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Gallai-Ramsey Number for Classes of Brooms

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GALLAI-RAMSEY NUMBERS FOR CLASSES OF BROOMS

by

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(Under the Direction of Hua Wang)

ABSTRACT

Given a graph G, we consider the problem of finding the minimum number n such that any k edge colored complete graph on n vertices contains either a rainbow colored triangle or a monochromatic copy of the graph G, denoted $gr_k(K_3 : G)$. More precisely we consider $G = B_{m,\ell}$ where $B_{m,\ell}$ is a broom graph with m representing the number of vertices on the handle and ℓ representing the number of bristle vertices. We develop a technique to reduce the difficulty of finding $gr_k(K_3 : B_{m,\ell})$, and use the technique to prove a few cases with a fixed handle length, but arbitrarily many bristles. Further, we find upper and lower bounds for any broom.

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INTRODUCTION

1.1 Basic Graph Theory

Graph Theory is a field whose foundations were birthed from the curiosity of Carl Leonhard Gottlieb Ehler, mayor of Danzig, in 1736. The famous inquiry was named the Königsberg Bridge problem or the Seven Bridges of Königsberg. To understand the problem we must understand the layout of the city of Königsberg, Prussia. A river flowed through the town where it split into two branches. In the town, there were four land masses separated by the river which were connected by seven bridges as shown in Figure 1.1. Ehler wondered if it was possible for a traveler to traverse each bridge exactly once. To solve this problem he enlisted the aide of the great mathematician Leonhard Euler; Euler agreed to study the problem and find a solution.
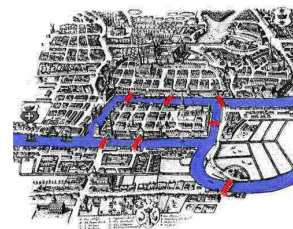
Figure 1.1: A Map of the Town of Königsberg in the 1800s with Bridges and River Highlighted.
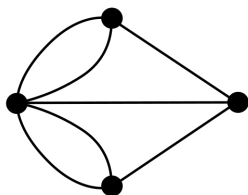
Figure 1.2: A Reimagining of the City of Königsberg as a Graph.

He began his invesitgation by recognizing that the starting land mass had no affect on the solution; only the sequence with which you cross the bridges must be considered. Next, he observed that the choice of route within the landmasses to move from bridge to bridge is irrelevant, so the problem may be condensed to imagining that the traveler instantly moves from bridge to bridge; similarly the route across the bridges is also irrelevant. Thus, we can imagine each land mass as a single point and each bridge as a single line as shown in

Figure 1.2. Euler also observed that each time you enter a land mass by a bridge you must also leave by a bridge; thus it follows that every land mass must have an even number of bridges attached in order to cross each bridge without repetition. This lead to the conclusion that it is not possible to traverse all bridges without crossing at least one bridge more than once. We now move on to provide some basic definitions. Any terms not defined here can be found in any introductory graph theory textbook such as [4] and [1].

De inition 1.1 ([4])**.** *A simple graph G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$ where each edge is a distinct unordered pair of vertices. We write uv for the edge $\{u, v\}$. If $uv \in E(G)$, then u and v are called* adjacent*. The vertices contained in an edge are its* endpoints*; and the vertices which are endpoints of an edge are said to be* incident *with that edge. The* degree *of a vertex v is the number of edges incident with v.*

Traversing the bridges and landmasses would require walking about the town and over the bridges, so we can name this type of wandering about a "walk." There are special types of walks that we can also define rigorously for use later in the chapter.

Definition 1.2 ([4])**.** *A walk of length k is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for all i. A* path *is a walk with no repeated vertex; a path of length k is denoted by P_k. A $u, v-$path is a path with end vertices u and v. A* cycle *is a closed walk of length at least 2 with no repeated edges and whose "endpoint" is the only repeated vertex; a cycle of length k is denoted by C_k.*

It is often interesting to consider the effects of removing certain edges or vertices from a graph. For instance, if any vertex from the graph in Figure 1.2 is removed along with all edges incident with it, the resulting graph - called a subgraph - yields a graph where it is possible to find a walk where each edge is used exactly once. More precisely we can define

subgraphs as below.

Definition 1.3 ([4]). *A* subgraph *of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write this as $H \subseteq G$ and say that "G contains H". An* induced subgraph *of G is a subgraph H such that every edge of G containing vertices in $V(H)$ must belong to $E(H)$. If H is an induced subgraph of G with vertex set S, then we write $H = G[S]$ and say that H is the subgraph of G "induced by S". A* spanning subgraph *of G is a subgraph with vertex set $V(G)$.*

If we consider going in the reverse direction, we can ask the question "Is there a largest graph?" There does exist a largest simple graph on a fixed set of vertices. However, if we do not set a particular number of vertices, another vertex may always be added along with a few edges. This clearly shows that there is no such graph if the number of vertices is left ambiguous. The definition for the largest graph on a set of vertices is defined below.

Definition 1.4 ([4]). *A* complete graph *is a simple graph in which every pair of vertices forms an edge. A complete graph with n vertices is denoted by K_n. The K_3 is referred to as a* triangle.

If a graph is not complete, there must exist vertices which are not adjacent to one another. A set of vertices with this property is called an independent set.

