Summer 2020

Explicit Pseudo-Kähler Metrics on Flag Manifolds

Thomas A. Mason III

EXPLICIT PSEUDO-KÄHLER METRICS ON FLAG MANIFOLDS

by

THOMAS ALBERT MASON, III

(Under the Direction of François Ziegler)

ABSTRACT

The coadjoint orbits of compact Lie groups each carry a canonical (positive definite) Kähler structure, famously used to realize the group's irreducible representations in holomorphic sections of certain line bundles (Borel-Weil theorem). Less well-known are the (indefinite) invariant *pseudo*-Kähler structures they also admit, which can be used to realize the same representations in higher cohomology of the sections (Bott), and whose analogues in a non-compact setting lead to new representations (Kostant-Langlands). The purpose of this thesis is to give an explicit description of these metrics in the case of the unitary group $\mathrm{G} = \mathrm{U}_n$.

INDEX WORDS: Coadjoint orbit, Complex manifold, Flag variety, Indefinite metric

2009 Mathematics Subject Classification: 14M15, 17B08, 32M10, 32Q15, 53C50.

EXPLICIT PSEUDO-KÄHLER METRICS ON FLAG MANIFOLDS

by

THOMAS ALBERT MASON, III

B.S., Mars Hill University, 2015

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

EXPLICIT PSEUDO-KÄHLER METRICS ON FLAG MANIFOLDS

by

THOMAS ALBERT MASON, III

Major Professor: François Ziegler
Committee: Yi Lin
Paul Sobaje

Electronic Version Approved:
July 2020

DEDICATION

This thesis is dedicated to my wife, Sydney, and our two beautiful children, Eleanor and Adelaide. Each of you has blessed me beyond measure.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

The coadjoint orbits of compact Lie groups each carry a canonical positive definite Kähler structure. When a coadjoint orbit carries more than two complex structures, that is, complex structures distinct from the canonical one and its negative, this orbit further admits indefinite pseudo-Kähler structures. In the case of the unitary group U_n, the existence of such pseudo-Kähler structures is well known, but their computation is often tedious and inflexible. The trouble is with computing the attendant complex structures on the orbit's tangent space. One finds that this computation yields non-explicit formulas for the complex structures. The objective of this thesis is to obtain explicit general formulas for these pseudo-Kähler structures. To this end, we begin by examining the Grassmannian.

We write Gr_m for the Grassmannian of complex m-planes in $\mathbf{C}^n$, each identified with the self-adjoint projector x upon it, i.e.

$$\mathrm{Gr}_m = \left\{ x \in \mathfrak{gl}_n(\mathbf{C}) : x \text{ is self-adjoint}, x^2 = x, \mathrm{Trace}(x) = m \right\}. \tag{1}$$

The canonical Kähler structure of Gr_m is (see (34)):

$$\begin{cases} \mathrm{I}\delta x = [\mathrm{i}x, \delta x] & \text{(2a)} \\ \mathrm{g}(\delta x, \delta' x) = \mathrm{Trace}(\delta x \delta' x) & \text{(2b)} \\ \omega(\delta x, \delta' x) = \mathrm{Trace}(\delta x \mathrm{I}\delta' x) = \mathrm{Trace}(\mathrm{i}x[\delta' x, \delta x]). & \text{(2c)} \end{cases}$$

That the Kähler structure of the Grassmannian is both simple and explicit is easy to see. We go on to utilize carefully chosen Grassmannians and their associated Kähler structures as the building blocks for the pseudo-Kähler structures we seek.

A coadjoint orbit of $\mathrm{G} = \mathrm{U}_n$ is a conjugacy class of self-adjoint matrices. For λ self-adjoint, G acts on λ by conjugation and the resulting orbit is called $\mathrm{G}(\lambda)$. The multiplicities and ordering of the eigenvalues of λ prove crucial to constructing the structures we seek. We show that a coadjoint orbit $\mathrm{X} = \mathrm{G}(\lambda)$ is isomorphic to a submanifold Y of a product of

Grassmannians; the exact composition of this product of Grassmannians is determined by a particular choice of the ordering of the eigenvalues of λ. This submanifold Y inherits the (pseudo-)Kähler structure of this product of Grassmanians.

As an example, let $\mathrm{G} = \mathrm{U}_4$ and take $\mathrm{X} = \mathrm{G}(\lambda)$ where $\lambda = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$. We will find that X is isomorphic to

$$\mathrm{Y} = \left\{ y = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathrm{Gr}_2 \times \mathrm{Gr}_3 \times \mathrm{Gr}_4 \; : \; \begin{matrix} y_3 y_2 = y_2 \\ y_4 y_3 = y_3 \end{matrix} \right\}. \tag{3}$$

Then, Y has metric

$$\mathrm{g}(\delta y, \delta' y) = \mathrm{Trace}(\delta y_2 \delta' y_2) - 2\,\mathrm{Trace}(\delta y_3 \delta' y_3) \tag{4}$$

and gives the symplectic form $\omega(\cdot,\cdot) = \mathrm{g}(\cdot, \mathrm{J}\cdot)$ with the product complex structure

$$\mathrm{J}\delta y = \begin{pmatrix} [\mathrm{i}y_2, \delta y_2] \\ [\mathrm{i}y_3, \delta y_3] \\ [\mathrm{i}y_4, \delta y_4] \end{pmatrix}. \tag{5}$$

This is obtained by taking y_2 to be the projector onto the eigenspace for eigenvalue 0, y_3 the projector onto the the sum of the eigenspaces for eigenvalues 0 and -1, and y_4 the projector onto the the sum of the eigenspaces for eigenvalues 0, -1, and 1. Taking these eigenvalues in different orders gives all other structures. For example, the order $1, 0, -1$ gives the canonical structure for the embedding into $\mathrm{Gr}_1 \times \mathrm{Gr}_3 \times \mathrm{Gr}_4$ having Riemannian metric

$$\mathrm{g}(\delta y, \delta' y) = \mathrm{Trace}(\delta y_1 \delta' y_1) + \mathrm{Trace}(\delta y_3 \delta' y_3). \tag{6}$$

CHAPTER 2

COADJOINT ORBITS

2.1 THE UNITARY GROUP AND ITS COMPLEXIFICATION

Throughout this thesis G will denote the unitary group

$$\mathrm{G} = \mathrm{U}_n = \left\{ g \in \mathrm{GL}_n(\mathbf{C}) : \overline{g} g = \underline{1} \right\}. \tag{7}$$

Here $\mathrm{GL}_n(\mathbf{C})$ is the multiplicative group of $n \times n$ complex matrices, and $\overline{\mathrm{M}}$ will always mean the ***adjoint*** (a.k.a. complex conjugate transpose) of any row, column or matrix M. The Lie algebra of G consists of all skew-adjoint matrices,

$$\mathfrak{g} = \mathfrak{u}_n = \left\{ \mathrm{Z} \in \mathfrak{gl}_n(\mathbf{C}) : \overline{\mathrm{Z}} + \mathrm{Z} = 0 \right\}, \tag{8}$$

inside the Lie algebra $\mathfrak{gl}_n(\mathbf{C})$ of all complex $n \times n$ matrices under the commutator bracket. We will also write $\mathrm{G}(\mathbf{C})$ and $\mathfrak{g}(\mathbf{C})$ for $\mathrm{GL}_n(\mathbf{C})$ and $\mathfrak{gl}_n(\mathbf{C})$. They are the ***complexifications*** of G and $\mathfrak{g}$, and $\mathfrak{g}(\mathbf{C})$ splits as the direct sum $\mathfrak{g} \oplus \mathrm{i}\mathfrak{g}$ of (8) and the self-adjoint matrices

$$\mathrm{i}\mathfrak{g} = \mathrm{i}\mathfrak{u}_n = \left\{ \mathrm{Z} \in \mathfrak{gl}_n(\mathbf{C}) : \overline{\mathrm{Z}} = \mathrm{Z} \right\} \tag{9}$$

where $\mathrm{i} = \sqrt{-1}$.

