Bohr Density of Simple Linear Group Orbits

Roger Howe
Yale University

Francois Ziegler
Georgia Southern University

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ROGER HOWE and FRANÇOIS ZIEGLER

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Bohr density of simple linear group orbits

ROGER HOWE† and FRANÇOIS ZIEGLER‡
† Department of Mathematics, Yale University, New Haven, CT 06520-8283, USA
(e-mail: howe@math.yale.edu)
‡ Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA
(e-mail: fziegler@georgiasouthern.edu)

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Abstract. We show that any non-zero orbit under a non-compact, simple, irreducible linear group is dense in the Bohr compactification of the ambient space.

1. Introduction
Let $V$ be a locally compact abelian group, $V^*$ its Pontryagin dual and $bV$ its Bohr compactification, that is, $bV$ is the dual of the discretized group $V^*$. On identifying $V$ with its double dual we have a dense embedding $V \hookrightarrow bV$, namely,

$$\{\text{continuous characters of } V^*\} \hookrightarrow \{\text{all characters of } V^*\}.$$

The relative topology of $V$ in $bV$ is known as the Bohr topology of $V$. Among its many intriguing properties (surveyed in [G07]) is the observation due to Katznelson [K73a] (see also [G79, §7.6]) that very ‘thin’ subsets of $V$ can be Bohr dense in very large ones.

While Katznelson was concerned with the case $V = \mathbb{Z}$ (the integers), we shall illustrate this phenomenon in the setting where $V$ is the additive group of a real vector space, and the subsets of interest are the orbits of a Lie group acting linearly on $V$. Indeed our aim is to establish the following result, which was conjectured in [Z96, p. 45].

**Theorem 1.** Let $G$ be a non-compact, simple real Lie group and $V$ a non-trivial, irreducible, finite-dimensional real $G$-module. Then every non-zero $G$-orbit in $V$ is dense in $bV$.

We prove this in §3 on the basis of four lemmas found in §2. Before that, let us record a similar property of nilpotent groups. In that case, orbits typically lie in proper affine subspaces, so we cannot hope for Bohr density in the whole space; but we have the following theorem.

**Theorem 2.** Let $G$ be a connected nilpotent Lie group and $V$ a finite-dimensional $G$-module of unipotent type. Then every $G$-orbit in $V$ is Bohr dense in its affine hull.
Proof. Recall that unipotent type means that the Lie algebra $\mathfrak{g}$ of $G$ acts by nilpotent operators. So $Z \mapsto \exp(Z)v$ is a polynomial map of $\mathfrak{g}$ onto the orbit of $v \in V$, and the claim follows immediately from [Z93, Theorem].

2. Four lemmas

Our first lemma gives several characterizations of Bohr density—each of which can also be regarded as providing a corollary of Theorem 1.

**Lemma 1.** Let $\mathcal{O}$ be a subset of the locally compact abelian group $V$. Then the following are equivalent:

1. $\mathcal{O}$ is dense in $bV$;
2. $\alpha(\mathcal{O})$ is dense in $\alpha(V)$ whenever $\alpha$ is a continuous morphism from $V$ to a compact topological group;
3. every almost periodic function on $V$ is determined by its restriction to $\mathcal{O}$;
4. Haar measure $\eta$ on $bV$ is the weak* limit of probability measures $\mu_T$ concentrated on $\mathcal{O}$.

**Proof.** (1) $\iff$ (2): Clearly (2) implies (1) as the special case where $\alpha$ is the natural inclusion $\iota: V \hookrightarrow bV$. Conversely, suppose (1) holds and $\alpha: V \to X$ is a continuous morphism to a compact group. By the universal property of $bV$ [D82, Theorem 16.1.1], $\alpha = \beta \circ \iota$ for a continuous morphism $\beta: bV \to X$. Now continuity of $\beta$ implies $\beta(\iota(\mathcal{O})) \subset \overline{\beta(\iota(\mathcal{O})))}$, which is to say that $\beta(bV) \subset \overline{\alpha(\mathcal{O})}$ and hence $\alpha(V) \subset \overline{\alpha(\mathcal{O})}$, as claimed.