Definition 1.5 ([4]). *An* independent set *in a graph G is a vertex subset $S \subseteq V(G)$ such that the induced subgraph $G[S]$ has no edges. A* maximal independent set *is an independent set that is not a subset of any other independent set.*

Often, we would like to consider if there are multiple independent subsets which have an empty intersection, or even if there are several maximally independent subsets with empty intersection. These types of graphs form special classes which have nice properties that will be helpful later on. A rigorous definition of this class of graphs follows.

Definition 1.6 ([4]). *A graph is* bipartite *if its vertex set can be partitioned into two inde-pendent sets where the intersection of the two sets is empty. A* complete bipartite *graph is a bipartite graph in which the edge set consists of all pairs having a vertex from each of the two independent sets in the vertex partition. We denote a complete bipartite graph as $K_{m,n}$ where m and n are the sizes of the two independent sets. A $K_{1,n}$ is called a* star *for any value of n.*

Notice that, for a complete bipartite graph, it is possible to begin at one vertex and reach any other vertex by some path. In fact, for complete bipartite graphs, it is not neces-sary to use a path of length more than 2. This is a very useful and interesting property; we can use it to describe a particularly nice class of graphs that have other nice properties.

Definition 1.7 ([4]). *A graph G is* connected *if it has a u,v-path for each pair $u.v \in V(G)$. Otherwise, G is* disconnected.

Some connected graphs have a property where it is possible to find a cycle for each vertex instead of simply a path. Sometimes it is interesting to consider when it is not possible to find such a cycle for any vertex. In this consideration, we can define yet another class of graphs.

Definition 1.8 ([4]). *A graph having no cycle is* acyclic. *A* forest *is an acyclic graph; a* tree *is a connected acyclic graph. A* leaf *is a vertex of degree* 1. *A* spanning tree *is a spanning subgraph that is a tree.*

In this thesis, we are particularly interested in a special type of tree called a broom.

Definition 1.9 ([4]). *A* broom *$B_{k,\ell}$ is a path of length ℓ with a star with k leaves joined at one of the path's leaf vertices and the root vertex of the star. It is said to have k* bristles, *denoted by $b(B) = k$, and a* handle *of length ℓ, denoted by $h(B) = \ell$.*

Figure 1.3: Examples of Brooms.

(a) $B_{2,5}$.   (b) $B_{m,6}$.
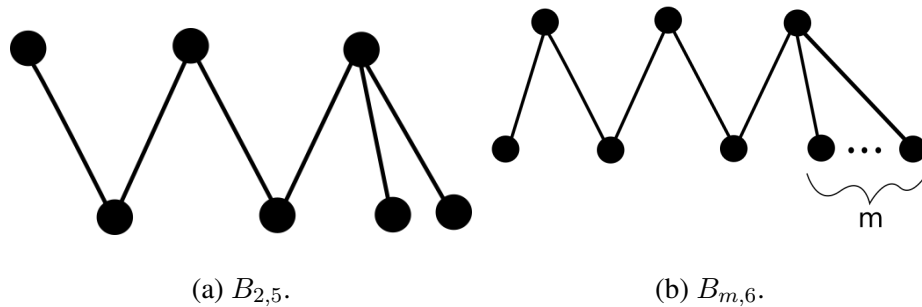
1.2   GRAPH COLORINGS

Throughout this thesis, we concern ourselves with coloring the edges of graphs and looking for certain colored subgraphs. To find these subgraphs we must first discuss color-ing these graphs, and why it is useful.

Definition 1.10 ([4])**.** *An assignment of colors to the edges of a nonempty graph G is an edge coloring of G. A coloring that uses k colors is a k-coloring. A graph G that is edge colored is called a* rainbow G *if its edges all have distinct colors.*

Imagine you and five others are at a p arty. You would like to determine if there is a group of three of you that are either all mutual acquaintances or mutual strangers. We can imagine that each person is a vertex, and each relationship is an edge. We will color an edge red if the adjacent vertices (the relationship between two particular people) are strangers, and blue otherwise. Notice, all we have to do now is find e ither a red t riangle o r a blue triangle. This is a famous problem called the party problem, and serves as one of the first interesting cases of Ramsey Theory.

Definition 1.11 ([4])**.** *For graphs $G_1, \ldots, G_k$, we write $n \to (G_1, \ldots, G_k)$ to mean that every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. The (graph) Ramsey number $R(G_1, \ldots, G_k)$ is the smallest integer n such that $n \to (G_1, \ldots, G_k)$. When $G_i = G$ for all i, we write $R_k(G) = R(G_1, \ldots, G_k)$.*

It becomes quite complicated to find monochromatic copies of several graphs as the number of colors increases. The simplest non-trivial case of Ramsey numbers is with only 2 colors, but these problems are already very difficult to solve once we start trying to find graphs such as K_5. It would be easier to find these numbers if we had more colors, but restricted the number of graphs for which we are searching. This can be realized by extending the theory to search for a rainbow graph.

Definition 1.12 ([3])**.** *For nonempty graphs G and H, the* Gallai-Ramsey number *$gr_k(G : H)$ is the smallest integer N such that for all $n \geq N$, every edge coloring of K_n, using at most k colors, contains either a rainbow colored copy of G or a monochromatic copy of H.*

The usual proof technique for Gallai-Ramsey numbers is to first assume that there is no rainbow triangle in the edge coloring of K_n.