2.2 THE TRACE FORM AND DUALITY

We write $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ for the symmetric, complex bilinear form $\mathfrak{g}(\mathbf{C}) \times \mathfrak{g}(\mathbf{C}) \to \mathbf{C}$ defined by

$$\langle\!\langle \mathrm{A}, \mathrm{B} \rangle\!\rangle = -\operatorname{Trace}(\mathrm{AB}). \tag{10}$$

This form is $\mathrm{G}(\mathbf{C})$-invariant, i.e. it satisfies $\langle\!\langle \mathrm{Ad}_g \mathrm{A}, \mathrm{Ad}_g \mathrm{B} \rangle\!\rangle = \langle\!\langle \mathrm{A}, \mathrm{B} \rangle\!\rangle$ and infinitesimally

$$\langle\!\langle \mathrm{ad}_\mathrm{Z} \mathrm{A}, \mathrm{B} \rangle\!\rangle + \langle\!\langle \mathrm{A}, \mathrm{ad}_\mathrm{Z} \mathrm{B} \rangle\!\rangle = 0 \tag{11}$$

where, we recall, $\mathrm{Ad}_g \mathrm{B} := g\mathrm{B}g^{-1}$ and infinitesimally $\mathrm{ad}_\mathrm{Z} \mathrm{B} := \frac{d}{dt} \mathrm{Ad}_{\mathrm{e}^{t\mathrm{Z}}} \mathrm{B}|_{t=0} = [\mathrm{Z}, \mathrm{B}]$. These formulas hold for all $g \in \mathrm{G}(\mathbf{C})$ and $\mathrm{A}, \mathrm{B}, \mathrm{Z} \in \mathfrak{g}(\mathbf{C})$, and we have

Proposition 2.2.1. *The restriction* $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathfrak{g}\times\mathfrak{g}}$ *is real-valued, real bilinear,* G*-invariant and positive definite. This allows us to* ***identify*** $\mathfrak{g}^*$ ***with*** $\mathrm{i}\mathfrak{g}$ (*and hence* $\mathfrak{g}$ *with* $\mathrm{i}\mathfrak{g}^*$) *so that duality and the coadjoint action read, for* $(g, x, \mathrm{Z}) \in \mathrm{G}\times\mathfrak{g}^*\times\mathfrak{g}$*,*

$$\langle x, \mathrm{Z}\rangle := \langle\!\langle \mathrm{i}x, \mathrm{Z}\rangle\!\rangle, \qquad g(x) = gxg^{-1}, \qquad \mathrm{Z}(x) = [\mathrm{Z}, x]. \tag{12}$$

(*The restriction* $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathrm{i}\mathfrak{g}\times\mathrm{i}\mathfrak{g}}$ *has the same properties, except it is negative definite.*) □

Remark 2.2.2. This identification is an arbitrary choice we make for convenience of the exposition.

2.3 THE ORBITS

A ***coadjoint orbit*** is an orbit X of the action (12) of G on $\mathfrak{g}^* = \mathrm{i}\mathfrak{g}$, or in other words, a conjugacy class of self-adjoint matrices. As such matrices have real eigenvalues and an orthonormal basis of eigenvectors, we have

Proposition 2.3.1. *Each orbit meets, exactly once, the* ***dominant Weyl chamber***

$$\mathrm{D} = \left\{ \lambda = \begin{pmatrix} \underline{\lambda_{s_1}} & & & \\ & \underline{\lambda_{s_2}} & & \\ & & \ddots & \\ & & & \underline{\lambda_{s_k}} \end{pmatrix} \in \mathfrak{g}^* : \lambda_{s_1} > \lambda_{s_2} > \cdots > \lambda_{s_k} \right\} \tag{13}$$

consisting of nonincreasing real diagonal matrices. □

Here $\underline{\lambda_{s_i}}$ denotes the scalar matrix $\lambda_{s_i}\underline{1}$ of a certain size $|s_i|$, i.e. we are lumping equal eigenvalues together: while the map $i \mapsto \lambda_i$ is nominally $\{1, \ldots, n\} \to \mathbf{R}$, it is constant on the members of a partition

$$\mathcal{S} = \{s_1, \ldots, s_k\} \tag{14}$$

of $\{1, \ldots, n\}$ into consecutive ***segments*** whose cardinalities are the $|s_i|$; hence it induces a map $\mathcal{S} \to \mathbf{R}$ which we write again $s \mapsto \lambda_s$.

2.4 THE STABILIZER AND ITS CENTER

One checks without trouble:

Proposition 2.4.1. *Under the coadjoint action* (12)*, the stabilizer* G_λ *of* λ *in* (13) *equals*

$$\mathrm{H} = \begin{pmatrix} \mathrm{U}_{|s_1|} & & & \\ & \mathrm{U}_{|s_2|} & & \\ & & \ddots & \\ & & & \mathrm{U}_{|s_k|} \end{pmatrix} \cong \mathrm{U}_{|s_1|} \times \cdots \times \mathrm{U}_{|s_k|}. \tag{15}$$

This subgroup is also the centralizer, $\mathrm{C(S)} = \{g \in \mathrm{G} : gsg^{-1} = s, \forall s \in \mathrm{S}\}$*, of its center*

$$\mathrm{S} = \mathrm{center(H)} = \begin{pmatrix} \underline{\mathrm{U}_1} & & & \\ & \underline{\mathrm{U}_1} & & \\ & & \ddots & \\ & & & \underline{\mathrm{U}_1} \end{pmatrix} \cong \underbrace{\underline{\mathrm{U}_1} \times \cdots \times \underline{\mathrm{U}_1}}_{k \text{ factors}}. \tag{16}$$

When we move to another point $x = g(\lambda)$ *in the coadjoint orbit* $\mathrm{X} = \mathrm{G}(\lambda)$*, the stabilizer and its center become* $\mathrm{G}_x = g\mathrm{H}g^{-1}$ *and* $g\mathrm{S}g^{-1}$. □

We note that $\mathrm{S} \subset \mathrm{T} \subset \mathrm{H}$, where T is the maximal torus of all diagonal matrices in G, and equality holds when all $|s_i| = 1$ (nondegenerate eigenvalues). Again the trace form (10) allows us to identify $(\mathfrak{s}^*, \mathfrak{t}^*, \mathfrak{h}^*)$ with $(\mathrm{i}\mathfrak{s}, \mathrm{i}\mathfrak{t}, \mathrm{i}\mathfrak{h})$; under this identification, the projections

$$\mathfrak{h}^* \longrightarrow \mathfrak{t}^* \longrightarrow \mathfrak{s}^* \tag{17}$$

consist in taking the diagonal part, resp. sending the ith segment $\mu = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{|s_i|} \end{pmatrix}$ of the diagonal to its average $\mathrm{avg}(\mu) = \underline{\mathrm{Trace}(\mu)/|s_i|}$.

2.5 THE TANGENT SPACE $\mathrm{T}_x\mathrm{X}$ AND ITS COMPLEXIFICATION

Under the identifications of Proposition 2.2.1, the last formula in (12) says that the tangent space $\mathrm{T}_x\mathrm{X} = \mathfrak{g}(x)$ to X at x is the image of the map

$$\mathrm{ad}_{\mathrm{i}x} = [\mathrm{i}x, \cdot] : \mathrm{i}\mathfrak{g} \to \mathrm{i}\mathfrak{g}. \tag{18}$$

Since $\mathrm{T}_x\mathrm{X} = [x, \mathfrak{g}]$ sits in the real part of $\mathfrak{g}(\mathbf{C}) = \mathfrak{g}^* \oplus \mathrm{i}\mathfrak{g}^*$, we can conveniently complexify it as

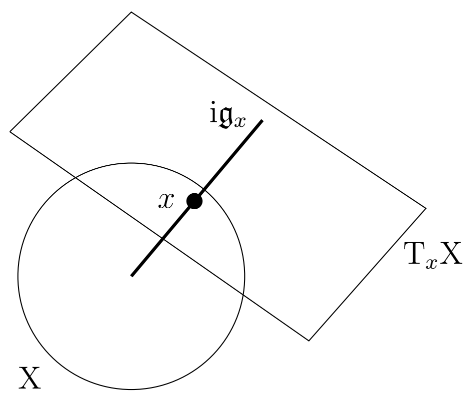
$$\mathrm{T}_x\mathrm{X} \oplus \mathrm{i}\mathrm{T}_x\mathrm{X} = [x, \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}] \subset \mathfrak{g}^* \oplus \mathrm{i}\mathfrak{g}^*. \tag{19}$$

As map (18) is skew-adjoint (see (11)), its image is also the orthogonal of its kernel $\mathrm{i}\mathfrak{g}_x$ relative to $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathrm{i}\mathfrak{g}\times\mathrm{i}\mathfrak{g}}$, i.e. we have

Proposition 2.5.1.

$$\mathrm{T}_x\mathrm{X} = \mathrm{i}\mathfrak{g}_x^\perp \qquad \textit{and in particular} \qquad \mathrm{T}_\lambda\mathrm{X} = \mathrm{i}\mathfrak{h}^\perp. \tag{20}$$

Remark 2.5.2. When G is U_2 (or SU_2, or SO_3), coadjoint orbits are just 2-spheres. Then (20) is the statement that the tangent space at a point is the orthogonal to the axis of rotations around that point — see Figure below.



