Pattern Containment in Circular Permutations

Charles Lanning

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by

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(Under the Direction of Hua Wang)

ABSTRACT

Pattern containment in permutations, as opposed to pattern avoidance, involves two aspects. The first is to contain every pattern at least once from a given set, known as finding superpatterns; while the second is to contain some given pattern as many times as possible, known as pattern packing. In this thesis, we explore these two questions in circular permutations and present some interesting observations. We also raise some questions and propose some directions for future study.

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CHAPTER 1

INTRODUCTION

We begin our discussion with an overview of important terminologies in the field of pattern containment.

1.1 Patterns in Permutations

Note that when we say pattern, we are referring to a permutation that is to be contained in another permutation. For example, in the permutation 21534, if we delete 2 and 3, we obtain the sequence 154. However, this is not a permutation, so we call it a subsequence of 21534.

Definition 1.1. Let $\pi = \pi_1\pi_2\ldots\pi_n$ be a permutation. A sequence formed by $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$, with $i_j < i_{j+1}$ for each $1 \leq j \leq k-1$, is called a *subsequence* of π.

This definition allows us to consider an idea of containment. However, since these subsequences are not necessarily permutations, we need to find some way to relate them to permutations. In the previous example, where 154 is contained in 21534, notice that $1 < 5$, $1 < 4$, and $5 > 4$. So, in words, the order goes low, high, middle. There is only one permutation that has the same order, and that is 132. We will say that 154 is order-isomorphic to 132.

Definition 1.2. Let $p = p_1p_2\ldots p_n$ and $q_1q_2\ldots q_n$ be sequences of natural numbers. If $p_i < q_k \iff q_i < q_k$, then we say that p and q are *order-isomorphic*.

We can easily see that for any sequence of non-repetitive natural numbers, there is only one permutation that is order-isomorphic to it. This allows us to define pattern containment in a convenient way.

Definition 1.3. Let π and τ be permutations of length n and k, respectively. Let $n \geq k$. If there is a subsequence of π that is order-isomorphic to τ, then we say that π contains τ.

So, in our example of 21534, since 132 is order-isomorphic to 154, which is a subsequence of 21534, we can say that 21534 contains 132. Alternatively, we can say that 132 occurs in 21534.

1.2 Superpatterns

In the study of pattern containment, a natural question might be to find permutations that contain all patterns of a certain length, say k. Note that there are $k!$ patterns of length k.Now, we can clearly construct a length $k \cdot (k!)$ permutation by having the first k entries representing a certain pattern, and so on. This is called a k-superpattern, and it is by no means the shortest k-superpattern.

Definition 1.4. Let P be a subset of permutations. We say that a permutation π is a *P-superpattern* if it contains at least one occurrence of every $\tau \in P$. Further, define $sp(P)$ to be the length of the shortest P-superpattern, i.e.

$$sp(P) = \min\{n : \text{there is a } P\text{-superpattern of length } n\}.$$

In the case that P consists of all permutations of length k, we employ the traditional notations k-superpattern and $sp(k)$.

As an illustration, suppose we wish to contain all permutations of length 3, i.e. we want a 3-superpattern. Then, $\pi = 25314$ accomplishes this; further, there is no permutation of length 4 that could possibly contain all permutations of length 3 since it would be impossible for such a permutation to contain both 123 and 321. Hence, $sp(3) = 5$.

Much of the research in superpatterns regards finding bounds for $sp(k)$. For cases $k \leq 3$, $sp(k)$ is known. However, for larger values of k, the problem becomes much harder.

1.3 Pattern Packing

Pattern packing refers to containing the most number of copies of a given pattern within a permutation. For a given pattern, there exists an (not necessarily unique) optimal permutation of a certain length.

Definition 1.5. For permutations π and τ (of length k), we let $f(\pi, \tau)$ be the number of occurences of τ in π, and we define

$$g(n, \tau) = \max\{f(\sigma, \tau) : \sigma \text{ is a permutation of length } n\}.$$

If π is of length n and $f(\pi, \tau) = g(n, \tau)$, then we say that π is τ-*optimal*. The *packing density* of τ is defined by

$$\delta(\tau) = \lim_{n\to\infty} \frac{g(n, \tau)}{\binom{n}{k}}.$$

We can think of packing density as a pattern's 'packability.' As a simple example, for the pattern $\tau = 123$ and the permutation $\pi = 123456$, it is not hard to see that π is τ-optimal and

$$g(6, \tau) = f(\pi, \tau) = \binom{6}{3}.$$

In general, if π is the monotone increasing permutation of length n, then π is τ-optimal. Hence, the packing density of τ is

$$\delta(\tau) = \lim_{n\to\infty} \frac{g(n, \tau)}{\binom{n}{3}} = \lim_{n\to\infty} \frac{\binom{n}{3}}{\binom{n}{3}} = 1.$$

1.4 Previous Work

The systematic study of pattern containment was first proposed by H. Wilf in his 1992 address to the SIAM meeting on Discrete Mathematics [11]. Until then, most research regarding patterns concerned pattern avoidance. For an excellent introduction into these results, consult [4].

Our research considers superpatterns and pattern packing, which can be seen as the dual problems to pattern avoidance.

In 1999, R. Arratia was the first to publish bounds for $sp(k)$. He found that $\frac{k^2}{e^2} \leq sp(k) \leq k^2$ [2]. This lower bound is still the best known. In 2007, H. Eriksson et al. improved Arratia's upper bound with $sp(k) \leq \frac{2k^2}{3} + O(k^{3/2}(\log(k))^{1/2})$ [7]. Alison Miller [12] then showed, in 2010, that $sp(k) \leq \frac{k(k+1)}{2}$, which strongly supports the conjecture by Eriksson et al that $sp(k) \sim \frac{k^2}{2}$.

Many results on $sp(P)$ for various sets of permutations P have also been found. As a brief list, bounds have been found for $sp(P)$ when P has been the set of layered permutations [8], the set of 321-avoiding permutations [3], the set of m-colored permutations [9], and the set of words [5]. For these last two items, the concept of pattern containment extends naturally to m-colored permutations and to the set of words. We will cover m-colored permutations, as well as the circular analogue, in Chapter 3.

