Summer 2021

New Results on Cyclic Compositions and Multicompositions

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NEW RESULTS ON CYCLIC COMPOSITIONS AND MULTICOMPOSITIONS

by

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(Under the Direction of Hua Wang)

ABSTRACT

Integer compositions, cyclic compositions, and lately k-compositions, are important topics in combinatorics and number theory. In this paper, we will explain, the general approach of using generating functions to study number sequences involving compositions, cyclic compositions, k-compositions, and the number of parts in each of them. After generating the data, some properties are observed and proved. Also, some interesting bijections involving Pell numbers and the Jacobsthal sequence are given.

INDEX WORDS: Compositions of n, Cyclic compositions, Primitive compositions, Primitive cyclic compositions, Multicompositions, Generating function

2009 Mathematics Subject Classification: 15A15, 41A10

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M.A., Polytechnic University of Tirana, Albania, 2014

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

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Electronic Version Approved:
July 2021

DEDICATION

Totus tuus ego sum et omnia mea tua sunt. I am all yours, and all that is mine is yours.

ACKNOWLEDGMENTS

I wish to express my gratitude to both my advisors Dr. Wang and Dr. Hopkins. I appreciated their dedications to helping me grow both academically and personally. I want to thank Dr. Lesaja and Dr. Gray for being apart of my thesis committee. I also want to express my gratitude to all the professors who taught and worked with me during my time here at Georgia Southern. Their support, wisdom, and encouragement were a vital part of my success. Finally, I would like to acknowledge Saint Matthew's Parish, Father John Johnson, uncle Mark Yocco, and all the friends I have made there, for their emotional and spiritual support.

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INTRODUCTION

In this chapter, we provide an introduction to the topics of integer compositions and generating functions. After a brief description of related applications, we present useful definitions and terminologies. We also illustrate the construction of a generating function for simple compositions with a small example.

1.1 BACKGROUND

Integer compositions and generating functions are important concepts in combinatorics and number theory. There are many studies and publications on the relationship between these concepts. We briefly introduce some of them in this section.

An introduction to the methods used in the combinatorics of pattern avoidance and pattern enumeration in compositions and words is given by Heubach and Mansour [8]. The relationships between sub-word patterns and residue classes of parts in the set of integer compositions of a given weight are introduced by Hopkins, Shattuck, Sills, Thanatipanonda, and Wang [11]. The On-Line Encyclopedia of Integer Sequences (OEIS) [16] has been an important reference in what is known about particular sequences generated in this paper.

In regards to colored compositions, n-colored compositions and their application in solving combinatorial problems are introduced by Agarwal [1]. Binary words, n-color compositions, and bijections of the Fibonacci numbers are introduced by Collins, Dedrickson, and Wang [5]. New combinatorial properties of n-color compositions, analogous to Euler's partition identity, are introduced by Agarwal [2]. The extension of MacMahon's definition of self-inverse composition to n-color self-inverse compositions and a new kind of weighted lattice paths using n-color compositions are given by Narang and Agarwalis in [14] and [15] respectively. The definition of n-color compositions as n-color ordered partitions and the introduction of nine new restricted n-color composition functions are given by Sachdeva and

Agarwal [17]. Previous and new results are proven by Hopkins [9] by using spotted tilings as a representation of a new combinatorial interpretation of the n-color compositions. A new binomial identity with combinatorial interpretations and new combinatorial properties of the walks are introduced by Agarwal [3].

Regarding cyclic compositions, a direct generating function construction for cycles of combinatorial structures is introduced by Flajolet and Soria [6]. The fact that two compositions of n into k parts are related if they differ only by a cyclic shift is introduced by Knopfmacher and Robbins [12]. Dynamic storage systems, which turn out to be related to cyclic colored compositions are mentioned by Mcllroy [13].

Special generating functions, exponential generating functions, and convolution are introduced by Graham, Knuth, and Patashnik [7]. Certain periodic properties of cyclic compositions of numbers are given by Sommerville [18]. Integer compositions with certain colored parts are introduced by Andrews [4]. A bijection between integer compositions with certain colored parts and integer compositions allowing zero as some parts and various combinatorial properties of these k-compositions are introduced by Hopkins and Ouvry [10].

1.2 Definitions and Examples

In this section, we establish all definitions accompanied by examples. This will help readers to have a good understanding of the concepts used in this paper.

1.2.1 Basic Concepts

First, we start with the definition of a basic concept, the composition of a positive integer n.

Definition 1. A sequence of positive integers $(a_1, a_2, ..., a_k)$ such that,

$$\sum_{i=1}^{k} a_i = n$$

is a composition of a positive integer n with k parts.

From the above definition, we can give some properties of the compositions of n. Negative integers do not have any compositions, since, according to the definition, there is no way we can express a negative number as a sum of positive integers. The composition of zero is the empty sequence and each positive integer n has 2^{n-1} distinct compositions. One way to count compositions is through generating functions, where the coefficient of x^j gives the total number of distinct compositions of j.

Definition 2. $F(x)$ is called the ordinary generating function for the sequence $(a_1, a_2, \cdots)$, if

$$F(x) = \sum_{n=1}^{\infty} a_n x^n.$$

The generating function for compositions having one part is given by

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x},$$

since, for all n in $\mathbb{P}$ the only composition of n into one part is to have the one part to be equal to n itself. The generating function for all compositions of n is given by

$$\frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{x}{1-2x} = x + 2x^2 + 4x^3 + 8x^4 + \cdots + 2^{n-1}x^n + \cdots .$$

Clearly, we read from the generating function that, the total number of distinct compositions of $n = 2$ is two from the coefficient of x^2. They are:

$$(1, 1), \quad (2).$$

The total number of distinct compositions of $n = 3$ is four and the total number of distinct compositions of $n = 4$ is eight. In general, the total number of distinct compositions of n is 2^{n-1}.

Example 1.1. *Let's consider all compositions of* $n = 4$.

The composition of 4 *with four parts is given by,*

$$1 + 1 + 1 + 1 = 4.$$

The compositions of 4 *with three parts are given by,*

$$1 + 2 + 1 = 4,$$

$$2 + 1 + 1 = 4,$$

$$1 + 1 + 2 = 4.$$

The compositions of 4 *with two parts are given by,*

$$1 + 3 = 4,$$

$$3 + 1 = 4,$$

$$2 + 2 = 4.$$

Finally, the composition of 4 *with one part is* 4 *itself.*

In total, there are eight distinct compositions of $n = 4$.

In the third subsection, we are going to study the total number of parts of size k and parts of size $k + 1$ among all compositions of n. In the following definition, we introduce their notations.

Definition 3. Let $N(n, k)$ denote the number of parts of size k among all compositions of n.

Let's consider the compositions of $n = 4$ listed in Example 1.1 to explain the meaning of the above notations.

Example 1.2. *From the above definition,* $N(4, 1)$ *means the total number of parts of size* 1 *among all compositions of* 4*. If we count them in the previous Example 1.1, we get:*

$$N(4, 1) = 12.$$

1.2.2 Cyclic Compositions

In this subsection, we will establish all the definitions and examples related to cyclic compositions. First, we start with the definition of the cyclic composition of a positive integer n.

Definition 4. A cyclic composition of n is an equivalent class of all linear compositions of n that can be obtained from each other by a cyclic shift.

Next, we will introduce in the following definition the property of cyclical equivalence between two compositions introduced in [12].

Definition 5. Two compositions of n with k parts $(a_0, a_1, \ldots, a_{k-1})$ and $(b_0, b_1, \ldots, b_{k-1})$ are cyclically equivalent if for some fixed non-negative integer t, we have

$$a_i = b_{(i+t)(\bmod k)},$$

for each $1 \leq i \leq k-1$. Note that this definition employs the convention that a multiple of k is equal to $k \pmod k$.

It can be easily demonstrated that this is a relation and is left up to the reader. Now let's explain this property in the following example.

Example 1.3. *Let's consider the following compositions of* $n = 6$ *with* $k = 3$ *parts:* $(1, 2, 3)$, $(2, 3, 1)$ *and* $(3, 1, 2)$. *We need to show that they are cyclically equivalent compositions:*

$$(1, 2, 3) \sim (2, 3, 1) \sim (3, 1, 2).$$

Now let's explain each of them by using the definition. Let's denote $(b_1, b_2, b_3) = (1, 2, 3)$ *and let* $t = 1$. *From the definition we get:*

$$a_1 = b_{(1+1) \bmod 3} = b_{2 \bmod 3} = b_2 = 2,$$

$$a_2 = b_{(2+1) \bmod 3} = b_{3 \bmod 3} = b_3 = 3,$$

$$a_3 = b_{(3+1) \bmod 3} = b_{4 \bmod 3} = b_1 = 1.$$

Thus $(a_1, a_2, a_3) = (2, 3, 1) \sim (1, 2, 3)$ *are cyclically equivalent compositions of* $n = 6$ *for a fixed* $t = 1$. *Next, let's denote* $(b_1, b_2, b_3) = (3, 1, 2)$ *for* $t = 1$. *We get the following:*

$$a_1 = b_{(1+1) \bmod 3} = b_{2 \bmod 3} = b_2 = 1,$$

$$a_2 = b_{(2+1) \bmod 3} = b_{3 \bmod 3} = b_3 = 2,$$

$$a_3 = b_{(3+1) \bmod 3} = b_{4 \bmod 3} = b_1 = 3.$$

Thus $(a_1, a_2, a_3) = (1, 2, 3) \sim (3, 1, 2)$ *are cyclically equivalent compositions of* $n = 6$ *for a fixed* $t = 1$. *Finally, by the transitivity property of the equivalence relation, we have:*

$$(2, 3, 1) \sim (3, 1, 2).$$

Therefore:

$$(1, 2, 3) \sim (2, 3, 1) \sim (3, 1, 2).$$

Now let's consider one case of two compositions of $n = 6$ *that are not cyclically equivalent with each other:*

$$(1, 2, 3) \nsim (2, 1, 3).$$

Let's denote $(a_1, a_2, a_3) = (2, 1, 3)$ *and* $(b_1, b_2, b_3) = (1, 2, 3)$. *From the definition we will have the following for* a_1*:*

$$a_1 = b_2 = 2.$$

This means that:

$$1 + t \equiv 2 \bmod 3,$$

$$t \equiv 1 \bmod 3.$$

Next, from the definition for a_2, *we will get the following relation:*

$$a_2 = b_1 = 1.$$

This implies:

$$2 + t \equiv 1 \mod 3,$$

$$t \equiv 2 \mod 3.$$

Since t *can not be congruent to two different values* $\mod 3$*, this means that there is not any fixed* t *that satisfies both equations. Therefore,* $(1, 2, 3)$ *and* $(2, 1, 3)$ *are not cyclically equivalent compositions of* $n = 6$*.*

Due to their complex structure, we will show that, to find the generating function of a cyclic composition, it is easier to first find the generating function of a primitive composition. The following definition gives the idea of a primitive composition.

Definition 6. A composition of n with k parts is primitive if there is no $d > 1$ such that it is d adjacent copies of some composition with $\frac{k}{d} = l$ parts.
If a composition of n with k parts is non-primitive, then there is a $d > 1$ such that the composition of n is formed by d adjacent copies of some primitive composition with $\frac{k}{d} = l$ parts, called the root of the composition.

In the following example, we will consider a composition of $n = 12$ to create a better understanding of how a primitive composition and a non-primitive composition are defined.

Example 1.4. *Consider a composition of* $n = 12$ *with* $k = 6$ *parts:*

$$(1, 2, 3, 1, 2, 3).$$

It is formed from $d = 2$ *adjacent copies of a composition* $(1, 2, 3)$ *with* $\frac{6}{2} = 3$ *parts. From the definition, this is not a primitive composition of* $n = 12$ *with* $k = 6$ *parts. Similarly,* $(2, 3, 1, 2, 3, 1)$ *and* $(3, 1, 2, 3, 1, 2)$ *are non-primitive compositions. Now let's consider another composition of* $n = 12$ *with* $k = 6$ *parts:*

$$(1, 3, 2, 3, 2, 1).$$

It is a primitive composition of $n = 12$ with $k = 6$ parts because it is not formed by adjacent copies of compositions, and if we cut the cyclic composition in between each number, we get six different linear compositions:

$$(1,3,2,3,2,1), \quad (3,2,3,2,1,1), \quad (2,3,2,1,1,3),$$

$$(3,2,1,1,3,2), \quad (2,1,3,2,3,2), \quad (1,1,3,2,3,2),$$

The repetition of $(3,2)$ in all linear compositions is not enough to keep the composition from being primitive.

