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The de Rham cohomology of a Lie group modulo a dense subgroup

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THE DE RHAM COHOMOLOGY OF A LIE GROUP MODULO A DENSE SUBGROUP

by

BRANT CLARK

(Under the Direction of François Ziegler)

ABSTRACT

Let H be a dense subgroup of a Lie group G with Lie algebra $\mathfrak{g}$. We show that the (diffeological) de Rham cohomology of G/H equals the Lie algebra cohomology of $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{h}$ is the ideal $\{\mathrm{Z} \in \mathfrak{g} : \mathrm{e}^{t\mathrm{Z}} \in \mathrm{H} \text{ for all } t \in \mathbf{R}\}$.

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CHAPTER 1

INTRODUCTION

The objective of this thesis is to prove the following theorem:

(1.1) Theorem. *Let* H *be a dense subgroup of a Lie group* G *with Lie algebra* $\mathfrak{g}$*. The (diffeological) de Rham cohomology of* G/H *equals the Lie algebra cohomology of* $\mathfrak{g}/\mathfrak{h}$*, where* $\mathfrak{h}$ *is the ideal* $\{\mathrm{Z} \in \mathfrak{g} : \mathrm{e}^{t\mathrm{Z}} \in \mathrm{H}$ *for all* $t \in \mathbf{R}\}$.

Done for 1-forms in [B24, 8.14], this theorem generalizes this result to k-forms.

We will go into some detail of Diffeology in chapter 7, but we will first give some motivation on why it is necessary for us. It is known that for a Lie group G and a subgroup H that the quotient space G/H is a manifold if and only if H is closed. So under the framework of differential geometry, our object of study G/H in Theorem (1.1) is not guaranteed to be a manifold, and hence will not have a defined de Rham cohomology. Diffeology is a generalization of differential geometry where given a set X one defines which maps from U to X are *smooth*, where U is any open subset $\mathbf{R}^n$ for any n. An important feature of diffeology is that quotients of diffeological spaces always inherit a diffeological structure. In addition, there is a notion of differential forms, the exterior derivative and thus a complex with a corresponding cohomology, the so called (diffeological) de Rham complex.

In this thesis we are inspired in large part by Claude Chevalley and Samuel Eilenberg's 1948 paper *Cohomology theory of Lie groups and Lie algebras* which introduced the so-called Chevalley-Eilenberg complex and Lie algebra cohomology. In what we take from their work, we do our best to simplify and update. The primary example of this being proof of Chevalley-Eilenberg coboundary formula [C48, 9.1], i.e, (5.5). In [C48] the formula is proven with a puzzling induction, which obfuscates how one can discover the formula oneself. With the help of a few simple lemmas, we give an easier, much more revealing proof of this important formula.

Part 1 will give some necessary background on homological algebra, exterior algebra, the de Rham Complex, Lie algebra cohomology and diffeology. Then in part 2, which constitutes the majority of the forthcoming [C24], we will prove (1.1).

Part I

Background

CHAPTER 2

HOMOLOGICAL ALGEBRA

(2.1) Definition. Let $\mathcal{C}$ be a sequence of abelian group homomorphisms

$$0 \longrightarrow \mathrm{C}^0 \xrightarrow{d_1} \mathrm{C}^1 \longrightarrow \cdots \longrightarrow \mathrm{C}^{n-1} \xrightarrow{d_n} \mathrm{C}^n \xrightarrow{d_{n+1}} \cdots \tag{2.2}$$

(i) The sequence $\mathcal{C}$ is a called a *cochain complex* if $d_{n+1} \circ d_n = 0$ for all n.

Members of C^n are called *n-cochains*. We call n-cochains that are in $\ker d_{n+1}$ n-*cocycles* and n-cochains that are in $\mathrm{Im}\, d_n$ are called n-*coboundaries*.

(ii) If $\mathcal{C}$ is a cochain complex, its *nth cohomology group* is the quotient group

$$\mathrm{H}^n(\mathcal{C}) = \frac{\mathrm{Ker}\, d_{n+1}}{\mathrm{Im}\, d_n}. \tag{2.3}$$

(2.4) Definition. Let $\mathcal{A} = \{\mathrm{A}^n\}$ and $\mathcal{B} = \{\mathrm{B}^n\}$ be cochain complexes. A *homomorphism of complexes* $\alpha : \mathrm{A} \to \mathrm{B}$ is a set of homomorphisms $\alpha_n : \mathrm{A}^n \to \mathrm{B}^n$ such that for every n the diagram commutes

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{A}^n \arrow[r, "d"] \arrow[d, "\alpha_n"] & \mathrm{A}^{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & \cdots \\ \cdots \arrow[r] & \mathrm{B}^n \arrow[r, "d"] & \mathrm{B}^{n+1} \arrow[r] & \cdots \end{tikzcd} \tag{2.5}$$

(2.6) Proposition. *A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of complexes induces group homomorphisms $\mathrm{H}^n(\mathcal{A}) \to \mathrm{H}^n(\mathcal{B})$ for $n \geq 0$ of the respective cohomology groups.*

Proof. See [D04] Proposition 17.1. □

CHAPTER 3

EXTERIOR ALGEBRA

(3.1) Definition. An *exterior k-form* on a vector space V over $\mathbf{R}$ is an alternating multilinear map $\omega : \underbrace{\mathrm{V} \times \cdots \times \mathrm{V}}_{\text{k times}} \to \mathbf{R}$. We denote the space of exterior k-forms by $\bigwedge^k(\mathrm{V}^*)$.

(3.2) Definition. The *wedge product* of exterior forms is the map $\wedge : \bigwedge^k(\mathrm{V}^*) \times \bigwedge^l(\mathrm{V}^*) \to \bigwedge^{k+l}(\mathrm{V}^*)$ defined by

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+l}) = \sum_{\sigma} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}), \tag{3.3}$$

where we are summing over (k, l)-*shuffles* σ, i.e., members of the symmetric group S_{k+l} such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+l)$.

(3.4) Proposition. *The wedge product is graded commutative, i.e., if* $\omega \in \bigwedge^k(\mathrm{V}^*)$ *and* $\eta \in \bigwedge^l(\mathrm{V}^*)$ *then*

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega. \tag{3.5}$$

In particular, notice that 2-forms commute with everything.

Proof. See [T11] Proposition 3.21. □

(3.6) Definition. For each v in V, we define the *interior product* with v as the map $\iota_v : \bigwedge^k(\mathrm{V}^*) \to \mathrm{V}^{k-1}(\mathrm{V}^*)$ defined by

$$\iota_v\omega(v_2, \ldots, v_k) := \omega(v, v_2, \ldots, v_k). \tag{3.7}$$

The following proposition, coupled with the linearity of various maps will streamline many of the proofs to come.

(3.8) Proposition. *Let* $\{b_1, \ldots, b_n\}$ *be a basis of* V*, and* $\{e_1, \ldots, e_n\}$ *be the dual basis of* V^**. Then products of the form* $e_{i_1} \wedge \cdots \wedge e_{i_k}$ *with* $i_1 < i_2 < \cdots < i_k$ *form a basis of* $\bigwedge^k(\mathrm{V}^*)$*. Thus we have* $\dim \bigwedge^k(\mathrm{V}^*) = \binom{n}{k}$.

Proof. See [T11] Proposition 3.29. □

The following lemma spells out the special cases k = 1 and 2 of (3.3).

(3.9) Lemma. (a) *If θ is a 1-form and ω is a k-form, then*

$$\theta \wedge \omega(v_0, \dots, v_k) = \sum_{m=0}^{k} (-1)^m \theta(v_m) \omega(v_0, \dots, \widehat{v_m}, \dots, v_k) \tag{3.10}$$

where the hat $\hat{}$ denotes a term to be omitted.

(b) *If α is a 2-form and β is a $k-1$-form, then*

(3.11)

$$(\alpha \wedge \beta)(v_0, \dots, v_k) = \sum_{0 \le i < j \le k} (-1)^{i+j-1} \alpha(v_i, v_j) \beta(v_0, \dots, \widehat{v}_j, \dots \widehat{v}_j, \dots, v_k).$$

(Proof of Lemma (3.9a)). By (3.3) we have that

$$(\theta \wedge \omega)(v_0, \dots, v_k) = \sum_{m=0}^{k} \operatorname{sgn}(\sigma) \theta(v_m)(v_0, \dots, \widehat{v_m}, \dots, v_k), \tag{3.12}$$

where σ is the permutation given by

$$\begin{pmatrix} 0 & 1 & \cdots & m-1 & m & m+1 & \cdots & k \\ m & 0 & 1 & \cdots & m-1 & m+1 & \cdots & k \end{pmatrix} \tag{3.13}$$

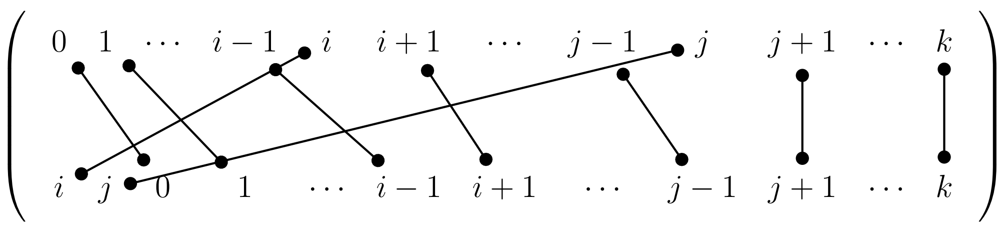
We count m crossings, hence $\operatorname{sgn}(\sigma) = (-1)^m$. Thus the claim follows. □

(Proof of Lemma (3.9b)). We have

(3.14)

$$(\alpha \wedge \beta)(v_0, \dots, v_k) = \sum_{0 \le i < j \le k} \operatorname{sgn}(\sigma) \alpha(v_i, v_j) \beta(v_0, \dots, \widehat{v}_i, \dots, \widehat{v}_j, \dots, v_k) \quad \text{by (3.3)}$$

where we are summing over $(2, k-1)$ shuffles in S_{k+1}, and σ is the permutation given by

(3.15) 

We count $i+j-1$ crossings, hence $\mathrm{sgn}(\sigma) = (-1)^{i+j-1}$. Thus the claim follows $\square$

(3.16) Lemma. *Let θ_i be 1-forms. Then it follows that*

(3.17)
$$(\theta_1 \wedge \cdots \wedge \theta_k)(v_1, \ldots, v_k) = \det[\theta_i(v_j)]_{i,j=1,\ldots,k}.$$

Proof. We prove by induction on k. The case when $k = 1$ is clear. Suppose that (3.17) holds (**I.H**) [1]. Now we show that the case of $k+1$ holds.