(1) $\iff$ (3): Recall that a function on $V$ is almost periodic if and only if it is the pull-back of a continuous $f: bV \to \mathbb{C}$ by the inclusion $V \hookrightarrow bV$. If two such functions coincide on $\mathcal{O}$ and $\mathcal{O}$ is dense in $bV$, then clearly they coincide everywhere. Conversely, suppose that $\mathcal{O}$ is not dense in $bV$. Then by complete regularity [H63, Theorem 8.4] there is a non-zero continuous $f: bV \to \mathbb{C}$ which is zero on the closure of $\mathcal{O}$ in $bV$. Now clearly this $f$ is not determined by its restriction to $\mathcal{O}$.

(1) $\iff$ (4) [K73a]: Suppose that $\eta$ is the weak* limit of probability measures $\mu_T$ concentrated on $\mathcal{O}$. So we have $\mu_T(f) \to \eta(f)$ for every continuous $f$, and the complement of $\mathcal{O}$ in $bV$ is $\mu_T$-null [B04, Definition V.5.7.4 and Proposition IV.5.2.5]. If $f$ vanishes on the closure of $\mathcal{O}$ in $bV$ then so do all $\mu_T(|f|)$ and hence also $\eta(|f|)$, which forces $f$ to vanish everywhere. So $\mathcal{O}$ is dense in $bV$. Conversely, suppose that $\mathcal{O}$ is dense in $bV$. We have to show that given continuous functions $f_1, \ldots, f_n$ on $bV$ and $\varepsilon > 0$, there is a probability measure $\mu$ concentrated on $\mathcal{O}$ such that $|\eta(f_i) - \mu(f_i)| < \varepsilon$ for all $i$. Writing

$$F = (f_1, \ldots, f_n) \quad \text{and} \quad \eta(F) = (\eta(f_1), \ldots, \eta(f_n)),$$

we see that this amounts to $\|\eta(F) - \mu(F)\| < \varepsilon$, where the norm is the sup norm in $\mathbb{C}^n$. Now by [B04, Corollary V.6.1] $\eta(F)$ lies in the convex hull of $F(bV)$ (which is compact by Carathéodory’s theorem [B87, Corollary 11.1.8.7]). So $\eta(F)$ is a convex combination $\sum_{i=1}^N \lambda_i F(w_i)$ of elements of $F(bV)$. But $F(\mathcal{O})$ is dense in $F(bV)$, so we can find $w_i \in \mathcal{O}$ such that $\|F(w_i) - F(w_i)\| < \varepsilon$. Putting $\mu = \sum_{i=1}^N \lambda_i \delta_{w_i}$, where $\delta_{w_i}$ is Dirac measure at $w_i$, we obtain the desired probability measure $\mu$. □
Remark 1. One might wonder if condition (2) is equivalent to the following a priori weaker but already interesting property:

(2') \(O\) has dense image in any compact quotient group of \(V\).

Here is an example showing that (2') does not imply (2). Let \(V = \mathbb{R}\) and \(O = \mathbb{Z} \cup 2\pi \mathbb{Z}\). Then clearly \(O\) has dense image in every compact quotient \(\mathbb{R}/a\mathbb{Z}\). On the other hand, considering the irrational winding \(\alpha : \mathbb{R} \to \mathbb{T}^2\) defined by \(\alpha(v) = (e^{iv}, e^{2\pi iv})\), one can check without difficulty that \(\alpha(O) = \mathbb{T} \times \{1\} \cup \{1\} \times \mathbb{T}\), which is strictly smaller than \(\overline{\alpha(V)} = \mathbb{T}^2\).

Remark 2. A net of probability measures \(\mu_T\) converging to Haar measure on \(bV\) as in (4) has been called a generalized summing sequence by Blum and Eisenberg [B74]. They observed, among others, the following characterization.

Lemma 2. The following conditions are equivalent:

1. \(\mu_T\) is a generalized summing sequence;
2. the Fourier transforms \(\hat{\mu}_T(u) = \int_{bV} \omega(u) d\mu_T(\omega)\) converge pointwise to the characteristic function of \(\{0\} \subset V^*\).