Every composition of n can be written uniquely as d copies of a primitive composition (now allowing $d = 1$). Let's consider three different compositions of $n = 4$ to have a better approach to the above statement.

Example 1.5. $3 + 1$ *is a composition of* 4 *formed from one copy of* $(3,1)$.

$2 + 2$ *is a composition of* 4 *formed from two copies of* (2).

$2 + 1 + 1$ *is a composition of* 4 *formed from one copy of* $(2,1,1)$.

A generating function describing compositions is an infinite polynomial. The total number of parts with a specific size in a composition of n is given by the coefficients of this infinite polynomial. In the following definition, we will introduce the notation used for the coefficients of the generating function of compositions and the coefficients of the generating function of a primitive composition.

Definition 7. Let $c(n,k)$ denote the number of general compositions with k parts that sum to n, and $prc(n,k)$ denote the number of primitive compositions of n with k parts.

If the composition is given, it is easy to identify if it is a primitive composition or it is formed by multiple primitive compositions. In terms of generating functions, there is a mathematical relation between coefficients of the general generating function of a composition $c(n, k)$ and the coefficients of the general generating function of a primitive composition $prc(n, k)$, which is given by the following equation:

$$c(n, k) = \sum_{d|(n,k)} prc\left(\frac{n}{d}, \frac{k}{d}\right),$$

where the notation $d|(n, k)$ means that d is a common divisor of n and k. In the following example, we will consider the composition of $n = 8$ with $k = 4$ parts to show how the above relation works.

Example 1.6. *Let $n = 8$ and $k = 4$. Then, $d = 1, 2, 4$, as these values are common divisors of 8 and 4. By doing the substitution in the above equation we get the following:*

$$c(8, 4) = prc\left(\frac{8}{1}, \frac{4}{1}\right) + prc\left(\frac{8}{2}, \frac{4}{2}\right) + prc\left(\frac{8}{4}, \frac{4}{4}\right),$$

which simplifies to:

$$c(8, 4) = prc(8, 4) + prc(4, 2) + prc(2, 1).$$

To calculate $prc(8, 4)$, we need to find out in how many ways we can express $n = 8$ as a primitive composition with $k = 4$ parts. Examples of this are $(1, 1, 1, 5)$, $(1, 1, 2, 4)$, $(1, 2, 2, 3)$ and $(1, 1, 3, 3)$; we notice that compositions such as $(1, 3, 1, 3)$ and $(3, 1, 3, 1)$ are not primitive. The combinations corresponding to $(1, 1, 1, 5)$ are determined by the position of the 5 among the four parts, so there are 4 of these. For $(1, 1, 2, 4)$, there are four possible positions for the 4 and then three possible positions for the 2, with the remaining spots filled by 1s, giving: $4 \cdot 3 = 12$ compositions. Then $(1, 2, 2, 3)$ follows the same enumeration as $(1, 1, 2, 4)$. The case $(1, 1, 3, 3)$ is a little trickier: not all of the

$$\binom{4}{2} = 6$$

ways of placing two 3s among four positions produce primitive compositions, given $(1, 3, 1, 3)$ and $(3, 1, 3, 1)$. Finally, we get the following:

$$prc(8, 4) = 4 + 12 + 12 + 4 = 32.$$

Next, to calculate $prc(4,2)$ *we need to find the total number of ways we can express* $n = 8$ *as a composition that is formed by two adjacent copies of a primitive composition with two parts that sum to four. These would be:*

$$(1,3,1,3), \quad (3,1,3,1).$$

Therefore, $prc(4,2) = 2$. *Now, to calculate* $prc(2,1)$ *we need to find the total number of ways we can express* $n = 8$ *as a composition that formed by four adjacent copies of a primitive composition with one parts which is two. That is:*

$$(2,2,2,2).$$

Therefore, $prc(2,1) = 1$. *Finally, we can calculate the total number of compositions of* $n = 8$ *with* $k = 4$ *parts by using the number of primitive compositions:*

$$c(8,4) = prc(8,4) + prc(4,2) + prc(2,1) = 32 + 2 + 1 = 35.$$

Definition 8. Let $prcc(n,k)$ denote the number of primitive cyclic compositions of n with k parts. These will be the coefficients of the generating function of primitive cyclic compositions. Also, let $PRCC$ denote the class of primitive cycles.

There is a relation between the coefficients of the primitive cyclic generating function and the coefficients of the primitive generating function given by the following mathematical equation:

$$prcc(m,l)x^m u^l = \frac{prc(m,l)}{l} x^m u^l.$$

The following example will help us to understand better the above relation.

Example 1.7. *Consider the total number of primitive cyclic compositions of* $m = 8$ *with* $l = 4$ *parts from Example 1.6:* $prc(8,4) = 32$. *For fixed values of* m *and* l*, we get the following:*

$$prcc(m,l) = \frac{prc(m,l)}{l}.$$

Thus, to find the total number of primitive cyclic compositions of $m = 8$ *with* $l = 4$ *parts, we can use the above relation:*

$$prcc(8,4) = \frac{prc(8,4)}{4} = \frac{32}{4} = 8.$$

Let's consider the primitive compositions $(1,1,1,5)$ *from Example 1.6. We have four different ways to represent this primitive composition with four parts that sum to eight. In terms of primitive cyclic compositions, we know that:*

$$(1,1,1,5) \sim (5,1,1,1) \sim (1,5,1,1) \sim (1,1,5,1).$$

Since they are all equivalent to each other, we have one way to represent this specific cyclic primitive composition with four parts that sum to eight.

1.2.3 Algebraic and Number Theoretical Concepts

In our main result of the general generating function for cyclic compositions, tools from algebra and number theory are used extensively. In this subsection, we will establish all relevant definitions and examples. In the following definition, we will introduce the multiplicative function, which leads to the inverse element in the set M. This property will be used later on to prove the relation between the generating function of primitive compositions and the generating function of cyclic compositions.

Definition 9. An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ when a and b are relatively prime.

Let's denote M as the set of all multiplicative functions. The set M forms an *abelian group* under *Dirichlet convolution*:

$$f * g = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) \qquad \text{for all } f, g \in M,$$

with identity,

$$i(n) = \begin{cases} 1, & n = 1, \\ 0, & otherwise. \end{cases}.$$

Now let's introduce the definition of the unit function $\overline{u}$, which is a multiplicative function and will be used to prove that $(M, *)$ is an *abelian group*.

Definition 10. Let $\overline{u}$ be the unit function such that

$$\overline{u}(n) = 1, \qquad \text{for all}\, n \in \mathbb{N}.$$

We will also use the well-known Möbius function.

Definition 11. The Möbius function $\mu(n)$ is a multiplicative function defined by:

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^k, & n = \text{product of } k \text{ distinct primes}, \\ 0, & otherwise. \end{cases}$$

The Möbius function and the unit function $\overline{u}(n)$ are inverse elements in M, and we will use this property more in Chapter 3. We show this property in the following proof.

Proof that μ is the inverse of $\overline{u}$. Clearly, if $n = 1$ we have $\mu(1) \cdot \overline{u}(1) = 1 = i(1)$. Assume $n > 1$. Then, from the fundamental theorem of arithmetic, every positive integer greater than 1 can be written as a product of prime numbers (or the integer is itself a prime number). Therefore,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where $p_1, p_2, \ldots, p_k$ are k distinct primes. Notice that in general

$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where $0 \leq \beta_j \leq \alpha_j$ for $1 \leq j \leq k$. However, by the definition of the Möbius function, $\mu(d) = 0$ for all such d having $\beta_j \geq 2$ for any j, and thus would have a contribution of zero in the *Dirichlet convolution* above. Hence, by the definition of the *Dirichlet function*, we get:

$$\mu * \overline{u} = \sum_{d|n} \mu(d)\overline{u}\left(\frac{n}{d}\right).$$

Now let's consider B as a subset of the index set $\{1, 2, \ldots, k\}$. Since d is a divisor of n, it must divide at least one p_j, for some j from the set B. Let's assume that $d = \prod p_j$, where p_j are distinct prime numbers for some j from the set B. By replacing d in the above relation and from the definition of the unit function $\overline{u}$, we get the following:

$$\mu * \overline{u} = \sum_{B \subseteq \{1,2,\ldots,k\}} \mu\left(\prod_{j \in B} p_j\right) \cdot 1.$$

By the definition of the Möbius function, we get:

$$\mu * \overline{u} = \sum_{B \subseteq \{1,2,\ldots,k\}} (-1)^{|B|},$$

where $|B|$ is the cardinality of the set B. Since we know that d must divide at least one of p_j, we can consider d as a product of some p_j with a random combination of the indexes j from the set $\{1, 2, \ldots, k\}$ which is represented by the set B. The way of determining the number of possible arrangements by selecting only j objects from our index set $\{1, 2, \ldots, k\}$ with k elements, and with no repetition, is given by:

$$\binom{k}{j}.$$

Therefore,

$$\mu * \overline{u} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j.$$

We recall the binomial theorem:

$$(a + b)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j.$$

Then, using this, we see:

$$\mu * \overline{u} = \sum_{k=0}^{j} \binom{j}{k} (-1)^j \cdot 1^{(k-j)} = (1 - 1)^k = 0,$$

for any $n > 1$. $\square$

Next, we will introduce the proof for the following identity:

$$\sum_{k|p} \frac{\mu(k)}{k} = \frac{\varphi(p)}{p},$$

where $\varphi(p)$ is the Euler's totient function. It counts all numbers $j < k$ such that j and k are relatively prime, that is $(j, k) = 1$.

Proof of the identity. Using the property establish by Gauss, we get:

$$\sum_{d|p} \varphi(d) = p.$$

Let I be an identity function such that $I(p) = p$ for all p. Now, using the definition of the function I and the property established by Gauss, we get:

$$I(p) = p = \sum_{d|p} \varphi(d) \cdot 1.$$

By expressing $1 = \overline{u}\left(\frac{p}{d}\right)$ and the *Dirichlet convolution* definition, we get the following:

$$\sum_{d|p} \varphi(d) \cdot 1 = \sum_{d|p} \varphi(d)\overline{u}\left(\frac{p}{d}\right) = \varphi * \overline{u}.$$

Finally, we get the following:

$$I = \varphi * \overline{u}.$$

Now, by multiplying both sides in the above relation by μ we get:

$$I * \mu = \varphi * \overline{u} * \mu.$$

Using the property of the inverse multiplicative functions under the *Dirichlet convolution*, we get the following:

$$I * \mu = \varphi.$$

Therefore, using the *Dirichlet convolution* definition and the definition of the I function, we have:

$$\varphi(p) = (\mu * I)(p) = \mu(p) * I(p) = \sum_{d|p} \mu(d) I\left(\frac{p}{d}\right) = \sum_{d|p} \mu(d)\frac{p}{d} = p\sum_{d|p} \frac{\mu(d)}{d}.$$

Finally,

$$\varphi(p) = p\sum_{d|p}\frac{\mu(d)}{d}.$$

And by dividing both sides by p, we get the following:

$$\frac{\varphi(p)}{p} = \sum_{d|p}\frac{\mu(d)}{d}.$$

□

1.2.4 MULTICOMPOSITIONS

In this subsection, we will establish all the definitions and examples related to multicompositions. Multicompositions were introduced for the first time by George Andrews in 2007. He introduced the compositions with k types of each part, with the restriction that the last part should have only one type. He used this concept in solving the problem of Emeric Deutch [4]. Now let's introduce the definition of multicompositions.

Definition 12. A k-composition is an integer composition with k types of each part except for the first part, which only has one type.

Next, let's consider compositions of $n = 4$ with $k = 3$ parts to find out the total number of 2-compositions. From the above definition, the first part can only be of type 1, but the other two parts can be of type 1 or type 2.