In 1993, W. Stromquist found the packing density of 132 to be $2\sqrt{3}-3$ [14]. Along with the packing density of 123 being trivially 1, this closed the length-3 case by symmetries (reversals, inverses, and complements of permutations preserve pattern containment). In 1997, A. Price found the packing densities of 1432, 2143, and 1324 [13]. Since there are 7 equivalence classes of length-4 patterns, this left 4 unsolved packing densities. In 2002, M. H. Albert et al. found the packing density of 1243 [1]. The remaining three cases are still unsolved.

CHAPTER 2

CIRCULAR PERMUTATIONS

2.1 Introduction

In this chapter, we generalize the study of pattern containment to circular permutations. We first introduce the related concepts in Section 2.2. In Section 2.3, we discuss circular superpatterns and R-rev superpatterns in general. We then study packing density in circular permutations and its R-rev analogue in Section 2.4. We briefly comment on our findings and raise some questions in Section 2.5.

2.2 Circular permutations and number of revolutions

In order to define the circular analogue of a permutation, we first introduce the circular shift of a permutation.

Definition 2.1. Let $\pi = \pi_1\pi_2\pi_3\cdots\pi_n$ be a permutation of length n. The *circular shift of* π, denoted $S(\pi)$, is given by

$$S(\pi) = \pi_n\pi_1\pi_2\cdots\pi_{n-1}.$$

For instance, with $\pi = 31524$, we have $S(\pi) = 43152$, $S^2(\pi) = 24315$, $S^3(\pi) = 52431$, $S^4(\pi) = 15243$, and $S^5(\pi) = 31524 = \pi$.

Definition 2.2. Let π be a permutation of length n. Then, the *circular permutation of* π, denoted π_c, is the permutation obtained by wrapping π clockwise around a circle in one revolution; so for $2 \leq i \leq n-1$, we have π_{i-1}, π_i, and π_{i+1} appear in clockwise ordering, with π_n being the counterclockwise neighbor of π_1. We say that two permutations are *equivalent* if one is the circular shift of the other.

Note that when a circular permutation is considered, circular shift does not change the permutation. As an example, Figure 2.1 shows the circular versions of a permutation and its circular shift, which are identical.

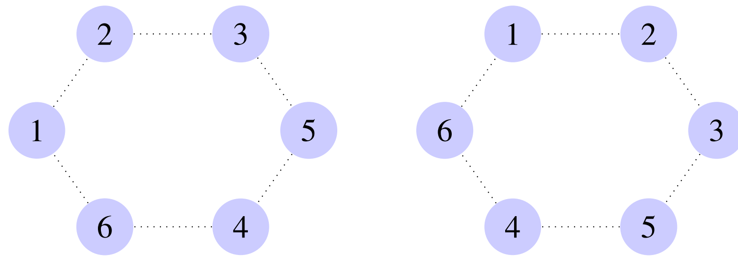

Figure 2.1: A circular permutation $\pi_c = (123546)_c$ (left) and $(S(\pi))_c = \pi_c$ (right).

Remark 2.3. When considering pattern containment in circular permutations, it is important and convenient to note that an occurrence of τ in π_c is the same as a linear occurrence of $S^i(\tau)$ in π for some i.

Definition 2.4. Let P be a set of permutations. We say that π is a *circular P-superpattern* if there is an occurrence of τ in π_c for every $\tau \in P$. Further, let $sp_c(P)$ be the length of the shortest circular P-superpattern.

In the case that P consists of all (regular) permutations of length k, similar to the traditional notations, we use terms circular k-superpattern and $sp_c(k)$.

Definition 2.5. For permutations π and τ (of length k), we define $f_c(\pi,\tau)$ to be the number of occurrences of τ in π wrapped around a circle, i.e.

$$f_c(\pi,\tau) = f(\pi,\tau) + f(\pi,S(\tau)) + f(\pi,S^2(\tau)) + \cdots + f(\pi,S^{k-1}(\tau)).$$

Likewise,

$$g_c(n,\tau) = \max\{f_c(\sigma,\tau) : \sigma \text{ is a permutation of length } n\}.$$

If π is of length n and $f_c(\pi,\tau) = g_c(n,\tau)$, then we say that π is *circular τ-optimal*.

Definition 2.6. Let τ be a permutation of length k. The *circular packing density* of τ, denoted $\delta_c(\tau)$, is defined by

$$\delta_c(\tau) = \lim_{n\to\infty} \frac{g_c(n,\tau)}{\binom{n}{k}}.$$

A circular permutation allows one to travel clockwise along the permutation with any choice of starting point, stopping our travel before we reach our starting point, i.e. we travel within one revolution of the circle. It is also interesting to explore the scenario when we can 'wrap' around the circle more than once. For instance, the pattern 321 occurs as 654 in the circular permutation in Figure 2.1, but not the pattern 4321. If we are allowed two revolutions, then we have 6541 as an occurrence of the pattern 4321.

Definition 2.7. Let π be a permutation of length n, and create a word $\pi_{(R)}$ by laying out R copies of π in a line, i.e.

$$\pi_{(R)} = \underbrace{\pi_1\pi_2\cdots\pi_n}_{R\text{ times}}.$$

Select a subsequence of distinct entries of $(\pi_{(R)})_c$, i.e. wrap $\pi_{(R)}$ clockwise around a circle and select a subsequence so that no two entries have the same value. Such a subsequence will be called an *R-revolution subsequence* (or just *R-rev* subsequence) of π_c.

Take $\pi = 1234$ for example, $(\pi_{(2)})_c$ is simply 12341234 wrapped around a circle. Then 421 is a 2-rev subsequence of π_c, but is clearly not a 1-rev subsequence of π_c.

Definition 2.8. Let π and τ be permutations of length n and k, respectively, with $n \geq k$, and let R be a positive integer. We say that there is an *R-rev occurrence* of τ in π if there is an R-rev subsequence of entries of π_c which is order isomorphic to τ. We may then analogously define *R-rev P-superpattern*, *R-rev τ-optimal*, and *R-rev packing density*. Further, let $sp_R(P)$ be the length of the shortest R-rev P superpattern. It is easy to see that $sp_c = sp_1$, $\delta_c = \delta_1$, etc.

2.3 Circular superpatterns

We start with some simple observations which bound $sp_c(k)$.