Proof. This characteristic function is the Fourier transform of Haar measure \(\eta\) on \(bV\). Thus, condition (2) says that \(\mu_T(f) \to \eta(f)\) for every continuous character \(f(\omega) = \omega(u)\) of \(bV\), whereas condition (1) says that \(\mu_T(f) \to \eta(f)\) holds for every continuous function \(f\) on \(bV\). Since linear combinations of continuous characters are uniformly dense in the continuous functions on \(bV\) (Stone–Weierstrass), the two conditions imply each other. \(\square\)

For our third lemma, let \(G\) be a group, \(V\) a finite-dimensional \(G\)-module, and write \(V^*\) for the dual module wherein \(G\) acts contragrediently: \(\langle gu, v \rangle = \langle u, g^{-1}v \rangle\).

Lemma 3. Suppose that \(V\) is irreducible and \(\phi(g) = \langle u, gv \rangle\) is a non-zero matrix coefficient of \(V\). Then every other matrix coefficient \(\psi(g) = \langle x, gy \rangle\) is a linear combination of left and right translates of \(\phi\).

Proof. Irreducibility of \(V\) and (therefore) \(V^*\) ensures that \(u\) and \(v\) are cyclic, that is, their \(G\)-orbits span \(V^*\) and \(V\). So we can write \(x = \sum_i \alpha_i g_i u\) and \(y = \sum_j \beta_j g_j v\), whence \(\psi(g) = \sum_i \sum_j \alpha_i \beta_j \phi(g_i^{-1}g_j)\). \(\square\)

Our fourth and final preliminary result is the following famous lemma.

Lemma 4. (Van der Corput) Suppose that \(F : [a, b] \to \mathbb{R}\) is differentiable, its derivative \(F'\) is monotone, and \(|F'| \geq 1\) on \((a, b)\). Then \(\int_a^b e^{iF(t)} dt \leq 3\).

Proof. See [S93, p. 332], or [R05, Lemma 3] which actually gives the sharp bound 2. \(\square\)

3. Proof of Theorem 1

By Lemma 1, it is enough to show that Haar measure on \(bV\) is the weak* limit of probability measures \(\mu_T\) concentrated on the orbit under consideration; or equivalently (Lemma 2), that the Fourier transforms of the \(\mu_T\) tend pointwise to the characteristic function of \(\{0\} \subset V^*\). (Here we identify the Pontryagin dual with the dual vector space or module.)
To construct such \( \mu_T \), we assume without loss of generality that the action of \( G \) on \( V \) is effective, so that we may regard \( G \subset \text{GL}(V) \). Let \( K \subset G \) be a maximal compact subgroup, \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) a Cartan decomposition, \( \mathfrak{a} \subset \mathfrak{p} \) a maximal abelian subalgebra, \( C \subset \mathfrak{a}^* \) a Weyl chamber, \( P \subset \mathfrak{a} \) the dual positive cone, and \( H \) an interior point of \( P \); thus we have that \( \langle v, H \rangle \) is positive for all non-zero \( v \in C \). (For all this structure see, for example, [K73b].) We fix a non-zero \( v \in V \), and for each positive \( T \in \mathbb{R} \) we let \( \mu_T \) denote the image of the product measure Haar \( \times \text{(Lebesgue}/T) \times \text{Haar} \) under the composed map

\[
K \times [0, T] \times K \longrightarrow Gv \longrightarrow bV
\]

\[
(k, t, k') \longmapsto k \exp(tH)k' \quad w \longmapsto e^{i\langle w, v \rangle}.
\]

Here \( \exp : \mathfrak{a} \to A \) is the usual matrix exponential with inverse \( \log : A \to \mathfrak{a} \), and the brackets \( \langle \cdot, \cdot \rangle \) denote both pairings, \( \mathfrak{a}^* \times \mathfrak{a} \to \mathbb{R} \) and \( V^* \times V \to \mathbb{R} \). By construction the \( \mu_T \) are concentrated on the subset \( Gv \) of \( bV \) [B04, Corollary V.6.2.3]. It remains to show that as \( T \to \infty \) we have, for every non-zero \( u \in V^* \),

\[
\int_{K \times K} dk \, dk' \frac{1}{T} \int_0^T e^{i\langle u, \exp(tH)k'v \rangle} \, dt \to 0. \tag{*}
\]