Example 1.8. *Let's consider all* 2*-compositions of* $n = 4$ *with* $k = 3$ *parts.*

$(1_1, 1_1, 2_1)$, $(1_1, 1_1, 2_2)$, $(1_1, 1_2, 2_1)$, $(1_1, 1_2, 2_2)$,

$(1_1, 2_1, 1_1)$, $(1_1, 2_2, 1_1)$, $(1_1, 2_1, 1_2)$, $(1_1, 2_2, 1_2)$,

$(2_1, 1_1, 1_1)$, $(2_1, 1_1, 1_2)$, $(2_1, 1_2, 1_1)$, $(2_1, 1_2, 1_2)$.

1.3 An Exemplary Study

In this section, we will explain, by means of a simple example, the general approach of using generating functions to study number sequences involving compositions and parts. After generating the data some properties are observed and proved. Our hope is to introduce the generating function approach and combinatorial observations through the study of two counting problems on n compositions and their parts of size k. First of all, for the generating function for each part, we have:

$$F(x) = x + x^2 + x^3 + \cdots + x^k + \cdots$$

since there is one copy of each part size, corresponding to the coefficient 1 of each power of x. Next, to label parts of size k, we multiply x^k by y and get the generating function

$$\begin{aligned} F(x,y) &= x + x^2 + \cdots + yx^k + x^{k+1} + \cdots \\ &= \frac{x}{1-x} - x^k + x^k y \\ &= \frac{x^k(1-x)(y-1) + x}{1-x}. \end{aligned}$$

It can be shown that the generating function for all compositions is given by

$$\begin{aligned} F_k(x,y) &= F(x,y) + (F(x,y))^2 + \cdots + (F(x,y))^k + \cdots \\ &= \frac{x^k(1-x)(y-1) + x}{(1-x) - x^k(1-x)(y-1) - x}. \end{aligned}$$

The coefficient of $x^n y^m$ in $F_k(x,y)$ represents the number of compositions of n with m parts of size k. Next, by partially differentiating $F_k(x,y)$ with respect to y and plugging in

$y = 1$, we get:

$$
\begin{aligned}
GF_k(x) &= \left.\frac{\partial F_k(x,y)}{\partial y}\right|_{y=1} \\
&= \frac{x^k(1-x)(1-2x) + x^{k+1}(1-x)}{(1-2x)^2} \\
&= \frac{(1-x)(x^k - x^{k+1})}{(1-2x)^2} \\
&= \frac{x^k - 2x^{k+1} + x^{k+2}}{(1-2x)^2}.
\end{aligned} \tag{1.1}
$$

Expanding the above generating function yields the results for $N(n,k)$ given in Table 1.1.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	5	12	28	64	144	320	704	1536	3328	7168	15360	32768
2	0	1	2	5	12	28	64	144	320	704	1536	3328	7168	15360
3	0	0	1	2	5	12	28	64	144	320	704	1536	3328	7168
4	0	0	0	1	2	5	12	28	64	144	320	704	1536	3328
5	0	0	0	0	1	2	5	12	28	64	144	320	704	1536
6	0	0	0	0	0	1	2	5	12	28	64	144	320	704
7	0	0	0	0	0	0	1	2	5	12	28	64	144	320
8	0	0	0	0	0	0	0	1	2	5	12	28	64	144
9	0	0	0	0	0	0	0	0	1	2	5	12	28	64
10	0	0	0	0	0	0	0	0	0	1	2	5	12	28
11	0	0	0	0	0	0	0	0	0	0	1	2	5	12
12	0	0	0	0	0	0	0	0	0	0	0	1	2	5
13	0	0	0	0	0	0	0	0	0	0	0	0	1	2
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1.1: The values of $N(n,k)$ for n=1:14 and k=1:14.

We can easily see from Table 1.1 that the values of $N(n,k)$ on the diagonals do not change. This result is given in the following Theorem 1.9.

Theorem 1.9. *For any n and k we have:*

$$N(n+1,k+1) = N(n,k).$$

We will use this observation as an example and present two different proofs, a proof through generating functions and a combinatorial proof.

Proof through generating functions. Consider the generating function (1.1). We have

$$GF_k(x) = \frac{x^k - 2x^{k+1} + x^{k+2}}{(1-2x)^2}$$

as the generating function for parts of size k.

Similarly, the generating function for parts of size $k+1$ will be

$$GF_{k+1}(x) = \frac{x^{k+1} - 2x^{k+2} + x^{k+3}}{(1-2x)^2}.$$

To prove $N(n+1,k+1) = N(n,k)$, we need to show that the coefficient of x^k in $GF_k(x)$ is equal to the coefficient of x^{k+1} in $GF_{k+1}(x)$. This can be seen through the following:

$$\begin{aligned} GF_{k+1}(x) &= \frac{x^{k+1} - 2x^{k+2} + x^{k+3}}{(1-2x)^2} \\ &= \frac{x(x^k - 2x^{k+1} + x^{k+2})}{(1-2x)^2} \\ &= xGF_k(x). \end{aligned}$$

□

Combinatorial proof. Consider all parts of size k among all compositions of n. If we add one to a part of size k in a composition of n, we end up with a part of size $k+1$ among a composition of $(n+1)$. It is easy to verify this operation, as a mapping between the parts of size k (in compositions of n) and the parts of size $k+1$ (in compositions of $n+1$) is a bijection. □

We can see this result in the following example demonstrating the bijection.

Example 1.10. *Consider all parts of size* $k = 2$ *in the compositions of* $n = 4$ *(on the left in the list below). For any parts of size* $k = 2$ *among compositions of* $n = 4$*, if we add one, we will end up with a part of size* $k + 1 = 3$ *among compositions of* $n + 1 = 5$*, as the following mapping shows:*

$$1 + 1 + 2 \longrightarrow 1 + 1 + 3,$$

$$2 + 1 + 1 \longrightarrow 3 + 1 + 1,$$

$$1 + 2 + 1 \longrightarrow 1 + 3 + 1,$$

$$2 + 2 \longrightarrow 3 + 2,$$

$$2 + 2 \longrightarrow 2 + 3.$$

Parts of size $k + 1 = 3$ *in compositions of* $n + 1 = 5$ *are on the right. This relation between the set of all parts of size* 2 *amongst all compositions of* 4 *and the set of all parts of size* 3 *amongst all compositions of* 5*, is one-to-one and onto. Therefore, it is a bijection.*

We know from [A045623, 16] that the number of 1s in all compositions of $n + 1$ is given by $(n + 3)2^{n-2}$. Then using Theorem 1.9 we can say that this is also the number of 2s in all compositions of $n + 2$, the number of 3s in all compositions of $n + 3$, and so on. In general we have:

$$\begin{aligned} N(n,k) &= [(n-k)+3]2^{(n-k)-2} \qquad (1.2) \\ &= (n-k+3)2^{n-k-2}, \\ N(n,k+1) &= [n-(k+1)+3]2^{n-(k+1)-2} \\ &= (n-k+2)2^{n-k-3}. \end{aligned}$$

The ratio between the total number of parts of size k and parts of size $k + 1$ among all compositions of n approaches two for larger values of n. This is another result given in Theorem 1.11.

Theorem 1.11. *For any k we have*

$$\lim_{n\to\infty} \frac{N(n,k)}{N(n,k+1)} = 2.$$

Proof. Using the formulas in (1.2), we get:

$$\begin{aligned}
\lim_{n\to\infty} \frac{N(n,k)}{N(n,k+1)} &= \lim_{n\to\infty} \frac{(n-k+3)2^{n-k-2}}{(n-k+2)2^{n-k-3}} \\
&= \lim_{n\to\infty} \frac{2(n-k+3)}{(n-k+2)} \\
&= \lim_{n\to\infty} \frac{2n-2k+6}{n-k+2} \\
&= 2.
\end{aligned}$$

□

Now, looking back at the Table 1.1, we can show that the numbers basically double as you go up the columns for large n:

$$\begin{aligned}
\frac{N(12,5)}{N(12,6)} &= \frac{704}{320} = 2.2, \\
\frac{N(14,4)}{N(14,5)} &= \frac{3328}{1536} = 2.166.
\end{aligned}$$

It isn't perfect but it is approximate.

CHAPTER 2

SOME NEW COMPOSITION RESULTS

In this chapter, we will show some new composition results involving the generating function for parts of size divisible by k among all compositions of n (Section 2.1) and some variants of compositions (Section 2.2).

2.1 Parts of Size Divisible by k.

In this section, we will establish some new results regarding the generating function approach through the study of the counting problem on compositions of n and their parts divisible by k. First, let's give the notation for the total number of parts divisible by k among compositions of n.

Definition 13. Let $\tilde{N}(n,k)$ denote the total number of parts divisible by k among all compositions of n.

Now, let's explain the meaning and calculate $\tilde{N}(4,2)$ in the following example. We will use Example 1.1 in the previous chapter as a reference.

Example 2.1. *From* (4), $(1,1,2)$, $(1,2,1)$, $(2,2)$, *and* $(2,1,1)$, *we will see:*

$$\tilde{N}(4,2) = 6.$$

Similar to the generating function with parts of size k given in the last section of the previous chapter, we start with the generating function of each part in order to find the generating function for all compositions of n:

$$H(x) = x + x^2 + x^3 + \cdots + x^k + \cdots + x^{2k} + x^{2k+1} + \cdots .$$

After labeling all parts of size divisible by k, we have:

$$\begin{aligned} H(x,y) &= x + x^2 + x^3 + \cdots + yx^k + \cdots + yx^{2k} + x^{2k+1} + \cdots \\ &= \frac{x(1-x^k) + x^k(1-x)(y-1)}{(1-x)(1-x^k)}. \end{aligned}$$

It can be shown that the generating function for all compositions of n is given by:

$$H_k(x,y) = \frac{x(1-x^k) + x^k(1-x)(y-1)}{(1-x^k)(1-2x) - x^k(1-x)(y-1)}.$$

The coefficient of $x^n y^m$ represents the total number of compositions of n with m parts divisible by k. Consequently, it can be shown that the generating function for all parts divisible by k amongst all compositions of n is

$$GH_k(x) = \left.\frac{\partial H_k(x,y)}{\partial y}\right|_{y=1} = \frac{x^k(1-x)^2}{(1-x^k)(1-2x)^2}. \tag{2.1}$$

Expanding this generating function yields the following Table 2.1. The first two rows of the

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
1	1	3	8	20	48	112	256	576	1280	2816	6144	13312	A001792
2	0	1	2	6	14	34	78	178	398	882	1934	4210	A059570
3	0	0	1	2	5	13	30	69	157	350	773	1693	
4	0	0	0	1	2	5	12	29	66	149	332	733	
5	0	0	0	0	1	2	5	12	28	65	146	325	
6	0	0	0	0	0	1	2	5	12	28	64	145	
7	0	0	0	0	0	0	1	2	5	12	28	64	
8	0	0	0	0	0	0	0	1	2	5	12	28	
9	0	0	0	0	0	0	0	0	1	2	5	12	

Table 2.1: The values of $\tilde{N}(n,k)$ for n=1:12 and k=1:9.

Table 2.1 are currently in The On-line Encyclopedia of Integer Sequences [16] as A001792, which counts the number of all parts in all compositions of n and A059570, which counts even parts in all compositions of n. Also, from a quick observation, we can tell that the values of $\tilde{N}(n,k)$ do not change on the diagonals starting from a specific position on the table and all the diagonals stabilizing at the sequence of numbers given in Table 1.1. These results are given in Theorem 2.2.