Definition 1.13 ([3])**.** *A* Gallai coloring *of a complete graph G is an edge coloring of G such that G does not contain a rainbow triangle as a subgraph.*

This assumption yields a nice structure in the edge coloring.

Theorem 1.14 ([3])**.** *Every Gallai coloring of a complete graph K_n has a non-trivial partition of the vertices such that between the parts, there are a total of at most two colors on the edges, and in between each pair of parts, there is only one color on the edges.*

Since the usual proof technique is to assume the absence of rainbow triangles, we will have this kind of partition in most studies of Gallai-Ramsey numbers.

Definition 1.15 ([3])**.** *A partition as described in Theorem 1.14 is called a* Gallai Partition*.*

1.3   Preliminaries and Main Results

Before we examine the main results of this thesis, we must first observe some preliminary results used in the proof of the main results. First, we cover some prerequisite

knowledge to assist in the proof of the first theorem.

Definition 1.16 ([2]). *For nonempty graphs G_1 and G_2 the* Ramsey bipartite number pair $B(G_1, G_2)$ *is the minimum ordered pair* (N, M) *(with $N \geq M$) such that for all ordered pairs* (n, m) *where $n \geq N$ and $m \geq M$, every 2-coloring of $K_{n,m}$ contains either a G_1 in color 1 or a G_2 in color 2.*

Next, a result that is useful in finding a portion of the graph in question.

Theorem 1.17 ([2]). *For $n, m \in \mathbb{Z}^+$,*

(i) $B(P_{2n}, P_{2m}) = (n + m - 1, n + m - 1),$

(ii) $B(P_{2n-1}, P_{2m}) = (n + m, n + m - 1) \qquad for \quad n \geq m - 1,$

(iii) $B(P_{2n+1}, P_{2m}) = (n + m - 1, n + m - 1) \qquad for \quad n < m - 1,$

(iv) $B(P_{2n+1}, P_{2m+1}) = (n + m, n + m - 1) \qquad for \quad n \neq m,$

(v) $B(P_{2n+1}, P_{2n+1}) = (2n + 1, 2n - 1).$

The next result is also useful in finding the same portion of the graph in question.

Theorem 1.18 ([2]). *If $|E(G)| \geq \frac{\ell - 1}{2} \cdot |V(G)|$, then G contains a path of length ℓ.*

The following result gives an immediate lower bound for the main theorems. This bound is, in fact, sharp for small cases.

Proposition 1.19 ([3]). *For any connected bipartite graph H, and for any integer k with $k \geq 2$, we have*

$$gr_k(K_3 : H) \geq R_2(H, H) + (k - 2)(s_H - 1)$$

where s_H denotes the size of the smaller part in the partition.

In order to utilize the above result, we require the knowledge provided below.

Theorem 1.20 ([6])**.** *Let m and ℓ be integers with $m \geq 2$ and $n = m + \ell$. Then*

$$R_2(B_{m,\ell}, B_{m,\ell}) = \begin{cases} n + \left\lceil \frac{\ell}{2} \right\rceil - 1 & if \ \ell \geq 2m - 1; \\ \\ 2n - 2\left\lceil \frac{\ell}{2} \right\rceil - 1 & if \ 4 \leq \ell \leq 2m - 2. \end{cases}$$

Theorem 1.21 ([3])**.** *In every Gallai-coloring of a complete graph, there exists a spanning monochromatic broom.*

Theorem 1.22 ([3])**.** *Every Gallai-coloring of a complete graph K_n contains a non-trivial partition (with at least two parts) of the vertices such that between the parts there is a total of at most two colors on the edges, and between each pair of parts there is only one color on the edges.*

Theorem 1.23 ([5])**.** *Given a bipartite graph H and a positive integer R with*

$$R \geq max\{R_2(H, H), 3|b(H)| - 2\},$$

where b_H denotes the size of the larger part in the partition, if every Gallai-coloring of K_R using 3 colors, in which all parts of a Gallai-partition have order at most $|s_H| - 1$, contains a monochromatic copy of H, then

$$gr_k(K_3 : H) \leq R + (|s_H| - 1)(k - 2).$$

We would like to determine when Theorem 1.23 is applicable for brooms. That is, when is $R_2(B_{m,\ell}, B_{m,\ell}) \geq 3|b(B_{m,\ell})| - 2$?

Lemma 1.3.1. *Given a broom $B_{m,\ell}$, we have*

$$R_2(B_{m,\ell}, B_{m,\ell}) < 3|b_{B_{m,\ell}}| - 2,$$

where $b_{B_{m,\ell}}$ denotes the size of the larger part in the partition

Proof. First, we will compute the number of vertices in $b_{B_{m,\ell}}$. We must set up the parts of the partition in order to determine their respective sizes. Let v_1 be the vertex of highest

degree, v_i be the handle vertex that is $i-1$ edges away from v_1 for all $i \in \{2, 3, \ldots, \ell\}$, and u_j for $j \in \{1, 2, \ldots, m\}$ be the star vertices. Now, let A_1 and A_2 be the two parts, and let v_1 be in A_1. This results in all u_j being contained in A_2. Notice that this decision results in $v_i \in A_1$ for all odd i and $v_i \in A_2$ for all even i. Notice that the part containing the star is inherently larger. This part also contains exactly half of the handle vertices for even length handles. However, if the handle is odd length, then the other part contains one more handle vertex than the large part. Hence, $|b_{B_{m,\ell}}| = |A_2| = m + \lfloor \frac{\ell}{2} \rfloor$. Notice, we also have that

$$|s_{B_{m,\ell}}| = |V(B_{m,\ell})| - |b_{B_{m,\ell}}|$$
$$= (m + \ell) - \left(m + \left\lfloor \frac{\ell}{2} \right\rfloor \right)$$
$$= \left\lceil \frac{\ell}{2} \right\rceil$$

