2.5 is also true.

In the diagram above in order to show $S, R$ are continuous mappings, we need the following well-known lemma.

Lemma 2.6. Let $h(x)$ be a monotonely nonincreasing, radial function in $L^1(\mathbb{R}^n)$. Let $h_j(x) = 2^{jn/2}h(2^{j/2}x)$ be its scaling. Then for all $f$ in $L^1_{loc}(\mathbb{R}^n)$

$$\left|\int h_j(x-y)f(y)dy\right| \leq c_n\|h\|_1M f(x),$$

where $Mf$ denotes the usual Hardy-Littlewood maximal function.

Evidently the decay estimate in [1] and Lemma 2.6 imply the continuity of $S$ and $R$, in light of the $L^p(\ell^q)$-valued maximal inequality.

The proof for $B^s_{p,q}(\mathcal{L})$ in part (b) proceeds in a similar way.

2.7. Real interpolation. Peetre’s $K$-functional [21] is defined as

$$K(t, a) := K(t, a; A_0, A_1) = \inf(\|a_0\|_{A_0} + t\|a_1\|_{A_1}),$$

where the infimum is taken over all representations of $a = a_0 + a_1$, $a_i \in A_i$. Let $0 < q \leq \infty, 0 < \theta < 1$. For a given interpolation couple $(A_0, A_1)$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is given by

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : \|a\|_{(A_0, A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta q}K(t, a)^q\frac{dt}{t}\right)^{1/q} < \infty\}$$

with usual modifications if $q = \infty$.

Proof of Theorem 1.3 is similar to [26, Subsection 2.4.2] and [3, Theorem 6.4.5]. Define $\ell^s_{\theta,q}(A) = \{a = \{a_j\} : \|a\|_{\ell^s_{\theta,q}(A)} = \|\{2^j a_j\|_{A_j}\|_{\ell^q} < \infty\}$. For Besov spaces it follows from

$$(\ell^{s_0}_{\theta,q}(A_0), \ell^{s_1}_{\theta,q}(A_1))_{\theta,q} = \ell^{s}_{\theta,q}((A_0, A_1)_{\theta,q}),$$

$s = (1 - \theta)s_0 + \theta s_1$, $q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$ and the commutative diagram for $B^s_{p,q}(\mathcal{L})$. Consult [26, 25] or [20, Chapter 5, Theorem 6]; both of their proofs rely on retraction method. Also see [3] for a different

For the $F$-space the proof follows from the commutative diagram for $F^{s,q}(\mathcal{L})$ and

$$(L^{pq}(A_0, w_0), L^{p_1}(A_1, w_1))_{\theta, p} = L^p((A_0, A_1)_{\theta, p}, w),$$

where $p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$, $w = w_0^{1-\theta}w_1^\theta$, $w_0, w_1$ being two weight functions [20, Chapter 5].

2.8. Schrödinger operators with magnetic potential. From [14], [28] or [18] we know that if the heat kernel of $\mathcal{L}$ satisfies the upper Gaussian bound

$$(5) \quad |e^{-t\mathcal{L}}(x, y)| \leq c_n t^{-n/2} e^{-c|x-y|^2/t}$$

then the kernel decay in Assumption 1.1 holds. Let

$$H = -\sum_{j=1}^n (\partial_{x_j} + ia_j)^2 + V,$$

where $a_j(x) \in L^2_{loc}(\mathbb{R}^n)$ is real-valued, $V = V_+ - V_-$ with $V_+ \in L^1_{loc}(\mathbb{R}^n)$, $V_- \in K_n$, the Kato class [23]. Proposition 5.1 in [7] showed that (5) is valid for $-\Delta + V$ if $V_+ \in K_n$ and $\|V_-\|_{K_n} < \gamma_n := \pi^{n/2}/\Gamma(n/2 - 1)$, $n \geq 3$, whose proof evidently works for $V_+ \in L^1_{loc}$. By the diamagnetic inequality [23, Theorem B.13.2], we see that (5) also holds for $H$ provided $\|V_-\|_{K_n} < \gamma_n$, $n \geq 3$.

As another example, a uniformly elliptic operator is given by

$$\mathcal{L} = -\sum_{j,k=1}^n \partial_{x_j}(a_{jk}\partial_{x_k}),$$

where $a_{jk}(x) = a_{kj}(x) \in L^\infty(\mathbb{R}^n)$ are real-valued and satisfy the ellipticity condition $(a_{jk}) \approx I_n$. Then [19, Theorem 1] tells that (5) is true provided that the infimum of its spectrum $\inf \sigma(\mathcal{L}) = 0$.

References


DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460-8093
E-mail address: szheng@georgiasouthern.edu
URL: http://math.georgiasouthern.edu/~szheng