(3.18)
$$(\theta_0 \wedge \theta_1 \wedge \cdots \wedge \theta_k) = \sum_{m=0}^{k} (-1)^m \theta_0(v_m)(\theta_1 \wedge \cdots \wedge \theta_k)(v_1, \ldots, \widehat{v_m}, \ldots, v_k) \qquad \text{by (3.10)}$$

(3.19)
$$= \sum_{m=0}^{k} (-1)^m \theta_0(v_m) \det \begin{pmatrix} \theta_1(v_0) & \cdots & \widehat{\theta(v_m)} & \cdots & \theta_1(v_k) \\ \vdots & & \vdots & & \vdots \\ \theta_k(v_0) & \cdots & \widehat{\theta(v_m)} & \cdots & \theta_k(v_k) \end{pmatrix} \qquad \text{by } (\mathbf{I.H})$$

(3.20)
$$= \det \begin{pmatrix} \theta_0(v_0) & \cdots & \theta_0(v_k) \\ \vdots & & \vdots \\ \theta_k(v_0) & \cdots & \theta_k(v_k) \end{pmatrix} \qquad \text{by cofactor expansion along row 1.}$$

$\square$

[1] When using induction, we will occasionally label the base case and induction hypothesis by **B.C.** and **I.H.** respectively.

CHAPTER 4

THE DE RHAM COMPLEX OF A EUCLIDEAN OPEN SET

By a *Euclidean open set*, we mean an open set inside $\mathbf{R}^n$ for some $n \in \mathbf{N}$.

(4.1) Definition. Let $\mathrm{U} \subset \mathbf{R}^n$ be an Euclidean open set. A *differential* k-form on U is a smooth map $x \mapsto \omega_x$ assigning to each $x \in \mathrm{U}$ an exterior k-from $\omega_x \in \bigwedge^k((\mathbf{R}^n)^*)$. We denote the space of such maps $\Omega^k(\mathrm{U})$. One typically drops the subscript in ω_x and just writes ω. When we need to emphasize that the form is evaluated at some point y other than x, however, we may denote that value by ω_y.

The wedge product of differential forms on U is defined pointwise, that is, for a k-form ω and an l-form η their wedge product is the $(k+l)$-form $\omega \wedge \eta$ such that for each $y \in \mathrm{U}$ we have that

$$(\omega \wedge \eta)_y = \omega_y \wedge \eta_y. \tag{4.2}$$

(4.3) Definition. Let $\mathrm{F} : \mathrm{U} \to \mathrm{V}$ be a smooth map of Euclidean open sets. The *pull-back* $\mathrm{F}^*\omega$ of a k-form ω on V by F is defined by

$$(\mathrm{F}^*\omega)_x(v_1, \ldots, v_k) := \omega_{\mathrm{F}(x)}(\mathrm{DF}(x)(v_1), \ldots, \mathrm{DF}(x)(v_k)). \tag{4.4}$$

for $v_i \in \mathbf{R}^n$.

(4.5) Remarks. We will occasionally denote $\mathrm{DF}(x)(v)$ using either $\mathrm{F}_*(v)$ or when $y = \mathrm{F}(x)$ by $\frac{\partial y}{\partial x}(v)$.

(4.6) Definition. Let ω be a k-form and V be a vector field on U. Then we define the *interior product of* ω by V to be the $k-1$-form defined by

$$(\iota_{\mathrm{V}}\omega)_x(v_2, \ldots, v_k) := \omega(\mathrm{V}(x), v_2, \ldots, v_k) \tag{4.7}$$

(4.8) Definition. Let ω be a k-form on U. The *exterior derivative* of ω is the $k+1$-form defined by

$$d\omega(v_0,\ldots,v_k) := \sum_{i=0}^{k}(-1)^i \frac{\partial\omega}{\partial x}(v_i)(v_0,\ldots\widehat{v_i},\ldots v_k). \tag{4.9}$$

Where the hat $\widehat{\cdot}$ indicates a term to be omitted. For ease of computation, we introduce the following notation $\delta\omega := \frac{\partial\omega}{\partial x}(\delta x)$. Hence

$$d\omega(\delta_0 x,\ldots,\delta_k x) = \sum_{i=0}^{k}(-1)^i(\delta_i\omega)(\delta_0 x,\ldots\widehat{\delta_i x},\ldots\delta_k x). \tag{4.10}$$

(4.11) Definition. If V and ω are a vector field and a k-form on U, then we define the *Lie derivative* of ω along V to be the k-from on U defined by

$$(\mathrm{L_V}\omega)(v_1,\ldots,v_k) := \frac{d}{dt}\left(e^{t\mathrm{V}*}\omega\right)(v_1,\ldots,v_k)\Big|_{t=0}. \tag{4.12}$$

where $e^{t\mathrm{V}}$ is the flow of the vector field V.

(4.13) Proposition. *The exterior derivative has the following properties:*

(a) (Graded Leibniz) *If* $\omega \in \Omega^k(\mathrm{U})$ *and* $\eta \in \Omega^l(\mathrm{U})$ *then*

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge d\eta. \tag{4.14}$$

(b) (Naturality) *The exterior derivative commutes with pull-backs, i.e.,* $\mathrm{F}^*[d\omega] = d[\mathrm{F}^*\omega]$.

(c) (Poincaré's Theorem) $\mathrm{Im}\, d \subset \mathrm{Ker}\, d$*, i.e.,* $d^2 = 0$.

Proof. See [T11] Proposition 4.7, Proposition 19.5. □

Poincaré's Theorem tell us that $(\Omega^\bullet(\mathrm{U}), d)$ is a complex hence we use $(2.1ii)$ to define the *kth de Rham cohomology group*

$$\mathrm{H}^k_{d\mathrm{R}}(\mathrm{U}) := \frac{k\text{-cocycles on U}}{k\text{-coboundaries on U}}. \tag{4.15}$$

As a corollary to (a), we have the following which will prove to be useful later.

(4.16) Corollary. *Let each θ_i be a 1-form. Then we have*

(4.17) $$d(\theta_1 \wedge \cdots \wedge \theta_k) = \sum_{m=1}^{k} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k.$$

Proof. We prove it by induction on k. The case of $k = 1$ is clear. Now as inductive hypothesis, suppose the following holds

(4.18) $$d(\theta_1 \wedge \cdots \wedge \theta_{k-1}) = \sum_{m=1}^{k-1} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_{k-1}.$$

We view $\theta_1 \wedge \cdots \wedge \theta_{k-1}$ as a $k-1$-form and θ_k as a 1-form. It then follows that

(4.19)

$$\begin{aligned}
d(\theta_1 \wedge \cdots \wedge \theta_{k-1} \wedge \theta_k) &= d(\theta_1 \wedge \cdots \wedge \theta_{k-1}) \wedge \theta_k && \text{by (4.14)} \\
&\quad + (-1)^{k-1} (\theta_1 \wedge \cdots \wedge \theta_{k-1}) \wedge d\theta_k \\
&= \left(\sum_{m=1}^{k-1} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_{k-1} \right) \wedge \theta_k && (4.20) \\
&\quad + (-1)^{k-1} (\theta_1 \wedge \cdots \wedge \theta_{k-1}) \wedge d\theta_k && \text{by (4.18)} \\
&= \sum_{m=1}^{k-1} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_{k-1} \wedge \theta_k && (4.21) \\
&\quad + (-1)^{k+1} (\theta_1 \wedge \cdots \wedge \theta_{k-1}) \wedge d\theta_k \\
&= \sum_{m=1}^{k-1} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_{k-1} \wedge \theta_k && (4.22) \\
&\quad + (-1)^{k+1} d\theta_k \wedge \theta_1 \wedge \cdots \wedge \theta_{k-1} && \text{by (3.5)} \\
&= \sum_{m=1}^{k} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k. && (4.23)
\end{aligned}$$

□

(4.24) Proposition. *If* $\mathrm{F} : \mathrm{U} \to \mathrm{V}$ *is a smooth map of Euclidean open sets and* $\omega \in \Omega^k(\mathrm{V})$ *and* $\eta \in \Omega^l(\mathrm{V})$, *then* $\mathrm{F}^*(\omega \wedge \eta) = \mathrm{F}^*\omega \wedge \mathrm{F}^*\eta$

Proof. See [T11] Proposition 18.11. □

(4.25) Proposition. *Let* $\omega \in \Omega^k(\mathrm{U})$ *and* $\eta \in \Omega^l(\mathrm{U})$ *and* V *be a vector field on* U. *Then the Lie derivative has the following properties :*

(a) *We have*

$$\mathrm{L}_{\mathrm{V}}(\omega \wedge \eta) = \mathrm{L}_{\mathrm{V}}\omega \wedge \eta + \omega \wedge \mathrm{L}_{\mathrm{V}}\eta \tag{4.26}$$

(b) *The Lie derivative commutes with the exterior derivative, i.e.,* $\mathrm{L}_{\mathrm{V}}(d\omega) = d(\mathrm{L}_{\mathrm{V}}\omega)$.