By Theorem 1.20 we must consider 2 cases; $\ell \geq 2m-1$ and $4 \leq \ell \leq 2m-2$.

$$R_2(B_{m,\ell}, B_{m,\ell}) = n + \left\lceil \frac{\ell}{2} \right\rceil - 1 \qquad\qquad R_2(B_{m,\ell}, B_{m,\ell}) = 2n - 2\left\lceil \frac{\ell}{2} \right\rceil - 1$$
$$= m + \ell + \left\lceil \frac{\ell}{2} \right\rceil - 1 \qquad\qquad < 2m + 3\left\lfloor \frac{\ell}{2} \right\rfloor - 1$$
$$< 3m + 3\left\lfloor \frac{\ell}{2} \right\rfloor - 2 \qquad\qquad < 3m + 3\left\lfloor \frac{\ell}{2} \right\rfloor - 2$$
$$= 3\left(m + \left\lfloor \frac{\ell}{2} \right\rfloor \right) - 2 \qquad\qquad = 3\left(m + \left\lfloor \frac{\ell}{2} \right\rfloor \right) - 2$$
$$= 3|b_{B_{m,\ell}}| - 2 \qquad\qquad = 3|b_{B_{m,\ell}}| - 2$$

Notice that neither case can ever occur for any possible combination of m and ℓ. $\qquad\square$

Lemma 1.3.1 that Theorem 1.23 cannot be used to obtain sharp results for Gallai-Ramsey numbers of brooms. However, upper bounds can still be obtained using Theorem 1.23 by choosing $N \in \mathbb{Z}$ with $gr_k(K_3 : B_{m,\ell}) \leq N$ such that $N \geq 3\left(m + \left\lceil \frac{\ell}{2} \right\rceil \right)$. See Theorem 1.25 for example. Our first result is a specific one, for the broom $B_{2,5}$.

Theorem 1.24. *For all $k \geq 1$ we have*

$$gr_k(K_3 : B_{2,5}) = 5 + 2k.$$

Our main result contains a general lower and upper bound.

Theorem 1.25. *If $m \geq \frac{7\ell}{2} + 3$, then*

$$2m + \ell - 2 + (k-2)\left\lceil \frac{\ell}{2} \right\rceil \leq gr_k(K_3 : B_{m,\ell}) \leq 3m - \left\lceil \frac{3\ell}{2} \right\rceil + (k-2)\left\lceil \frac{\ell}{2} \right\rceil$$

Finally, we list sharp results for two classes of brooms.

Theorem 1.26. *For $m \geq 2$ and $k \geq 2$ we have*

$$gr_k(K_3 : B_{m,5}) = \begin{cases} m + 2k + 3 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k - 1 & \text{if } m \geq 4. \end{cases}$$

Theorem 1.27. *For $m \geq 2$ and $k \geq 2$ we have*

$$gr_k(K_3 : B_{m,6}) = \begin{cases} m + 2k + 4 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k + 1 & \text{if } m \geq 4. \end{cases}$$

For the remainder of this thesis, we will prove Theorems 1.24– 1.27. In Chapter 2, we explore an example of a Gallai-Ramsey number of a triangle and a specific broom in order to understand in greater detail the type of problems explored in Gallai-Ramsey Theory, and to attain an intuition of the problem at hand. In Chapter 3, we find loose bounds for $gr_k(K_3 : B_{m,\ell})$ for any $m, \ell \in \mathbb{N}^+$ which is the most general case we explore. Finally, in Chapter 4, we find sharp results for two classes of brooms where we fix the handle length and allow any number of bristles.

Conjecture 1.28. *For any broom $B_{m,\ell}$ with $m, \ell \in \mathbb{Z}$ and $n = m + \ell$, and for any integer k with $k \geq 2$, we have*

$$gr_k(K_3 : B_{m,\ell}) = \begin{cases} n + (k-1)\left(\left\lceil \frac{\ell}{2} \right\rceil - 1\right) & \text{if } \ell \geq 2m - 1; \\ 2n - 3 + (k-4)\left(\left\lceil \frac{\ell}{2} \right\rceil - 1\right) & \text{if } 4 \leq \ell \leq 2m - 2. \end{cases}$$

CHAPTER 2

GALLAI-RAMSEY NUMBER OF A TRIANGLE AND THE BROOM $B_{2,5}$.

Before we can prove Theorem 1.24 we need a lemma. This Lemma is analogous to Theorem 1.23, but is applicable to the broom we are interested in for this chapter's result.

Lemma 2.0.1. *If every Gallai-coloring of $K_{R_2(B_{2,5},B_{2,5})}$ using 3 colors, in which all parts of a Gallai-partition have order at most $|s(B_{2,5})| - 1$, contains a monochromatic copy of $B_{2,5}$, then*

$$gr_k(K_3 : B_{2,5}) \leq R_2(B_{2,5}, B_{2,5}) + (|s(B_{2,5})| - 1)(k - 2).$$

The proof follows by easy case analysis for $B_{2,5}$.