Theorem 2.9. *Given a positive integer k, we have*

$$sp_c(k) \geq g(k)\frac{k^2}{e^2},$$

where $g(k) \to 1$ *as* $k \to \infty$.

Proof. Let $n = sp_c(k)$. Note that the circular shift partitions the set of permutations of length k into $(k-1)!$ equivalence classes, each of size k. Any circular k-superpattern must contain at least one representative of each of these equivalence classes. Hence, the number of subsequences of length k in a circular k-superpattern must be at least $(k-1)!$. Then,

$$\binom{n}{k} \geq (k-1)!.$$

Notice that $\frac{n^k}{k!} \geq \binom{n}{k}$ and, by Stirling's Approximation, $k! \geq \sqrt{2\pi k}\frac{k^k}{e^k}$. So,

$$\begin{aligned} n^k &\geq k!(k-1)! \\ n &\geq \sqrt[k]{\frac{(k!)^2}{k}} \\ &\geq \sqrt[k]{2\pi \cdot \frac{k^{2k}}{e^{2k}}} \\ &= (2\pi)^{1/k}\frac{k^2}{e^2}. \end{aligned}$$

Letting $g(k) = (2\pi)^{1/k}$, we clearly see that $g(k) \to 1$ as $k \to \infty$, which proves our result. $\square$

To state our next observation we first recall that the *direct sum* $\pi \oplus \pi'$ of two permutations $\pi = \pi_1 \ldots \pi_\ell$ and $\pi' = \pi'_1 \ldots \pi'_m$ is defined as

$$\pi_1 \ldots \pi_\ell(\pi'_1 + \ell) \ldots (\pi'_m + \ell).$$

Theorem 2.10. *For any k-superpattern* π*,* $(\pi \oplus 1)_c$ *is a circular* $(k+1)$*-superpattern. Consequently*

$$sp_c(k+1) \leq sp(k) + 1$$

for any $k \geq 1$.

Proof. Suppose $\pi = \pi_1 \ldots \pi_n$, then $\pi \oplus 1 = \pi_1 \ldots \pi_n(n+1)$. Given any permutation/pattern τ of length $k+1$, $S^i(\tau)$ is of the form $\tau_1 \ldots \tau_k(k+1)$ for some i. If $= \pi$ is a k-superpattern, some subsequence $\pi_{i_1}\dots\pi_{i_k}$ of π is order isomorphic to $\tau_1\dots\tau_k$, then,

$$\pi\oplus 1=\pi_1\dots\pi_n(n+1)$$

contains the subsequence

$$\pi_{i_1}\dots\pi_{i_k}(n+1)$$

that is order isomorphic to

$$\tau_1\dots\tau_k(k+1)=S^i(\tau).$$

Thus, $(\pi\oplus 1)_c$ contains τ as a pattern, implying that $(\pi\oplus 1)_c$ is a circular $(k+1)$-superpattern.

□

When allowing extra revolutions, the following result states that every sufficiently long circular permutation, with enough revolutions allowed, contains every pattern.

Theorem 2.11. *Given any $k\geq 2$ and $R\geq k-1$, any permutation of length at least k is an R-rev k-superpattern.*

Proof. It is sufficient to show the statement for $R=k-1$. We proceed by induction on k. The initial case is trivial.

Assume now, that any permutation of length at least k is a $(k-1)$-rev k-superpattern and let π be a permutation of length at least $k+1$.

Given any pattern τ of length $k+1$, we show that there is a k-rev occurrence of some circular shift of τ in π. Note that some circular shift $S^i(\tau)$ is of the form $\tau_1\dots\tau_k(k+1)$. Let π' be the permutation of length at least k obtained by removing the largest entry n of π. Then π' is a $(k-1)$-rev k-superpattern by the induction hypothesis. That is, some subsequence of distinct entries of

$$\underbrace{\pi'\dots\pi'}_{k-1\text{ times}}$$

is order isomorphic to $\tau_1 \ldots \tau_k$. This subsequence, with n appended at the end, is order isomorphic to $\tau_1 \ldots \tau_k(k+1) = S^i(\tau)$. Such a sequence is a subsequence of distinct entries of

$$\underbrace{\pi \ldots \pi}_{k \text{ times}},$$

which is to say that there is a k-rev occurrence of $S^i(\tau)$ in π.

Hence, π is a k-rev $(k+1)$-superpattern. □

Remark 2.12. In some sense Theorem 2.11 is the best possible. For instance, let $\pi = 12\ldots k$ and $\tau = k\ldots 21$, indeed it requires $k-1$ revolutions for τ to occur as a subsequence of π_c.

Since a k-superpattern (regardless of the number of revolutions allowed) has to contain at least k entries, Theorem 2.11 immediately implies the following.

Corollary 2.13. *For any $k \geq 2$ and $R \geq k-1$, $sp_R(k) = k$.*

2.4 Packing density in circular permutations

For regular permutations, it is easy to see that the monotonic patterns are the only ones with packing density 1, with corresponding optimal permutations being monotonic as well. It is interesting to see that this is also the case for circular permutations. For brevity, we will say that a circular pattern τ_c is monotonic if $S^i(\tau)$ is monotonic for some i.

Theorem 2.14. *The circular packing density of a pattern τ is at most 1, with equality if and only if τ_c is monotonic. In this case, the circular τ-optimal permutation must also be monotonic. Further, if τ_c is not monotonic, then $\delta_c(\tau) \leq \frac{2}{3}$.*

Proof. It is obvious that $\delta_c(\tau) \leq 1$ for any pattern τ. In the case that τ is a monotonic pattern (say $12\ldots k$) and π is a monotonic permutation $12\ldots n$ of length n, it is easy to see that

$$f_c(\pi,\tau) = \binom{n}{k} = g_c(n,\tau)$$

and hence $\delta_c(\tau) = 1$. If π_c is not monotonic, then some subsequence of length k is not order isomorphic to τ, hence π_c would not be τ-optimal. The case of τ and π being decreasing is similar.

Suppose, then, that τ_c is not monotonic. We will show that the number of subsequences of length k which are not occurrences of τ in any τ-optimal circular permutation is relatively large with respect to n, i.e. the probability that a randomly selected subsequence of length k is not an occurrence of τ is non-zero as $n \to \infty$.

