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Shijun Zheng
Georgia Southern University, szheng@georgiasouthern.edu

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INTERPOLATION THEOREMS FOR SELF-ADJOINT OPERATORS

SHIJUN ZHENG

Abstract. We prove a complex and a real interpolation theorems on Besov spaces and Triebel-Lizorkin spaces associated with a self-adjoint operator \( L \), without assuming the gradient estimate for its spectral kernel. The result applies to the cases where \( L \) is a uniformly elliptic operator or a Schrödinger operator with electromagnetic potential.

1. Introduction and main result

Interpolation of function spaces has played an important role in classical Fourier analysis and PDEs \([1, 6, 12, 15, 18]\). Let \( L \) be a selfadjoint operator in \( L^2(\mathbb{R}^n) \). Then, for a Borel measurable function \( \phi: \mathbb{R} \to \mathbb{C} \), we define \( \phi(L) \) using functional calculus. In \([15, 11, 2, 17]\) several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with Schrödinger operators. In this note we present an interpolation result on these spaces for \( L \).

Let \( \{\varphi_j\}_{j=0}^{\infty} \subset C_0^\infty(\mathbb{R}) \) be a dyadic system satisfying (i) \( \text{supp} \varphi_0 \subset \{x: |x| \leq 1\} \), \( \text{supp} \varphi_j \subset \{x: 2^{j-2} \leq |x| \leq 2^j\}, j \geq 1 \), (ii) \( |\varphi_j^{(k)}(x)| \leq c_k 2^{-kj} \) for all \( j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), (iii) \( \sum_{j=0}^{\infty} |\varphi_j(x)| \approx 1, \ \forall x \). Let \( \alpha \in \mathbb{R}, 1 \leq p, q \leq \infty \). The inhomogeneous Besov space associated with \( L \), denoted by \( B_\alpha^{p,q}(L) \), is defined to be the completion of \( S(\mathbb{R}^n) \), the Schwartz class, with respect to the norm

\[
\|f\|_{B_\alpha^{p,q}(L)} = \left( \sum_{j=0}^{\infty} 2^{j\alpha q} \|\varphi_j(L)f\|_{L_p}^q \right)^{1/q}.
\]

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Similarly, the inhomogeneous Triebel-Lizorkin space associated with \( \mathcal{L} \), denoted by \( F^s_p,q(\mathcal{L}) \), is defined by the norm
\[
\|f\|_{F^s_p,q(\mathcal{L})} = \left( \sum_{j=0}^{\infty} 2^{j\alpha q} |\varphi_j(\mathcal{L}) f|^q \right)^{1/q} \|L^p\).
\]

The following assumption on the kernel of \( \phi_j(\mathcal{L}) \) is fundamental in the study of function space theory. Let \( \phi(\mathcal{L})(x,y) \) denote the integral kernel of \( \phi(\mathcal{L}) \).

**Assumption 1.1.** Let \( \phi_j \in C^\infty_0(\mathbb{R}) \) satisfy conditions (i), (ii) above. Assume that there exist some \( \varepsilon > 0 \) and a constant \( c_n > 0 \) such that for all \( j \)
\[
|\phi_j(\mathcal{L})(x,y)| \leq c_n \frac{2^{n\varepsilon/2}}{(1 + 2^{j\varepsilon}|x-y|)^{n+\varepsilon}}.
\]

This is the same condition assumed in [28, 18] except that we drop the gradient estimate condition on the kernel. This is the case when \( \mathcal{L} \) is a Schrödinger operator \( -\Delta + V, V \geq 0 \) belonging to \( L^1_{\text{loc}}(\mathbb{R}^n) \) [11, 10] or \( \mathcal{L} \) is a uniformly elliptic operator in \( L^2(\mathbb{R}^n) \) [8, Theorem 3.4.10].

In what follows, \( [A, B]_\theta \) denotes the usual complex interpolation between two Banach spaces; \( (A, B)_{\theta,r} \) the real interpolation, see Section 2. The notion \( T: X \to Y \) means that the linear operator \( T \) is bounded from \( X \) to \( Y \).