Proof of Theorem 1.24. For the lower bound, the case $k = 1$ follows from considering a 1-colored K_6 which clearly does not contain a monochromatic $B_{2,5}$. For $k \geq 2$, by Proposition 1.19 and Theorem 1.20, we have $gr_k(K_3 : B_{2,5}) \geq 2k + 5$.

For the upper bound, suppose G is a k-colored copy of K_n with $n = 2k + 5$. Assume there is no rainbow triangle in G and no monochromatic copy of $B_{2,5}$. By Theorem 1.22, there is a Gallai-partition of G. By Theorem 1.21, there is a spanning monochromatic broom, say a spanning blue broom B. If $k = 1$, the result is trivial and if $k = 2$, the result follows from Theorem 1.21, so suppose $k \geq 3$. If $h(B) \geq 5$ and $b(B) \geq 2$, then this contains the desired $B_{2,5}$, so suppose this is not the case. Thus, either $h(B) \leq 4$ or $b(B) = 1$. Since $k \geq 3$, we have $n \geq 11$, so these two cases cannot occur at the same time. We break the remainder of the proof into these two cases.

Case 1. $b(B) = 1$.

This means that B is a spanning monochromatic path $P = v_1 v_2 \ldots v_n$. We claim that there are no extra blue edges except possibly for the edge $v_1 v_n$. Indeed, any extra edge would result in a monochromatic copy of $B_{2,5}$, as seen in the following claim.

Claim 1. *Other than the edges of P and possibly the edge v_1v_n, G contains no blue edges.*

Proof of Claim 1. Suppose there is an extra non-path blue edge e between two vertices on the path other than the edge v_1v_n. Let u and v be the ends of e, say with $u = v_i$ and $v = v_j$ where $i < j$ where $i \neq 1$ or $j \neq n$. If $i \geq 5$, then the path $v_{i-4}v_{i-3}v_{i-2}v_{i-1}v_i$ is the handle of a blue $B_{2,5}$ with bristle edges v_iv_{i+1} and v_iv_j, so we may assume $i \leq 4$ and symmetrically, $j \geq n-3$. If $i = 4$ and $j = n-3$, then the path $v_1v_2v_3v_4v_j$ is the handle of a blue $B_{2,5}$ with bristle edges v_jv_{j-1} and v_jv_{j+1}, so we may assume either $i < 4$ or $j < n-3$. Without loss of generality, say $i < 4$. Since $n \geq 11$, the path $v_{j-4}v_{j-3}v_{j-2}v_{j-1}v_j$ is the handle of a blue $B_{2,5}$ with bristle edges v_jv_{j+1} and v_jv_i, completing the proof of Claim 1. $\square$

Next a claim that provides even more structure.

Claim 2. *Each part of the Gallai-partition of G has order 1.*

Proof of Claim 2. By Lemma 2.0.1 each part has order at most 2. Suppose there is a part $A = \{u, v\}$ of order 2. Since P is spanning, it must contain both u and v. Without loss of generality, suppose $u = v_i$ and $v = v_j$ where $i < j$. If $i \geq 3$, then since A is a part of the Gallai-partition, $v_{i-1}u$ and $v_{i-1}v$ must both be blue, contradicting Claim 1, so $i \leq 2$ and symmetrically, $j \geq n-1$. With $n \geq 11$, there are at least 7 vertices on P in between u and v. As above, $v_{i+1}u$ and $v_{i+1}v$ must both be blue, a contradiction to Claim 1, completing the proof of Claim 2. $\square$

By Claim 2, each part of the Gallai-partition has order 1, meaning that G is a 2-coloring. Since $n \geq 11 > R(B_{2,5}, B_{2,5})$, this graph contains a monochromatic copy of $B_{2,5}$.

Case 2. $h(B) \leq 4$.

Choose the spanning broom B so that $h(B)$ is as large as possible. Let u be the center vertex of the star part and let v be the vertex at the other end of the handle of B. Let $A = G\backslash\{u,v\}$. Since Lemma 2.0.1 says all parts of the Gallai-partition have order at most 2 and we may assume that red and blue are the colors appearing in the partition, this means $\{u,v\}$ is a part of the Gallai-partition.

Claim 3. *The vertex v has at least $|A| - 2$ red edges to A.*

Proof of Claim 3. If v has any blue edges to A in the star part of B, then we could find a blue broom with a longer handle, a contradiction. This means that v has at most 3 blue edges to A.

Suppose v has an edge of another color, say green, to a vertex $w \in A$. Now v must have red edges to the rest of A, at least $n - 2 - 2 - 1 \geq 6$ vertices. Since $\{v,w\}$ is a part of the partition, w must also have red edges to all of these vertices. To avoid a red copy of $B_{2,5}$, there can be no red edges within those vertices, but all edges between those vertices form a matching so there is a blue P_4 in A, providing a blue copy of $B_{2,5}$ for a contradiction. $\qquad\square$

Since $|A| \geq n - 2 \geq 9$, there are at least 5 parts of the Gallai-partition within A. With $R(P_4, P_4) = 5$, we know there is either a red P_4 or a blue P_4 appearing between the parts. If it is blue, we are done as above, so it must be red. If an end is adjacent to v in red, we're done again so both ends must have blue edges to v.