In τ_c, there exists a smallest i such that i and $i-1$ are not neighbors (where 0 is equivalent to k). Without loss of generality, suppose that i is clockwise of $i-1$. Let $n >> k$ and let π be a circular τ-optimal permutation of length n.

Suppose that T is an occurrence of τ in π with T_r and T_s playing the roles of $i-1$ and i, respectively. Notice that if $T_r < \gamma < T_s$, then γ cannot play any role in T.

Since either $T_r = T_s - 1$ or $T_r < T_s - 1$, consider two cases:

- Let P be the set of τ-occurrences, T, where $T_r = T_s - 1$.

  Clearly, $|P| \leq \binom{n}{k-1}$. Indeed, once we select x to play the role of $i-1$ in such a τ-occurrence, the role of i will automatically be assigned to $x+1$.

- Let Q be the set of τ-occurrences, T, where $T_r < T_s - 1$.

  Let $\gamma = \lfloor \frac{T_r+T_s}{2} \rfloor$, and recall that γ plays no role in T. Let T' be obtained from T by inserting γ into T and removing T_s and let T'' be obtained by inserting γ into T and removing T_r. We will show that at most one of T' and T'' is a τ-occurrence.

  To see this, let T' be an occurrence of τ. recall that $T_r < \gamma < T_s$ with T_r playing the role of $i-1$ and T_s playing the role of i. Hence, the only possible role that γ can play in T' is the role of i. If T' is an occurrence of τ, we may further say that γ occupies the same position in T' as T_s did in T.

Hence, the number of entries of T between T_r and T_s (clockwise) is exactly the same as the number of entries between T_r and γ in T', and the number of entries of T between T_s and T_r in T is exactly the same as the number of entries between γ and T_r in T'. Thus, if γ is clockwise (resp. counter-clockwise) of T_s, then there are no entries of T between T_s and γ (resp. γ and T_s). This means in T'', γ and T_s are neighbors. Since γ is now playing the role of $i-1$ (because T_r was removed) and T_s is playing the role of i, and i and $i-1$ are not neighbors in τ_c by assumption, T'' cannot be an occurrence of τ.

Thus, for every occurrence, T, of τ in Q, at least one of T' and T'' is not an occurrence of τ. Hence, the number of subsequences of π which are not occurrences of τ is at least $\frac{1}{2}|Q|$. This is because it is immediately obvious whether the non-occurrence is a T' or a T'' since T' and T'' are identical to T except for the placement of one entry. If the non-occurrence is a T' then T_r and γ are known, and there are only two possible values of T_s which will give us the same γ; similarly, if it was a T'' then T_s and γ are known, and there are only two possible values of T_r which will give the same γ). Therefore, each non-occurrence which resulted as a T' or a T'' is counted at most twice by members of Q.

Summarizing the above, we have

$$\begin{aligned} |P|+|Q|+\frac{1}{2}|Q| &\leq \binom{n}{k} \\ \frac{\frac{3}{2}|Q|}{\binom{n}{k}} &\leq 1 \\ \frac{|Q|}{\binom{n}{k}} &\leq \frac{2}{3}. \end{aligned}$$

Also notice that $\frac{|P|}{\binom{n}{k}} \leq \frac{\binom{n}{k-1}}{\binom{n}{k}} \to 0$ as $n \to \infty$. Therefore,

$$\delta_c(\tau) = \lim_{n\to\infty} \frac{g(n,\tau)}{\binom{n}{k}} \leq \lim_{n\to\infty} \frac{|P|+|Q|}{\binom{n}{k}} \leq \frac{2}{3},$$

which proves the result. □

Remark 2.15. As a consequence of Proposition 2.14, noticing that all length 3 patterns are equivalent to a monotonic pattern through circular shift, we see that all length 3 patterns have circular packing density 1. This is certainly not the case for the traditional packing densities.

On the other hand, it is easy to see the following relationship between the packing density of a pattern τ and the circular packing density of τ.

Proposition 2.16. *Let τ be a pattern of any length. Then, $\delta_c(\tau) \geq \delta(S^i(\tau))$ for all i.*

Proof. Suppose that τ is of length k, and recall that

$$g_c(n,\tau) = \max_{\pi\in S_n}\{f(\pi,\tau) + f(\pi,S(\tau)) + \cdots + S^{k-1}(\tau)\}.$$

Let i be such that $g(n,S^i(\tau)) \geq g(n,S^j(\tau))$ for any $0 \leq j \leq k-1$. Then $g_c(n,\tau) \geq g(n,S^i(\tau))$. Thus, $\delta_c(\tau) \geq \delta(S^i(\tau))$. □

When multiple revolutions are allowed, the situation changes dramatically. For the sake of our argument, we first introduce another notation.

Definition 2.17. For permutations π and τ, we define $f_R(\pi,\tau)$ to be the number of R-rev occurrences of τ in π. Likewise,

$$g_R(n,\tau) = \max\{f_R(\sigma,\tau) : \sigma \text{ is a permutation of length } n\}.$$

The following is an analogue of Theorem 2.11 in terms of packing density.

Proposition 2.18. *Given $k \geq 2$ and $R \geq k-1$, every permutation π of length at least k is R-rev τ-optimal for any pattern τ of length k.*

Proof. We only consider the case $R = k-1$. Take any k-subsequence of π (of length $n \geq k$), say ϕ. Theorem 2.11 claims that ϕ is a $(k-1)$-rev k-superpattern, implying that ϕ contains τ as a pattern when $k-1$ revolutions are allowed. Hence, ϕ is an occurrence of τ.

Thus, $f_R(\pi,\tau) = \binom{n}{k} = g_R(n,\tau)$, showing that π is $(k-1)$-rev τ-optimal. □

2.5 Concluding remarks

In this chapter, we generalized the study of pattern containment to circular permutations where a permutation or pattern is allowed to wrap around a circle one or more times. We provided some basic observations on the minimum length of a circular superpattern. Furthermore, we showed that any long enough permutation, when enough revolutions are allowed, is a superpattern. This also implies that the classic pattern avoidance problem should not be considered for circular permutations when too many revolutions are allowed. While there has been some study on pattern avoidance in circular permutations [6], it is not clear how it plays out when multiple (but not too many) revolutions are allowed. In terms of packing patterns in a circular permutation, all patterns equivalent to monotonic patterns still have the highest packing density of 1. By Proposition 2.16, we have that $\delta_c(\tau) \geq \delta(S^i(\tau))$ for any pattern τ and for any i. It is still unknown whether there exists a pattern τ for which this inequality is strict for all i, however, we make the following conjecture:

Conjecture 2.19. *If τ is not a layered pattern, then $\delta_c(\tau) > \delta(\tau)$.*

Using brute force, we have found for the pattern $\tau = 3142$, that $g_c(n,\tau) > g(n,\tau)$ for $6 \leq n \leq 10$. We have outlined our results in the Table 2.1. However, for all layered patterns of length 3 or 4, we have found no difference between the circular and linear packing densities. While this is not proof, it does lend some support to Conjecture 2.19.