(c) (Cartan's magic formula) *We have the following formula for computing the Lie derivative*

$$\mathrm{L}_{\mathrm{V}} = d\iota_{\mathrm{V}} + \iota_{\mathrm{V}}d. \tag{4.27}$$

Proof. See [T11] Proposition 20.10. □

As an easy corollary to (c) we have the following,

(4.28) Corollary. *If* ω *is a cocyle form, then* $\mathrm{L}_{\mathrm{V}}\omega$ *is a coboundary.*

Proof. If ω satisfies $d\omega = 0$, then

$$\mathrm{L}_{\mathrm{V}}\omega = (d\iota_{\mathrm{V}} + \iota_{\mathrm{V}}d)\omega = d\iota_{\mathrm{V}}\omega + \iota_{\mathrm{V}}d\omega = d\iota_{\mathrm{V}}\omega. \tag{4.29}$$

Hence $\mathrm{L}_{\mathrm{X}}\omega$ is the coboundary of $\iota_{\mathrm{V}}\omega$. □

CHAPTER 5

SUBCOMPLEXES OF INVARIANT FORMS

The theory of differential forms, consequently the de Rham complex, and vector fields applies to general manifolds thanks to propositions (4.13b) and (4.24) where F is the transition map between charts on the manifold. In particular, we want to consider the case when the manifolds are Lie groups.

Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{T}_e\mathrm{G}$, and let $g \in \mathrm{G}$. We define the left translation map $\mathrm{L}_g : \mathrm{G} \to \mathrm{G}$ by $\mathrm{L}_g(q) = gq$. For a tangent vector v at q, we will use gv to denote $\mathrm{DL}_g(q)(v)$ where $\mathrm{DL}_g(q) : \mathrm{T}_q\mathrm{G} \to \mathrm{T}_{gq}\mathrm{G}$. We follow a similar notation for the right translation map $\mathrm{R}_g : \mathrm{G} \to \mathrm{G}$ defined by $\mathrm{R}_g(q) = qg$. For a tangent vector v, we will use vg to denote $\mathrm{DR}_g(q)(v)$ where $\mathrm{DR}_g(q) : \mathrm{T}_q\mathrm{G} \to \mathrm{T}_{qg}\mathrm{G}$. A k-form on G is called *left-invariant* if $\mathrm{L}_g^*\omega = \omega$ for each $g \in \mathrm{G}$. If ω is a left-invariant k form, and $\delta_i g$ are members of $\mathrm{T}_g\mathrm{G}$, then it follows that

$$\omega(\delta_1 g, \ldots, \delta_k g) = (\mathrm{L}_{g^{-1}})^*\omega(\delta_1 g, \ldots, \delta_k g) \tag{5.1}$$

$$= \omega_e(g^{-1}\delta_1 g, \ldots, g^{-1}\delta_k g), \tag{5.2}$$

telling us that ω is uniquely determined by its value at $\mathrm{T}_e\mathrm{G} = \mathfrak{g}$. Hence we have the following proposition,

(5.3) Proposition. *The ring of left-invariant forms on* G*, denoted* $\Omega^\bullet(\mathrm{G})^{\mathrm{G}}$ *and the ring* $\bigwedge^\bullet(\mathfrak{g}^*)$ *(with multiplication given by the wedge product) are isomorphic as graded algebras via the map* $\omega \mapsto \omega_e$.

Now we will compute what becomes of the exterior derivative d under this isomorphism.

(5.4) Proposition. *Let $\omega \in \Omega^k(\mathrm{G})^{\mathrm{G}}$, then we have*

$$d\omega(\mathrm{Z}_0, \dots, \mathrm{Z}_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \dots, \widehat{\mathrm{Z}}_i, \dots, \widehat{\mathrm{Z}}_j, \dots, \mathrm{Z}_k) \tag{5.5}$$

for $\mathrm{Z}_i \in \mathfrak{g}$.

(5.6) Remarks. Proposition (5.4) is [B72, III.3.14, Prop.51], or [M08, Lemma 14.14] or with different normalization [C48, Thm.9.1]. In [B72] and [M08] it is proven using "Palais' formula" while [C48] proves it using induction on k, the base case being the well known *Maurer-Cartan formula*

$$d\omega(\mathrm{Z}_0, \mathrm{Z}_1) = -\omega([\mathrm{Z}_0, \mathrm{Z}_1]). \tag{5.7}$$

Restricted to the case of matrix groups we will give an easier, more direct proof based on the Maurer-Cartan formula for which we have the following.

Proof of Proposition (5.4). We first prove (5.7). To prove it we compute the exterior derivative of the $\mathfrak{g}$-valued Maurer-Cartan 1-form Θ defined by $\Theta(\delta g) = g^{-1}\delta g$. Let's first compute the derivative of the inversion map $g \mapsto g^{-1}$. Deriving both sides of $e = g.g^{-1}$ we get that

$$0 = \delta(g.g^{-1}) = \delta g.g^{-1} + g\delta[g^{-1}], \tag{5.8}$$

where in the second equality follows by the product rule. Hence we see that

$$\delta[g^{-1}] = -g^{-1}\delta g.g^{-1}. \tag{5.9}$$

Now with this, using (4.10) we compute the exterior derivative of Θ as

$$d\Theta(\delta_0 g, \delta_1 g) = \delta_0[g^{-1}]\delta_1 g - \delta_1[g^{-1}]\delta_0 g \tag{5.10}$$

$$= -g^{-1}\delta_0 g.g^{-1}\delta_1 g + g^{-1}\delta_1 g.g^{-1}\delta_0 g \tag{5.11}$$

$$= [g^{-1}\delta_1 g, g^{-1}\delta_0 g] \tag{5.12}$$

$$= [\Theta(\delta_1 g), \Theta(\delta_0 g)]. \tag{5.13}$$

Let $\omega(\delta g) = \langle \omega_e, g^{-1}\delta g\rangle = \langle \omega_e, \Theta(\delta g)\rangle$ (5.2) be a left-invariant 1-form. Taking $g = e$ and $\delta_i g = \mathrm{Z}_i \in \mathfrak{g}$, (5.13) implies that

$$d\omega(\mathrm{Z}_0, \mathrm{Z}_1) = \langle \omega_e, d\Theta(\mathrm{Z}_0, \mathrm{Z}_1)\rangle = \langle \omega_e, [\mathrm{Z}_1, \mathrm{Z}_0]\rangle = -\omega([\mathrm{Z}_0, \mathrm{Z}_1]). \tag{5.14}$$

Now we argue it is enough to prove (5.4) when $\omega = \theta_1 \wedge \cdots \wedge \theta_k$ where each θ_i is a left-invariant 1-form on G. Indeed by Proposition (3.8) each member of $\bigwedge^k \mathfrak{g}^*$ may be expressed as a linear combination of products of the form $e_{i_1} \wedge \cdots \wedge e_{i_k}$. By linearity of the exterior derivative it is enough to prove (5.4) on the k-form $\omega = \theta_1 \wedge \cdots \wedge \theta_k$ obtained when ω_e in (5.3) is a single such product. Then

$$
\begin{aligned}
&(d\omega)(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) \\
&\quad = \sum_{m=1}^{k} (-1)^{m+1} (d\theta_m \wedge \theta_1 \wedge \ldots \wedge \widehat{\theta}_m \wedge \ldots \wedge \theta_k)(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) && \text{by (4.16)} \\
&\quad = \sum_{m=1}^{k} (-1)^{m+1} \sum_{0 \le i < j \le k} (-1)^{i+j-1} d\theta_m(\mathrm{Z}_i, \mathrm{Z}_j)(\theta_1 \wedge \ldots \wedge \widehat{\theta}_m \wedge \ldots \wedge \theta_k) \\
&\qquad\qquad (\mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k) && \text{by (3.9b)} \\
&\quad = \sum_{0 \le i < j \le k} (-1)^{i+j} \sum_{m=1}^{k} (-1)^{m+1} \theta_m([\mathrm{Z}_i, \mathrm{Z}_j])(\theta_1 \wedge \ldots \wedge \widehat{\theta}_m \wedge \ldots \wedge \theta_k) \\
&\qquad\qquad (\mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k) && \text{by (5.14)} \\
&\quad = \sum_{0 \le i < j \le k} (-1)^{i+j} \det \begin{pmatrix} \theta_1([\mathrm{Z}_i,\mathrm{Z}_j]) & \cdots & \widehat{\theta_1(\mathrm{Z}_i)} & \cdots \widehat{\theta_1(\mathrm{Z}_j)} & \cdots \theta_1(\mathrm{Z}_k) \\ \vdots & & \vdots & \vdots & \vdots \\ \theta_k([\mathrm{Z}_i,\mathrm{Z}_j]) & \cdots & \widehat{\theta_k(\mathrm{Z}_i)} & \cdots \widehat{\theta_k(\mathrm{Z}_j)} & \cdots \theta_k(\mathrm{Z}_k) \end{pmatrix} && \text{by cofactor expansion along col. 1 and (3.16)} \\
&\quad = \sum_{0 \le i < j \le k} (-1)^{i+j} (\theta_1 \wedge \cdots \wedge \theta_k)([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k) && \text{by (3.16)} \\
&\quad = \sum_{0 \le i < \le j} (-1)^{i+j} \omega([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k).
\end{aligned}
$$

Hence the proposition is proved. □

CHAPTER 6

LIE ALGEBRA COHOMOLOGY

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$. We use $\bigwedge^k(\mathfrak{g}^*)$ to denote the space of all k-linear, alternating real valued functions $\underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{\text{k times}} \to \mathbf{R}$, and we will call such functions *k-cochains*. The definitions and results of Chapter 4 still apply since $\mathfrak{g}^*$ is a vector space over $\mathbf{R}$. In particular, we have a notion of a wedge and interior product of cochains from (3.3) and (3.6).