Theorem 2.2. *For any n and any $k \geq \lceil \frac{n+1}{2} \rceil$, we have:*

$$\tilde{N}(n,k) = \tilde{N}(n+1,k+1) = N(n,k) = N(n+1,k+1).$$

Proof. For k in the specified range, the only multiple of k that can occur in the composition is k itself, so $\tilde{N}(n,k) = N(n,k)$. As we will see below, this can also be seen through the generating functions. Let's consider the generating function for parts of size divisible by k in compositions of n (2.1):

$$GH_k(x) = \frac{x^k(1-x)^2}{(1-x^k)(1-2x)^2}.$$

Similarly, the generating function for parts of size divisible by $k+1$ in compositions of n will be:

$$\begin{aligned} GH_{k+1}(x) &= \frac{x^{k+1}(1-x)^2}{(1-x^{k+1})(1-2x)^2} \\ &= (x^{k+1} + x^{2(k+1)} + \cdots)\frac{(1-x)^2}{(1-2x)^2}. \end{aligned}$$

Since $k \geq \lceil \frac{n+1}{2} \rceil$ we have $2(k+1) > n+1$. Thus, to consider the coefficient of x^n, we can ignore terms with exponent equal to and greater than $2(k+1)$. Therefore, between the "truncated" versions of the generating functions, we have:

$$GH_{k+1}(x) = \frac{(1-x)^2}{(1-2x)^2}(x^{k+1}) = x\frac{(1-x)^2}{(1-2x)^2}x^k = xGH_k(x).$$

Now, we can use the result of the Theorem 1.9 to prove all equations given in Theorem 2.2:

$$\begin{aligned} xGF_k(x) = GF_{k+1}(x) &= \frac{x^{k+1} - 2x^{k+2} + x^{k+3}}{(1-2x)^2} \\ &= \frac{x^{k+1}(1-2x+x^2)}{(1-2x)^2} \\ &= \frac{x^{k+1}(1-x)^2}{(1-2x)^2} \\ &= x\frac{x^k(1-x)^2}{(1-2x)^2} \\ &= GH_{k+1}(x) = xGH_k(x). \end{aligned}$$

□

2.2 VARIANTS OF COMPOSITIONS

In this section, we will show the generating function approach and some tables of some possible variants of compositions of n. Let's consider the case of one type of odd parts and k types of even parts. The generating function for one part made of one type of odd parts and k types of even parts is given by:

$$\begin{aligned} M(x) &= x + kx^2 + x^3 + kx^4 + \cdots \\ &= (x + x^3 + \cdots) + k(x^2 + x^4 + \cdots) \\ &= \frac{x}{1-x^2} + \frac{kx^2}{1-x^2} \\ &= \frac{x+kx^2}{1-x^2}. \end{aligned}$$

It can be shown that the generating function for all compositions of n made of one type of odd parts and k types of even parts is given by:

$$\begin{aligned} GM(x) &= \frac{M(x)}{1-M(x)} \\ &= \frac{\frac{x+kx^2}{1-x^2}}{1-\frac{x+kx^2}{1-x^2}} \\ &= \frac{x+kx^2}{1-x^2-x-kx^2}. \end{aligned}$$

The coefficient of x^n gives the total number of compositions of n made of one type of odd parts and k types of even parts. Expanding this generating function yields the following Table 2.2. Only the last two rows of the Table 2.2 are not currently in The On-line Encyclopedia of Integer Sequences [16].

$k \backslash n$	1	2	3	4	5	6	7	8	9	OEIS
1	1	2	4	8	16	32	64	128	256	A000079
2	1	3	6	15	33	78	177	411	942	A105476
3	1	4	8	24	56	152	376	984	2488	A159612
4	1	5	10	35	85	260	685	1985	5410	A189732
5	1	6	12	48	120	408	1128	3576	10344	
6	1	7	14	63	161	602	1729	5943	18046	

Table 2.2: The number of compositions of n with one type of odd parts and k types of even parts.

After we label each part with y, it can be shown that the generating function for the total number of parts in compositions of n with one type of odd parts and k types of even parts is given by:

$$GM(x,y) = \frac{yx + ykx^2}{1 - x^2 - yx - ykx^2}.$$

The coefficient of $x^n y^m$ is the number of compositions of n with m parts. Partially differentiating $GM(x,y)$ with respect to y and plugging in $y = 1$ gives the following:

$$\left.\frac{\partial GM(x,y)}{\partial y}\right|_{y=1} = \frac{(x + kx^2)(1 - x^2)}{(1 - x^2 - x - kx^2)^2}.$$

Expanding the above generating function, we get the results given in Table 2.3. Only the first row of the Table 2.3 is currently in The On-line Encyclopedia of Integer Sequences [16].

$k\backslash n$	1	2	3	4	5	6	7	8	9	OEIS
1	1	3	8	20	48	112	256	576	1280	A001792
2	1	4	12	36	99	270	711	1854	4752	
3	1	5	16	56	168	520	1512	4424	12584	
4	1	6	20	80	255	880	2755	8880	27380	
5	1	7	24	108	360	1368	4536	15912	52344	
6	1	8	28	140	483	2002	6951	26306	91280	

Table 2.3: The total number of parts in compositions of n with one type of odd parts and k types of even parts.

In general, it can be shown that the generating function for a composition of n made with a types of odd parts and b types of even parts is:

$$N(x) = \frac{\frac{ax+bx^2}{1-x^2}}{1-\frac{ax+bx^2}{1-x^2}} = \frac{ax+bx^2}{1-x^2-ax-bx^2}.$$

The coefficient of x^n gives the total number of compositions of n made of a types of odd parts and b types of even parts. Expanding the generating function we get the results given in Table 2.4. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$a, b\backslash n$	1	2	3	4	5	6	7	8	9
a=2,b=3	2	11	50	209	828	3172	11856	43520	157504
a=3,b=2	3	20	108	532	2481	11166	49005	211086	896184
a=3,b=4	3	22	132	720	3717	18512	89889	428336	2011620
a=4,b=3	4	35	244	1541	9192	52840	295968	1626224	8804096

Table 2.4: The total number of compositions of n having a types of odd parts and b types of even parts for selected a and b.

After we label all parts with y, it can be shown that the generating function for the number of compositions of n into k parts where each odd part has a types and each even part has b types is given by:

$$\begin{aligned} GN(x,y) &= \frac{\frac{y(ax+bx^2)}{1-x^2}}{1-\frac{y(ax+bx^2)}{1-x^2}} \\ &= \frac{yax+ybx^2}{1-x^2-yax-ybx^2}. \end{aligned}$$

Partially differentiating $H(x,y)$ with respect to y and plugging in $y=1$ and gives the following:

$$\left.\frac{\partial GN(x,y)}{\partial y}\right|_{y=1} = \frac{ax-ax^3+bx^2-bx^4}{(1-x^2-ax-bx^2)^2}.$$

Expanding the above result, we get the results shown in Table 2.5. Only the first row is currently in The On-Line Encyclopedia of Integer Sequences [16].

$a,b\backslash n$	1	2	3	4	5	6	7	8	9	OEIS
a=2,b=3	2	7	22	72	232	752	2432	7872	25472	A162770
a=3,b=2	3	11	42	159	603	2286	8667	32859	124578	
a=3,b=4	3	13	54	227	951	3988	16719	70097	293886	
a=4,b=3	4	19	92	444	2144	10352	49984	241344	1165312	

Table 2.5: The total number of parts in compositions of n having a types of odd parts and b types of even parts for selected a and b.

CHAPTER 3

CYCLIC COMPOSITIONS

In this chapter, we will explain, by means of a simple example, and applications, the direct generating function construction for cycles of combinatorial structure given in [6]. Flajolet and Soria gave in their paper [6] a general guide (without details) on how we can obtain the generating function of a cyclic composition from the generating function of regular compositions and primitive compositions based on three principles. The first principle states that every non-empty composition has a unique root which is a primitive composition.

Example 3.1. *Let $n = 18$. Then $(1, 2, 3, 1, 2, 3, 1, 2, 3)$ is made by repetition of the primitive composition $(1, 2, 3)$ three times, which represents the root of the composition.*

The second principle states that every primitive k-cycle has k-distinct primitive representations. A cycle is said to be primitive if and only if any associated linear composition is primitive.

Example 3.2. *The 3-cycle: $(1, 2, 3) \sim (2, 3, 1) \sim (3, 1, 2)$ is primitive, and if we cut the cyclic composition in between each number, we get three different linear primitive compositions:*

$$(1, 2, 3), (2, 3, 1), (3, 1, 2).$$

And the third principle states that every cycle has a root which is a primitive cycle.

Example 3.3. *The cycle $(1, 2, 3, 1, 2, 3, 1, 2, 3)$ has a unique root defined up to cycle length that is here:*

$$(1, 2, 3) \sim (2, 3, 1) \sim (3, 1, 2).$$

After the introduction of the main result in terms of a theorem, we will provide some further illustration with an example as well as applications.

3.1 Construction of the Generating Function for Cyclic Compositions

In this section, we will establish the main result given in Flajolet and Soria's paper as a theorem that shows the process of the construction of the cyclic generating function and its proof. The notations used in this chapter are given in Definition 14.

Definition 14. Let $A(x)$ be the generating function of a composition with exactly one part, $A(x, y)$ be the generating function of a composition with exactly one part but with a specific part size labeled by y, and $CC(x, y, u)$ be the generating function of a cyclic composition where y marks a specific part size, u marks the length of the cycle and x marks the sum of the parts.

The following Theorem 3.4 gives the way to get the generating function of a cyclic compositions from the ordinary generating function of linear compositions.

Theorem 3.4. *If the ordinary generating function of each part of some constrained composition is $A(x, y) = f(x, y)/g(x)$, then the ordinary generating function of the cyclic compositions, under the same constraints, is*

$$CC(x, y, u) = \sum_{p \geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{g(x^p)}{g(x^p) - u^p f(x^p, y^p)}\right),$$

where y is the variable used to label a specific part, u is the variable used to mark the length of a cycle, and $\varphi(p)$ is Euler's totient function.

Proof. Let

$$A(x, y) = \frac{f(x, y)}{g(x)}.$$

Then, to measure the length of a sequence or cycle, we use the variable u to label each part of the generating function and get:

$$uA(x, y) = \frac{uf(x, y)}{g(x)}.$$

The coefficient of $x^n y^l u^m$ in a generating function represents the number of compositions of n having length m with l parts of a specific size. Now, we introduce the multivariate generating function $C(x, y, u)$ for all regular compositions and get the following:

$$\begin{aligned} C(x,y,u) &= uA(x,y) + (uA(x,y))^2 + (uA(x,y))^3 + \cdots + (uA(x,y))^k + \cdots \\ &= \frac{uA(x,y)}{1 - uA(x,y)} \\ &= \frac{u\frac{f(x,y)}{g(x)}}{1 - u\frac{f(x,y)}{g(x)}} \\ &= \frac{uf(x,y)}{g(x) - uf(x,y)}. \end{aligned}$$

According to the first principle used in [6], every composition of n can be written uniquely as d copies of primitive compositions:

$$C(x,y,u) = \frac{uf(x,y)}{g(x) - uf(x,y)} = \sum_{d=1}^{\infty} PRC(x^d, y^d, u^d), \tag{3.1}$$

where $PRC(x^d, y^d, u^d)$ counts the compositions with d repetitions of a primitive composition. Now that $PRC(x^d, y^d, u^d)$ is represented implicitly, we will find a direct expression for it in the following. First, expand both generating functions, and we have:

$$\sum_{n,p,k} c(n,p,k) x^n y^p u^s = \sum_{d \geq 1} \sum_{m,q,l} prc(m,q,l) x^{md} y^{qd} u^{ld}.$$

By replacing $n = md$, $p = qd$ and $s = ld$, we get the following:

$$\sum_{n,p,s} c\,(n,p,s)\, x^n y^p u^s = \sum_{n,p,s} \sum_{d|(n,p,s)} prc\left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right) x^n y^p u^s.$$

Now, we compare the coefficients of these two infinite polynomials. For fixed n, p, and s, we get:

$$c(n,p,s) = \sum_{d|(n,p,s)} prc\left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right) = \sum_{d|(n,p,s)} prc\left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right) \overline{u}(d),$$

where $\overline{u}(d) = 1$ for all $d \in N$ (i.e.,$\overline{u}$ is the unit function). Next, by the definition of the *Dirichlet convolution* given in Section 1.2.3 in Chapter 1, the above statement is equivalent to the following:

$$c = prc * \overline{u}.$$

By multiplying both sides by μ, we get the following:

$$c * \mu = prc * \overline{u} * \mu.$$

Using the fact that $\overline{u}$ and μ are inverse multiplicative functions under *Dirichlet convolution*, as shown in Section 1.2.3; we have:

$$prc = c * \mu.$$

Going back to the *Dirichlet convolution* definition, we can get the following:

$$prc(n, p, s) = \sum_{d|(n,p,s)} \mu(d) c \left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right).$$