Table 2.1: Optimal permutations for 3142 and its circular shifts

Length		n=10	n=9	n=8
Circular 3142-optimal permutations		6 7 2 8 3 9 4 5 1 10	4 5 6 1 7 2 8 3 9	5 6 2 7 3 4 1 8
			5 6 7 2 8 3 4 1 9	
			6 7 2 8 3 4 5 1 9	
	$g_c(n, 3142)$	73	45	27
Linear permutations	3142-optimal	6 7 1 8 2 9 3 10 4 5	5 6 1 7 2 8 9 3 4	6 4 1 7 2 8 5 3
		6 7 1 8 2 9 3 10 5 4	5 6 1 7 2 8 9 4 3	
		7 5 1 8 2 9 3 10 6 4	6 4 1 7 2 8 9 5 3	
		7 6 1 8 2 9 3 10 4 5	6 5 1 7 2 8 9 3 4	
		7 6 1 8 2 9 3 10 5 4	6 5 1 7 2 8 9 4 3	
		8 6 4 1 2 9 10 7 5 3	6 7 4 1 2 8 9 5 3	
		8 6 4 2 1 9 10 7 5 3	6 7 4 2 1 8 9 5 3	
		8 6 4 1 9 2 10 7 5 3	7 6 4 1 2 8 9 5 3	
		8 6 4 1 2 10 9 7 5 3	7 6 4 2 1 8 9 5 3	
		8 6 4 2 1 10 9 7 5 3	6 7 1 2 8 3 9 4 5	
			6 7 2 1 8 3 9 4 5	
			6 7 1 2 8 3 9 5 4	
			6 7 2 1 8 3 9 5 4	
			6 7 4 1 8 2 9 5 3	
			7 5 1 2 8 3 9 6 4	
			7 5 2 1 8 3 9 6 4	
			7 6 1 2 8 3 9 4 5	
			7 6 2 1 8 3 9 4 5	
			7 6 1 2 8 3 9 5 4	
			7 6 2 1 8 3 9 5 4	
			7 6 4 1 8 2 9 5 3	
			5 6 1 7 2 9 8 3 4	
			5 6 1 7 2 9 8 4 3	
			6 4 1 7 2 9 8 5 3	

			6 5 1 7 2 9 8 3 4	
			6 5 1 7 2 9 8 4 3	
			6 7 4 1 2 9 8 5 3	
			6 7 4 2 1 9 8 5 3	
			7 6 4 1 2 9 8 5 3	
			7 6 4 2 1 9 8 5 3	
			7 5 1 2 8 9 6 3 4	
			7 5 2 1 8 9 6 3 4	
			7 5 1 2 8 9 6 4 3	
			7 5 2 1 8 9 6 4 3	
			7 5 1 8 2 9 6 3 4	
			7 5 1 8 2 9 6 4 3	
			7 5 1 2 9 8 6 3 4	
			7 5 2 1 9 8 6 3 4	
			7 5 1 2 9 8 6 4 3	
			7 5 2 1 9 8 6 4 3	
	$g(n,3142)$	41	26	17
Linear permutations	2314-optimal	3 4 5 6 7 2 8 1 9 10	3 4 5 6 2 7 1 8 9	3 4 5 6 1 2 7 8
		4 5 6 7 2 3 8 1 9 10	3 4 5 6 7 1 2 8 9	3 4 5 6 2 1 7 8
		4 5 6 7 2 8 3 1 9 10	3 4 5 6 7 2 1 8 9	3 4 5 6 1 2 8 7
		4 5 6 7 3 8 1 2 9 10	3 4 5 6 2 7 1 9 8	3 4 5 6 2 1 8 7
		4 5 6 7 3 8 2 1 9 10	3 4 5 6 7 1 2 9 8	
		4 5 6 7 8 2 3 1 9 10	3 4 5 6 7 2 1 9 8	
		3 4 5 6 7 2 8 1 10 9		
		4 5 6 7 2 3 8 1 10 9		
		4 5 6 7 2 8 3 1 10 9		
		4 5 6 7 3 8 1 2 10 9		
		4 5 6 7 3 8 2 1 10 9		
		4 5 6 7 8 2 3 1 10 9		
	$g(n,2314)$	62	40	24
Linear permutations	4231-optimal	10 8 9 4 5 6 7 2 3 1	8 9 4 5 6 7 2 3 1	7 8 3 4 5 6 1 2

			9 7 8 3 4 5 6 1 2	7 8 3 4 5 6 2 1
			9 7 8 3 4 5 6 2 1	8 5 6 7 2 3 4 1
			9 8 4 5 6 7 2 3 1	8 7 3 4 5 6 1 2
				8 7 3 4 5 6 2 1
	$g(n,4231)$	68	42	24
Linear permutations	1423-optimal	1 2 9 10 3 8 4 5 6 7	1 2 8 9 3 4 5 6 7	1 2 7 8 3 4 5 6
		2 1 9 10 3 8 4 5 6 7	2 1 8 9 3 4 5 6 7	2 1 7 8 3 4 5 6
		1 2 10 3 8 9 4 5 6 7	1 2 9 3 8 4 5 6 7	1 2 8 7 3 4 5 6
		2 1 10 3 8 9 4 5 6 7	2 1 9 3 8 4 5 6 7	2 1 8 7 3 4 5 6
		1 2 10 8 3 9 4 5 6 7	1 2 9 8 3 4 5 6 7	
		2 1 10 8 3 9 4 5 6 7	2 1 9 8 3 4 5 6 7	
		1 2 10 3 9 4 5 6 7 8		
		2 1 10 3 9 4 5 6 7 8		
		1 2 10 8 9 3 4 5 6 7		
		2 1 10 8 9 3 4 5 6 7		
		1 2 10 9 3 8 4 5 6 7		
		2 1 10 9 3 8 4 5 6 7		
	$g(n,1423)$	62	40	24

Analogous to the study of superpatterns, we also showed that enough number of revolutions will automatically achieve the maximum packing density. It would also be interesting to continue this study for limited number of revolutions.

