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ADRC Based Control of Nonlinear Dynamical System with Multiple Sources of Disturbance and Multiple Inputs

Chan Mi Park

ADRC BASED CONTROL OF NONLINEAR DYNAMICAL SYSTEM WITH

MULTIPLE SOURCES OF DISTURBANCE AND MULTIPLE INPUTS

by

CHAN MI PARK

(Under the Direction of Yan Wu)

ABSTRACT

In this thesis, we study the stability of Active Disturbance Rejection Control (ADRC) applied to controlling the Lorenz system. The Lorenz system is a nonlinear dynamical system that we attempt to control. In fact, the system is used to model convection flow such as that found in thermosyphons, electric circuits, and lasers. We are stabilizing the Lorenz system along with a few disturbances. Thus, to stabilize this chaotic system, a robust controller is required. The ADRC system is known as as effective method to stabilize a dynamical system. With the help of the Extended State Observer (ESO), the system can be stabilized with the least information about the disturbances. In particular, when the model of the plant is given the system converges asymptotically. Since most physical plants are highly uncertain in the real world, we also establish a second case. When the dynamics of the plant is largely unknown, the errors of the ADRC Controlled Lorenz system is bounded by the observer gains and feedback control gains, which is Lyapunov stable.

INDEX WORDS: Lorenz system, ADRC, ESO, Control theory, MIMO, SISO

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CHAN MI PARK

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CHAN MI PARK

Major Professor: Yan Wu
Committee: James Braselton
                   Adel El Shahat

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DEDICATION

This thesis is dedicated to my parents. My father, Sang Young Park, has always encouraged me to seek my goal and also supported achieving it. My mother, Yun Sook Kwon, has helped me grow strongly in my spiritual development.

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CHAPTER 1

INTRODUCTION

A butterfly flapping its wings in Brazil can cause a tornado in Texas. We have heard this phrase before, and in fact this is some of the chaotic behavior we deal with in real life. It describes a small change to the initial state in a nonlinear system could lead to a significant variation in the state over time. In some applications, it is important to control these behaviors in certain way to mitigate the effects. The field of research that deals with managing the behavior of dynamical systems is called control theory. Dynamical systems are mathematical objects used to model physical phenomena whose state changes over time. See Chen, [6].

1.1 PRELIMINARIES

The focus of this paper is to control the chaotic behavior of dynamical systems using a field of research called control theory. A dynamical system is given by

$$\dot{x} = f(x(t),t),\ x \in \mathbb{R}^n,\ f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n.$$

The dot is the derivative with respect to time, where x is a function of time with an initial point x_0. Assume that the system is chaotic. Before we continue, we shall first give several important definitions.

Definition 1.1. *An $n \times n$ constant matrix A is a **Hurwitz matrix** if all its eigenvalues lie in the left half of the complex plane.*

One objective to stabilize a system could be that, we want our solution to be closer to the equilibrium point.

Definition 1.2. *Let $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$. Then a point $x^* \in \mathbb{R}^n$ is an **equilibrium point** of the system if $f(x^*) = 0$.*

Our goal for this thesis is to stabilize a given system. Thus, we should define the term stability.

Definition 1.3. *A solution $\bar{x}(t)$ of $\dot{x} = f(x,t)$ is **stable** if for each $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\delta = f(\varepsilon, t_0) > 0$ such that if $x(t)$ is a solution of $\dot{x} = f(x,t)$ and $|x(t_0) - \bar{x}(t_0)| < \delta$ then $|x(t) - \bar{x}(t)| < \varepsilon$ for all $t \geq t_0$.*

There are different types of stability for a system at the equilibrium point. One of them is called asymptotic stability.

Definition 1.4. *A dynamical system is said to be **asymptotically stable** about its equilibrium point x^*, if it is stable and there exists a $\delta > 0$, such that $||x - x^*|| < \delta$ implies $\lim_{t \to \infty} ||x(t) - x^*|| = 0$.*

It is not easy to show the stability of a system at a point through the definition. Another method, called the Lyapunov directed method is often used to determine asymptotic stability.

Definition 1.5. *Let $\dot{x} = f(x)$ where $x \in \mathbb{R}^n$. Then the trivial equilibrium point $x = 0$ is **Lyapunov stable** if there exists a scalar function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that the following conditions hold*

1. $V(x) = 0$ if and only if $x = 0$

2. $V(x) > 0$ if and only if $x \neq 0$

3. $\dot{V}(x) < 0$ for $x \neq 0$

Proof. See Hokayem, [10]. □

In the next section, we will provide some background information of the Lorenz system and the origin of disturbances that should be considered when designing controllers.

1.2 THE LORENZ SYSTEM

We will be using the Lorenz system as an example to explain how to control a chaotic dynamical system. Therefore, before we stabilize the system, we must first introduce the Lorenz system. The Lorenz system was originally used to model the fluid flow of the atmosphere. In fact, the convection flow in thermosyphons and nuclear reactors are modeled using the Lorenz system. However, the Lorenz system was formed from the basis of another set of equations.