CHAPTER 3

CANONICAL KÄHLER STRUCTURE

3.1 The canonical complex structure

Let $\mathrm{X} = \mathrm{G}(\lambda)$ be the coadjoint orbit with dominant element λ as in (13). The restriction of ad_{ix} (18) to its tangent space (20) has kernel $\mathrm{i}\mathfrak{g}_x \cap \mathrm{i}\mathfrak{g}_x^\perp = \{0\}$, so it is a (still skew-adjoint) linear bijection we shall denote

$$\mathrm{A}_x : \mathrm{T}_x\mathrm{X} \to \mathrm{T}_x\mathrm{X}. \tag{21}$$

Theorem 3.1.1. *The Kirillov-Kostant-Souriau* 2*-form of* X *is given by*

$$\omega(\delta x, \delta' x) = \langle\!\langle \delta x, \mathrm{A}_x^{-1}\delta' x \rangle\!\rangle \tag{22}$$

where A_x *is the map* (21)*. Moreover the formulas*

$$\mathrm{I}_x = |\mathrm{A}_x|^{-1}\mathrm{A}_x, \qquad\qquad \mathrm{g}(\delta x, \delta' x) = -\langle\!\langle \delta x, |\mathrm{A}_x|^{-1}\delta' x \rangle\!\rangle, \tag{23}$$

where $|\mathrm{A}_x| = \sqrt{-\mathrm{A}_x^2}$*, make* ω *part of a* G*-invariant* ***Kähler structure*** $(\mathrm{I}, \mathrm{g}, \omega)$, *i.e.*:

(a) I *is an (integrable) complex structure,*

(b) g *is a positive-definite metric,*

(c) *we have* $\omega(\,\cdot\,,\,\cdot\,) = \mathrm{g}(\,\cdot\,, \mathrm{I}_x\,\cdot\,)$ *and* $\mathrm{g}(\,\cdot\,,\,\cdot\,) = \omega(\mathrm{I}_x\,\cdot\,,\,\cdot\,)$.

Proof. Fix $\delta x, \delta' x \in \mathrm{T}_x\mathrm{X}$ and put $\mathrm{iZ} = \mathrm{A}_x^{-1}\delta' x \in \mathrm{i}\mathfrak{g}$. Then (21), (18) and (12) give

$$\delta' x = \mathrm{A}_x\mathrm{A}_x^{-1}\delta' x = [\mathrm{i}x, \mathrm{iZ}] = \mathrm{Z}(x), \tag{24}$$

whence we get (22) by definition of the KKS 2-form [S70; K76; Z17, 3.15] and (12):

$$\omega(\delta x, \delta' x) = \langle \delta x, \mathrm{Z} \rangle = \langle\!\langle \delta x, \mathrm{iZ} \rangle\!\rangle = \langle\!\langle \delta x, \mathrm{A}_x^{-1}\delta' x \rangle\!\rangle. \tag{25}$$

Next we note that $|\mathrm{A}_x|$ and I_x are by construction the (commuting) positive-definite and unitary part of the *polar decomposition* of A_x. So they depend smoothly on A_x [S70, 6.70] and I_x, being again skew-adjoint, is an ***almost complex structure***: $\mathrm{I}_x^2 = -\mathrm{I}_x^*\mathrm{I}_x = -\underline{1}$. Now (c) is clear by plugging $\mathrm{A}_x = |\mathrm{A}_x|\mathrm{I}_x$ in (22) and (23); (b) is clear if we remember that $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathrm{i}\mathfrak{g}\times\mathrm{i}\mathfrak{g}}$ is negative-definite; so there remains to see (a). For a G-invariant I (such as ours is by construction), this is equivalent to either of

(26) sections of the bundle of +i-eigenspaces of I in $\mathrm{TX}\oplus\mathrm{iTX}$ are *closed under Lie bracket* (Frobenius-Newlander-Nirenberg [N57]);

(27) at $x = \lambda$, the preimage of the +i-eigenspace of I_λ under the infinitesimal action $\mathfrak{g}(\mathbf{C}) \to \mathrm{T}_\lambda\mathrm{X}\oplus\mathrm{iT}_\lambda\mathrm{X}$ (see (19)) is a *Lie subalgebra* (Frölicher [F55, §20]).

We prove (27). First observe that if u and v are eigenvectors of $x\in\mathrm{X}$ for eigenvalues λ_r and λ_s, then the matrix $u\overline{v}$ is an eigenvector of ad_x for eigenvalue $\lambda_r-\lambda_s$:

$$[x, u\overline{v}\,] = xu\overline{v} - u\overline{xv} = (\lambda_r-\lambda_s)u\overline{v}. \tag{28}$$

It follows that $\mathrm{ad}_{\mathrm{i}x}$ is diagonalizable with spectrum $\Delta = \{\mathrm{i}(\lambda_r-\lambda_s) : r,s\in\mathcal{S}\}$, and so is A_x with spectrum $\Delta\smallsetminus\{0\}$. And indeed A_λ explicitly *is* "diagonal" with eigenvectors the elementary matrices $\mathrm{E}_{ij} = e_i\overline{e_j}$: in more detail, writing tangent vectors $\mathrm{V}\in\mathrm{T}_\lambda\mathrm{X} = \mathrm{i}\mathfrak{h}^\perp$ (20) as self-adjoint matrices with blocks $\mathrm{V}_{r|s}$ in the shape (15), formula (28) gives

$$\mathrm{A}_\lambda\left(\begin{array}{c|c|c} & \mathrm{V}_{p|q} & \mathrm{V}_{p|r} \\ \hline \mathrm{V}_{q|p} & & \mathrm{V}_{q|r} \\ \hline \mathrm{V}_{r|p} & \mathrm{V}_{r|q} & \end{array}\right) = \mathrm{i}\left(\begin{array}{c|c|c} & (\lambda_p-\lambda_q)\mathrm{V}_{p|q} & (\lambda_p-\lambda_r)\mathrm{V}_{p|r} \\ \hline (\lambda_q-\lambda_p)\mathrm{V}_{q|p} & & (\lambda_q-\lambda_r)\mathrm{V}_{q|r} \\ \hline (\lambda_r-\lambda_p)\mathrm{V}_{r|p} & (\lambda_r-\lambda_q)\mathrm{V}_{r|q} & \end{array}\right) \tag{29}$$

(we only write out the case where the partition (14) is into 3 segments p, q, r, but the general pattern should be clear). Hence we obtain by definition of $|\mathrm{A}_\lambda|$ and I_λ

$$|\mathrm{A}_\lambda|\left(\begin{array}{c|c|c} & \mathrm{V}_{p|q} & \mathrm{V}_{p|r} \\ \hline \mathrm{V}_{q|p} & & \mathrm{V}_{q|r} \\ \hline \mathrm{V}_{r|p} & \mathrm{V}_{r|q} & \end{array}\right) = \left(\begin{array}{c|c|c} & |\lambda_p-\lambda_q|\mathrm{V}_{p|q} & |\lambda_p-\lambda_r|\mathrm{V}_{p|r} \\ \hline |\lambda_q-\lambda_p|\mathrm{V}_{q|p} & & |\lambda_q-\lambda_r|\mathrm{V}_{q|r} \\ \hline |\lambda_r-\lambda_p|\mathrm{V}_{r|p} & |\lambda_r-\lambda_q|\mathrm{V}_{r|q} & \end{array}\right) \tag{30}$$

and

$$\mathrm{I}_\lambda\left(\begin{array}{c|c|c} & \mathrm{V}_{p|q} & \mathrm{V}_{p|r} \\ \hline \mathrm{V}_{q|p} & & \mathrm{V}_{q|r} \\ \hline \mathrm{V}_{r|p} & \mathrm{V}_{r|q} & \end{array}\right) = \left(\begin{array}{c|c|c} & \mathrm{iV}_{p|q} & \mathrm{iV}_{p|r} \\ \hline -\mathrm{iV}_{q|p} & & \mathrm{iV}_{q|r} \\ \hline -\mathrm{iV}_{r|p} & -\mathrm{iV}_{t|q} & \end{array}\right). \tag{31}$$

Thus we see that the +i-eigenvectors of I_λ, and likewise their preimages under (18) or ad_x, are the block upper triangular matrices — hence a Lie subalgebra in $\mathfrak{g}(\mathbf{C})$. $\square$

Remark 3.1.2. We could have shortened the proof by using the fact that in presence of (b) and (c), (a) is equivalent to $\mathrm{d}\omega = 0$. But this is a “delicate” fact [B87, 2.29], whereas (31) is both easy and useful for the sequel. Alternatively, the integrability of I will follow from Corollary 5.1.2 and knowing it on Grassmanians (34a).