**Theorem 1.2** (complex interpolation). Suppose that \( \mathcal{L} \) is a selfadjoint operator satisfying Assumption [11]. Let \( 0 < \theta < 1 \), \( s = (1-\theta)s_0 + \theta s_1 \), \( s_0, s_1 \in \mathbb{R} \) and
\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.
\]

(a) If \( 1 < p_i < \infty \), \( 1 < q_i < \infty \), \( i = 0, 1 \), then
\[
[F^{s_0,\theta q_0}(\mathcal{L}), F^{s_1,\theta q_1}(\mathcal{L})]_\theta = F^{s,q}(\mathcal{L}).
\]

(b) If \( 1 \leq p_i \leq \infty \), \( 1 \leq q_i \leq \infty \), \( i = 0, 1 \), then
\[
[B^{s_0,\theta q_0}(\mathcal{L}), B^{s_1,\theta q_1}(\mathcal{L})]_\theta = B^{s,q}(\mathcal{L}).
\]

c) If \( T: F^{s_0,\theta q_0}(\mathcal{L}) \to F^{s_0,\theta q_0}(\mathcal{L}) \) and \( T: F^{s_1,\theta q_1}(\mathcal{L}) \to F^{s_1,\theta q_1}(\mathcal{L}) \), then \( T: F^{s,q}(\mathcal{L}) \to F^{s,q}(\mathcal{L}) \), where \( \bar{s}, \bar{p}, \bar{q} \) and \( \bar{s}_i, \bar{p}_i, \bar{q}_i \), satisfy the same relations as those for \( s, p, q \) and \( s_i, p_i, q_i \), \( 1 < p_i, q_i < \infty \). Similar statement holds for \( B^{s,q}(\mathcal{L}) \).

Complex interpolation method originally was due to Calderón [4] and Lions and Peetre [10]; see also [13, 24]. The classical interpolation theory for Besov and Triebel-Lizorkin spaces on \( \mathbb{R}^n \) has been given
systematic treatments in [20], [3], and [25, 26]. There are interesting discussions on interpolation theory in [20] and [26, 25, 22] for generalized Besov spaces associated with differential operators, which requires certain Riesz summability for $\mathcal{L}$ that seems a nontrivial condition to verify. Nevertheless, we would like to mention that the Riesz summability, the spectral multiplier theorem and the decay estimate in (1) are actually intimately related [10, 18].

The real interpolation result for $B^\alpha_{p,q}(\mathbb{R}^n)$, $F^\alpha_{p,q}(\mathbb{R}^n)$ can be found in [20] and [26, 25]. Following the proof as in the classical case, but applying the estimate in (1) in stead of spectral multiplier result, we obtain

**Theorem 1.3** (real interpolation). Suppose that $\mathcal{L}$ satisfies Assumption 1.1. Let $0 < \theta < 1$, $1 \leq p \leq \infty$, $s = (1 - \theta)s_0 + \theta s_1$, $s_0 \neq s_1$.

(a) If $1 \leq p \leq \infty$, $1 \leq q_1, q_2 \leq \infty$, then

$$(F^{s_0,q_0}_p(\mathcal{L}), F^{s_1,q_1}_p(\mathcal{L}))_{\theta,r} = B^{s,r}_p(\mathcal{L}).$$

(b) If $1 \leq p, q_1, q_2 \leq \infty$, then

$$(B^{s_0,q_0}_p(\mathcal{L}), B^{s_1,q_1}_p(\mathcal{L}))_{\theta,r} = B^{s,r}_p(\mathcal{L}).$$

The homogeneous spaces $\dot{B}^{\alpha,q}_p(\mathcal{L})$ and $\dot{F}^{\alpha,q}_p(\mathcal{L})$ can be defined using $\{\varphi_j\}_{j=-\infty}^\infty$ in (i) to (iii), instead of $\{\varphi_j\}_{j=0}^\infty$. Then the analogous results of Theorem 1.2 and Theorem 1.3 hold.

2. INTERPOLATION FOR $\mathcal{L}$

Theorem 1.2 and Theorem 1.3 are part of the abstract interpolation theory for $\mathcal{L}$. In this section we present the outline of their proofs. It was mentioned in [22] that the interpolation associated with $\mathcal{L}$ is a "subtle and difficult" subject, which normally relies on the very property of $\mathcal{L}$.

2.1. **Complex interpolation.** The proof of Theorem 1.2 is similar to that given in [25] in the Fourier case. The insight is that the three line theorem (involved in Riesz-Thorin or Calderón’s constructive proof for $L^p$ spaces) reflects the fact that the value of an analytic function in the interior of a domain is determined by its boundary values.