(6.1) Definition. For each cochain $f \in \bigwedge^k(\mathfrak{g}^*)$ we take inspiration from (5.5) to define the *coboundary* of f to be the $k+1$ cochain df defined by

$$df(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} f([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k). \tag{6.2}$$

When $k = 0$, we define df to be 0.

(6.3) Proposition. *When $\mathfrak{g}$ is a Lie algebra of a Lie group G, the coboundary operator d in (6.2) satisfies*

(a) (Graded Leibniz) *Let $f \in \bigwedge^k(\mathfrak{g}^*)$ and $g \in \bigwedge^l(\mathfrak{g}^*)$ then*

$$d(f \wedge g) = df \wedge g + (-1)^k f \wedge dg \tag{6.4}$$

(b) (Poincaré's Theorem) $\mathrm{Im}\, d \subset \mathrm{Ker}\, d$*, i.e.,* $d^2 = 0$.

(c) *The cohomology ring of $(\Omega^\bullet(\mathrm{G})^{\mathrm{G}}, d)$ is isomorphic to the cohomology ring of $(\bigwedge^\bullet(\mathfrak{g}), d)$, the so-called **Lie algebra cohomology ring*** $\mathrm{H}^\bullet(\mathfrak{g}) := \frac{\mathrm{Z}^\bullet(\mathfrak{g})}{\mathrm{B}^\bullet(\mathfrak{g})}$.

Proof. These properties follow immediately from results of earlier chapters. In particular, (4.24) shows that $(\Omega^\bullet(\mathrm{G})^{\mathrm{G}}, d)$ is a subcomplex of $(\Omega^\bullet(\mathrm{G}), d)$, which by (5.4) and (2.6) is isomorphic to $(\bigwedge^\bullet(\mathfrak{g}), d)$. Hence (a) and (b) follow from (4.13a) and (4.13c).

To prove (c) we note by (5.5) and (6.2) the following diagram commutes

$$
\begin{tikzcd}
\Omega^k(\mathrm{G})^{\mathrm{G}} \arrow[r, "d"] \arrow[d, "(\cdot)_e"] & \Omega^{k+1}(\mathrm{G})^{\mathrm{G}} \arrow[d, "(\cdot)_e"] \\
\bigwedge^k(\mathfrak{g}^*) \arrow[r, dotted, "d"] & \mathrm{C}^{k+1}(\mathfrak{g}).
\end{tikzcd} \tag{6.5}
$$

Hence $\omega \mapsto \omega_e$ defines a isomorphism of complexes and hence by (2.6) it induces a isomorphism between the cohomoloiges. □

The above proof relies on the assumption of existence of a Lie group G. We choose to give direct proofs of (6.3a) and (6.3b) without the assumption of G. We first establish an analogue of (4.16).

(6.6) Lemma. *Let each θ_i be in $\mathfrak{g}^*$. Then* (6.2) *satisfies*

$$d(\theta_1 \wedge \cdots \wedge \theta_k) = \sum_{m=1}^{k} (-1)^{m+1} d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k \tag{6.7}$$

$$= \sum_{m=1}^{k} (-1)^{m+1} \theta_1 \wedge \cdots \wedge d\theta_m \wedge \cdots \wedge \theta_k. \tag{6.8}$$

The following proof reverses the argument of proving (5.5) using (4.16) and (5.7).

Proof of the lemma. Write $h = \theta_1 \wedge \cdots \wedge \theta_k$. Then (6.2) gives

$$
\begin{aligned}
dh(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) &= \sum_{0 \le i < j \le k} (-1)^{i+j} (\theta_1 \wedge \cdots \wedge \theta_k)([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k) \\
&= \sum_{0 \le i < j \le k} (-1)^{i+j} \det \begin{pmatrix} \theta_1([\mathrm{Z}_i,\mathrm{Z}_j]) & \theta_1(\mathrm{Z}_0) & \cdots & \widehat{\theta_1(\mathrm{Z}_i)} & \cdots & \widehat{\theta_1(\mathrm{Z}_j)} & \cdots & \theta_1(\mathrm{Z}_k) \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \theta_k([\mathrm{Z}_i,\mathrm{Z}_j]) & \theta_k(\mathrm{Z}_0) & \cdots & \widehat{\theta_k(\mathrm{Z}_i)} & \cdots & \widehat{\theta_k(\mathrm{Z}_j)} & \cdots & \theta_k(\mathrm{Z}_k) \end{pmatrix} \\
&= \sum_{0 \le i < j \le k} (-1)^{i+j} \sum_{m=1}^{k} (-1)^{m+1} \theta_m([\mathrm{Z}_i, \mathrm{Z}_j]) (\theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k) \\
&\qquad (\mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k) \\
&= \sum_{m=1}^{k} (-1)^{m+1} \sum_{0 \le i < j \le k} (-1)^{i+j-1} d\theta_m(\mathrm{Z}_i, \mathrm{Z}_j) (\theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k) \\
&\qquad (\mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k)
\end{aligned}
$$

$$= \sum_{m=1}^{k} (-1)^{m+1} (d\theta_m \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k)(\mathrm{Z}_0, \ldots, \mathrm{Z}_k),$$

where the second equality is the exterior algebra lemma (3.16), the third is cofactor expansion along the first column, the fourth is the $k = 1$ case $d\theta(\mathrm{Z}_0, \mathrm{Z}_1) = -\theta([\mathrm{Z}_0, \mathrm{Z}_1])$, and the fifth is the exterior algebra lemma (3.9). This proves (6.7). Then (6.8) follows the fact that the 2-forms $d\theta_m$ wedge-commute with everything by (3.5). □

Proof of graded Leibniz (6.3a). By the usual argument (3.8), it is enough to prove it for monomials $f = \theta_1 \wedge \cdots \wedge \theta_k$ and $g = \theta_{k+1} \wedge \cdots \wedge \theta_{k+l}$. Then (6.8) gives

$$\begin{aligned}
d(f \wedge g) &= d(\theta_1 \wedge \cdots \wedge \theta_{k+l}) \\
&= \sum_{m=1}^{k+l} (-1)^{m+1} \theta_1 \wedge \cdots \wedge d\theta_m \wedge \cdots \wedge \theta_{k+l} \\
&= \sum_{m=1}^{k} (-1)^{m+1} \theta_1 \wedge \cdots \wedge d\theta_m \wedge \cdots \wedge \theta_k \wedge g \\
&\qquad + \sum_{m=k+1}^{k+l} (-1)^{m+1} f \wedge \theta_{k+1} \wedge \cdots \wedge d\theta_m \wedge \cdots \wedge \theta_{k+l} \\
&= df \wedge g + f \wedge \sum_{i=1}^{l} (-1)^{k+i+1} \theta_{k+1} \wedge \cdots \wedge d\theta_{k+i} \wedge \cdots \wedge \theta_{k+l} \\
&= df \wedge g + (-1)^k f \wedge dg.
\end{aligned}$$

□

Proof of (6.3b). We will induct on k. If $f \in \mathrm{C}^1(\mathfrak{g})$ we have that

(6.9)

$$d^2 f(\mathrm{Z}_0, \mathrm{Z}_1, \mathrm{Z}_2) = -df([\mathrm{Z}_0, \mathrm{Z}_1], \mathrm{Z}_2) + df([\mathrm{Z}_0, \mathrm{Z}_2], \mathrm{Z}_1) - df([\mathrm{Z}_1, \mathrm{Z}_2], \mathrm{Z}_0) \quad \text{by (6.2)}$$

(6.10) $$= f([\mathrm{Z}_2, [\mathrm{Z}_0, \mathrm{Z}_1]]) - f([\mathrm{Z}_1, [\mathrm{Z}_0, \mathrm{Z}_2]]) + f([\mathrm{Z}_0, [\mathrm{Z}_1, \mathrm{Z}_2]]) \quad \text{by (6.2)}$$

(6.11) $$= f([\mathrm{Z}_0, [\mathrm{Z}_1, \mathrm{Z}_2]] + [\mathrm{Z}_1, [\mathrm{Z}_2, \mathrm{Z}_0]] + [\mathrm{Z}_2, [\mathrm{Z}_0, \mathrm{Z}_1]]) \quad \text{by antisymm of } [\cdot,\cdot]$$

(6.12) $$= f(0) \quad \text{by Jacobi id.}$$

$$(6.13) \qquad = 0.$$

Hence if $f \in \mathrm{C}^1(\mathfrak{g})$, $d^2 f = 0$ (**B.C**). Since the exterior derivative d is a linear map, by Proposition (3.8) it is enough to prove $d^2 f = 0$ when $f = e \wedge h$ where $e \in \mathrm{C}^1(\mathfrak{g})$ and $h = e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \in \mathrm{C}^{k-1}(\mathfrak{g})$. Now suppose that $d^2 h = 0$ (**I.H**). Then we have that

$$
\begin{aligned}
&(6.14) \qquad d^2(e \wedge h) = d(de \wedge h - e \wedge dh) && \text{by } (6.4)\\
&(6.15) \qquad = d^2 e \wedge h + de \wedge dh - de \wedge dh + de \wedge d^2 h && \text{by } (6.4)\\
&(6.16) \qquad = 0 + de \wedge dh - de \wedge dh + 0 && \text{by } (\textbf{B.C}), (\textbf{I.H})\\
&(6.17) \qquad = 0.
\end{aligned}
$$

Hence by induction (b) is proved. □

6.1. Relative Lie Algebra Cohomology

As we did with the exterior derivative d (4.8) in (6.2), we recast the Lie Derivative (4.11) into the setting of $\bigwedge^k(\mathfrak{g}^*)$ with the following definition.