Remark 3.1.3. Using the diagonalizability (28) and Lagrange interpolation [H71, §6.7] one can give an explicit formula for I_x at any point, viz.

$$\mathrm{I}_x = \sum_{\delta\in\Delta\smallsetminus\{0\}} \mathrm{i}\,\mathrm{sign}(\delta)\mathcal{E}_\delta \qquad \text{with} \qquad \mathcal{E}_\delta = \prod_{\varepsilon\in\Delta\smallsetminus\{0,\delta\}} \frac{(\mathrm{ad}_{\mathrm{i}x}-\varepsilon)}{(\delta-\varepsilon)}, \tag{32}$$

which confirms e.g. the G-invariance and smoothness (indeed algebraicity) of I. Unfortunately this formula seems rather less enlightening than (23).

Remark 3.1.4. The idea of using the polar decomposition to produce almost complex structures occurs in a general context in [W77, p. 8]. Other, less direct descriptions of the canonical complex structure are found in Borel [S54, §2; B54, §4; B58, 14.6], Guillemin-Sternberg [G82, p. 522], Besse [B87, §8.34], Vogan [V87, 5.8].

3.2 Example: The Grassmannians

Let Gr_m be the Grassmannian of complex m-planes in $\mathbf{C}^n$, each identified with the self-adjoint projector y upon it, i.e.

$$\mathrm{Gr}_m = \left\{ y \in \mathfrak{g}^* : y^2 = y, \mathrm{Trace}(y) = m \right\} = \mathrm{G}\begin{pmatrix} 1_m & 0 \\ 0 & 0_{n-m} \end{pmatrix}. \tag{33}$$

Its dominant element ϖ_m is the highest weight of the "fundamental" module $\wedge^m \mathbf{C}^n$.

Proposition 3.2.1. *In this case $|\mathrm{A}_y| = \underline{1}$ so that the canonical structure of Gr_m is simply*

$$\begin{cases} \mathrm{I}\delta y = [\mathrm{i}y, \delta y] & \text{(34a)} \\ \mathrm{g}(\delta y, \delta' y) = \mathrm{Trace}(\delta y \delta' y) & \text{(34b)} \\ \omega(\delta y, \delta' y) = \mathrm{Trace}(\delta y \mathrm{I} \delta' y) = \mathrm{Trace}(\mathrm{i}y[\delta' y, \delta y]). & \text{(34c)} \end{cases}$$

Proof. Deriving and reusing the relations $y = y^2 = y^3$ gives $\delta y = \delta y.y + y.\delta y = \delta y.y + y.\delta y.y + y.\delta y$. This implies $y.\delta y.y = 0$ and $-\mathrm{A}_y^2 \delta y = [y, [y, \delta y]] = y.\delta y - 2y.\delta y.y + \delta y.y = \delta y$. So $-\mathrm{A}_y^2$ and hence its square root are the identity. $\square$

Remark 3.2.2. The Hermitian metric $\mathrm{g} + \mathrm{i}\omega$ in (34) can be seen as Kähler reduction of the flat metric $(s, s') := 2\,\mathrm{Trace}(\overline{s}s')$ on $\mathbf{C}^{n\times m} \cong \mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n)$.[1] Indeed U_m acts there by $u(s) = su^{-1}$, preserving $\Omega = \mathrm{Im}(\cdot, \cdot)$ with moment map $\psi(s) = -\overline{s}s$, and (34) obtains on passing to the quotient $\mathrm{Gr}_m = \psi^{-1}(-\underline{1})/\mathrm{U}_m$ [G73, §V.5; T06, p. 240]. E.g. for $m = 1$ one recovers the ***Fubini-Study metric*** on projective space, i.e.

$$2\left[\frac{(\delta s, \delta' s)}{\|s\|^2} - \frac{(\delta s, s)(s, \delta' s)}{\|s\|^4}\right] \qquad \text{on} \qquad \mathrm{Gr}_1 = \left\{ y = \frac{s(s, \cdot)}{\|s\|^2} : s \in \mathbf{C}^n \smallsetminus \{0\} \right\}. \tag{35}$$

See [S05, §5], and for $m > 1$, [C29; E34]. Formulas (34) are emblematic of the explicitness we'd like to have in general.

[1] Or alternatively on the dual $\mathbf{C}^{m\times n}$, if we insist on obtaining (34a) and not its opposite $\frac{1}{\mathrm{i}}[y, \delta y]$.

CHAPTER 4

PSEUDO-KÄHLER STRUCTURES CLASSIFIED: THE COSET MODEL

In this section we review the standard classification of complex structures which results from the principle: a G-invariant J on $\mathrm{X} = \mathrm{G}(\lambda) = \mathrm{G}/\mathrm{H}$ amounts to an H-invariant $\mathrm{J}_\lambda \in \mathrm{End}(\mathrm{T}_\lambda \mathrm{X})$.

4.1 The decomposition of the isotropy representation

Let $\mathfrak{g}_{r|s}(\mathbf{C})$ denote, for segments $r \neq s$ in the partition $\mathcal{S}$ (14), the matrices (29) where all blocks vanish except perhaps $\mathrm{V}_{r|s}$, i.e.

$$\mathfrak{g}_{r|s}(\mathbf{C}) = \{\mathrm{Z} \in \mathfrak{g}(\mathbf{C}) : \mathrm{Z}_{ij} = 0 \text{ for } (i,j) \notin r \times s\}\,, \tag{36}$$

and let $\mathfrak{X}_{r|s}$ (resp. $\mathrm{i}\mathfrak{X}_{r|s}$) denote the intersection of $\mathfrak{g}_{r|s}(\mathbf{C}) \oplus \mathfrak{g}_{s|r}(\mathbf{C})$ with $\mathfrak{g}$ (resp. $\mathrm{i}\mathfrak{g}$).

Theorem 4.1.1. *The isotypic decomposition of the isotropy representation of* $\mathrm{H} = \mathrm{G}_\lambda$ *in the complexified tangent space* (19) *at* λ *into inequivalent irreducibles is*

$$\mathrm{T}_\lambda \mathrm{X} \oplus \mathrm{i}\mathrm{T}_\lambda \mathrm{X} = \bigoplus_{r \neq s \text{ in } \mathcal{S}} \mathfrak{g}_{r|s}(\mathbf{C}) \tag{37}$$

Consequently,

(a) *Every* G*-invariant almost complex structure* J *on* $\mathrm{X} = \mathrm{G}(\lambda)$ *is obtained by flipping the sign of* I (*and hence* g) *on some summands in* $\mathrm{T}_\lambda \mathrm{X} = \bigoplus_{r<s \text{ in } \mathcal{S}} \mathrm{i}\mathfrak{X}_{r|s}$.

(b) *As* g *coincides with* $\frac{-1}{|\lambda_r - \lambda_s|}\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ *on* $\mathrm{i}\mathfrak{X}_{r|s}$*, each such flip affects its signature by turning a block of* $|r| \times |s|$ *pluses into minuses.*

(c) *If* $\mathcal{S}$ *has* k *parts, then* X *admits* $2^{k(k-1)/2}$ *different* G*-invariant almost complex structures.*

Proof. Using the notation of (15) and (29), one checks without trouble that the isotropy action of $h = \mathrm{diag}(u_{s_1}, \dots, u_{s_k}) \in \mathrm{H}$ takes block $\mathrm{V}_{r|s}$ of $\mathrm{V} \in \mathrm{T}_\lambda\mathrm{X}$ to

$$h(\mathrm{V}_{r|s}) = u_r \mathrm{V}_{r|s} u_s^{-1}. \tag{38}$$

So the $\mathfrak{g}_{r|s}(\mathbf{C})$ are H-invariant and the representation on each factors through the natural representation of $\mathrm{U}_{|r|} \times \mathrm{U}_{|s|}$ on $\mathrm{Hom}(\mathbf{C}^{|s|}, \mathbf{C}^{|r|}) \cong \mathbf{C}^{|r|} \otimes \overline{\mathbf{C}^{|s|}}$. As these are irreducible and different for different pairs (r, s), we obtain (37). Now J_λ is determined by its $\pm$i-eigenspaces

$$\mathrm{T}_\lambda\mathrm{X}^\pm = \mathrm{Im}(\mathrm{J}_\lambda \pm \underline{\mathrm{i}}\,) = \mathrm{Ker}(\mathrm{J}_\lambda \mp \underline{\mathrm{i}}\,) \tag{39}$$

which are (complex conjugate) H-invariant subspaces of (37), hence are each the sum of *some* $\mathfrak{g}_{r|s}(\mathbf{C})$ [B12, §4, Prop. 4d] — one per pair $(\mathfrak{g}_{r|s}(\mathbf{C}), \mathfrak{g}_{s|r}(\mathbf{C}))$. So they can only differ from those of I_λ (31) by the indicated sign flips, and we obtain (a, b, c). □

4.2 The invariant complex structures

There remains to determine which of the almost complex structures of Theorem 4.1.1 are integrable.