Letting x, y be the vertices on the handle in this order from u to v, we see that y must have an additional red edge to an interior vertex of the red path to avoid a blue $B_{2,5}$. this red edge allows us to reroute the P_4 so v is adjacent to an end in red. making a red $B_{2,5}$ and completing the proof. $\qquad\square$

CHAPTER 3

GALLAI-RAMSEY NUMBER BOUNDS FOR A TRIANGLE AND ANY BROOM.

Next, we prove the general bound for of Gallai-Ramsey numbers for any broom.

Proof of Theorem 1.25. For the lower bound, we simply apply Proposition 1.19. This yields

$$
\begin{aligned}
gr_k(K_3 : B_{m,\ell}) &\geq R_2(B_{m,\ell}) + (k-2)(s_{B_{m,\ell}} - 1) \\
&= 2n - 2\left\lceil \frac{\ell}{2} \right\rceil - 1 + (k-2)\left(\left\lceil \frac{\ell}{2} \right\rceil - 1\right) \\
&= 2m + 2\ell - 2\left\lceil \frac{\ell}{2} \right\rceil - 1 + (k-2)\left(\left\lceil \frac{\ell}{2} \right\rceil - 1\right) \\
&\geq 2m + \ell - 2 + (k-2)\left(\left\lceil \frac{\ell}{2} \right\rceil - 1\right).
\end{aligned}
$$

For the upper bound, we begin by supposing G is a colored complete graph of order $n = 3m - \left\lceil \frac{3\ell}{2} \right\rceil + (k-2)\left\lceil \frac{\ell}{2} \right\rceil$ with no rainbow triangle and no monochromatic $B_{m,\ell}$.

Claim 1. *Every vertex has at least $m + \ell$ incident edges in one color.*

Proof of Claim 1. By assumption, v is adjacent to $\frac{\ell}{2} - 1$ vertices in its partition. Therefore, there are at most $\frac{\ell}{2} - 1$ edges incident with v in its partition's color, say green. Since there are only two remaining colors, at least half of the edges incident with v must be in one color. Recall that $m \geq \frac{7\ell}{2} + 3$. Using this fact, we will show that the number of edges to each vertex in the graph from a vertex, say v, is at least $2m + \frac{5\ell}{2} + 2$. Then

$$
\begin{aligned}
3m - \ell &\geq 2m + \left(\frac{7\ell}{2} + 3\right) - \ell \\
&= 2m + \frac{5\ell}{2} + 3.
\end{aligned}
$$

Notice, we have

$$\frac{\left(2m + \frac{5\ell}{2} + 3\right) - \left(\frac{\ell}{2} - 1\right)}{2} = m + \ell + 2$$

$$> m + \ell.$$

Hence, there are at least $m + \ell$ edges incident with v in one color. $\qquad\square$

Suppose the colors used in the coloring of G are red, blue, and green where green is the color used within the parts. Partition $V(G)$ into the red vertices and blue vertices; if any are both blue and red, the vertex is assigned arbitrarily to either the red or blue set. If two vertices belong to the same part of the Gallai-partition, they must both go into the same set. If both sets have at least ℓ vertices, then by Theorem 1.17 there exists a monochromatic path using the edges in between. This results in either a red or blue broom depending on the color of the path. Therefore, one of the sets must have less than ℓ vertices and so the other part has at least $3m - 2\ell - 2$. Suppose the larger set is the red set. We will find the largest path that is guaranteed to be in the red set. Let $q \leq 3m - 2\ell - 4$. Then

$$\begin{aligned}
\frac{q-1}{2} \cdot |V(G)| &= (3m - 2\ell - 2)\left(\frac{q-1}{2}\right) \\
&\leq (3m - 2\ell - 2)\left(\frac{3m - 2\ell - 4 - 1}{2}\right) \\
&= (3m - 2\ell - 2)\left(\frac{3m - 2\ell - 3}{2} - 1\right) \\
&= \frac{(3m - 2\ell - 2)(3m - 2\ell - 3)}{2} - (3m - 2\ell - 2) \\
&= \frac{(3m - 2\ell - 2)!}{2! \cdot (3m - 2\ell - 4)!} - (3m - 2\ell - 2) \\
&= \binom{3m - 2\ell - 2}{2} - (3m - 2\ell - 2) \\
&= |E(G)|.
\end{aligned}$$

By Theorem 1.18, we know that there exists a red path of length $3m - 2\ell - 4$ in the red set. Notice $3m - 2\ell - 4 \geq m + 7\ell - 2\ell - 4 = m + 5\ell - 4 \gg m + \ell$. Hence, any section

of path with length $\ell - 1$ combined with an edge to the blue set and a star with degree m back to the red set gives the desired broom. □

CHAPTER 4

GALLAI-RAMSEY NUMBERS OF A TRIANGLE AND CLASSES OF BROOMS.

Again, before we can prove Theorem 1.26 or Theorem 1.27 we need a lemma. This Lemma is again analogous to Theorem 1.23,and is simply an extension of Lemma 2.0.1.