**Definition 2.2.** Let $(A_0, A_1)$ be an interpolation couple, i.e., $A_0, A_1$ are (complex) Banach spaces, linearly and continuously embedded in a Hausdorff space $\mathcal{H}$. The space $A_0 \cap A_1$ is endowed with the norm $\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_j}, j = 0, 1\}$. The space $A := A_0 + A_1$ is endowed with the norm

$$\|a\|_A = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1\}.$$
Let \( S = \{ z \in \mathbb{C} : 0 \leq \Re z \leq 1 \} \) and \( \bar{S} \) its closure. Denote \( F \) the class of all \( A \)-valued functions \( f(z) \) on \( S \) such that \( z \mapsto f(z) \in A \) is analytic in \( S \) and continuous on \( \bar{S} \), satisfying

(i) \[ \sup_{z \in \bar{S}} \| f(z) \|_A \text{ is finite.} \]

(ii) The mapping \( t \mapsto f(j + it) \in A_j \) are continuous from \( \mathbb{R} \) to \( A_j \), \( j = 0, 1 \).

Then \( F \) is a Banach space with the norm

\[ \| f \|_F = \max_j \{ \sup_t \| f(j + it) \|_{A_j} \} . \]

For \( 0 < \theta < 1 \) we define the interpolation space \([A_0, A_1]_{\theta}\) as

\[ [A_0, A_1]_{\theta} := \{ a \in A : \exists f \in F \text{ with } f(\theta) = a \} \]

Then \([A_0, A_1]_{\theta}\) is a Banach space equipped with the norm

\[ \| a \|_{\theta} := \inf \{ \| f \|_F : f \in F \text{ and } f(\theta) = a \} . \]

2.3. Outline of the proof of Theorem 1.2. Let \( \{ \phi_j \}, \{ \psi_j \} \) satisfy the conditions in (i)-(iii) and \( \sum_j \psi_j(x) \phi_j(x) = 1 \). Define the operators \( S : f \mapsto \{ \phi_j(\mathcal{L}) f \} \), and \( R : g \mapsto \sum_j \psi_j(\mathcal{L}) g \). The proof for part (a) follows from the commutative diagram

\[
\begin{CD}
F_{p,q}^s(\mathcal{L}) @>S>> L^p(\ell^q)
\end{CD}
\]

\[
\begin{CD}
Id @>>> Id
\end{CD}
\]

\[
\begin{CD}
F_{p,q}^s(\mathcal{L}) @<R<< L^p(\ell^q)
\end{CD}
\]

and Lemma 2.4 and Lemma 2.5 which are interpolation results for Banach space valued \( L^p \) and \( \ell^q \) spaces [25].

**Lemma 2.4.** Let \( 0 < \theta < 1 \), \( 1 \leq p_0, p_1 < \infty \) and \( p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1} \). Let \( A_0, A_1 \) be Banach spaces. Then

\[ \[ L^{p_0}(A_0), L^{p_1}(A_1) \]_{\theta} = L^p([A_0, A_1]_{\theta}). \]

If \( p_1 = \infty \), then \( (2) \) holds with \( L^{p_1}(A_1) \) replaced by \( \ell_0^\infty(A_1) \), the completion of simple \( A_1 \)-valued functions with the esssup norm.

As in [25], denote \( \ell^q(A_j) \) the space of functions consisting of \( a = \{ a_j \} \), \( a_j \in A_j \) \((A_j \text{ being Banach spaces})\) equipped with the norm

\[ \| a \|_{\ell^q(A_j)} = \left( \sum_j \| a_j \|_{A_j}^q \right)^{1/q} . \]
Lemma 2.5. Let $0 < \theta < 1$, $1 \leq q_0, q_1 < \infty$ and $q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$. Let $A_j$ be Banach spaces, $j \in \mathbb{N}$. Then
\begin{equation}
[\ell^{q_0}(A_j), \ell^{q_1}(B_j)]_\theta = \ell^q([A_j, B_j]_\theta).
\end{equation}
If $q_1 = \infty$, then
\begin{equation}
[\ell^{q_0}(A_j), \ell^\infty(B_j)]_\theta = \ell^q([A_j, B_j]_\theta) = [\ell^{q_0}(A_j), \ell^\infty(B_j)]_\theta,
\end{equation}
where $\ell^\infty(B_j) := \{\{c_j\} \in \ell^\infty(B_j) : \|c_j\|_{B_j} \to 0 \text{ as } j \to \infty\}$.

If $1 \leq q_0, q_1 < \infty$, (3) also follows from Lemma 2.3 as a special case where the underlying measure space can be taken as $(X, \mu) = \mathbb{Z}$. If $q_1 = \infty$, then the remark in [25, Subsection 1.18.1] shows that the second statement in Lemma 2.5 is also true.