(6.18) Definition. Let f be in $\bigwedge^k(\mathfrak{g}^*)$. For each $\mathrm{X} \in \mathfrak{g}$ we define the *Lie derivative* of f by X to be the linear map $\mathrm{L}_\mathrm{X} : \bigwedge^k(\mathfrak{g}^*) \to \bigwedge^k(\mathfrak{g}^*)$ defined by

$$(6.19) \qquad (\mathrm{L}_\mathrm{X} f)(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) = \sum_{j=1}^{k} (-1)^j f([\mathrm{X}, \mathrm{Z}_j], \mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k).$$

(6.20) Remarks. As in Proposition (6.3), the upcoming propositions follow from the analogous results in chapter 3. In particular (6.21), (6.30), (6.33), (6.35) follow from (4.25c), (4.25b), (4.28), (4.25a) respectively. In addition we choose to give direct proofs.

(6.21) Proposition. *We have the following formula for computing the Lie derivative*

$$(6.22) \qquad \mathrm{L}_\mathrm{X} = d\iota_\mathrm{X} + \iota_\mathrm{X} d.$$

Proof. We have that

(6.23)

$$d(\iota_{Z_0} f)(Z_1, \dots, Z_k) = \sum_{1\le i<j\le k} (-1)^{i+j} \iota_{Z_0} f([Z_i, Z_j], Z_1, \dots, \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_k) \quad \text{by (6.2)}$$

(6.24)
$$= \sum_{1\le i<j\le k} (-1)^{i+j} f(Z_0, [Z_i, Z_j], Z_1, \dots, \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_k) \quad \text{by (3.6)}$$

(6.25)
$$= -\sum_{1\le i<j\le k} (-1)^{i+j} f([Z_i, Z_j], Z_0, Z_1, \dots, \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_k) \quad \text{by anti-sym. of } f,$$

while on the other hand

(6.26)

$$\iota_{Z_0} df(Z_1, \dots, Z_k) = df(Z_0, \dots, Z_k) \quad \text{by (3.6)}$$

(6.27)
$$= \sum_{0\le i<j\le k} (-1)^{i+j} f([Z_i, Z_j], Z_0, \dots, \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_j) \quad \text{by (6.2)}$$

(6.28)
$$= \sum_{j=1}^{k} (-1)^j f([Z_0, Z_j], Z_1 \dots, \widehat{Z}_j, \dots, Z_k)$$

(6.29)
$$+ \sum_{1\le i<j\le k} (-1)^{i+j} f([Z_i, Z_j], Z_0, Z_1, \dots, \widehat{Z}_i, \dots, \widehat{Z}_j, \dots, Z_k).$$

Adding the results together we see that (6.24) and (6.29) cancel out, leaving us with (6.28) which is equal to $L_{Z_0} f(Z_1, \dots, Z_k)$. This proves (6.22). □

(6.30) Proposition. *The Lie derivative commutes with the coboundary, i.e.,* $L_X d = dL_X$.

Proof. Let f be a cochain. Recalling that $d^2 = 0$, by (6.22) we have that

(6.31)
$$L_X d = (d\iota_X + \iota_X d)df = d\iota_X d + \iota_X d^2 = d\iota_X d,$$

while at the same time

(6.32)
$$dL_X = d(d\iota_X + \iota_X d) = d^2\iota_X + d\iota_X d = d\iota_X d.$$

Hence we see that $L_X d = dL_X$. □

(6.33) Proposition. *If f is a cocycle then $\mathrm{L}_{\mathrm{X}} f$ a coboundary.*

Proof. This also follows immediately from (6.22). Indeed if f satisfies $df = 0$, then

$$\mathrm{L}_{\mathrm{X}} f = (d\iota_{\mathrm{X}} + \iota_{\mathrm{X}} d) f = d\iota_{\mathrm{X}} f + \iota_{\mathrm{X}} df = d\iota_{\mathrm{X}} f. \tag{6.34}$$

Hence $\mathrm{L}_{\mathrm{X}} f$ is the coboundary of $\iota_{\mathrm{X}} f$. □

(6.35) Proposition. *For cochains f and g, we have that*

$$\mathrm{L}_{\mathrm{X}}(f \wedge g) = \mathrm{L}_{\mathrm{X}}(f) \wedge g + f \wedge \mathrm{L}_{\mathrm{X}}(g). \tag{6.36}$$

To prove the previous proposition, first we have a lemma.

(6.37) Lemma. *Let each θ_i be a one form, then we have*

$$\mathrm{L}_{\mathrm{X}}(\theta_1 \wedge \ldots \wedge \theta_k) = \sum_{m=1}^{k} \theta_1 \wedge \ldots \wedge \mathrm{L}_{\mathrm{X}} \theta_m \wedge \ldots \wedge \theta_k. \tag{6.38}$$

Proof of the lemma. Write $h = \theta_1 \wedge \cdots \wedge \theta_k$. Then by (6.19) we have

$$\begin{aligned}
\mathrm{L}_{\mathrm{X}} h(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) &= \sum_{m=1}^{k} (\theta_1 \wedge \cdots \wedge \theta_k)([\mathrm{X}, \mathrm{Z}_m], \mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_m, \ldots, \mathrm{Z}_k) \\
&= \sum_{m=1}^{k} \det \begin{pmatrix} \theta_1([\mathrm{X},\mathrm{Z}_m]) & \theta_1(\mathrm{Z}_1) & \cdots & \widehat{\theta_1(\mathrm{Z}_m)} & \cdots & \theta_1(\mathrm{Z}_k) \\ \vdots & \vdots & & \vdots & & \vdots \\ \theta_k([\mathrm{X},\mathrm{Z}_m]) & \theta_k(\mathrm{Z}_1) & \cdots & \widehat{\theta_k(\mathrm{Z}_m)} & \cdots & \theta_k(\mathrm{Z}_k) \end{pmatrix} \\
&= \sum_{m=1}^{k} \sum_{j=1}^{k} (-1)^{j+1} \theta_j([\mathrm{X}, \mathrm{Z}_m]) (\theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k) \\
&\qquad\qquad (\mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_m, \ldots, \mathrm{Z}_k) \\
&= \sum_{j=1}^{k} (-1)^j \sum_{m=1}^{k} \mathrm{L}_{\mathrm{X}} \theta_j(\mathrm{Z}_m) (\theta_1 \wedge \cdots \wedge \widehat{\theta}_m \wedge \cdots \wedge \theta_k) \\
&\qquad\qquad (\mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_m, \ldots, \mathrm{Z}_k) \\
&= \sum_{j=1}^{k} (-1)^j (\mathrm{L}_{\mathrm{X}} \theta_j \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_j \wedge \cdots \wedge \theta_k)(\mathrm{Z}_1, \ldots, \mathrm{Z}_k), \\
&= \sum_{j=1}^{k} \theta_1 \wedge \ldots \wedge \mathrm{L}_{\mathrm{X}} \theta_j \wedge \ldots \wedge \theta_k(\mathrm{Z}_1, \ldots, \mathrm{Z}_k).
\end{aligned}$$

where the second equality is the exterior algebra lemma (3.16), the third is cofactor expansion along the first column, the fourth is the $k = 1$ case $\mathrm{L_X}\theta(\mathrm{Z}_1) = -\theta([\mathrm{X}, \mathrm{Z}_1])$, the fifth is the exterior algebra lemma (3.9), and the sixth follows from the graded commutativity of the wedge product. This proves (6.38). $\square$

Now with this lemma we can prove (6.35)

Proof of (6.35). By the usual argument (3.8), it is enough to prove it for monomials $f = \theta_1 \wedge \cdots \wedge \theta_k$ and $g = \theta_{k+1} \wedge \cdots \wedge \theta_{k+l}$. Then (6.38) gives

$$\begin{aligned}
\mathrm{L_X}(f \wedge g) &= \mathrm{L_X}(\theta_1 \wedge \cdots \wedge \theta_{k+l}) \\
&= \sum_{m=1}^{k+l} \theta_1 \wedge \cdots \wedge \mathrm{L_X}\theta_m \wedge \cdots \wedge \theta_{k+l} \\
&= \sum_{m=1}^{k} \theta_1 \wedge \cdots \wedge \mathrm{L_X}\theta_m \wedge \cdots \wedge \theta_k \wedge g \\
&\qquad + \sum_{m=k+1}^{k+l} f \wedge \theta_{k+1} \wedge \cdots \wedge \mathrm{L_X}\theta_m \wedge \cdots \wedge \theta_{k+l} \\
&= \mathrm{L_X} f \wedge g + f \wedge \sum_{i=1}^{l} \theta_{k+1} \wedge \cdots \wedge \mathrm{L_X}\theta_{k+i} \wedge \cdots \wedge \theta_{k+l} \\
&= \mathrm{L_X} f \wedge g + f \wedge \mathrm{L_X} g.
\end{aligned}$$

$\square$

(6.39) Definition. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. We call a cochain $f \in \bigwedge^k(\mathfrak{g}^*)$ an $\mathfrak{h}$-*basic* if it is $\mathfrak{h}$-*horizontal* and $\mathfrak{h}$-*invariant.* Respectively this means that

(6.40) $\qquad f(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) = 0 \qquad$ whenever one of the $\mathrm{Z}_j \in \mathfrak{g}$ belongs to $\mathfrak{h}$.

(6.41) $\qquad \mathrm{L_X} f = 0 \qquad$ for $\mathrm{X} \in \mathfrak{h}$.