In Proposition 2.18, we showed that if $R \geq k-1$ then every permutation π of length at least k is R-rev τ-optimal for any pattern τ of length k, without specifically stating the packing density. Let $a(k,R)$ be the number of R-rev subsequences of length k in a permutation of length k. Clearly the number of these arrangements does not depend on the permutation itself. Then, the total number of R-rev subsequences of π of length k is just $\binom{n}{k} \cdot a(k,R)$, and as a consequence

$$\delta_R(\tau) = \lim_{n\to\infty} \frac{g_R(n,\tau)}{\binom{n}{k} \cdot a(k,R)} = \lim_{n\to\infty} \frac{\binom{n}{k}}{\binom{n}{k} \cdot a(k,R)} = \frac{1}{a(k,R)}$$

for any τ of length $k \leq R+1$. It seems to be worthwhile to further study the combinatorics related to $a(k,R)$, which appears to be interesting in its own right.

CHAPTER 3

COLORED PERMUTATIONS

3.1 SUPERPATTERNS

We will now turn our attention to the topic of colored permutations. By this, we mean that each entry will be assigned a color. When considering pattern containment, both the numerical ordering and the colors will need to match.

For example, let $p = 1_r 3_b 2_r$. Here, 1 and 2 are red, and 3 is blue. So, in $1_r 2_b 6_b 5_r 4_r 3_b$, p only occurs twice, as $1_r 6_b 5_r$ and $1_r 6_b 4_r$.

Some results have been obtained in the linear case, and we will attempt to extend those results to the circular case. We will also note some differences between them. We begin with some definitions.

- $S(k,m)$ will be used to denote the set of all permutations of length k in m colors.
- $NMS(k,m)$ is the set of nonmonochromatic m-colored patterns of length k, and $MS(k,m)$ is the set of all monochromatic m-colored patterns of length k.
- We define $sp(k,m)$ to be the length of the shortest $S(k,m)$-superpattern.
- Similarly, we define $nmsp(k,m)$ to be the length of the shortest $NMS(k,m)$-superpattern and $msp(k,m)$ to be the length of the shortest $MS(k,m)$-superpattern.

We now review some results in the linear case. The first of which provides a link between colored superpatterns and non-colored superpatterns.

Theorem 3.1 ([9]). *For any positive integers k and m, we have*

$$sp(k,m) = m \cdot sp(k).$$

Because of this result, we can bound $sp(k,m)$ using the bounds for $sp(k)$. Another interesting relationship between these values is the following.

Theorem 3.2 ([9]).

$$msp(k,m) = m \cdot sp(k) = sp(k,m) \geq nmsp(k,m)$$

This statement is quite interesting considering that $NMS(k,m)$ may be much larger than $MS(k,m)$, as we will see later when we find their cardinalities. We have another result which allows us to bound $nmsp(k,m)$ from below.

Theorem 3.3 ([9]). *For any positive integers $k \geq 2$ and m, we have*

$$msp(k-1,m) \leq nmsp(k,m) \leq msp(k,m).$$

Now we consider the analogous questions in the circular case. We first define the analogues of $msp(k,m)$ and $nmsp(k,m)$.

- $sp_c(k,m)$ is defined to be the length of the shortest circular $S(k,m)$-superpattern.
- $msp_c(k,m)$ is defined to be the length of the shortest circular $MS(k,m)$-superpattern.
- $nmsp_c(k,m)$ is defined to be the length of the shortest circular $NMS(k,m)$-superpattern.

Theorem 3.4. *For any positive integers k and m, we have that*

$$sp_c(k,m) = m \cdot sp_c(k).$$

Proof. Let p' be a circular $S(k,m)$-superpattern. Let p'_i be the longest monochromatic subsequence in p' in color i. We can see that p' is then a circular k-superpattern. Thus, $|p'_i| \geq sp_c(k)$ for any i. Also,

$$|p'| = \sum_{i=1}^{m} |p'_i| \geq m \cdot sp_c(k).$$

Now, let p be a circular permutation of length $sp_c(k)$ that contains all noncolored patterns of length k. Construct circular m-colored permutation p'' by replacing each $1 \leq j \leq sp(k)$ in p by the sequence

$$s_j = [m(j-1)+1]_1[m(j-1)+2]_2 \ldots [m(j-1)+m]_m.$$

Note that $|p''| = m \cdot |p| = m \cdot sp_c(k)$. For any pattern in $S(k,m)$, the noncolored version is contained in p, and the colored pattern can be found in p'' by choosing corresponding digits in s_j with the required color. Thus,

$$sp_c(k,m) \leq |p''| = m \cdot sp_c(k).$$

$\square$

This is a very nice result, as we can now use our results from Chapter 2 for $sp_c(k)$ to bound $sp_c(k,m)$. We have

$$sp_c(k,m) = m \cdot sp_c(k) \geq m \cdot g(k)\frac{k^2}{e^2}$$

where $g(k) \to 1$ as $k \to \infty$, and

$$sp_c(k,m) = m \cdot sp_c(k) \leq m(sp(k-1)+1) \leq m\left(\frac{k(k-1)}{2}+1\right).$$

Putting these together, we have

$$m \cdot g(k)\frac{k^2}{e^2} \leq sp_c(k,m) \leq m\left(\frac{k(k-1)}{2}+1\right)$$

where $g(k) \to 1$ as $k \to \infty$.

It is a very interesting fact that when considering the color of entries, the bounds for the size of the smallest circular superpattern grows at the same rate as those of the non-colored case. Next, we will consider restricting the superpattern to only monochromatic and nonmonochromatic patterns. First, we note the sizes of $MS(k,m)$ and $NMS(k,m)$, as in [9].