In the diagram above in order to show $S, R$ are continuous mappings, we need the following well-known lemma.

Lemma 2.6. Let $h(x)$ be a monotonely nonincreasing, radial function in $L^1(\mathbb{R}^n)$. Let $h_j(x) = 2^{jn/2}h(2^{j/2}x)$ be its scaling. Then for all $f$ in $L^1_{loc}(\mathbb{R}^n)$
\[ |\int h_j(x - y)f(y)dy| \leq c_n\|h\|_1Mf(x), \]
where $Mf$ denotes the usual Hardy-Littlewood maximal function.

Evidently the decay estimate in (1) and Lemma 2.6 imply the continuity of $S$ and $R$, in light of the $L^p(\ell^q)$-valued maximal inequality.

The proof for $B_p^{s,q}(\mathcal{L})$ in part (b) proceeds in a similar way.

2.7. Real interpolation. Peetre’s $K$-functional [21] is defined as
\[ K(t,a) := K(t,a; A_0, A_1) = \inf(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \]
where the infimum is taken over all representations of $a = a_0 + a_1$, $a_i \in A_i$. Let $0 < q \leq \infty, 0 < \theta < 1$. For a given interpolation couple $(A_0, A_1)$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is given by
\[ (A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : \|a\|_{(A_0, A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta q}K(t,a)^q dt\right)^{1/q} < \infty\} \]
with usual modifications if $q = \infty$.

Proof of Theorem 1.3 is similar to [26, Subsection 2.4.2] and [3, Theorem 6.4.5]. Define $\ell^{s,q}(A) = \{a = \{a_j\} : \|a\|_{\ell^{s,q}(A)} = \|\{2^j a_j\}_{j=1}^\infty\|_{\ell^q} < \infty\}$. For Besov spaces it follows from
\[ (\ell^{s_0,q_0}(A_0), \ell^{s_1,q_1}(A_1))_{\theta,q} = \ell^{s,q}(\{A_0, A_1\}_{\theta,q}), \]
$s = (1 - \theta)s_0 + \theta s_1, q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$ and the commutative diagram for $B_p^{s,q}(\mathcal{L})$. Consult [26, 25] or [20, Chapter 5, Theorem 6]; both of their proofs rely on retraction method. Also see [3] for a different

For the $F$-space the proof follows from the commutative diagram for $F^s,q(L)$ and $(L^{p0}(A_0, w_0), L^{p1}(A_1, w_1))_{\theta, p} = L^p((A_0, A_1)_{\theta, p}, w)$, where $p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$, $w = w_0^{1-\theta} w_1^\theta$, $w_0, w_1$ being two weight functions [20, Chapter 5].

2.8. Schrödinger operators with magnetic potential. From [14], [28] or [18] we know that if the heat kernel of $\mathcal{L}$ satisfies the upper Gaussian bound

$$|e^{-t\mathcal{L}(x, y)}| \leq c n t^{-n/2} e^{-c|x-y|^2/t}$$

then the kernel decay in Assumption [3] holds. Let

$$H = -\sum_{j=1}^n (\partial_{x_j} + ia_j)^2 + V,$$

where $a_j(x) \in L^2_{\text{loc}}(\mathbb{R}^n)$ is real-valued, $V = V_+ - V_-$ with $V_+ \in L^1_{\text{loc}}(\mathbb{R}^n)$, $V_- \in K_n$, the Kato class [23]. Proposition 5.1 in [7] showed that (5) is valid for $-\Delta + V$ if $V_+ \in K_n$ and $\|V_+\|_{K_n} < \gamma_n := \pi^{n/2}/\Gamma(n/2) - 1$, $n \geq 3$, whose proof evidently works for $V_+ \in L^1_{\text{loc}}$. By the diamagnetic inequality [23, Theorem B.13.2], we see that (5) also holds for $H$ provided $\|V_-\|_{K_n} < \gamma_n$, $n \geq 3$.

As another example, a uniformly elliptic operator is given by

$$\mathcal{L} = -\sum_{j,k=1}^n \partial_{x_j}(a_{jk}\partial_{x_k})$$

where $a_{jk}(x) = a_{kj}(x) \in L^\infty(\mathbb{R}^n)$ are real-valued and satisfy the ellipticity condition $(a_{jk}) \approx I_n$. Then [19] Theorem 1 tells that (5) is true provided that the infimum of its spectrum $\inf \sigma(\mathcal{L}) = 0$.

**References**