The space of $\mathfrak{h}$-basic cochains $\bigwedge^k(\mathfrak{g}^*)_{\mathrm{basic}}$ forms a subspace of $\bigwedge^k(\mathfrak{g}^*)$. We note that the wedge product of $\mathfrak{h}$-basic cochains is $\mathfrak{h}$-basic cochain. Indeed if $f \in \bigwedge^k(\mathfrak{g}^*)_{\mathrm{basic}}$

and $g \in \bigwedge^l(\mathfrak{g}^*)_{\mathrm{basic}}$ then clearly $f \wedge g$ satisfies (6.40) and it satisfies (6.41) by (6.35). Furthermore the coboundary of a $\mathfrak{h}$-basic cochain is also $\mathfrak{h}$-basic, for if $\mathrm{X} \in \mathfrak{h}$ we have by (6.30)

(6.42) $$\mathrm{L}_{\mathrm{X}}(df) = d(\mathrm{L}_{\mathrm{X}} f) = d(0) = 0.$$

hence df is $\mathfrak{h}$-invariant. If $\mathrm{Z}_0 \in \mathfrak{h}$ then

(6.43)
$$df(\mathrm{Z}_0, \ldots, \mathrm{Z}_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} f([\mathrm{Z}_i, \mathrm{Z}_j], \mathrm{Z}_0, \ldots, \widehat{\mathrm{Z}}_i, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k)$$

Each summand with $i \neq 0$ evaluates to 0 since f is $\mathfrak{h}$-horizontal, hence

(6.44) $$= \sum_{1 \le j \le k} (-1)^j f([\mathrm{Z}_0, \mathrm{Z}_j], \mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k)$$

(6.45) $$= \mathrm{L}_{\mathrm{Z}_0} f(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) \qquad \text{by (6.19)}$$

(6.46) $$= 0$$

where the last equality follows from f being $\mathfrak{h}$-invariant and $\mathrm{Z}_0 \in \mathfrak{h}$. Hence df is $\mathfrak{h}$-horizontal.

In other words $(\bigwedge^\bullet(\mathfrak{g}^*)_{\mathrm{basic}}, d)$ is a subcomplex of $(\bigwedge^\bullet(\mathfrak{g}), d)$ and its cohomology is called the relative cohomology $\mathrm{H}^k(\mathfrak{g}, \mathfrak{h})$ with

(6.47) $$\mathrm{H}^k(\mathfrak{g}, \mathfrak{h}) := \frac{\mathrm{Z}^k(\mathfrak{g}) \cap \mathrm{C}^k(\mathfrak{g}, \mathfrak{h})}{\mathrm{B}^k(\mathfrak{g}) \cap \mathrm{C}^k(\mathfrak{g}, \mathfrak{h})}.$$

Let G be a Lie group and H be a closed connected subgroup of G. Then $(\Omega^\bullet(\mathrm{G}/\mathrm{H})^{\mathrm{G}}, d)$ is a subcomplex of $(\Omega^\bullet(\mathrm{G}/\mathrm{H}), d)$; while we won't use it we note that the following proposition is analogous to (6.3c).

(6.48) Proposition. *The cohomology ring of* $(\Omega^\bullet(\mathrm{G}/\mathrm{H})^{\mathrm{G}}, d)$ *is isomorphic to the relative Lie algebra cohomology* $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h})$.

Proof. See [B00, 1.6] □

(6.49) Proposition. *When $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, all $\mathfrak{h}$-horizontal cochains are automatically $\mathfrak{h}$-basic.*

Proof. Suppose that f is a $\mathfrak{h}$-horizontal cochain and let $\mathrm{X} \in \mathfrak{h}$. Then by the definition of $\mathrm{L}_{\mathrm{X}} f$

$$(\mathrm{L}_{\mathrm{X}} f)(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) = \sum_{j=1}^{k} (-1)^{j+1} f([\mathrm{X}, \mathrm{Z}_j], \mathrm{Z}_1, \ldots, \widehat{\mathrm{Z}}_j, \ldots, \mathrm{Z}_k). \tag{6.50}$$

Since $\mathfrak{h}$ is an ideal we have that $[\mathrm{X}, \mathrm{Z}_j] \in \mathfrak{h}$ for each j, and since f is $\mathfrak{h}$-horizontal, each summand will evaluate to 0. Hence $\mathrm{L}_{\mathrm{X}} f = 0$. □

(6.51) Proposition. *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the relative Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{h})$ and the Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{g}/\mathfrak{h})$ are isomorphic as graded rings.*

Proof. We will use $\overline{\mathrm{Z}}_i$ to denote the coset $\mathrm{Z}_i + \mathfrak{h}$. Consider the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} : \mathrm{Z}_i \mapsto \overline{\mathrm{Z}}_i$, then dual to this we have the injective map $\pi^* : \bigwedge^k (\mathfrak{g}/\mathfrak{h})^* \to \bigwedge^k (\mathfrak{g}^*)$ defined by

$$\pi^*(f)(\mathrm{Z}_1, \ldots, \mathrm{Z}_k) := f(\pi(\mathrm{Z}_1), \ldots, \pi(\mathrm{Z}_k)). \tag{6.52}$$

We will prove that the image of π^* is exactly $\bigwedge^k (\mathfrak{g})_{\mathrm{basic}}$, whence π^* defines an isomorphism of complexes between $(\bigwedge^\bullet (\mathfrak{g}/\mathfrak{h})^*, d)$ and $(\bigwedge^\bullet (\mathfrak{g}^*)_{\mathrm{basic}}, d)$. For any $f \in \bigwedge^k (\mathfrak{g}/\mathfrak{h})^*$ we must show that $\pi^*(f)$ is $\mathfrak{h}$-basic. By the previous proposition this can be achieved by showing that $\pi^*(f)$ is $\mathfrak{h}$-horizontal. Indeed, if $\mathrm{Z}_i \in \mathfrak{h}$, then $\pi(\mathrm{Z}_i) = 0$. It then follows that

$$\pi^*(f)(\mathrm{Z}_1, \ldots, \mathrm{Z}_i, \ldots, \mathrm{Z}_k) = f(\pi(\mathrm{Z}_1), \ldots, \pi(\mathrm{Z}_i), \ldots, \pi(\mathrm{Z}_k)) \tag{6.53}$$

$$= f(\overline{\mathrm{Z}}_1, \ldots, 0, \ldots, \overline{\mathrm{Z}}_k) \tag{6.54}$$

$$= 0. \tag{6.55}$$

Hence we have proven that $\pi^*(f)$ is $\mathfrak{h}$-horizontal, and thus $\mathfrak{h}$-basic. We have then proven that the image of π^* is exactly $\bigwedge^k(\mathfrak{g})_{\text{basic}}$. So $\pi^* : \bigwedge^k(\mathfrak{g}/\mathfrak{h})^* \to \bigwedge^k(\mathfrak{g}^*)_{\text{basic}}$ defines a isomorphism of the complexes $(\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*, d)$ and $(\bigwedge^\bullet(\mathfrak{g}^*)_{basic}, d)$. $\square$

(6.56) Remarks. As an isomorphism, π^* has a well defined inverse that we will denote by $\pi_* : \bigwedge^k(\mathfrak{g}^*)_{\text{basic}} \to \bigwedge^k(\mathfrak{g}/\mathfrak{h})^*$, and which we will be use to prove (1.1).

CHAPTER 7

DIFFEOLOGY

For any set X, we call a map $\mathrm{P} : \mathrm{U} \to \mathrm{X}$ a *parameterization* of X if U is a Euclidean open set, which we recall is any open subset of $\mathbf{R}^n$ for any n. We denote the set of all parameterizations of X by $\mathcal{P}(\mathrm{X})$.

(7.1) Definition. A *diffeology* on a set X is a subset $\mathcal{D}$ of $\mathcal{P}(\mathrm{X})$ that satisfies the following axioms:

1. (*Covering*) $\mathcal{D}$ contains each constant parameterization. For each $x \in \mathrm{X}$ and $n \in \mathbf{N}$, $\mathcal{D}$ contains $\mathrm{P} : \mathbf{R}^n \to \{x\}$.
2. (*Locality*) If $(\mathrm{P} : \mathrm{U} \to \mathrm{X}) \in \mathcal{P}(\mathrm{X})$ is such that for each point $u \in \mathrm{U}$ there is an open neighborhood $\mathrm{V} \subset \mathrm{U}$ of u such that $\mathrm{P}|_{\mathrm{V}} \in \mathcal{D}$, then $\mathrm{P} \in \mathcal{D}$.
3. (*Smooth compatibility*) Let $(\mathrm{P} : \mathrm{U} \to \mathrm{X}) \in \mathcal{D}$. Then for each $n \in \mathbf{N}$ and open subset $\mathrm{V} \subset \mathbf{R}^n$, and every smooth map $\mathrm{F} : \mathrm{V} \to \mathrm{U}$, we have that $\mathrm{P} \circ \mathrm{F} \in \mathcal{D}$.

The elements of $\mathcal{D}$ are called *plots*. A *diffeological space* is a pair $(\mathrm{X}, \mathcal{D})$ where X is any set and $\mathcal{D}$ is a diffeology on X.

(7.2) Definition. Let $\mathcal{D}$ and $\mathcal{D}'$ be two diffeologies on a set X. We say that $\mathcal{D}'$ is *finer* that $\mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$. Also in this case we would call $\mathcal{D}$ *coarser* than $\mathcal{D}'$.

Each diffeology on X is coarser than the discrete diffeology, that is the diffeology consisting of all locally constant parameterizations. Each diffeology on X is finer than the trivial diffeology, that is the diffeology consisting of all parameterizations.

(7.3) Example. Every manifold X is naturally a diffeological space with its plots being the smooth maps from an Euclidean open set to X.

(7.4) Definition. Let $(\mathrm{X}, \mathcal{D})$ and $(\mathrm{X}', \mathcal{D}')$ be diffeological spaces, and let $\mathrm{F} : \mathrm{X} \to \mathrm{X}'$ be a map. We call F (diffeologically) *smooth* if for each $\mathrm{P} \in \mathcal{D}$, we also have $\mathrm{F} \circ \mathrm{P} \in \mathcal{D}'$.