For any $S(k,m)$-permutation, each entry can be any one of m colors. Since there are $k!$ permutations of length k, we have that

$$|S(k,m)| = m^k k!.$$

Also, since any $MS(k,m)$-permutation has all entries simultaneously one of m entries, it is easy to see that

$$|MS(k,m)| = m \cdot k!.$$

Since each colored permutation is either monochromatic or nonmonochromatic, then $S(k,m)$ is the disjoint union of $MS(k,m)$ and $NMS(k,m)$. So,

$$\begin{aligned}
|NMS(k,m)| &= |S(k,m)| - |MS(k,m)| \\
&= m^k k! - m \cdot k! \\
&= (m^k - m)k! \\
&= (m^{k-1} - 1)m \cdot k! \\
&= (m^{k-1} - 1)|MS(k,m)|.
\end{aligned}$$

We will use the cardinalities of these sets to provide a bound on $nmsp_c(k,m)$.

Proposition 3.5. *For positive integers k and m,*

$$nmsp_c(k,m) \geq m \cdot g(k,m)\frac{k^2}{e^2},$$

where $g(k,m) \to 1$ *as* $k \to \infty$.

Proof. Let $n = nmsp_c(k,m)$. Each equivalence class of permutations in $NMS(k,m)$ is of size k, since we may take $k-1$ circular shifts of any element and obtain an equivalent permutation. So, since an NMS-superpattern must include each at least one element from each equivalence class of $NMS(k,m)$, then

$$\binom{n}{k} \geq \frac{|NMS(k,m)|}{k} = \frac{(m^k - m)k!}{k} = (m^k - m)(k-1)!.$$

So, by the fact that $\frac{n^k}{k!} \geq \binom{n}{k}$ and by Stirling's approximation, $k! \geq \sqrt{2\pi k}\frac{k^k}{e^k}$, we have that

$$\begin{aligned} n^k &\geq (m^k - m)k!(k-1)!. \\ n &\geq \sqrt[k]{(m^k - m)\frac{(k!)^2}{k}} \\ &\geq \sqrt[k]{(m^k - m)2\pi\frac{k^{2k}}{e^{2k}}} \\ &= \sqrt[k]{m^k(1-m^{-1})2\pi}\frac{k^2}{e^2} \\ &= m\sqrt[k]{(1-m^{-1})2\pi}\frac{k^2}{e^2} \\ &= m \cdot g(k,m)\frac{k^2}{e^2}, \end{aligned}$$

with $g(k,m) \to 1$ as $k \to \infty$. □

This gives us a similar lower bound as that for $sp_c(k,m)$. In order to bound $nmsp_c(k,m)$ from above, we first note that $nmsp_c(k,m) \leq sp_c(k,m)$. From earlier, we know that $sp_c(k,m) = m \cdot sp_c(k)$. Now, consider a circular $MS(k,m)$-superpattern. It must have m copies of a circular k-superpattern, one for each color. Then, $m \cdot sp_c(k) = msp(k,m)$. Putting this all together, we have

$$nmsp_c(k,m) \leq sp_c(k,m) = m \cdot sp_c(k) = msp_c(k,m).$$

This gives us an upper bound for $nmsp_c(k,m)$. It is very interesting that $nmsp_c(k,m)$ and $msp_c(k,m)$ are related in this way. Another interesting relation between them is the following proposition.

Proposition 3.6. *For any positive integers $k \geq 2$ and m, we have*

$$msp_c(k-1,m) \leq nmsp_c(k,m) \leq msp_c(k,m).$$

Proof. We already have the second inequality from above. So, it remains to show the first one. Let q be an m-colored pattern of length k with the last entry being color j, and the others being color i. So, q must be contained in all circular $NMS(k,m)$-superpatterns. Now, note that the first $k-1$ entries form a monochromatic pattern in color i. Since we may choose any color for these $k-1$ entries and we still must have q contained in a circular NMS-superpattern, then it follows that all length $k-1$ monochromatic patterns are also contained in any circular NMS-superpattern. Thus, $msp_c(k-1,m) \leq nmsp_c(k,m)$. □

3.2 Packing Densities

Now we will discuss optimal permutations for packing colored patterns. We start with a definition.

Definition 3.7. In a colored permutation π, a colored block is a maximal monochromatic segment $\pi_i^{(a)}$ in which every entry in this segment has color a and every entry not in this segment is either larger or smaller that each entry in $\pi_i^{(a)}$.

For example, the permutation $\pi = 1_r2_r6_b5_b3_b4_r$ has four colored blocks. From left to right, they are $\pi_1^{(r)} = 1_r2_r$, $\pi_2^{(b)} = 6_b5_b$, $\pi_3^{(b)} = 3_b$, $\pi_4^{(r)} = 4_r$.

When comparing the numerical values between different blocks, we say that $\pi_i^{(r)} < \pi_j^{(b)}$ when all entries of $\pi_i^{(r)}$ are less than all entries of $\pi_j^{(b)}$.

The linear case when there are 2 or 3 colored blocks has been studied. We will give a brief review over these results.

In the case of having only 2 colored blocks, we can see that a pattern must be of the form $\pi = \pi_1^{(r)}\pi_2^{(b)}$. Note also that for the purpose of packing patterns, we may assume that $\pi_1^{(r)} < \pi_2^{(b)}$.

Theorem 3.8. *For a pattern p with two blocks of the form rb with $r < b$, there is an optimal length n permutation π of the form RB with $R < B$.*

We will omit the proof, but when we extend this result to the circular case, the proof is similar. Essentially, taking a permutation and sliding all of its red blocks to the left preserves each instance of p.

Now, when we consider patterns with 3 colored blocks, we have two cases to consider. One is when we have 3 colors, so our pattern will be of the form rbg. The other is when we still have only 2 colors, so our pattern will be of the form rbb, brb, or bbr.

Theorem 3.9. *Given a pattern p of the form rbg, there is an optimal permutation π of the form RBG with the same numerical ordering as rbg.*

The proof goes similarly to that of the previous theorem. Essentially, we may move all the red blocks to the front, then blue to the middle, then green to the end.

Now, for 3 colored blocks in 2 colors, we note that the rb_1b_2 case is the same as the b_1b_2r case since reversing does not affect optimality. So, there are two cases: rb_1b_2 and b_1rb_2. As expected, both cases yield results similar to the previous two theorems.

For 3 colored blocks in 2 colors, there is an optimal permutation that has the same arrangement of blocks with the same numerical ordering.