The multivariate generating function for primitive compositions is

$$PRC(x, y, u) = \sum_{m\geq 1, q\geq 1, l\geq 1} prc(m, q, l) x^m y^q u^l,$$

where $prc(m, q, l)$ counts the number of primitive compositions of m with length l and q parts of specific size. The multivariate generating function for general compositions is

$$C(x, y, u) = \sum_{n\geq 1, p\geq 1, s\geq 1} c(n, p, s) x^n y^p u^s,$$

where u marks the length of the composition, y marks the number of parts of a specific size, x the sum of parts. Now, we can reconstruct the multivariate generating functions for primitive compositions from Möbius inversion applied in (3.1). We get:

$$\sum_{n,p,s} prc(n, p, s) x^n y^p u^s = \sum_{n,p,s} \sum_{d|(n,p,s)} \mu(d) c \left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right) x^n y^p u^s.$$

From the definition of the general generating function, we get the following:

$$PRC(x, y, u) = \sum_{n,p,s} \sum_{d|(n,p,s)} \mu(d) c \left(\frac{n}{d}, \frac{p}{d}, \frac{s}{d}\right) x^n y^p u^s.$$

Now, using back substitution, we get:

$$PRC(x,y,u) = \sum_{d\geq 1}\sum_{m,q,l} \mu(d)c(m,q,l)x^{md}y^{qd}u^{ld}.$$

Similarly, by the definition of the multivariate generating function, we get the following:

$$PRC(x,y,u) = \sum_{d\geq 1} \mu(d)C(x^d,y^d,u^d) = \sum_{d\geq 1} \mu(d)\frac{u^d f(x^d,y^d)}{g(x^d) - u^d f(x^d,y^d)}.$$

Our goal is to find the multivariate generating function for primitive cyclic compositions. By the definition of the multivariate generating function, we have the following:

$$PRCC(x,y,u) = \sum_{m,q,l} prcc(m,q,l)x^m y^q u^l.$$

According to the second principle in [6], to obtain the multivariate generating functions for primitive cyclic compositions $PRCC(x,y,u)$, we need to divide each term of the generating function for primitive compositions $PRC(x,y,u)$ by l:

$$\sum_{m,q,l} prcc(m,q,l)x^m y^q u^l = \sum_{m,q,l} \frac{prc(m,q,l)}{l} x^m y^q u^l.$$

Integrating according to u with limits from 0 to u gives:

$$\sum_{m,q,l} \frac{prc(m,q,l)}{l} x^m y^q u^l = \sum_{m,q,l} \left(\int_0^u prc(m,q,l)x^m y^q u^{l-1} du\right).$$

Moving the integral sign inside the summation and factoring out $\frac{1}{u}$ gives:

$$\sum_{m,q,l} \left(\int_0^u prc(m,q,l)x^m y^q u^{l-1} du\right) = \int_0^u \frac{1}{u}\left(\sum_{m,q,l} prc(m,q,l)x^m y^q u^l\right) du.$$

From the general definition of a generating function, we now have:

$$\int_0^u \frac{1}{u}\left(\sum_{m,q,l} prc(m,q,l)x^m y^q u^l\right) du = \int_0^u \frac{PRC(x,y,u)}{u} du.$$

Thus:

$$PRCC(x,y,u) = \int_0^u \frac{PRC(x,y,u)}{u} du.$$

Since

$$PRC(x,y,u) = \sum_{d\geq 1}\mu(d)\frac{u^d f(x^d,y^d)}{g(x^d)-u^d f(x^d,y^d)},$$

we have:

$$\begin{aligned}\int_0^u \frac{PRC(x,y,u)}{u}du &= \int_0^u \sum_{d\geq 1}\mu(d)\frac{u^d f(x^d,y^d)}{g(x^d)-u^d f(x^d,y^d)}\frac{du}{u}\\ &= \sum_{d\geq 1}\mu(d)\int_0^u \frac{u^{d-1} f(x^d,y^d)}{g(x^d)-u^d f(x^d,y^d)}du.\end{aligned}$$

Solving the integral through substitution, we have:

$$\sum_{d\geq 1}\mu(d)\int_0^u \frac{u^{d-1} f(x^d,y^d)}{g(x^d)-u^d f(x^d,y^d)}du = \sum_{d\geq 1}\mu(d)\frac{1}{d}\ln\left(\frac{g(x^d)}{g(x^d)-u^d f(x^d,y^d)}\right).$$

Finally, by the above integration, we derive the generating function of primitive cyclic compositions:

$$PRCC(x,y,u) = \sum_{d\geq 1}\frac{\mu(d)}{d}\ln\left(\frac{g(x^d)}{g(x^d)-u^d f(x^d,y^d)}\right).$$

According to the third principle given in [6], every cyclic composition is composed of l adjacent copies of primitive cyclic compositions and the multivariate generating functions for cyclic compositions are given by the following:

$$CC(x,y,u) = \sum_{l\geq 1} PRCC(x^l,y^l,u^l) = \sum_{l\geq 1}\sum_{d\geq 1}\frac{\mu(d)}{d}\ln\left(\frac{g(x^{ld})}{g(x^{ld})-u^{ld} f(x^{ld},y^{ld})}\right).$$

By using the variable substitution $p = ld$, we get the following:

$$CC(x,y,u) = \sum_{l\geq 1} PRCC(x^l,y^l,u^l) = \sum_{p\geq 1}\sum_{d|p}\frac{\mu(d)}{d}\ln\left(\frac{g(x^{p})}{g(x^{p})-u^{p} f(x^{p},y^{p})}\right).$$

Recall the adjacent relation from Chapter 1, Section 1.2.3:

$$\sum_{d|p}\frac{\mu(d)}{d} = \frac{\varphi(p)}{p},$$

where $\varphi(p)$ is the Euler's totient function. We get the generating function for cyclic compositions:

$$CC(x,y,u) = \sum_{p\geq 1}\frac{\varphi(p)}{p}\ln\left(\frac{g(x^{p})}{g(x^{p})-u^{p} f(x^{p},y^{p})}\right).$$

$\square$

3.2 An Example

Now, let's consider the total number of parts with size k among all cyclic compositions of n as an illustrative example of Theorem 3.4. We already have the generating function for the number of parts of size k in the compositions of n with exactly one part:

$$F(x,y) = x + x^2 + \cdots + yx^k + \cdots = \frac{x^k(1-x)(y-1)+x}{1-x}.$$

According to Theorem 3.4, $F(x,y) = A(x,y)$. The generating function for all compositions of n is given by:

$$\begin{aligned} C(x,y,u) &= uA(x,y) + (uA(x,y))^2 + \cdots + (uA(x,y))^k + \cdots \\ &= \frac{uA(x,y)}{1-uA(x,y)} \\ &= \frac{u\frac{x^k(1-x)(y-1)+x}{1-x}}{1-u\frac{x^k(1-x)(y-1)+x}{1-x}} \\ &= \frac{u[x^k(1-x)(y-1)+x]}{1-x-u[x^k(1-x)(y-1)+x].} \end{aligned}$$

After we calculate the general generating function implicitly, we can easily distinguish $f(x,y)$ and $g(x)$ in accordance with Theorem 3.4 as

$$f(x,y) = x^k(1-x)(y-1)+x,$$

and

$$g(x) = 1-x.$$

Now we have the form:

$$C(x,y,u) = \frac{u[x^k(1-x)(y-1)+x]}{1-x-u[x^k(1-x)(y-1)+x]} = \frac{uf(x,y)}{g(x)-uf(x,y)}.$$

Following the guide in [6], now we can express, implicitly, the generating function for primitive compositions, the generating function for primitive cyclic compositions, and finally the generating function for cyclic compositions. The generating function for a primitive compositions is given by:

$$PRC(x,y,u) = \frac{u^d f(x^d,y^d)}{g(x^d) - u^d f(x^d,y^d)} = \frac{u^d[x^{kd}(1-x^d)(y^d-1)+x^d]}{1-x^d-u^d[x^{kd}(1-x^d)(y^d-1)+x^d]}.$$

The generating function for primitive cyclic compositions is given by:

$$\begin{aligned} PRCC(x,y,u) &= \sum_{d\geq 1} \frac{\mu(d)}{d} \ln\left(\frac{g(x^d)}{g(x^d)-u^d f(x^d,y^d)}\right) \\ &= \sum_{d\geq 1} \frac{\mu(d)}{d} \ln\left(\frac{1-x^d}{1-x^d-u^d[x^{kd}(1-x^d)(y^d-1)+x^d]}\right). \end{aligned}$$

In accordance with Theorem 3.4, we can find out the generating function for the cyclic compositions as follows:

$$\begin{aligned} CC(x,y,u) &= \sum_{p\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{g(x^p)}{g(x^p)-u^p f(x^p,y^p)}\right) \\ &= \sum_{p\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1-x^p}{1-x^p-u^p\left[x^{pk}(1-x^p)(y^p-1)+x^p\right]}\right). \end{aligned}$$

Before we may showcase some enumeration results from the previous generating functions, let's recall the fact that y is marking all parts of size k and u is marking all parts in a composition of n. If we plug in $y = u = 1$ into $C(x,y,u)$, we get the following:

$$C(x,1,1) = \frac{x}{1-2x}.$$

After we do the Taylor expansion, the coefficient of x^n is the total number of distinct linear compositions of n. If we take the derivative of $C(x,y,u)$ with respect to y and plug in $y = u = 1$, we get the following:

$$\left.\frac{\partial C(x,y,1)}{\partial y}\right|_{y=1} = \frac{x^k - 2x^{k+1} + x^{k+2}}{(1-2x)^2} = GF_k(x).$$

After we do the Taylor expansion of $GF_k(x)$, the coefficient of $x^n y^p$, gives the composition of n with p parts of size k. If we plug in $y = u = 1$ into $CC(x,y,u)$, we get the following:

$$CC(x,1,1) = \sum_{p\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1-x^p}{1-2x^p}\right).$$

This expands to the following polynomial:

$$P(x) = x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 13x^6 + 19x^7 + 35x^8 + 59x^9 + 107x^{10}$$
$$+ 187x^{11} + 351x^{12} + 631x^{13} + 1181x^{14} + \cdots .$$

The coefficient of x^n gives the total number of cyclic compositions of n. Some results are shown in the Table 1.1. Let's denote it with $b(n)$. The values of $b(n)$ are listed in Table 3.1. The sequence can be found in The On-Line Encyclopedia of Integer Sequences [16] as A008965, which represents the number of necklaces of sets of beads containing a total of n beads.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	OEIS
$b(n)$	1	2	3	5	7	13	19	35	59	107	187	351	631	1181	A008965

Table 3.1: Values of $b(n)$ for n = 1:16.

Example 3.5. *Let* $n = 5$*. The cyclic compositions of* 5 *are listed below.*

$$(1, 1, 1, 1, 1),$$

$$(1, 1, 1, 2) \sim (1, 1, 2, 1) \sim (1, 2, 1, 1) \sim (2, 1, 1, 1),$$

$$(1, 1, 3) \sim (1, 3, 1) \sim (3, 1, 1),$$

$$(1, 4),$$

$$(2, 3),$$

$$(2, 2, 1) \sim (2, 1, 2) \sim (1, 2, 2),$$

$$(5).$$

The number of distinct cyclic compositions is $b(5) = 7$.

Now, to find the total number of parts of a specific size k, we plug in $u = 1$:

$$CC(x, y, 1) = \sum_{p \geq 1} \frac{\varphi(p)}{p} \ln \left(\frac{1 - x^p}{1 - 2x^p - x^{pk}(1 - x^p)(y^p - 1)} \right).$$

The coefficient of $x^n y^p$ is the number of cyclic compositions of n with p parts of size k. Next, take the partial derivative of $CC(x, y, 1)$ with respect to y and plug in $y = 1$:

$$\left. \frac{\partial CC(x, y, 1)}{\partial y} \right|_{y=1} = \sum_{p \geq 1} \varphi(p) \frac{x^{pk}(1 - x^p)}{1 - 2x^p}.$$

The coefficient of x^n yields the number of parts of size k in all cyclic compositions. For various values of k our results are listed in Table 3.2. Here we use $b(n, k)$ to denote the number of parts of size k among cyclic compositions of n. None of the rows and columns of the table can be found in The On-Line Encyclopedia of Integer Sequences [16].