Theorem 4.2.1. *We have*

$$\big[\mathfrak{g}_{p|q}(\mathbf{C}), \mathfrak{g}_{r|s}(\mathbf{C})\big] = \begin{cases} 0 & \text{if } s \neq p;\ q \neq r & \text{(40a)} \\ \mathfrak{g}_{p|s}(\mathbf{C}) & \text{if } s \neq p;\ q = r & \text{(40b)} \\ \mathfrak{g}_{r|q}(\mathbf{C}) & \text{if } s = p;\ q \neq r & \text{(40c)} \\ \big(\mathfrak{g}_{p|p}(\mathbf{C}) + \mathfrak{g}_{q|q}(\mathbf{C})\big) \cap \mathfrak{sl}_n(\mathbf{C}) & \text{if } s = p;\ q = r. & \text{(40d)} \end{cases}$$

Consequently,

(a) *An almost complex structure* J *obtained as in Theorem* 4.1.1 *is integrable iff it respects the "Chasles" rule: if the sign is flipped on* $\mathrm{i}\mathfrak{X}_{r|s}$ *and* $\mathrm{i}\mathfrak{X}_{s|t}$ $(r < s < t)$, *then it is also flipped on* $\mathrm{i}\mathfrak{X}_{r|t}$.

(b) *Such is the case iff the preimage of* $\mathrm{T}_\lambda \mathrm{X}^+$ *under the infinitesimal action* (27) *is a* ***parabolic subalgebra*** *of* $\mathfrak{g}(\mathbf{C})$, *containing* $\mathfrak{h}(\mathbf{C})$.

Proof. Relations (40) follow from noting that $\mathfrak{g}_{r|s}(\mathbf{C})$ is the span of elementary matrices $\mathrm{E}_{ij} = e_i\overline{e_j}$ for $(i,j) \in r \times s$, and computing $[e_i\overline{e_j}, e_k\overline{e_l}]$. Next (a) translates condition (27) that the preimage in (b) be a subalgebra; and (b) translates, via [B75, Déf. VIII.3.2], the observation made in the proof of Theorem 4.1.1 that each E_{ij} not in $\mathfrak{h}(\mathbf{C})$ is in either $\mathrm{T}_\lambda \mathrm{X}^+$ or $\mathrm{T}_\lambda \mathrm{X}^-$. □

Remark 4.2.2. Versions of Theorems 4.1.1 and/or 4.2.1 valid beyond U_n can be found in [B58; S69; B82; N84; A86; A97; K10; Y14]. There, the roots $\alpha_{ij} = \mathrm{E}_{ii} - \mathrm{E}_{jj} \in \mathfrak{t}^*$ whose root space $\mathbf{C}\mathrm{E}_{ij}$ lies in $\mathrm{T}_\lambda \mathrm{X}^+$ are called ***roots of*** J.

4.3 THE RESIDUAL ACTION OF THE NORMALIZER

A systematic way to produce new complex structures (or "Chasles" block flips) from old ones is to let diffeomorphisms act on them. This is the place to recall that any homogeneous space $\mathrm{X} = \mathrm{G}/\mathrm{H}$ carries a residual "right" action,

$$\underset{\sim}{n}(g\mathrm{H}) = g\mathrm{H}n^{-1} = gn^{-1}\mathrm{H}, \tag{41}$$

of the normalizer $\mathrm{N} = \{n \in \mathrm{G} : n\mathrm{H}n^{-1} = \mathrm{H}\}$. As H itself (clearly) acts trivially, this is really an action of the Weyl-like quotient $\mathrm{W} = \mathrm{N}/\mathrm{H}$ — which is also a subset of X.

When $k = n$ (nondegenerate case, the stabilizer (15) is the maximal torus T), W is the actual Weyl group $\mathfrak{S}_n$ of G and the $n!$ complex structures thus obtained turn out to exhaust all of them, as was observed in [B58, 13.8; B82, Exerc. IX.4.8e] and will be recovered here in §5.2.

CHAPTER 5

PSEUDO-KÄHLER STRUCTURES REALIZED: THE FLAG MODELS

Theorem 4.2.1 only spells out complex structures by giving the effect of J at the base point λ. At any other point $x = g(\lambda)$, computation of $\mathrm{J}_x = g\mathrm{J}_\lambda g^{-1}$ requires use of some $g \in \mathrm{G}$, on whose nonuniqueness the outcome is *known not to depend*. Our goal below is a more intrinsic picture where J_x can be explicit in terms of x alone, as in (22–23, 32, 34). We freely use the notation introduced in (13–14, 33).

5.1 The canonical complex structure, redux

A first idea is to note that ***spectral decomposition*** expresses each $x \in \mathrm{X}$ as linear combination of eigenprojectors, $\mathrm{E}_r \in \mathrm{Gr}_{|r|}$, belonging to the (fixed) eigenvalues λ_r:

$$x = \sum_{r\in\mathcal{S}} \lambda_r \mathrm{E}_r, \qquad \text{where} \qquad \mathrm{E}_r = \prod_{s\in\mathcal{S}\smallsetminus\{r\}} \frac{(x-\lambda_s)}{(\lambda_r-\lambda_s)} \tag{42}$$

(Lagrange interpolation [H71, §6.7]). So sending x to $y = (\mathrm{E}_r)_{r\in\mathcal{S}}$ embeds X G-equivariantly as a submanifold Y of a product $\prod_{r\in\mathcal{S}} \mathrm{Gr}_{|r|}$ of Grassmannians (33), hopefully pulling product structures back to useful ones on X. Alas, Proposition 5.1.1 below dashes this hope: Y *isn't a complex submanifold* of the product, so there is no complex structure to transport back. Fortunately, the same Proposition will also indicate the way out.

To state it, note that the E_r are just a small part of x's ***spectral measure*** $\mathrm{A} \mapsto \mathrm{E}_\mathrm{A}$ which maps *subsets of* $\mathcal{S}$ (or alternatively, of the spectrum $\{\lambda_s : s \in \mathcal{S}\}$) to projectors

$$\mathrm{E}_\mathrm{A} = \sum_{r\in\mathrm{A}} \mathrm{E}_r \in \mathrm{Gr}_{|\mathrm{A}|}, \qquad\qquad |\mathrm{A}| = \sum_{r\in\mathrm{A}} |r|, \tag{43}$$

with the property that $\mathrm{E}_{\mathrm{A}\cap\mathrm{B}} = \mathrm{E}_\mathrm{A}\mathrm{E}_\mathrm{B}$ (so the E_A all commute). Thus, not only the singletons but any subfamily $\mathcal{A} \subset 2^\mathcal{S}$ gives rise to a G-equivariant map, $x \mapsto (\mathrm{E}_\mathrm{A})_{\mathrm{A}\in\mathcal{A}}$, from X to a product of Grassmannians.

Proposition 5.1.1. *The image* Y *of this map is a complex submanifold of* $\prod_{\mathrm{A}\in\mathcal{A}} Gr_{|\mathrm{A}|}$ *(for the product complex structure) if and only if* $\mathcal{A}$ *is totally ordered by inclusion.*

Proof. First note that as G is transitive on X, the map's equivariance (visible on (42)) ensures that Y is an orbit of a smooth group action, hence as always an ("initial") submanifold [O04, Prop. 2.3.12].