Unfortunately, the proofs require some more work as we consider more blocks. The cases with four or more blocks are still open.

We will now look at the analogues of these results in the circular case. If there are only 2 colored blocks, then we note that there is only one case, where p is of the form rb, $r < b$.

Theorem 3.10. *For a pattern p with two colored blocks of the form rb with $r < b$, there is an optimal circular permutation π of the form RB with $R < B$.*

Proof. Let π be a p-optimal permutation of length n with colored block $\pi_1\pi_2\dots\pi_k$. We can assume without loss of generality that π_1 is red. Otherwise, we could take some number of circular shifts of π until it is.

Now, if we take all the red blocks $\pi_{r_1}\pi_{r_2}\dots\pi_{r_s}$ and blue blocks $\pi_{b_1}\pi_{b_2}\dots\pi_{b_t}$, and we form a new circular permutation $\pi' = \pi_{r_1}\dots\pi_{r_s}\pi_{b_1}\dots\pi_{b_t}$, then we can see that any occurrence of p in π is also in π'. So, $f_c(\pi', p) \geq f_c(\pi, p)$.

Now, since p is of the form rb with $r < b$, then our optimal permutation π' must have that each red entry is less than each blue entry. Then, each red block together form one block, and each blue block together form one blue block. So, π' is of the form RB with $R < B$. □

Our result in the case with 2 colored blocks is the same as that of the linear case. However, when considering patterns with 3 blocks in 2 colors, we note a key difference. That is, for p of the form rb_1b_2, we may take a circular shift and obtain b_2rb_1. We also note that r must be numerically between b_1 and b_2, otherwise, b_1 and b_2 forms one blue block. So, the only case to consider is rb_1b_2, and we may assume $b_1 < r < b_2$.

Theorem 3.11. *For a pattern p with three colored blocks of the form rb_1b_2, there is an optimal circular permutation π that is also of the form RB_1B_2 (or some circular shift thereof). Also, the numerical ordering of the colored blocks in π is the same as that of the colored blocks in p.*

Proof. If r is not numerically between b_1 and b_2, then b_1 and b_2 form one colored block. Then, we would be in the same case as the previous theorem. So, we must have that r is between b_1 and b_2. Without loss of generality, we assume that $b_1 < r < b_2$.

Let π be a p-optimal circular permutation of length n with colored blocks $\pi_1\pi_2\ldots\pi_k$. First, we will show that we can put all the blue blocks in increasing order. Let π_r be the set of red blocks of π. Fix a red pattern R in π. Let $\pi_{b_{<R}}$ be the set of all blue blocks less than R, and let $\pi_{b_{>R}}$ be the set of all blue blocks greater than R. So, any occurrence of an rb_1b_2 pattern in these blocks must have the b_1 pattern contained in $\pi_{b_{<R}}$, and the b_2 pattern must be contained in $\pi_{b_{>R}}$. Then, the maximum contribution of this pattern R to $f_c(\pi, p)$ is

$$f(\pi_{b_{<R}}, b_1) \cdot f(\pi_{b_{>R}}, b_2).$$

This can be achieved by arranging the blue blocks into increasing order. So, let $\pi' = \pi_r\pi_{b_1}\dots\pi_{b_t}$, with $\pi_{b_i} < \pi_{b_{i+1}}$.

Next, we will show that π_r must be one block. Consider a 'cut' in the blue blocks. Let $\pi_{b_{\geq j}}$ be all the blue blocks greater than and including some π_{b_j}, and let $\pi_{b_{<j}}$ be all those less than π_{b_j}. Then, there must exist some j_0 that maximizes the occurrences of b_1 in $\pi_{b_{<j}}$ and b_2 in $\pi_{b_{\geq j}}$. In other words,

$$f(\pi_{b_{<j_0}}, b_1)\cdot f(\pi_{b_{\geq j_0}}, b_2) \geq f(\pi_{b_{<j}}, b_1)\cdot f(\pi_{b_{\geq j}}, b_2),$$

for any $1 < j \leq t$. So,

$$f_c(\pi', p) \leq f(\pi_r, r)\cdot f(\pi_{b_{<j_0}}, b_1)\cdot f(\pi_{b_{\geq j_0}}, b_2).$$

We obtain equality when $\pi_{b_{<j_0}} < \pi_r < \pi_{b_{\geq j_0}}$, which means they are each colored blocks. Then, we have that $\pi' = \pi_r\pi_{b_{<j_0}}\pi_{b_{\geq j_0}}$, which is of the form RB_1B_2 with $B_1 < B_2$. □

For the case when we have 3 colored blocks in 3 colors, we note that we only have one numerical ordering to consider. To see this, assume that $r < b < g$. Then, we may take circular shifts of rbg to obtain grb and bgr. We can reverse each of those to obtain gbr, brg, and rgb. These give us every numerical ordering possible.

Theorem 3.12. *For a pattern p of the form rbg, there is an optimal circular permutation π of the form RBG. Also, the numerical ordering of the blocks of π is the same as that of the blocks of p.*

Proof. Let π be a p-optimal circular permutation of length n with colored blocks $\pi_1\pi_2\dots\pi_k$. For the reasons noted above, we first assume, without loss of generality, that $r < b < g$. Let π_R, π_B, and π_G be the maximum length monochromatic subsequence of color red, blue, and green in π, respectively. Let $\pi' = \pi_r\pi_b\pi_g$. Then,

$$f(\pi', p) \leq f(\pi_r, r)\cdot f(\pi_b, b)\cdot f(\pi_g, g),$$

with equality when $\pi_r < \pi_b < \pi_g$. □

The preceding results raise a good question. As asked in [10], is it true that the optimal colored permutation with respect to a given colored pattern always shares the same number and arrangement of colored blocks as those of the pattern?

While the answer is unknown in the linear case, we can show an explicit example where we can find an optimal circular permutation that is not of the same form as the pattern.

Let $p = 1_r 2_b$. Then, $\pi = (1_r 3_b 2_r 4_b)_c$ is an optimal length-4 circular permutation for p. Of course, this is not a counterexample for our theorems in this section. There still exists a p-optimal length-4 permutations of the form RB, $R < B$, which is $\pi' = (1_r 2_r 3_b 4_b)$. The key fact about our proofs in this section is that, while we were not losing optimality, we were not necessarily increasing optimality either, as this example showed.

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