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	4	7	12	22	38	74	138	272	522	1058	2060	4140	8224
2	0	1	1	3	4	11	16	36	66	136	256	528	1024	2070	4108
3	0	0	1	1	2	5	8	17	34	66	128	264	512	1032	2056
4	0	0	0	1	1	2	4	9	16	33	64	132	256	516	1026
5	0	0	0	0	1	1	2	4	8	17	32	65	128	258	514
6	0	0	0	0	0	1	1	2	4	8	16	33	64	129	256
7	0	0	0	0	0	0	1	1	2	4	8	16	32	65	128
8	0	0	0	0	0	0	0	1	1	2	4	8	16	32	64
9	0	0	0	0	0	0	0	0	1	1	2	4	8	16	32
10	0	0	0	0	0	0	0	0	0	1	1	2	4	8	16

Table 3.2: Values of $b(n, k)$ for n = 1:15 and k = 1:10.

Example 3.6. *Again, we consider all cyclic compositions of $n = 5$ represented in Example 3.5. To find the total parts of size $k = 1$ among all cyclic compositions of $n = 5$, $b(5, 1)$, we count parts one in Example 3.5 and get:*

$$b(5, 1) = 12.$$

Similarly, by counting all parts of size two among all cyclic compositions of five, we have:

$$b(5, 2) = 4.$$

The total number of parts equal to three among all cyclic compositions of five will be:

$$b(5, 3) = 2.$$

Finally, the total number of parts of size five among all cyclic compositions of five is:

$$b(5, 4) = b(5, 5) = 1.$$

Similarly, to find the total number of parts among all cyclic compositions, we plug in $y = 1$:

$$CC(x, 1, u) = \sum_{p \geq 1} \frac{\varphi(p)}{p} \ln \left(\frac{1 - x^p}{1 - x^p - u^p x^p} \right).$$

The coefficient of $x^n u^m$ gives the number of cyclic compositions of n with m parts. Next, we partially differentiate $CC(x, 1, u)$ with respect to u and plug in $u = 1$:

$$\left. \frac{\partial CC(x, 1, u)}{\partial u} \right|_{u=1} = \sum_{p \geq 1} \varphi(p) \frac{x^p}{1 - 2x^p}.$$

By expanding the above generating function, we obtain the number $\tilde{b}(n)$ of parts among all cyclic compositions, shown in Table 3.3. The sequence can be found in The On-Line Encyclopedia of Integer Sequences [16] as A034738, which represents the *Dirichlet convolution* of $b_n = 2^{n-1}$ with $\phi(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	OEIS
$\tilde{b}(n)$	1	3	6	12	20	42	70	144	270	540	1034	2112	4108	A034738

Table 3.3: Values of $\tilde{b}(n)$ for n = 1:13.

Example 3.7. *With $n = 5$, we can easily see that:*

$$\tilde{b}(5) = b(5,1) + b(5,2) + b(5,3) + b(5,4) + b(5,5) = 12 + 4 + 2 + 1 + 1 = 20.$$

3.3 Applications

In this section, we will present some tables showing our enumeration results, as applications of Theorem 3.4. We will give some explanation about each table, such as the generating function used to find the values, the choices for the values of k, and initial values of n. Table 3.4 represents the total number of parts divisible by k among all cyclic compositions of n. The generating function for one part, referring to Chapter 2, is:

$$H(x,y) = \frac{x(1-x^k) + x^k(1-x)(y-1)}{(1-x)(1-x^k)}.$$

In this case, $H(x,y)$ represents $A(x,y)$. It can be shown that the generating function used to generate the numbers given in Table 3.4 is given by:

$$\left.\frac{\partial CCH(x,y,1)}{\partial y}\right|_{y=1} = \sum_{p,k\geq 1} \varphi(p) \frac{x^{pk}(1-x^p)^2}{(1-2x^p)^2(1-x^{pk})}.$$

The first and the second rows are the only rows of the table currently in The On-Line Encyclopedia of Integer Sequences [16]. The first row is the same as the sequence given in Table 3.3.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	OEIS
1	1	3	6	12	20	42	70	144	270	540	1034	2112	4108	A034738
2	0	1	1	4	5	14	21	48	87	180	341	704	1365	A306897
3	0	0	1	1	2	6	9	19	39	75	146	302	585	
4	0	0	0	1	1	2	4	10	17	35	68	141	273	
5	0	0	0	0	1	1	2	4	8	18	33	67	132	
6	0	0	0	0	0	1	1	2	4	8	16	34	65	
7	0	0	0	0	0	0	1	1	2	4	8	16	32	
8	0	0	0	0	0	0	0	1	1	2	4	8	16	
9	0	0	0	0	0	0	0	0	1	1	2	4	8	
10	0	0	0	0	0	0	0	0	0	1	1	2	4	

Table 3.4: The number of parts divisible by k among all cyclic compositions of n.

The total numbers of cyclic compositions made with only parts of size $\equiv 1 \pmod{k}$ are given in Table 3.5, and it can be shown that the generating function used to find those values given is by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1-x^{pk}}{1-x^{pk}-u^p x^p}\right).$$

The values of k start from 2, since, for $k = 1$, we will get the total number of cyclic compositions of n, which are already given in Table 3.1. The first row is the only sequence of numbers found in [16] (as A032189) which represents the number of ways to partition n elements into pie slices, each with an odd number of elements.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	OEIS
2	1	1	2	2	3	4	5	7	10	14	19	30	41	63	94	142	A032189
3	1	1	1	2	2	2	3	4	4	6	7	10	12	17	22	31	
4	1	1	1	1	2	2	2	2	3	4	4	5	6	8	10	12	
5	1	1	1	1	1	2	2	2	2	2	3	4	4	5	5	7	
6	1	1	1	1	1	1	2	2	2	2	2	2	3	4	4	5	
7	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	4	
8	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	
9	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	
10	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	
11	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	
12	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	
13	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	

Table 3.5: The number of cyclic compositions of n with only parts $\equiv 1 \pmod{k}$.

The total number of parts in cyclic compositions made with only parts of size $\equiv 1 \pmod{k}$ are given in Table 3.6, and it can be shown that the generating function used to obtain the values is given by:

$$\sum_{p,k\geq 1} \varphi(p) \left(\frac{x^p}{1 - x^{pk} - x^p} \right).$$

We start from $k = 2$, since, for $k = 1$, we get the total number of parts among all cyclic compositions of n which are already given in the Table 3.3. Only the first row is currently in The On-Line Encyclopedia of Integer Sequences [16] (as A034748), which represents the *Dirichlet convolution* of *Fibonacci numbers* with primes (with 1).

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	OEIS
2	1	2	4	6	9	14	19	30	44	68	99	168	245	402	636	A034748
3	1	2	3	5	7	9	12	17	21	30	38	57	72	106	147	
4	1	2	3	4	6	8	10	12	15	20	24	32	38	52	66	
5	1	2	3	4	5	7	9	11	13	15	18	23	27	35	40	
6	1	2	3	4	5	6	8	10	12	14	16	18	21	26	30	
7	1	2	3	4	5	6	7	9	11	13	15	17	19	21	24	
8	1	2	3	4	5	6	7	8	10	12	14	16	18	20	22	
9	1	2	3	4	5	6	7	8	9	11	13	15	17	19	21	
10	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20	
11	1	2	3	4	5	6	7	8	9	10	11	13	15	17	19	
12	1	2	3	4	5	6	7	8	9	10	11	12	14	16	18	
13	1	2	3	4	5	6	7	8	9	10	11	12	13	15	17	
14	1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	
15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	

Table 3.6: The number of parts in cyclic compositions of n made of only parts $\equiv 1 \pmod{k}$.

The total number of cyclic compositions made with only parts of size $\equiv 2 \pmod{k}$ are given in Table 3.7, and it can be shown that the generating function used to obtain these numbers is given by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1-x^{pk}}{1-x^{pk}-u^p x^{2p}}\right).$$

The values of k start from 3 and include a number that is relatively prime with 2, since, for $k=1$, we get the total number of cyclic compositions of n given in Table 3.1, and for all even values of k, we get the total number of cyclic compositions of n made by only even parts which are similar with the sequence given in Table 3.1. The only the difference is that, for each k multiple of 2, we get $k-1$ zeros in between each number. The values of n start from 2 because for $n=1$ the result is zero for all ks. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	1	0	1	1	1	1	2	1	3	2	3	3	5	5	7	7	10
5	1	0	1	0	1	1	1	1	1	1	2	1	3	1	3	2	4
7	1	0	1	0	1	0	1	1	1	1	1	1	1	1	2	1	3
9	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1
11	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1
13	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1
15	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1

Table 3.7: The number of cyclic compositions of n with only parts $\equiv 2 \pmod{k}$.

The total number of parts of size $\equiv 2 \pmod{k}$ among all cyclic compositions made of parts of size $\equiv 2 \pmod{k}$ are given in Table 3.8, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)x^{2p}}{1 - x^{pk} - x^{2p}}.$$

The values of k in Table 3.8 start from 3 and include numbers that are relatively prime with 2, because for $k = 1$, we get the total number of parts in cyclic compositions of n given in Table 3.3, and for all even values of k we get the total number of parts in cyclic compositions of n made by only even parts. This is similar to the sequence given in Table 3.3 but for each k multiple of 2 we get $k-1$ zeros in between each numbers. The values of n start from 2 for the same reason as in the previous table. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	1	0	2	1	3	2	5	3	9	5	12	9	20	18	29	28	48
5	1	0	2	0	3	1	4	2	5	3	7	4	11	5	14	7	21
7	1	0	2	0	3	0	4	1	5	2	6	3	7	4	9	5	13
9	1	0	2	0	3	0	4	0	5	1	6	2	7	3	8	4	9
11	1	0	2	0	3	0	4	0	5	0	6	1	7	2	8	3	9
13	1	0	2	0	3	0	4	0	5	0	6	0	7	1	8	2	9
15	1	0	2	0	3	0	4	0	5	0	6	0	7	0	8	1	9

Table 3.8: The number of parts in cyclic compositions of n made of only parts $\equiv 2 \pmod{k}$.

The number of cyclic compositions of n made by only parts of size $\equiv 3 \pmod{k}$ is given in Table 3.9, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1-x^{pk}}{1-x^{pk}-u^p x^{3p}}\right).$$

The values of k start at $k=4$ and include only numbers that are relatively prime with 3 because the sequence generated for $k=1$ gives the same sequence generated at Table 3.1. The sequence generated for $k=2$ is the same as the sequence generated for $k=2$ in Table 3.5 since the generated function in both cases is made by parts of the same size, and for all k multiples of 3, we get the total number of cyclic compositions made only by parts of size that are multiples of 3; additionally, the sequence generated will be of the same numbers as the general sequence given in Table 3.1, but for each k we get $k-1$ zeros in between each number of the original sequence. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	1	0	0	1	1	0	1	1	1	1	1	2	2	1	2	3	2
5	1	0	0	1	0	1	1	0	1	1	1	1	1	2	1	2	2
7	1	0	0	1	0	0	1	1	0	1	1	0	1	1	1	1	1
8	1	0	0	1	0	0	1	0	1	1	0	1	1	0	1	1	1
10	1	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
11	1	0	0	1	0	0	1	0	0	1	0	1	1	0	1	1	0
13	1	0	0	1	0	0	1	0	0	1	0	0	1	1	0	1	1
14	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1	1	0

Table 3.9: The number of cyclic compositions of n with only parts $\equiv 3 \pmod{k}$.