Lemma 4.0.1. *Given a broom* $B = B_{m,\ell}$ *with* $\ell = 5$ *or* 6, *if every Gallai-coloring of* $K_{R_2(B,B)}$ *using 3 colors, in which all parts of a Gallai-partition have order at most* $|s(B)| - 1$, *contains a monochromatic copy of* B, *then*

$$gr_k(K_3 : B) \leq R_2(B,B) + (|s(B)| - 1)(k - 2).$$

The proof follows by easy case analysis for the specific graphs in question, and is simply a restatement of Lemma 1.23 in [5] for these graphs.

4.1 GALLAI-RAMSEY NUMBER OF A TRIANGLE AND THE CLASS OF BROOMS $B_{m,5}$.

Now, we move on to prove the first sharp Gallai-Ramsey number.

Proof of Theorem 1.26. For the lower bound, by Proposition 1.19 and Theorem 1.20, we have

$$gr_k(K_3 : B_{m,5}) \geq \begin{cases} m + 2k + 3 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k - 1 & \text{if } m \geq 4. \end{cases}$$

For the upper bound, suppose G is a k-colored K_n with

$$n = \begin{cases} m + 2k + 3 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k - 1 & \text{if } m \geq 4. \end{cases}$$

If $k = 2$, then $|G| = R_2(B_{m,5}, B_{m,5})$ so G clearly contains a monochromatic copy of $B_{m,5}$, so assume $k \geq 3$.

Assume there is no monochromatic copy of $B_{m,5}$ and no rainbow triangle in G. By Theorem 1.22, there is a Gallai-partition. If all parts have order 1, then $k \leq 2$, so we may

assume that there is a part of order 2. Let $H_1, H_2, \ldots, H_t$ be the parts of this Gallai-partition with $|H_i| \geq |H_j|$ whenever $i \leq j$, so $|H_1| = 2$. By Lemma 4.0.1, we may assume $k = 3$.

Let A (and B) be the vertices in $G \setminus H_1$ with all red (respectively blue) edges to H_1, say with $|A| \geq |B|$. Note that this means $|A| \geq \frac{n-2}{2}$.

First assume $2 \leq m \leq 3$, so $n = m + 9$. In this case, $|A| \geq 5$ and we first suppose $|A| \leq n - 5$, so $|B| \geq 3$. If there is a red edge uv from A to B, then a red path of the form $vu - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,5}$ so this means that all edges between A and B must be blue. Then since $|B| \geq 3$, a blue path of the form $B - H_1 - B - H_1 - B$ along with m blue edges from the last vertex on the path to A, produces a blue $B_{m,5}$, a contradiction. Next suppose $|A| \geq n - 4 = m + 5$. If there is a red edge uv within A, then a red path of the form $uv - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,5}$, so there can be no red edge within A. Since the parts of the Gallai-partition have order at most 2, this means that A is a blue complete graph minus a matching on at least $m + 5$ vertices, so there is clearly a blue copy of $B_{m,5}$ within A, again a contradiction.

Thus, we may assume $m \geq 4$ so $n = 2m+5$ and $|A| \geq m+2$. First suppose $|A| \leq 2m$. If there is a red edge uv from A to B, then a red path of the form $vu - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,5}$ so this means that all edges between A and B must be blue. Then since $|B| = n - |A| - 2 \geq 3$, a blue path of the form $B - H_1 - B - H_1 - B$ along with m blue edges from the last vertex on the path to A, produces a blue $B_{m,5}$, a contradiction.

Next suppose $|A| \geq 2m + 1$ so $|B| \leq 2$. If there is a red edge uv within A, then a red path of the form $uv - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,5}$, so there can be no red edge within A. Since the parts of the Gallai-partition have order at most 2, this means that A is a blue complete graph

minus a matching on at least $2m+1$ vertices. Since $m \geq 4$, we know that $2m+1 \geq m+5$ so there is clearly a blue copy of $B_{m,5}$ within A, again a contradiction. $\qquad\square$

4.2 Gallai-Ramsey Number of a Triangle and the Class of Brooms $B_{m,6}$.

Finally, we prove the final result which is another sharp Gallai-Ramsey number.

Proof of Theorem 1.27. The proof of this result is very similar to the proof of Theorem 1.26. For the lower bound, by Proposition 1.19 and Theorem 1.20, we have

$$
gr_k(K_3 : B_{m,6}) \geq \begin{cases} m + 2k + 4 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k + 1 & \text{if } m \geq 4. \end{cases}
$$

For the upper bound, suppose G is a k-colored copy of K_n with

$$
n = \begin{cases} m + 2k + 4 & \text{if } 2 \leq m \leq 3, \\ 2m + 2k + 1 & \text{if } m \geq 4. \end{cases}
$$

If $k = 2$, then $|G| = R(B_{m,6}, B_{m,6})$ so G clearly contains a monochromatic copy of $B_{m,6}$.

Assume there is no monochromatic copy of $B_{m,6}$ and no rainbow triangle in G. By Theorem 1.22, there is a Gallai-partition. If all parts have order 1, then $k \leq 2$, so we may assume that there is a part of order 2. Let $H_1, H_2, \ldots, H_t$ be the parts of this Gallai-partition with $|H_i| \geq |H_j|$ whenever $i \leq j$, so $|H_1| = 2$. By Lemma 4.0.1, we may assume $k = 3$.

Let A (and B) be the vertices in $G\backslash H_1$ with all red (respectively blue) edges to H_1, say with $|A| \geq |B|$. Note that this means $|A| \geq \frac{n-2}{2}$.