Assume that $\mathcal{A} \subset 2^{\mathcal{S}}$ is not totally ordered by inclusion. So there are $\mathrm{A}, \mathrm{B} \in \mathcal{A}$ such that $\mathrm{A} \not\subset \mathrm{B}$ and $\mathrm{B} \not\subset \mathrm{A}$. Pick $r \in \mathrm{A} \smallsetminus \mathrm{B}$ and $s \in \mathrm{B} \smallsetminus \mathrm{A}$ and nonzero eigenvectors $u, v \in \mathbf{C}^n$ for eigenvalues λ_r, λ_s of x. Thus we have (with (P, Q) short for $(\mathrm{E_A}, \mathrm{E_B})$):

$$\mathrm{P}u = u, \qquad \mathrm{P}v = 0, \qquad \mathrm{Q}u = 0, \qquad \mathrm{Q}v = v. \tag{44}$$

Now put $\mathrm{Z} = u\overline{v} - v\overline{u} \in \mathfrak{g}$ and consider the image $\delta y \in \mathrm{T}_y\mathrm{Y}$ of $\delta x = [\mathrm{Z}, x] \in \mathrm{T}_x\mathrm{X}$. By equivariance and (44), its components in $\mathrm{T_P X}_{|\mathrm{A}|}$ and $\mathrm{T_Q X}_{|\mathrm{B}|}$ are respectively

$$\begin{aligned} \delta\mathrm{P} &= [\mathrm{Z}, \mathrm{P}] = [u\overline{v} - v\overline{u}, \mathrm{P}] = -u\overline{v} - v\overline{u}, \\ \delta\mathrm{Q} &= [\mathrm{Z}, \mathrm{Q}] = [u\overline{v} - v\overline{u}, \mathrm{Q}] = u\overline{v} + v\overline{u}. \end{aligned} \tag{45}$$

They (of course) satisfy the relation $[\delta\mathrm{P}, \mathrm{Q}] + [\mathrm{P}, \delta\mathrm{Q}] = 0$ which any tangent vector to Y must, as one sees by deriving $[\mathrm{P}, \mathrm{Q}] = 0$. On the other hand, we claim that $\mathrm{I}\delta\mathrm{P}$ and $\mathrm{I}\delta\mathrm{Q}$ *fail* that relation. Indeed (34a) gives

$$\begin{aligned} \mathrm{I}\delta\mathrm{P} &= [\mathrm{iP}, \delta\mathrm{P}] = \mathrm{i}(v\overline{u} - u\overline{v}) = \mathrm{iZ}, \\ \mathrm{I}\delta\mathrm{Q} &= [\mathrm{iQ}, \delta\mathrm{Q}] = \mathrm{i}(v\overline{u} - u\overline{v}) = \mathrm{iZ}, \end{aligned} \tag{46}$$

whence (using (45))

$$\begin{aligned} [\mathrm{I}\delta\mathrm{P}, \mathrm{Q}] + [\mathrm{P}, \mathrm{I}\delta\mathrm{Q}] &= [\mathrm{iZ}, \mathrm{Q} - \mathrm{P}] \\ &= \mathrm{i}(\delta\mathrm{Q} - \delta\mathrm{P}) \\ &= 2\mathrm{i}(u\overline{v} + v\overline{u}) \neq 0. \end{aligned} \tag{47}$$

This shows that $\mathrm{T}_y\mathrm{Y}$ is not preserved by the product complex structure, as claimed.

Conversely, assume that $\mathcal{A}$ is totally ordered by inclusion. Then a tuple $(\mathrm{E_A})_{\mathrm{A}\in\mathcal{A}}$ of projectors is in Y if and only if it satisfies

$$\mathrm{E_A E_B} = \mathrm{E_B E_A} = \mathrm{E_A} \tag{48}$$

for all $\mathrm{A} \subset \mathrm{B}$ in $\mathcal{A}$; and $\delta y = (\delta\mathrm{E_A})_{\mathrm{A}\in\mathcal{A}}$ is in $\mathrm{T}_y\mathrm{Y}$ if and only if we also have, for all $\mathrm{A} \subset \mathrm{B}$ in $\mathcal{A}$, the derived relations (where again we write $(\mathrm{P},\mathrm{Q}) = (\mathrm{E_A},\mathrm{E_B})$ for short):

$$\delta\mathrm{P}.\mathrm{Q} + \mathrm{P}.\delta\mathrm{Q} = \delta\mathrm{P}, \tag{49a}$$

$$\delta\mathrm{Q}.\mathrm{P} + \mathrm{Q}.\delta\mathrm{P} = \delta\mathrm{P}. \tag{49b}$$

Assume (49a). Multiplying it on the right by Q gives $\mathrm{P}.\delta\mathrm{Q}.\mathrm{Q} = 0$ and hence

$$\begin{aligned}\mathrm{I}\delta\mathrm{P}.\mathrm{Q} + \mathrm{P}.\mathrm{I}\delta\mathrm{Q} &= [\mathrm{iP}, \delta\mathrm{P}]\mathrm{Q} + \mathrm{P}[\mathrm{iQ}, \delta\mathrm{Q}] \\ &= \mathrm{i}(\mathrm{P}.\delta\mathrm{P}.\mathrm{Q} + \mathrm{P}.\mathrm{Q}.\delta\mathrm{Q} - \delta\mathrm{P}.\mathrm{P}.\mathrm{Q}) \\ &= \mathrm{iP}.(\delta\mathrm{P}.\mathrm{Q} + \mathrm{P}.\delta\mathrm{Q}) - \mathrm{i}\delta\mathrm{P}.\mathrm{P} \\ &= \mathrm{iP}.\delta\mathrm{P} - \mathrm{i}\delta\mathrm{P}.\mathrm{P} \\ &= \mathrm{I}\delta\mathrm{P}.\end{aligned} \tag{50}$$

Thus we see that $\mathrm{I}\delta y$ also satisfies (49a); and arguing similarly for (49b) completes the proof that I preserves $\mathrm{T}_y\mathrm{Y}$. □

Choosing $\mathcal{A} = \{\{s_1, \ldots, s_i\} : i = 1, \ldots, k\}$ and $m_i = |s_1| + \cdots + |s_i|$ now gives an independent reconstruction of the Kähler structure (22–23):

Corollary 5.1.2. *The coadjoint orbit* $(\mathrm{X}, \mathrm{I}, \mathrm{g} + \mathrm{i}\omega)$ *is isomorphic to the orbit* Y *of* $(\varpi_{m_i})_{i=1}^k$ *in* $\prod_{i=1}^k Gr_{m_i}$ *with the product complex structure and hermitian metric*

$$\sum_{i=1}^k \ell_{s_i}(\mathrm{g}_{m_i} + \mathrm{i}\omega_{m_i}) \tag{51}$$

where $(Gr_m, \mathrm{I}, \mathrm{g}_m + \mathrm{i}\omega_m)$ *is the Grassmannian* (34) *and* $\ell_{s_i} = \lambda_{s_i} - \lambda_{s_{i+1}}$ $(\lambda_{s_{k+1}} := 0)$*. The map from* X *to* Y *and the (moment) map from* Y *to* X *are respectively*

$$x \mapsto \big(\mathrm{E}_{\{s_1,\ldots,s_i\}}\big)_{i=1}^k \qquad \text{and} \qquad (y_{m_i})_{i=1}^k \mapsto \sum_{i=1}^k \ell_{s_i} y_{m_i}. \tag{52}$$

Proof. We have just seen that Y is a complex submanifold of the product. Thus it is (homogeneous) symplectic, hence isomorphic via its moment map, which is clearly (52), to a coadjoint orbit which is X since $\sum_{i=1}^{k}\ell_{s_i}\varpi_{m_i} = \sum_{i=1}^{k}(\lambda_{s_i}-\lambda_{s_{i+1}})\varpi_{m_i} = \lambda_{s_1}\varpi_{m_1} + \lambda_{s_2}(\varpi_{m_2}-\varpi_{m_1}) + \cdots + \lambda_{s_k}(\varpi_{m_k}-\varpi_{m_{k-1}}) = \lambda$. $\square$

5.2 THE $k!$ FLAG MODELS AND COMPLEX STRUCTURES

The point now is that the previous construction (51, 52) ***works just as well*** when the λ_{s_i} are not in decreasing order. In other words, we may preface it by rearranging the k diagonal blocks of (13) according to any permutation π of $\{1,\dots,k\}$: as some $\lambda_{s_{\pi(i)}} - \lambda_{s_{\pi(i+1)}}$ become negative, this should get us the indefinite metrics (see (51)) and attendant complex structures (see (29)) we have been seeking.