The number of parts in cyclic compositions of n made by only parts of size $\equiv 3 \pmod{k}$ is given in Table 3.10, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)x^{3p}}{1 - x^{pk} - x^{3p}}.$$

According to Table 3.10, k values start at $k = 4$ and include only numbers that are relatively prime with 3 because the sequence generated for $k = 1$ is the same as the sequence generated in Table 3.3. The sequence generated for $k = 2$ in Table 3.10 is the same as the sequence generated for $k = 2$ in Table 3.6 since the generated function in both cases is made by parts of the same size, and for all k multiples of 3, we get the total number of parts in cyclic compositions made only by parts of size that are multiples of 3 which is the general sequence given in Table 3.3 with $k - 1$ zeros in between each number. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	1	0	0	2	1	0	3	2	1	4	3	4	6	4	6	10	6
5	1	0	0	2	0	1	3	0	2	4	1	3	5	4	4	7	6
7	1	0	0	2	0	0	3	1	0	4	2	0	5	3	1	6	4
8	1	0	0	2	0	0	3	0	1	4	0	2	5	0	3	6	1
10	1	0	0	2	0	0	3	0	0	4	1	0	5	2	0	6	3
11	1	0	0	2	0	0	3	0	0	4	0	1	5	0	2	6	0
13	1	0	0	2	0	0	3	0	0	4	0	0	5	1	0	6	2
14	1	0	0	2	0	0	3	0	0	4	0	0	5	0	1	6	0

Table 3.10: The number of parts in cyclic compositions of n made of only parts $\equiv 3 \pmod{k}$.

The number of cyclic compositions of n made by only parts of size 1 and k is given in Table 3.11, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p,k\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1}{1-u^p(x^p+x^{pk})}\right).$$

We start with $k=2$ because $k=1$ means that we are interested in the number of cyclic compositions made by only parts of size one and there is one way we can get that combination for all n. Therefore the result will be a sequence made of only ones. Only the first row is currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	OEIS
2	1	2	2	3	3	5	5	8	10	15	19	31	41	64	94	A000358
3	1	1	2	2	2	3	3	4	5	6	7	11	12	17	23	
4	1	1	1	2	2	2	2	3	3	4	4	6	6	8	10	
5	1	1	1	1	2	2	2	2	2	3	3	4	4	5	6	
6	1	1	1	1	1	2	2	2	2	2	2	3	3	4	4	
7	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	
8	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	
9	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	
10	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	
11	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	
12	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	
13	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	
14	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	

Table 3.11: The number of cyclic compositions of n with only parts 1 and k.

The number of parts in cyclic compositions of n, made by only parts of size 1 and k, is given in Table 3.12, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p,k\geq 1} \frac{\phi(p)(x^p + x^{pk})}{1 - x^p - x^{pk}}.$$

We start from $k = 2$ because $k = 1$ means we are interested in the total number of parts in cyclic compositions of n made by only parts of size one. Therefore the sequence generated will be the sequence of natural numbers. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	3	5	9	12	22	27	47	67	109	154	270	389	649
3	1	2	4	6	8	12	15	22	29	40	51	82	100	150
4	1	2	3	5	7	9	11	15	18	25	29	42	48	67
5	1	2	3	4	6	8	10	12	14	18	21	28	32	42
6	1	2	3	4	5	7	9	11	13	15	17	21	24	31
7	1	2	3	4	5	6	8	10	12	14	16	18	20	24
8	1	2	3	4	5	6	7	9	11	13	15	17	19	21
9	1	2	3	4	5	6	7	8	10	12	14	16	18	20
10	1	2	3	4	5	6	7	8	9	11	13	15	17	19
11	1	2	3	4	5	6	7	8	9	10	12	14	16	18
12	1	2	3	4	5	6	7	8	9	10	11	13	15	17
13	1	2	3	4	5	6	7	8	9	10	11	12	14	16
14	1	2	3	4	5	6	7	8	9	10	11	12	13	15
15	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Table 3.12: The number of parts in cyclic compositions of n made of only parts 1 and k.

The number of cyclic compositions of n made by only parts 1 through k is given in Table 3.13, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p\geq 1} \frac{\varphi(p)}{p} \ln\left(\frac{1}{1-u^p\sum_{j\geq 1} x^{pj}}\right).$$

All the rows of the table can be found in The On-Line Encyclopedia of Integer Sequences [16]. A000358 represents the number of binary necklaces of length n with no subsequence 00, excluding the necklace 0. A093305 represents the number of binary necklaces of length n with no subsequence 000, and A280218 represents the number of binary necklaces of length n with no subsequence 0000. The values of k start from 2, because for $k = 1$, we are interested in the number of cyclic compositions made by only parts of size one. Therefore the result will be a sequence made of only ones.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	OEIS
2	1	2	2	3	3	5	5	8	10	15	19	31	41	64	A000358
3	1	2	3	4	5	9	11	19	29	48	75	132	213	369	A093305
4	1	2	3	5	6	11	15	27	43	75	125	228	391	707	A280218

Table 3.13: The number of cyclic compositions of n made of only parts 1 up to k.

The number of parts in cyclic compositions of n made by only parts 1 through k are given in Table 3.14, and it can be shown that the generating function used to find these values is given by:

$$\sum_{p\geq 1} \frac{\varphi(p)\sum_{j\geq 1} x^{pj}}{1-\sum_{j\geq 1} x^{pj}}.$$

The values of k start from $k = 2$, since $k = 1$ means we are interested in the total number of parts in cyclic compositions of n made by only parts of size one. Therefore the sequence generated will be the sequence of natural numbers. None of the rows or columns are currently in The On-Line Encyclopedia of Integer Sequences [16].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	3	5	9	12	22	27	47	67	109	154	270	389	649
3	1	3	6	11	17	34	50	96	163	299	514	981	1717	3198
4	1	3	6	12	19	39	62	124	222	428	783	1551	2884	5610

Table 3.14: The number of parts in cyclic compositions made of only parts 1 up to k.

CHAPTER 4

MULTICOMPOSITIONS

In this chapter we will explain the general approach of using generating functions to obtain the number sequences involving multicompositions and parts in multicompositions. We, also will present a few interesting bijections involving multicompositions.

4.1 Definition and Combinatorial Representation

In this section we will present two different combinatorial representations of compositions of n, which are going to be used to build the bijections. Let's recall Definition 12, the definition of a k-compositions given in Chapter 1:

Definition 15. A k-composition is an integer composition with k types of each part except for the first part, which only has one type.

Most of the time, we will use subscripts to denote the type of a part. For instance, $(3_1, 4_2, 1_3, 2_1)$ is a 3-composition of 10 with first part of size 3 and type 1, second part of size 4 and type 2, third part of size 1 and type 3, and fourth part of size 2 and type 1. Let's also recall Example 1.8 given in the first chapter:

Example 4.1. *The* 2*-compositions of* $n = 4$ *with* $k = 3$ *parts are:*

$$(1_1, 1_1, 2_1), \quad (1_1, 1_1, 2_2), \quad (1_1, 1_2, 2_1), \quad (1_1, 1_2, 2_2),$$
$$(1_1, 2_1, 1_1), \quad (1_1, 2_2, 1_1), \quad (1_1, 2_1, 1_2), \quad (1_1, 2_2, 1_2),$$
$$(2_1, 1_1, 1_1), \quad (2_1, 1_1, 1_2), \quad (2_1, 1_2, 1_1), \quad (2_1, 1_2, 1_2).$$

We will show now two equivalent combinatorial representations of compositions of n used in this chapter. The regular compositions of n can be represented by a board of length n with $n-1$ internal integer positions. There are two possibilities for the relation of two adjacent squares. They are joined (J) or they are separated so a new part starts at the right (S). Each new part is assigned one of k colors. This combinatorial representation of a regular composition of n is given in [10]. Finally, there are $k+1$ choices at each interval denoted by J, S_1 for separate and start a new part with color 1, S_2 for separate and start a new part with color 2, and in general, S_k for separate and start a new part with color k. See Figure 4.1 for an illustrative example of this J, S_i map.

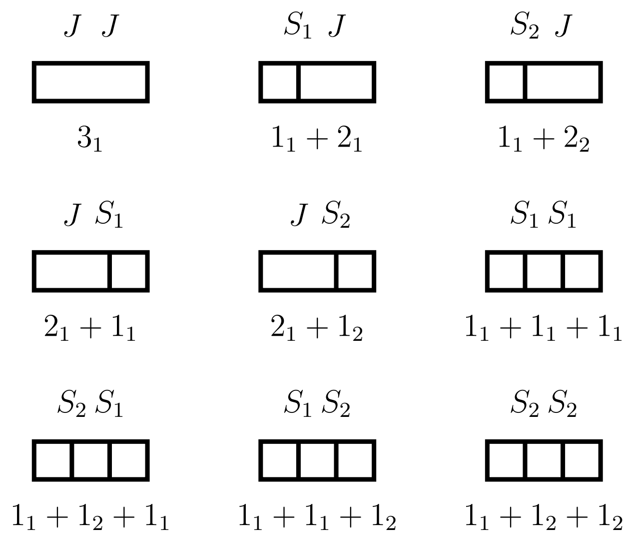


Figure 4.1: J, S_i map in 2-compositions of $n = 3$.

We have $n-1$ choices with $k+1$ options each from the tilings. Let's consider the generating function for all k-compositions of n:

$$\begin{aligned}
GF_k(x) &= \frac{x + x^2 + \cdots}{1 - k(x + x^2 + \cdots)} \\
&= \frac{\frac{x}{1-x}}{1 - \frac{kx}{1-x}} \\
&= \frac{x}{1 - x(k+1)} \\
&= x + x^2(k+1) + x^3(k+1)^2 + \cdots + x^n(k+1)^{n-1} \\
&= \sum_{n=1}^{\infty} (k+1)^{n-1} x^n.
\end{aligned}$$

The coefficient of x^n is $(k+1)^{n-1}$, and it gives the total number of k-compositions of n. Also, for any two distinct k-compositions, there is no way we can get the same choice. It is easy to verify this relation is one-to-one and onto, so it is a bijection between the choices and k-compositions of n. Another way to conceptualize the tilings of a $1 \times n$ board is by using height map. Here we employ a similar approach, where a part of size m and type t is represented by a $t \times m$ tile, and a k-composition of n is a covering of the $1 \times n$ board with blocks of height up to k, with the additional constraint that the first block is always of height one. See Figure 4.2 for an illustration.

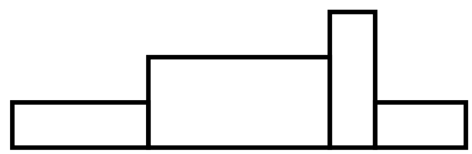


Figure 4.2: Block representation of the composition $(3_1, 4_2, 1_3, 2_1)$.

One way to view such a tiling is through the vertical lines separating adjacent unit squares, as shown in Figure 4.3. We will call this the line representation of the multicomposition.

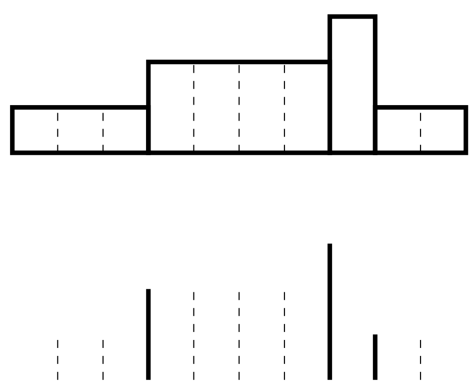


Figure 4.3: Block representation with lines (top) and line representation (bottom) of the composition $(3_1, 4_2, 1_3, 2_1)$.

The k color options from the J, S_i map are the same as the k heights options of the vertical line separating adjacent unit squares from height map. Therefore, the height map and the J, S_i map are equivalent. In the next sections, we will be using this representation to establish some interesting combinatorial connections between multicompositions and other objects.

4.2 Generating Functions and Counting Sequences

In this section, we establish the generating function for multicompositions under various constraints, presenting the counting sequences and checking them against established integer sequences on The On-Line Encyclopedia of Integer Sequence [16]. Also, we will give bijections involving Pell numbers in Subsection 4.2.1 and Jacobsthal sequences in Subsection 4.2.2.