To this end, let

\[
F_{kk'}(t) = \langle u, k \exp(tH)k'v \rangle
\]

denote the exponent in (*). We will show that Lemma 4 applies to almost every \( F_{kk'} \). In fact, it is well known (see, for example, [K73b, Proposition 2.4 and proof of Proposition 3.4]) that \( a \) acts diagonalizably (over \( \mathbb{R} \)) on \( V \). Thus, letting \( E_v \) be the projector of \( V \) onto the weight \( v \) eigenspace of \( a \), we can write

\[
F_{kk'}(t) = \sum_{v \in \mathfrak{a}^*} \langle u, k E_v k'v \rangle e^{i\langle v, H \rangle t}.
\]

Now we claim that there are non-zero \( v \) such that the coefficient \( f_v(k, k') = \langle u, k E_v k'v \rangle \) is not identically zero on \( K \times K \). (Then \( f_v \), being analytic, will be non-zero almost everywhere.) Indeed, suppose otherwise. Then, writing any \( g \in G \) in the form \( kak' \) (\( KAK \) decomposition [K02]), we would have

\[
\langle u, gv \rangle = \sum_{v \in \mathfrak{a}^*} \langle u, k E_v k'v \rangle e^{i\langle v, \log(a) \rangle} = \langle u, k E_0 k'v \rangle.
\]

In particular, the matrix coefficient \( \langle u, gv \rangle \) would be bounded. Hence so would all matrix coefficients, since they are linear combinations of translates of this one (Lemma 3); and this would contradict the non-compactness of \( G \subset \text{GL}(V) \).

So the set \( N = \{ v \in \mathfrak{a}^* : v \neq 0, f_v \neq 0 \} \) is not empty. It is also Weyl group invariant, hence contains weights \( v \in C \) for which we know that \( \langle v, H \rangle \) is positive. Therefore, maximizing \( \langle v, H \rangle \) over \( N \) produces a positive number \( \langle v_0, H \rangle \), in terms of which our exponent and its derivatives can be written

\[
\frac{d^n}{dt^n} F_{kk'}(t) = e^{i\langle v_0, H \rangle t} \sum_{v \in \mathfrak{a}^*} f_v(k, k') \langle v, H \rangle^n e^{i\langle v - v_0, H \rangle t},
\]

where \( \langle v - v_0, H \rangle < 0 \) in all non-zero terms except the one indexed by \( v_0 \). (Here we assume, as we may, that \( H \) was initially chosen outside the kernels of all pairwise
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From this it is clear that for almost all \((k, k')\) there is a \(T_0\) beyond which the first two derivatives of \(F_{kk'}\) are greater than 1 in absolute value. So Lemma 4 applies and gives

\[
\left| \int_{T_0}^{T} e^{i F_{kk'}(t)} \, dt \right| \leq 3 \quad \text{for all } T.
\]

Therefore, \(\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} e^{i F_{kk'}(t)} \, dt = 0\) for almost all \((k, k')\), whence the conclusion (\(*\)) by dominated convergence. This completes the proof.

4. Outlook

Theorem 1 says that the \(G\)-action on \(V \setminus \{0\}\) is minimal [P83] in the Bohr topology. It would be interesting to determine if it is still minimal, and/or uniquely ergodic, on \(bV \setminus \{0\}\).

It is also natural to speculate whether our theorems have a common extension to more general group representations. Here we shall content ourselves with noting two obstructions. First, Theorem 1 clearly fails for semisimple groups with compact factors. Secondly, Theorem 2 fails for \(V\) not of unipotent type, as one sees by observing that the orbits of \(R\) acting on \(R^2\) by \(t_{0}^t \cdot 0_{t}^t\) (i.e., hyperbolas) already have non-dense images in \(R^2/Z^2\).

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