Conjecture 5.2.1. *The explicit pseudo-Kähler structures so obtained are exactly those of Theorem* 4.2.1*. In particular* X *carries precisely* $k!$ G*-invariant complex structures, realized in* $k!$ *different flag models.*

Sketch of proof. As we haven't checked all details, we shall be content to prove this when $k = 3$ (writing out matrices of size $n = 4$ for simplicity) and indicate why we believe it stays true in general. The left column below shows all $3! = 6$ possible rearrangements of 3 blocks with eigenvalues $a > b > c$:

$$\begin{pmatrix} \begin{array}{c|cc|c} a & + & + & + \\ \hline & & & + \\ & \multicolumn{2}{c|}{\underline{b}} & + \\ \hline & & & c \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{c|cc|c} 1 & & & \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline & & & 1 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc|c} a & + & + & + \\ \hline & & & + \\ & \multicolumn{2}{c|}{\underline{b}} & + \\ \hline & & & c \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc|c} 1 & & & \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline & & & 1 \end{array} \end{pmatrix} \tag{53a}$$

$$\begin{pmatrix} \begin{array}{c|c|cc} a & + & + & + \\ \hline & c & + & + \\ \hline & & & \\ & & \multicolumn{2}{c}{\underline{b}} \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{c|cc|c} 1 & & & \\ \hline & & & 1 \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc|c} a & + & + & + \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{b}} & \\ \hline & + & + & c \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|c|cc} 1 & & & \\ \hline & & & \\ & & \multicolumn{2}{c}{\underline{1}} \\ \hline & 1 & & \end{array} \end{pmatrix} \tag{53b}$$

$$\begin{pmatrix} \begin{array}{cc|c|c} & & + & + \\ \multicolumn{2}{c|}{\underline{b}} & + & + \\ \hline & & a & + \\ \hline & & & c \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{c|cc|c} & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline 1 & & & \\ \hline & & & 1 \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc|c} a & & & + \\ \hline + & & & + \\ + & \multicolumn{2}{c|}{\underline{b}} & + \\ \hline & & & c \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{cc|c|c} & & 1 & \\ \hline & & & \\ \multicolumn{2}{c|}{\underline{1}} & & \\ \hline & & & 1 \end{array} \end{pmatrix} \tag{53c}$$

$$\left(\begin{array}{cc|c|c} & & + & + \\ \multicolumn{2}{c|}{\underline{b}} & + & + \\ \hline & & c & + \\ \hline & & & a \end{array}\right) = \left(\begin{array}{c|cc|c} & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline & & & 1 \\ \hline 1 & & & \end{array}\right) \left(\begin{array}{c|cc|c} a & & & \\ \hline + & & & + \\ + & \multicolumn{2}{c|}{\underline{b}} & + \\ \hline + & & & c \end{array}\right) \left(\begin{array}{cc|c|c} & & & 1 \\ \hline & & & \\ \multicolumn{2}{c|}{\underline{1}} & & \\ \hline & & 1 & \end{array}\right) \tag{53d}$$

$$\left(\begin{array}{c|c|cc} c & + & + & + \\ \hline & a & + & + \\ \hline & & & \\ & & \multicolumn{2}{c}{\underline{b}} \end{array}\right) = \left(\begin{array}{c|cc|c} & & & 1 \\ \hline 1 & & & \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \end{array}\right) \left(\begin{array}{c|cc|c} a & + & + & \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{b}} & \\ \hline + & + & + & c \end{array}\right) \left(\begin{array}{c|c|cc} & 1 & & \\ \hline & & & \\ & & \multicolumn{2}{c}{\underline{1}} \\ \hline 1 & & & \end{array}\right) \tag{53e}$$

$$\left(\begin{array}{c|cc|c} c & + & + & + \\ \hline & & & + \\ & \multicolumn{2}{c|}{\underline{b}} & + \\ \hline & & & a \end{array}\right) = \left(\begin{array}{c|cc|c} & & & 1 \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline 1 & & & \end{array}\right) \left(\begin{array}{c|cc|c} a & & & \\ \hline + & & & \\ + & \multicolumn{2}{c|}{\underline{b}} & \\ \hline + & + & + & c \end{array}\right) \left(\begin{array}{c|cc|c} & & & 1 \\ \hline & & & \\ & \multicolumn{2}{c|}{\underline{1}} & \\ \hline 1 & & & \end{array}\right) \tag{53f}$$

Each is brought back to decreasing order by a unique ***block permutation*** matrix [T61; A08] as shown on the right. Our construction (52) produces 6 complex structures having as +i-eigenspaces the upper triangular blocks marked $+$ on the left. These make subalgebras as prescribed by (27), and the permutations take them to precisely the 6 "block flips" allowed by Theorem 4.2.1. Indeed the remaining two (of $2^{3(3-1)/2} = 8$) possible flips are those that fail the test to give subalgebras, and thus give only non-integrable almost complex structures:

$$\left(\begin{array}{c|cc|c} & + & + & \\ \hline & & & + \\ & & & + \\ \hline + & & & \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|cc|c} & & & + \\ \hline + & & & \\ + & & & \\ \hline & + & + & \end{array}\right) \tag{54}$$

This argument works unchanged with $k = 3$ diagonal blocks in any size n. We believe it keeps working in general because the blocks flipped to $+$ on the right always define parabolic subalgebras (4.2.1b) and these are necessarily conjugate to one of the upper triangular ("standard") parabolics on the left. The conjugating matrix should provide the desired block permutation. $\square$

Remark 5.2.2. For completeness we spell out the explicit flag model Y, moment map (52), and metric signature (from flipped blocks) under all six permutations:

case:	manifold Y:	moment map, $y \mapsto$:	sig.:
(53a) (53f)	$\left\{ y = \begin{pmatrix} y_1 \\ y_3 \\ y_4 \end{pmatrix} \in \mathrm{Gr}_1 \times \mathrm{Gr}_3 \times \mathrm{Gr}_4 \ : \ \begin{matrix} y_3 y_1 = y_1 \\ y_4 y_3 = y_3 \end{matrix} \right\}$	$(a-b)y_1 + (b-c)y_3 + cy_4$ $(c-b)y_1 + (b-a)y_3 + ay_4$	(5,0) (0,5)
(53b) (53e)	$\left\{ y = \begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix} \in \mathrm{Gr}_1 \times \mathrm{Gr}_2 \times \mathrm{Gr}_4 \ : \ \begin{matrix} y_2 y_1 = y_1 \\ y_4 y_2 = y_2 \end{matrix} \right\}$	$(a-c)y_1 + (c-b)y_2 + by_4$ $(c-a)y_1 + (a-b)y_2 + by_4$	(3,2) (2,3)
(53c) (53d)	$\left\{ y = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathrm{Gr}_2 \times \mathrm{Gr}_3 \times \mathrm{Gr}_4 \ : \ \begin{matrix} y_3 y_2 = y_2 \\ y_4 y_3 = y_3 \end{matrix} \right\}$	$(b-a)y_2 + (a-c)y_3 + cy_4$ $(b-c)y_2 + (c-a)y_3 + ay_4$	(3,2) (2,3)

The point $\mathrm{Gr}_4 = \{\underline{1}\}$ could of course be mostly omitted from the notation. It would be nice if we could interpret the equations above as appropriate moment map levels, and thus exhibit each Y as a symplectic reduction (cf. Remark 3.2.2)[2].

Remark 5.2.3. Our count of $k!$ complex structures seems to have escaped the attention of both [B58][3] and later authors who studied the problem by means of *closed subsystems* [S69, Thm 7.1] or S-*root systems* [A86, Cor. 3.1, Prop. 5.1; A97, §IV.5] (see (16) and also [K10]). It is worth noting that while in case $k = n$ they all arise geometrically from the residual action (41), ***this is not true in general***. In fact, when $\mathrm{G} = \mathrm{U}_4$ and X is the so-called *adjoint variety* studied in [B61; K98; L02],

$$\mathrm{X} = \mathrm{G} \left(\begin{array}{c|c|c} 1 & & \\ \hline & \underline{0} & \\ \hline & & -1 \end{array} \right) \tag{55}$$

one finds that N/H boils down to a $\mathbf{Z}_2$ group whose representatives

$$\left(\begin{array}{c|c|c} 1 & & \\ \hline & \underline{1} & \\ \hline & & 1 \end{array} \right) \quad \text{and} \quad \left(\begin{array}{c|c|c} & & 1 \\ \hline & \underline{1} & \\ \hline 1 & & \end{array} \right) \tag{56}$$

exchange pairs in the above table, and Borel-Hirzebruch [B58, 13.9; H05] observed that the complex structures in (53a) and (53b) are related by *no* diffeomorphism, because they

[2]And also the first sentence of [W78, §2].

[3]Despite the title of their §13.7!

have different characteristic numbers. So the "action" of block permutations we have found remains a little mysterious.