4.2.1 Multicompositions Made of Odd Parts

According to [10], the general generating function for k-compositions made of only odd parts is:

$$GF_{odd} = \frac{x}{1 - x^2 - kx}.$$

All the sequences generated by expanding the generating function yield sequences that are currently in The On-Line Encyclopedia of Integer Sequences [16]. Note that one of the sequences generated is $A000129$, the well-known Pell numbers. Let's now give a bijection involving Pell numbers using the map given in Figure 4.1.

Example 4.2. *Referring to Figure 4.1: All* 2*-compositions of* 3 *made with only odd parts are given below.*

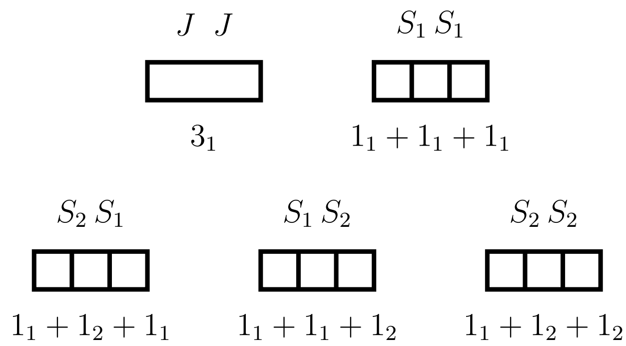


Figure 4.4: The J, S_1, S_2 map in 2-compositions of $n = 3$ with only odd parts.

Also, according to The On-Line Encyclopedia of Integer Sequences [16], Pell numbers count the number of lattice paths from $(0, 0)$ to the line $x = n - 1$ consisting of steps $U = (1, 1)$ $D = (1, -1)$ and $H = (2, 0)$. Let's consider the bijection map J, S_i from 2-compositions of n and consider the following correspondence given in Figure 4.5. We get the equivalent map to the lattice paths, which is also a bijection map. Note to remove the first step which is always U since the first part has only one type.

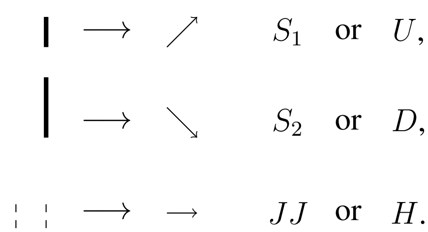
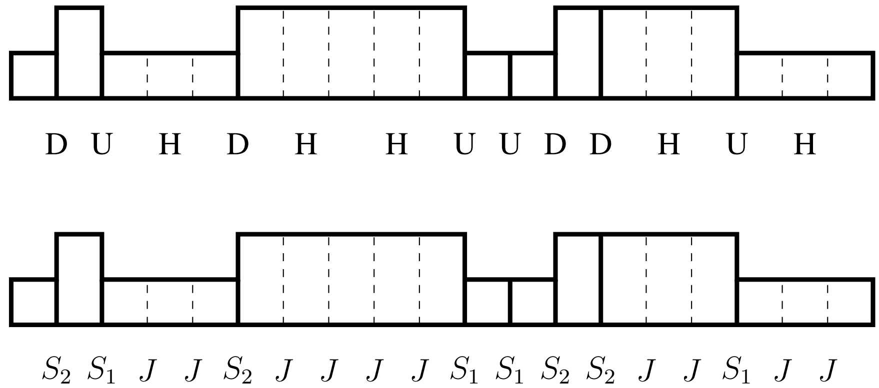


Figure 4.5: The height map and the J, S_i map for the lattice paths.

Now, let's consider a random tiling representation as an example to show the application of the above maps.

Example 4.3. *According to Figure 4.5 we get the following,*



$$1_1 + 1_2 + 3_1 + 5_2 + 1_1 + 1_1 + 1_2 + 3_2 + 3_1.$$

Figure 4.6: Application of the height map and J, S_i map for the lattice paths.

4.2.2 Multicompositions with no Part of Size 1

Similar to the previous subsection, according to [10], the generating function for k-compositions with no part of size 1 is:

$$GF_{\hat{1}}(x) = \frac{x^2}{1 - x - kx^2}.$$

All the sequences generated by expanding the generating function yield the sequences of numbers that are currently in The On-Line Encyclopedia of Integer Sequences [16]. Note that one of the sequences generated is A001045, the Jacobsthal sequence. Now let's give an application of the J, S_1, S_2 map through an example.

Example 4.4. *Let's consider the same 2-compositions from the previous example.*

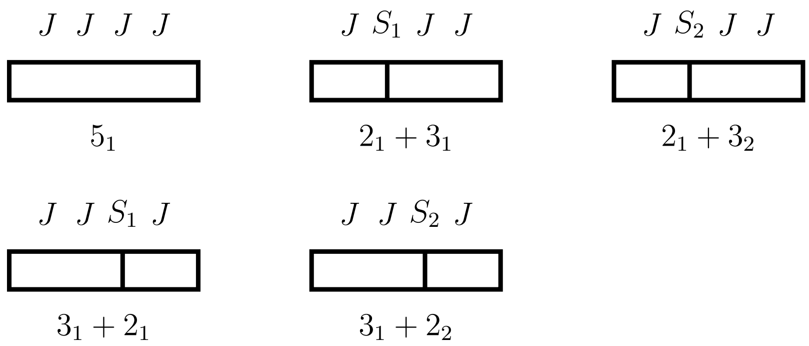


Figure 4.7: The J, S_i map in 2-compositions of $n = 5$ with no part of size one.

Also, according to The On-Line Encyclopedia of Integer Sequences [16], the Jacobsthal sequence counts the number of ways to tile a $2 \times (n-1)$ rectangular board with dominoes and 2×2 squares. We start from the block representations. For each block of size $1 \times k$, map it to a pair of horizontal dominoes followed by $k-2$ vertical dominoes, and for each block of size $2 \times k$, map it to a 2×2 square followed by $k-2$ vertical dominoes. Remove the two horizontal dominoes at the beginning. See Figure 4.8 for an illustration.

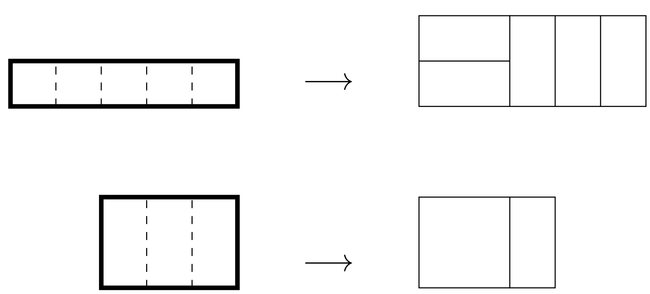


Figure 4.8: The mapping from block representation to tiling.

In the following example we will illustrate the map given in Figure 4.8 and later on we will explain why it is a bijection map referring to the J, S_i map.

Example 4.5. *Let's consider all* 2*-compositions of* $n = 5$ *with no part one.*

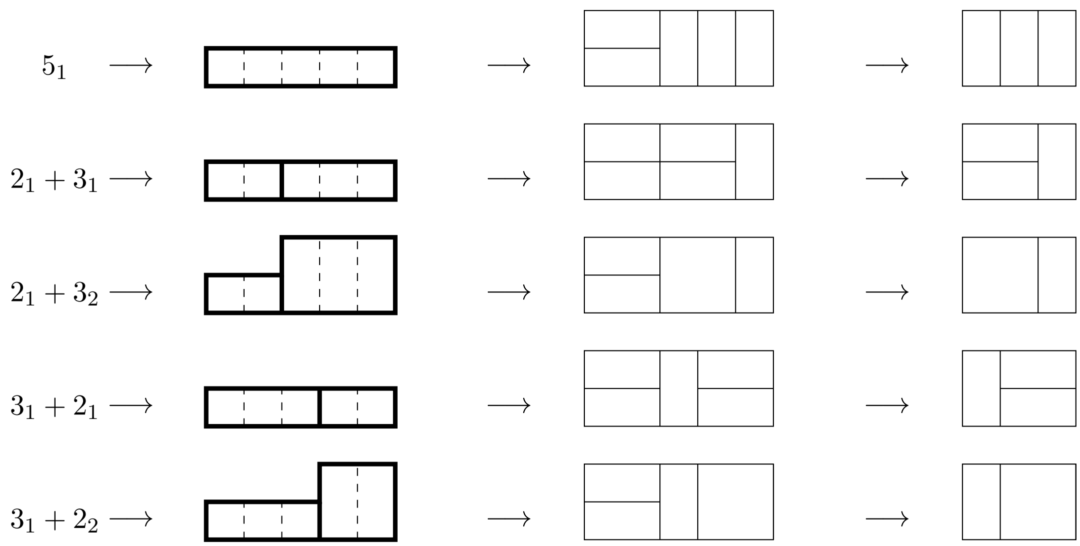


Figure 4.9: A bijection between the 2-compositions of $n = 5$ with no part one and tilings of a 2×4 board.

The J, S_i map is a bijection, and by comparing Figure 4.7 with Figure 4.9, we can easily see the equivalence between two maps by the substitutions given in Figure 4.10 without considering the first J. Since J, S_i is a bijection, the mapping from block representation to tiling is also a bijection.

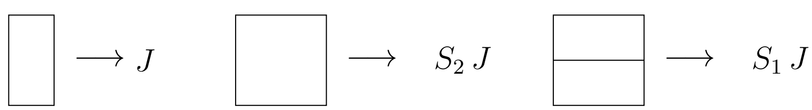


Figure 4.10: The equivalence between the J, S_i map and tilings of the block representations.

4.3 Parts in Multicompositions

In this section we will explain, by means of a simple example, the general approach of using generating functions to obtain number sequences involving parts in k-compositions. Let's consider the case of the total number of parts in all k-compositions of n. After we label all parts with y, it can be shown that the generating function is:

$$\begin{aligned} GF(x,y,k) &= \frac{y(x+x^2+x^3+\ldots)}{1-ky(x+x^2+x^3+\ldots)} \\ &= \frac{yx(1+x+x^2+\ldots)}{1-kyx(1+x+x^2+\ldots)} \\ &= \frac{yx\frac{1}{1-x}}{1-kyx\frac{1}{1-x}} \\ &= \frac{yx}{1-x-kyx}. \end{aligned}$$

The coefficient of $x^n y^m$ is the number of k-compositions of n with m parts. Next, by partially differentiating $GF(x,y,k)$ with respect to y and plugging in $y=1$, it can be shown that the final result is:

$$\left.\frac{\partial GF(x,y,k)}{\partial y}\right|_{y=1} = \frac{x-x^2}{(1-x-kx)^2}.$$

Expanding this generating function yields the following Table 4.1. All the sequences

$k\backslash n$	1	2	3	4	5	6	7	8	9	OEIS
1	1	3	8	20	48	112	256	576	1280	A049610
2	1	5	21	81	297	1053	3645	12393	41553	A081038
3	1	7	40	208	1024	4864	22528	102400	458752	A081039
4	1	9	65	425	2625	15625	90625	515625	2890625	A081040
5	1	11	96	756	5616	40176	279936	1912896	12877056	A081041
6	1	13	133	1225	10633	88837	722701	5764801	45294865	A081042

Table 4.1: Total number of parts in all multicompositions.

generated can be found in The On-Line Encyclopedia of Integer Sequences [16].

CHAPTER 5

CONCLUDING REMARKS AND FUTURE WORKS

In this chapter, we will establish some summaries about the results shown in each chapter, as well as concluding remarks and future works.

In Chapter 1, we gave definitions along with examples of the concepts and notations used in this study. Also, through a single example, we derived the result relating the connection between the total number of parts of size k among all compositions of n and the total number of parts of size $k+1$ among all compositions of $n+1$.

In Chapter 2, we gave the relation between the total number of parts divisible by k among all compositions of n and the total number of parts divisible by $k+1$ among all compositions of $n+1$ under a specific constraint. Then, we showed the connection between this result and the result given in Chapter 1. Also in this chapter we considered some variants of compositions of n.

In Chapter 3, we gave a detailed example and some applications of the important result given in the Flajolet and Soria paper [6]. It showed a direct way to construct the generating function of the cyclic compositions from the ordinary generating function of each part.

In Chapter 4, we gave a few interesting bijections involving multicompositions made of only odd parts and multicompositions with no part of size one. These results are part of the further work I would pursue on multicompositions.

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