(7.5) Proposition. *Let $(\mathrm{X}, \mathcal{D})$ be a diffeological space, X' be a set, and $\mathrm{F} : \mathrm{X} \to \mathrm{X}'$ be a map. There exist a finest diffeology on X' that will make F smooth. Denoted $\mathrm{F}_*(\mathcal{D})$, we call it the pushforward diffeology of $\mathcal{D}$.*

Proof. See [I13, 1.43] $\square$

(7.6) Definition. Let $\pi : \mathrm{X} \to \mathrm{X}'$ be a surjective map of diffeological spaces. If the pushforward of $\mathcal{D}$ coincides with the diffeology on X', then π is called a *subduction*.

(7.7) Proposition. *Let $(\mathrm{X}', \mathcal{D}')$ be a diffeological space, X be a set, and $\mathrm{F} : \mathrm{X} \to \mathrm{X}'$ be a map. There exist a coarsest diffeology on X that will make F smooth. Denoted $\mathrm{F}^*(\mathcal{D}')$, we call it the pull-back diffeology of $\mathcal{D}'$.*

Proof. See [I13, 1.26] $\square$

(7.8) Definition. Let $\psi : \mathrm{X} \to \mathrm{X}'$ be a injective map of diffeological spaces. If the pull-back of $\mathcal{D}'$ coincides with the diffeology of X, then ψ is called a *induction*.

(7.9) Definition. Let X be a diffeological space and let $\sim$ be an equivalence relation on X. The *quotient diffeology* on $\mathrm{X}/{\sim}$ is the pushforward of the diffeology of X by the natural map $\pi : \mathrm{X} \to \mathrm{X}/{\sim}$. By [I13, 1.43], its plots are the maps $\mathrm{P} : \mathrm{U} \to \mathrm{X}/{\sim}$ such that around each point in U there is a neighborhood $\mathrm{V} \subset \mathrm{U}$ and a plot $\mathrm{Q} : \mathrm{V} \to \mathrm{X}$ such that $\mathrm{P}|_{\mathrm{V}} = \pi \circ \mathrm{Q}$.

(7.10) Definition. Let X be a diffeological space and let Y be a subset of X. The *subset diffeology* on Y is the pull-back of the diffeology of X by the inclusion map i. By [I13, 1.26], its plots are the maps $\mathrm{P} : \mathrm{U} \to \mathrm{Y}$ such that $i \circ \mathrm{P}$ is a plot of X.

CHAPTER 8

THE DE RHAM COMPLEX OF A DIFFEOLOGICAL SPACE

Let us call *ordinary* the k-forms on Euclidean open sets and operations on them (pull-back, exterior derivative).

(8.1) Definition. Let X be a diffeological space. A (diffeological) *differential k-form* on X is a functional α, that associates to each plot $(\mathrm{P} : \mathrm{U} \to \mathrm{X})$ an ordinary k-form (4.1) on U, denoted $\mathrm{P}^*\alpha$, with the compatibility condition:

$$(\mathrm{P} \circ \mathrm{F})^*\alpha = \mathrm{F}^*\mathrm{P}^*\alpha \tag{8.2}$$

for all smooth $\mathrm{F} : \mathrm{V} \to \mathrm{U}$ where V is any other Euclidean open set, and where F^* denotes the pullback as in (4.3) Note that $\mathrm{P} \circ \mathrm{F}$ is also a plot by Definition (7.1). We denote the set of differential k-forms on X by $\Omega^k(\mathrm{X})$.

(8.3) Definition. Let X and Y be diffeological spaces, and α be a k-form on Y. Its *pull-back* $\mathrm{F}^*\alpha$ by a smooth map $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ is the k-form on X defined by: if P is a plot of X (so $\mathrm{F} \circ \mathrm{P}$ is a plot of Y), then

$$\mathrm{P}^*\mathrm{F}^*\alpha = (\mathrm{F} \circ \mathrm{P})^*\alpha. \tag{8.4}$$

(8.5) Proposition. *If* $\mathrm{G} : \mathrm{W} \to \mathrm{X}$ *is another smooth map, then* $(\mathrm{F} \circ \mathrm{G})^*\alpha = \mathrm{G}^*\mathrm{F}^*\alpha$.

Proof. See [I13, 6.32]. $\square$

(8.6) Remarks. If X is a manifold (7.3), forms (8.1) correspond 1-to-1 to forms of Chapter 4, by using ordinary pull-back. Furthermore, conditions (8.2) and (8.4) become special cases of (8.5).

The following is a criterion for when a k-form descends to a quotient.

(8.7) Theorem. *Let* X, X' *be diffeological spaces,* $\pi : \mathrm{X} \to \mathrm{X}'$ *be a subduction, and* $\alpha \in \Omega^k(\mathrm{X})$*. Then* α *is the pull-back of some* $\beta, i.e., \alpha = \pi^*(\beta)$ *if and only if for any two plots* P, Q *of* X *such that* $\pi \circ \mathrm{P} = \pi \circ \mathrm{Q}$*, we have that* $\mathrm{P}^*(\alpha) = \mathrm{Q}^*(\alpha)$*.*

Proof. See [S85, 2.5c] or [I13, 6.38]. □

(8.8) Proposition. *Let* X *and* X' *be diffeological spaces and let* $\pi : \mathrm{X} \to \mathrm{X}'$ *be a subduction. Then the pull-back* $\pi^* : \Omega^k(\mathrm{X}') \to \Omega^k(\mathrm{X})$ *is injective.*

Proof. See [S85, 2.5b] or [I13, 6.39]. □

(8.9) Definition. Let X be a diffeological space. The *exterior derivative* is the linear map $d : \Omega^k(\mathrm{X}) \to \Omega^{k+1}(\mathrm{X})$ defined as follows: If α is a k-form on X, then

$$\mathrm{P}^*(d\alpha) = d(\mathrm{P}^*\alpha) \tag{8.10}$$

for all plots P of X. Note that on the right hand side of we are taking the exterior derivative of the ordinary k-form $\mathrm{P}^*\alpha$ as defined in definition (4.8).

(8.11) Proposition. *The exterior derivative* d *and* F^* *commute for all smooth* F*.*

Proof. See [I13, 6.34] □

(8.12) Remarks. When X is a manifold (7.3), the exterior derivative d (8.10) is the same as the ordinary exterior derivative d (4.8) of chapter 4.

On ordinary differential forms, by Proposition (4.13), the exterior derivative commutes with pull-backs and satisfies $d^2 = 0$. These properties extend to the exterior derivative of differential forms, and we can thus define the (diffeological) kth de Rham cohomology group of X by

$$\mathrm{H}^k_{d\mathrm{R}}(\mathrm{X}) = \frac{\text{Kernel } d \cap \Omega^k(\mathrm{X})}{\text{Image } d \cap \Omega^k(\mathrm{X})}. \tag{8.13}$$

Part II

Main Result

CHAPTER 9

DIFFERENTIAL FORMS ON $\mathrm{X} = \mathrm{G}/\mathrm{H}$ FOR A DENSE SUBGROUP H

Let G be a Lie group and H a dense subgroup. It is known that H is a Lie group, with Lie algebra given by $\mathfrak{h} = \{\mathrm{Z} \in \mathfrak{g} : e^{t\mathrm{Z}} \in \mathrm{H}\ \forall\ t \in \mathbf{R}\}$ (see [B72, III.4.5] or [H12, 9.6.13]). Endow $\mathrm{X} = \mathrm{G}/\mathrm{H}$ with the quotient diffeology, and write $\Pi : \mathrm{G} \to \mathrm{X}$ for the natural projection, $\Pi(q) = q\mathrm{H}$.

(9.1) Proposition. *Pull-back via* Π *defines a bijection* Π^* *from* $\Omega^k(\mathrm{X})$ *onto the set of those* $\mu \in \Omega^k(\mathrm{G})$ *that are*

(a) right-invariant: $\mathrm{R}_g^*\mu = \mu$ *for all* $g \in \mathrm{G}$, *where* $\mathrm{R}_g : \mathrm{G} \to \mathrm{G}$ *maps* q *to* qg;

(b) horizontal: $\mu(g\mathrm{Z}_1, \dots, g\mathrm{Z}_k) = 0$ *whenever one of the* $\mathrm{Z}_j \in \mathfrak{g}$ *belongs to* $\mathfrak{h}$.

Proof. First, X having the quotient diffeology means that Π is a subduction, and this implies that Π^* is one-to-one by (8.8). Next we recall that H is canonically a Lie group, with Lie algebra $\mathfrak{h} = \{\mathrm{Z} \in \mathfrak{g} : e^{t\mathrm{Z}} \in \mathrm{H}$ for all $t \in \mathbf{R}\}$. As in [B24, 8.11], a key property is that

$$\text{(9.2)} \qquad \mathrm{G} \text{ normalizes } \mathfrak{h}: \qquad g\mathfrak{h}g^{-1} = \mathfrak{h} \quad \text{for all } g \in \mathrm{G}.$$

Indeed one knows that the normalizer $\mathrm{N}_\mathrm{G}(\mathfrak{h})$ is always a closed subgroup containing H [B72, III.9.4, Prop. 10], so it must be G by our density assumption. By deriving (9.2) at e one deduces that $\mathfrak{h}$ is an ideal, i.e., $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Suppose $\mu = \Pi^*\alpha$ for some $\alpha \in \Omega^k(\mathrm{X})$. We must prove (a) and (b). Now, the relation $\Pi \circ \mathrm{R}_h = \Pi$ implies $\mathrm{R}_h^*\Pi^*\alpha = \Pi^*\alpha$ for all $h \in \mathrm{H}$, and since H is dense, the same follows for all $g \in \mathrm{G}$: so μ is right-invariant. To see that it is horizontal, fix $g \in \mathrm{G}$ and consider the two plots $\mathrm{P}, \mathrm{Q} : \mathfrak{g} \times \mathfrak{h} \to \mathrm{G}$ sending $u = (\mathrm{Z}, \mathrm{W})$ to

$$\text{(9.3)} \qquad \mathrm{P}(u) = g e^{\mathrm{Z}} e^{\mathrm{W}}, \qquad \text{resp.} \qquad \mathrm{Q}(u) = g e^{\mathrm{Z}}.$$