CHAPTER 6

CONCLUSION

Conjecture 5.2.1 has now been proved in [M20]. Recall that a coadjoint orbit X of $\mathrm{G} = \mathrm{U}_n$ is simply the conjugacy class of all self-adjoint matrices x sharing the spectrum and multiplicities of a decreasing diagonal

$$\lambda = \begin{pmatrix} \lambda_1 \underline{1}_{n_1} & & \\ & \ddots & \\ & & \lambda_k \underline{1}_{n_k} \end{pmatrix}, \qquad \lambda_1 > \cdots > \lambda_k. \tag{57}$$

When the spectrum is $\{1, 0\}$ with multiplicities $(m, n-m)$ this is the Grassmannian Gr_m with its well-known Kähler structure $(\mathrm{I}_m, \mathrm{g}_m, \omega_m)$:

$$\begin{cases} \mathrm{I}_m \delta x &= [\mathrm{i}x, \delta x] \\ \mathrm{g}_m(\delta x, \delta' x) &= \mathrm{Trace}(\delta x \delta' x) \\ \omega_m(\delta x, \delta' x) &= \mathrm{Trace}(\delta x \mathrm{I}_m \delta' x). \end{cases} \tag{58}$$

In general, for every permutation π of $\{1, \ldots, k\}$ consider the ***eigenflag*** map EF_π sending x to $(y_i)_{i=1}^k$ where $y_i \in \mathrm{Gr}_{m_i(\pi)}$ is the sum of the eigenprojectors of x for eigenvalues $\{\lambda_{\pi(1)}, \ldots, \lambda_{\pi(i)}\}$ and we set $m_i(\pi) := n_{\pi(1)} + \cdots + n_{\pi(i)}$. Then the main result of [M20] can be stated:

Theorem 6.0.1. EF_π *embeds* X *as a complex submanifold of* $\prod_{i=1}^k \mathrm{Gr}_{m_i(\pi)}$, *thereby endowing it with a* G*-invariant complex structure* J_π. *The map* $\pi \mapsto \mathrm{J}_\pi$ *is one-to-one onto the set of such structures* (*of which there are therefore exactly* $k!$). *The Kirillov-Kostant-Souriau* 2*-form* ω *of* X *and resulting pseudo-Kähler metric* $\mathrm{g}_\pi = \omega(\mathrm{J}_\pi \cdot, \cdot)$ *are explicitly the pull-backs of*

$$\sum_{i=1}^k (\lambda_{\pi(i)} - \lambda_{\pi(i+1)}) \omega_{m_i(\pi)} \qquad \textit{and} \qquad \sum_{i=1}^k (\lambda_{\pi(i)} - \lambda_{\pi(i+1)}) \mathrm{g}_{m_i(\pi)} \tag{59}$$

on $\prod_{i=1}^k \mathrm{Gr}_{m_i(\pi)}$, *where* (ω_m, g_m) *are as in* (58) *and we set* $\lambda_{\pi(k+1)} := 0$.

A remarkable feature of this result is that unlike the cases $k = 2$ or n detailed in [B58, 13.8], the various complex structures ***are not*** in general related by a geometrical action of $\mathfrak{S}_k$ (and complex conjugation) on X.

REFERENCES

[A08] Marcelo Aguiar and Rosa C. Orellana, *The Hopf algebra of uniform block permutations*. J. Algebraic Combin. **28** (2008) 115–138. (21)

[A86] Dmitri V. Alekseevskii and Askold M. Perelomov, *Invariant Kähler-Einstein metrics on compact homogeneous spaces*. Funktsional. Anal. i Prilozhen. **20** (1986) 1–16, 96. (Translation: Funct. Anal. Appl. **20** (1986), 171–182.) (16, 22)

[A97] Dmitri V. Alekseevskii, *Flag manifolds*. Zbornik Radova (N.S.) **6(14)** (1997) 3–35. (16, 22)

[B87] Arthur L. Besse, *Einstein manifolds*. Springer-Verlag, Berlin, 1987. (12)

[B61] William M. Boothby, *Homogeneous complex contact manifolds*. Proc. Symp. Pure Math. **3** (1961) 144–154. (22)

[B54] Armand Borel, *Kählerian coset spaces of semisimple Lie groups*. Proc. Nat. Acad. Sci. U. S. A. **40** (1954) 1147–1151. (12)

[B58] Armand Borel and Friedrich Hirzebruch, *Characteristic classes and homogeneous spaces. I*. Amer. J. Math. **80** (1958) 458–538. (12, 16, 22, 24)

[B75] Nicolas Bourbaki, *Groupes et algèbres de Lie. Chapitres 7 et 8*. Hermann, Paris, 1975. (16)

[B82] ———, *Groupes et algèbres de Lie. Chapitre 9. Groupes de Lie réels compacts*. Masson, Paris, 1982. (16)

[B12] ———, *Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Springer-Verlag, Berlin, 2012. (15)

[C29] Élie Cartan, *Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces*. Ann. Soc. Polon. Math. **8** (1929) 181–225. (13)

[E34] Charles Ehresmann, *Sur la topologie de certains espaces homogènes*. Ann. of Math. (2) **35** (1934) 396–443. (13)

[F55] Alfred Frölicher, *Zur Differentialgeometrie der komplexen Strukturen*. Math. Ann. **129** (1955) 50–95. (11)

[G73] Werner Greub, Stephen Halperin, and Ray Vanstone, *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes*. Academic Press, New York-London, 1973. (13)

[G82] Victor Guillemin and Shlomo Sternberg, *Geometric quantization and multiplicities of group representations*. Invent. Math. **67** (1982) 515–538. (12)

[H05] Friedrich Hirzebruch, *The projective tangent bundles of a complex three-fold*. Pure Appl. Math. Q. **1** (2005) 441–448. (22)

[H71] Kenneth Hoffman and Ray A. Kunze, *Linear algebra*, Second edition. Prentice-Hall Inc., Englewood Cliffs, N.J., 1971. (12, 17)

[K98] Hajime Kaji, *Secant varieties of adjoint varieties*. Matemática Contemporânea **14** (1998) 75–87. (22)

[K76] Aleksandr A. Kirillov, *Elements of the theory of representations*. Springer-Verlag, Berlin, Heidelberg, 1976. (10)

[K10] Bertram Kostant, *Root systems for Levi factors and Borel-de Siebenthal theory*. In: H. E. A. Campbell, Aloysius G. Helminck, Hanspeter Kraft, and David Wehlau (eds.) Symmetry and spaces, pp. 129–152. Birkhäuser, Boston, 2010. (16, 22)

[L02] Joseph M. Landsberg and Laurent Manivel, *Construction and classification of complex simple Lie algebras via projective geometry*. Selecta Math. (N.S.) **8** (2002) 137–159. (22)

[M20] Thomas Mason and François Ziegler, *Explicit pseudo-Kähler metrics on flag manifolds* (2020). `arXiv:2007.09302`. (24)

[N57] August Newlander and Louis Nirenberg, *Complex analytic coordinates in almost complex manifolds*. Ann. of Math. (2) **65** (1957) 391–404. (11)

[N84] Musubi Nishiyama, *Classification of invariant complex structures on irreducible compact simply connected coset spaces*. Osaka J. Math. **21** (1984) 39–58. (16)

[O04] Juan-Pablo Ortega and Tudor S. Raţiu, *Momentum maps and Hamiltonian reduction*. Birkhäuser, Boston, 2004. (18)

[S54] Jean-Pierre Serre, *Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (d'après Armand Borel et André Weil)*. In: Séminaire Bourbaki, Vol. 2, Exp. No. 100, 1–8. Secrétariat mathématique, Paris, 1954. (12)

[S69] Jean de Siebenthal, *Sur certains modules dans une algèbre de Lie semisimple*. Comment. Math. Helv. **44** (1969) 1–44. (16, 22)

[S70] Jean-Marie Souriau, *Structure des systèmes dynamiques*. Dunod, Paris, 1970. (Reprint: Éditions Jacques Gabay, Sceaux, 2008. Translation: *Structure of dynamical systems*. Birkhäuser, Boston, 1997.) (10, 11)

[S05] Eduard Study, *Kürzeste Wege im komplexen Gebiet*. Math. Ann. **60** (1905) 321–378. (13)

[T61] Olga Taussky, *Commutators of unitary matrices which commute with one factor*. J. Math. Mech. **10** (1961) 175–178. (21)

[T06] Richard P. Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*. In: Shing-Tung Yau (ed.) Surveys in differential geometry, Vol. X, pp. 221–273. Int. Press, Somerville, MA, 2006. (13)

[V87] David A. Vogan Jr., *Representations of reductive Lie groups*. In: Proc. Internat. Congr. Math. (Berkeley, 1986) Vol. 1, pp. 245–266. Amer. Math. Soc., Providence, RI, 1987. (12)

[W77] Alan Weinstein, *Lectures on symplectic manifolds*. Amer. Math. Soc., Providence, RI, 1977. (12)

[W78] ———, *A universal phase space for particles in Yang-Mills fields*. Lett. Math. Phys. **2** (1978) 417–420. (22)

[Y14] Takumi Yamada, *Invariant pseudo-Kähler metrics on generalized flag manifolds*. Differential Geom. Appl. **36** (2014) 44–55. (16)

[Z17] François Ziegler, *Elements of differential geometry*. Georgia Southern University lecture notes. 2017. (10)