First assume $2 \leq m \leq 3$, so $n = m + 10$. In this case, $|A| \geq 5$ and we first suppose $|A| \leq n - 5$, so $|B| \geq 3$. If there is a red edge uv from A to B with an adjacent blue edge either within A or within B, then a red path of the form $B - vu - H_1 - A - H_1$ or $vu - A - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to all of the remaining vertices in A produces a red $B_{m,6}$ so this means that any red edge

between A and B must be adjacent with only blue edges within A and within B. Suppose there is at least 1 red edge between A and B. If there is a blue edge of the form $v - A$, then a blue path of the form $H_1 - B - H_1 - v - A - u$ along with all remaining blue edges from u to A produces a blue $B_{m,6}$. Therefore, all edges from v to A are red. Then a red path of the form $A - H_1 - uv - A - H_1$ along with all remaining red edges from the last vertex on the path to A gives a red $B_{m,6}$. Now suppose there is no red edge between A and B. Then a blue path of the form $A - B - H_1 - B - H_1 - B$ along with all remaining red edges from the last vertex on the path to A gives a red $B_{m,6}$.

Thus, we may assume $m \geq 4$ so $n = 2m + 7$. First suppose $|A| = m + 5$, so $|B| = m$. If there is a red path uvw in A, then $uvw - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,6}$ so this means that there is no red P_3 in A. Therefore, A is blue minus a matching and there is a blue P_4 in A. Furthermore, every edge between A and B is red. If this was not the case, then there exists some vertex v in A with a blue edge to B, and $P_4 - B - H_1$ along with all remaining blue edges from the last vertex on the path to B gives a blue $B_{m,6}$. Now we have that $A - B - A - H_1 - A - H_1$ along with all remaining red edges to A results in a red $B_{m,6}$.

Now suppose $|A| = m + 4$, so $|B| = m + 1$. If there is a red path uvw in A, then $uvw - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,6}$ so this means that there is no red P_3 in A. Therefore, A has a blue P_4. Furthermore, every vertex in A has a blue edge to B. If this was not the case, then there exists some vertex v in A with all red edges to B, and $B - A - H_1 - A - H_1 - v$ along with all remaining red edges from the last vertex on the path to B gives a red $B_{m,6}$. Hence, $P_4 - B - H_1$ along with all remaining blue edges to B gives a blue $B_{m,6}$.

Next suppose $|A| = m + 3$, so $|B| = m + 2$. Suppose there exists some vertex v in A with all red edges to B, and $B - A - H_1 - A - H_1 - v$ along with all remaining red edges from the last vertex on the path to B gives a red $B_{m,6}$. Therefore, every vertex

in A has a blue edge to B. If there is a blue path uv in A, then $uv - B - H_1 - B - H_1$ along with all remaining blue edges to B results in a blue $B_{m,6}$. So we may assume A is completely red minus a matching. Suppose u and v are vertices in A with red edges to B. Then $B - u - H_1 - A - H_1 - v$ along with all remaining red edges from the last vertex to A and B form a red $B_{m,6}$. Hence, only one vertex in A can have any red edges to B, but since every vertex in B has a red edge to A they must all go to that vertex v. Let uw be a red path in A. Notice $uw - H_1 - A - H_1 - v$ along with all remaining edges to B gives a red $B_{m,6}$.

Finally suppose $|A| \geq m + 6$, so $|B| \leq m - 1$. If there is a red path uvw in A, then $uvw - H_1 - A - H_1$ along with all remaining red edges from the last vertex on the path to A produces a red $B_{m,6}$ so this means that there is no red P_3 in A. Hence, A is complete in blue minus a matching on $m + 6$ vertices. □

CHAPTER 5

CONCLUSION.

We considered the problem of finding $gr_k(K_3 : B_{m,\ell})$ for a sharp result, but we discovered that we do not yet have the tools to tackle this problem. Instead, we found loose upper and lower bounds for $gr_k(K_3 : B_{m,\ell})$ in Theorem 1.25. We developed a technique to reduce the difficulty of finding $gr_k(K_3 : B_{m,\ell})$ for classes of short handled brooms. We used the technique to prove a few cases with a fixed handle length, but arbitrarily many bristles in Theorem 1.26 and Theorem 1.27. We also conjectured that the lower bound found in Theorem 1.25 is, in fact, a sharp result in Conjecture 1.28.

REFERENCES

[1] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs & Digraphs, Fifth Edition*. Chapman & Hall/CRC, 5th edition, 2010.

[2] R. J. Faudree and R. H. Schelp. Path-path Ramsey-type numbers for the complete bipartite graph. *J. Combinatorial Theory Ser. B*, 19(2):161–173, 1975.

[3] Colton Magnant and Pouria Salehi Nowbandegani. *Ramsey Theory for Rainbow or Monochromatic Subgraphs, Submitted Manuscript*. submitted.

[4] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, September 2000.

[5] Haibo Wu, Colton Magnant, Pouria Salehi Nowbandegani, and Suman Xia. All partitions have small parts — Gallai–Ramsey numbers of bipartite graphs. *Discrete Appl. Math.*, 254:196–203, 2019.

[6] Pei Yu and Yusheng Li. All Ramsey numbers for brooms in graphs. *Electron. J. Combin.*, 23(3):Paper 3.29, 8, 2016.