(For these to be literally plots, use bases to identify $\mathrm{U} := \mathfrak{g} \times \mathfrak{h}$ with some $\mathbf{R}^n$.) Then clearly $\Pi \circ \mathrm{P} = \Pi \circ \mathrm{Q}$, so by the criterion of theorem (8.7) we have $\mathrm{P}^*\mu = \mathrm{Q}^*\mu$, i.e.,

$$\mu(\mathrm{P}_*(\delta_1 u), \ldots, \mathrm{P}_*(\delta_k u)) = \mu(\mathrm{Q}_*(\delta_1 u), \ldots, \mathrm{Q}_*(\delta_k u)) \tag{9.4}$$

for all choices of tangent vectors $\delta_i u \in \mathrm{T}_u\mathrm{U}$. Taking $u = (0,0)$, $\delta_1 u = (0, \mathrm{W}_1)$ and $\delta_i u = (\mathrm{Z}_i, 0)$ for $i \geqslant 2$, we obtain $\mathrm{P}_*(\delta_1 u) = g\mathrm{W}_1$, $\mathrm{Q}_*(\delta_1 u) = 0$ and $\mathrm{P}_*(\delta_i u) = \mathrm{Q}_*(\delta_i u) = g\mathrm{Z}_i$. So (9.4) says that $\mu(g\mathrm{W}_1, g\mathrm{Z}_2, \ldots, g\mathrm{Z}_k) = 0$, whence (by antisymmetry) our claim that μ is horizontal.

Conversely, suppose that $\mu \in \Omega^k(\mathrm{G})$ satisfies (a) and (b), and let $\mathrm{P}, \mathrm{Q} : \mathrm{U} \to \mathrm{G}$ be any two plots with $\Pi \circ \mathrm{P} = \Pi \circ \mathrm{Q}$. By (8.7) we must show that $\mathrm{P}^*\mu = \mathrm{Q}^*\mu$. Since $\Pi \circ \mathrm{P} = \Pi \circ \mathrm{Q}$ it follows that $\mathrm{R}(u) := \mathrm{P}(u)^{-1}\mathrm{Q}(u)$ defines a plot $\mathrm{R} : \mathrm{U} \to \mathrm{H}$. So $(g, gh, h) := (\mathrm{P}(u), \mathrm{Q}(u), \mathrm{R}(u))$ are ordinary smooth functions of u, and given tangent vectors $\delta_i u \in \mathrm{T}_u\mathrm{U}$ we may compute e.g. $\mathrm{Q}_*(\delta_i u)\mathrm{Q}(u)^{-1} \in \mathfrak{g}$ as

$$\begin{aligned} \delta_i[gh].(gh)^{-1} &= [\delta_i g.h + g\delta_i h](gh)^{-1} \\ &= \delta_i g.g^{-1} + g\delta_i h.h^{-1}g^{-1}. \end{aligned} \tag{9.5}$$

By (9.2), the second term here (call it W_i) is in $\mathfrak{h}$. Therefore we obtain

$$\begin{aligned} (\mathrm{Q}^*\mu)(\delta_1 u, \ldots, \delta_k u) &= \mu(\mathrm{Q}_*(\delta_1 u), \ldots, \mathrm{Q}_*(\delta_k u)) \\ &= \mu(\delta_1[gh], \ldots, \delta_k[gh]) \\ &= \mu(\delta_1[gh].(gh)^{-1}, \ldots, \delta_k[gh].(gh)^{-1}) && \text{by (a)} \\ &= \mu(\delta_1 g.g^{-1} + \mathrm{W}_1, \ldots, \delta_k g.g^{-1} + \mathrm{W}_k) && \text{by (9.5)} \\ &= \mu(\delta_1 g.g^{-1}, \ldots, \delta_k g.g^{-1}) && \text{by (b)} \\ &= \mu(\delta_1 g, \ldots, \delta_k g) && \text{by (a)} \\ &= \mu(\mathrm{P}_*(\delta_1 u), \ldots, \mathrm{P}_*(\delta_k u)) \\ &= (\mathrm{P}^*\mu)(\delta_1 u, \ldots, \delta_k u) \end{aligned} \tag{9.6}$$

as desired. □

CHAPTER 10

PASSAGE TO LEFT-INVARIANT FORMS

In chapter 6 we constructed the Lie algebra cohomology using left-invariant rather than right-invariant forms. Here we pass from right-invariant forms to left-invariant forms. To accomplish this we simply pull back by the inversion map, $\mathrm{inv}(g) = g^{-1}$:

(10.1) Corollary. *In the setting of* (9.1), *pull-back via* $\tilde{\Pi} = \Pi \circ \mathrm{inv}$ *defines a bijection* $\tilde{\Pi}^* = \mathrm{inv}^* \, \Pi^*$ *from* $\Omega^k(\mathrm{X})$ *onto the set of those* $\omega \in \Omega^k(\mathrm{G})$ *that are*

(a) left-invariant: $\mathrm{L}_g^*\omega = \omega$ *for all* $g \in \mathrm{G}$, *where* $\mathrm{L}_g : \mathrm{G} \to \mathrm{G}$ *maps* q *to* gq;

(b) horizontal: $\omega(g\mathrm{Z}_1, \ldots, g\mathrm{Z}_k) = 0$ *whenever one of the* $\mathrm{Z}_j \in \mathfrak{g}$ *belongs to* $\mathfrak{h}$.

Proof. This is simply a matter of checking that $\mu \in \Omega^k(\mathrm{G})$ is right-invariant and horizontal (9.1a,b) iff $\omega := \mathrm{inv}^* \mu$ is left-invariant and horizontal (10.1a,b). Now the elementary relation $\mathrm{inv} \circ \mathrm{L}_g = \mathrm{R}_{g^{-1}} \circ \mathrm{inv}$ show and (8.5) that (9.1a) implies (10.1a):

$$\mathrm{L}_g^*\omega = \mathrm{L}_g^* \, \mathrm{inv}^* \mu = \mathrm{inv}^* \, \mathrm{R}_{g^{-1}}^* \mu = \mathrm{inv}^* \mu = \omega \tag{10.2}$$

(and conversely). Also, the relation $\mathrm{inv}_*(\mathrm{Z}g) = \frac{d}{dt} \mathrm{inv}(\mathrm{e}^{t\mathrm{Z}}g)\big|_{t=0} = -g^{-1}\mathrm{Z}$ shows that (9.1b) implies

$$\omega(\mathrm{Z}_1 g, \ldots, \mathrm{Z}_k g) = \mu(-g^{-1}\mathrm{Z}_1, \ldots, -g^{-1}\mathrm{Z}_k) = 0 \tag{10.3}$$

whenever one of the Z_j belongs to $\mathfrak{h}$; whence (10.1b) since we have $g\mathfrak{h} = \mathfrak{h}g$ (9.2) (and conversely). $\square$

CHAPTER 11

$\mathrm{H}^\bullet_{\mathrm{DR}}(\mathrm{X})$ AND $\mathrm{H}^\bullet(\mathfrak{g}/\mathfrak{h})$

(11.1) Theorem. *Let* H *be a dense subgroup of a Lie group* G *with Lie algebra* $\mathfrak{g}$*. The (diffeological) de Rham cohomology of* G/H *equals the Lie algebra cohomology of* $\mathfrak{g}/\mathfrak{h}$*, where* $\mathfrak{h}$ *is the ideal* $\{\mathrm{Z} \in \mathfrak{g} : \mathrm{e}^{t\mathrm{Z}} \in \mathrm{H}$ *for all* $t \in \mathbf{R}\}$.

Proof. We will use $\Omega^\bullet(\mathrm{G})^{\mathrm{G}}_{\mathrm{Hor}}$ to denote the set

$$\Big\{\omega \in \Omega^\bullet(\mathrm{G}) : \omega \text{ satisfies } (10.1a, b)\Big\}. \tag{11.2}$$

Then $(\Omega^\bullet(\mathrm{G})^{\mathrm{G}}_{\mathrm{Hor}}, d)$ forms a subcomplex of $(\Omega^\bullet(\mathrm{G})^{\mathrm{G}}, d)$, and we have the following isomorphisms of complexes

$$\tilde{\Pi}^* : (\Omega^\bullet(\mathrm{X}), d) \to (\Omega^\bullet(\mathrm{G})^{\mathrm{G}}_{\mathrm{Hor}}, d) \qquad \text{by (10.1) and (8.11)} \tag{11.3}$$

$$\omega \mapsto \omega_e : (\Omega^\bullet(\mathrm{G})^{\mathrm{G}}_{\mathrm{Hor}}, d) \to (\textstyle\bigwedge^\bullet(\mathfrak{g}^*)_{\mathrm{basic}}, d) \qquad \left(\begin{array}{l}\text{by (6.3), (6.49), and} \\ \text{(10.1b) for } \omega \text{ implies (6.40) for} \\ \omega_e\end{array}\right) \tag{11.4}$$

$$\pi_* : (\textstyle\bigwedge^\bullet(\mathfrak{g}^*)_{\mathrm{basic}}, d) \to (\textstyle\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*, d) \quad \text{by (6.51).} \tag{11.5}$$

Composing these together we get an isomorphism of complexes $(\Omega^\bullet(\mathrm{X}), d)$ to $(\bigwedge^\bullet(\mathfrak{g}/\mathfrak{h})^*, d)$ which induces an isomorphism between $\mathrm{H}^\bullet_{d\mathrm{R}}(\mathrm{X})$ and $\mathrm{H}^\bullet(\mathfrak{g}/\mathfrak{h})$ by (2.6). $\square$

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