Claude-Louis Navier and George Gabriel Stokes developed the Navier-Stokes equation which describes the motion of viscous fluid substances. The natural convection in closed loops can be described by a two-dimensional pipe loop filled with a fluid in a two-dimensional cell being heated from below and cooled from above. See Ehrhard, [11]. The resulting convection motion is modeled by a partial differential equation as follows,

$$\frac{1}{l}\frac{\partial u}{\partial \varphi} = 0$$

$$\rho_0\left(\frac{\partial u}{\partial t}\right) = -\frac{1}{l}\frac{\partial p}{\partial \varphi} - \rho T g \sin(\varphi) - f_W \tag{1.1}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + \frac{u}{l}\frac{\partial T}{\partial \varphi}\right) = h_W[T_W(\varphi) - T]$$

where $u = u(t)$ and $T = T(\varphi,t)$ are the the cross-sectionally averaged velocity and temperature respectively. See Ehrhard, [11]. In these equations; c_p is the heat capacity of the fluid, f_W is the friction force at the pipe wall, h_W is the coefficient of heat transfer at the pipe wall, l is the radius of the loop of pipe, p is the total temperature, g is the gravitation constant, $T_W(\varphi)$ is the pipe wall temperature of the loop, φ is the position coordinate, and ρ_0 is the average density of fluid.

The partial differential equations (1.1) are then transformed into a system of ordinary differential equations using the Fourier series expansion,

$$T(\varphi,t) = T_0 + \sum_{n=1}^{\infty}[S_n(t)\sin(n\varphi) + C_n(t)\cos(n\varphi)]$$

where S_j and C_j are the sine and cosine coefficients of the temperature in the loop. We have now obtained a dimensionless set of ordinary differential equations in time,

$$\dot{x}_1 = \alpha(x_2 - x_1)$$

$$\dot{x}_2 = \beta x_1 - x_1 x_3 - x_2$$

$$\dot{x}_3 = x_1 x_2 - b x_3$$

where x_1 is the velocity and x_2, x_3 are the leading coefficients of the temperature. The three coupled nonlinear partial differential equations, in fact describe the flow in the thermosyphon. See Bourroughs, [4]. For this reason, the Lorenz system can be constructed as the following,

$$\dot{x} = p(y - x)$$

$$\dot{y} = Rx - y - xz \qquad (1.2)$$

$$\dot{z} = xy - bz$$

See Singer, [9]. The variables x, y, and z are the three states and are functions of the time variable. The remaining p, R, and b are dimensionless positive parameters known respectively as the Prandtl number, the Rayleigh number, and a geometric factor. If the Rayleigh number gets high enough it will cause the system to become chaotic therefore it is known as one of the most influential parameters. The Rayleigh number for the instability of steady convection is the following:

$$R = \frac{p(p + b + 3)}{(p - b - 1)}$$

If $p < b + 1$, no positive value of R satisfies this equation, and steady convection is stable. But if $p > b + 1$, steady convection is unstable. Since we are trying to stabilize the chaotic behavior, we therefore will focus on the second case.

Let us now assume that the system is chaotic. There are three equilibrium points in Lorenz system. The first will be the origin $(0, 0, 0)$. The other two points are $(\pm\sqrt{b(R-1)},$

$\pm\sqrt{b(R-1)},R-1)$. See Tran, [1]. In this paper we shall only focus on stabilizing at the origin, however, instead of just the original Lorenz system, we will include an external disturbance on both the *y*- and *z*-state. Under these conditions, control methods will be used to stabilize the system since having these external disturbances will affect the system to behave outside the structure of the original model. The new Lorenz system, possibly with multiple sources of disturbance, can be formed. Those possible sources are the higher-order terms from the Fourier series being dropped and the disturbance due to heat source and ambient condition. The following is the new Lorenz system,

$$\dot{x} = p(y-x)$$
$$\dot{y} = Rx - y - xz + w_1 \qquad (1.3)$$
$$\dot{z} = xy - bz + w_2$$

where *x* is analogous to velocity, *y* is the difference in temperature between ascending and descending currents, and *z* is proportional to the distortion of the vertical temperature, based on the modeling of a single-loop thermosyphon system. See Lorenz, [5]. The two unknowns, w_1 and w_2, are used to represent disturbances which include the un-modeled components of a physical system due to approximation or idealization.

1.3   EXTENDED STATE OBSERVER

Note that our goal is to control a chaotic system with at least one disturbance. However we might not be able to model or write the disturbance in a mathematical expression. There-fore, we need a mechanism to track a disturbance that cannot be modeled. For the last couple of decades, there were several approaches to estimate disturbances, including the unknown input observer, the disturbance observer, the perturbation observer, the equiv-alent input disturbance based estimation,and the extended state observer, which shall be referred to as ESO for the remainder of this paper. In fact, ESO required the least informa-

tion about the disturbance to still track the disturbance. Thus, ESO became the best method to track the disturbance.

The extended state observer (ESO) is a method to estimate any disturbances by treating them as state variables. The main advantage in using ESO, as stated above, is that it does not require an accurate plant model. An ESO can easily estimate several disturbances without changing the observer's structure and parameters. Later in this paper, we will use ESO with a controller to stabilize the given system in Lorenz equations.

Now, let us take a look at an ESO along with a given general nth order dynamical system. First, let $w(t)$ be a external disturbance and $u(t)$ be a controller dependent on t, the time variable.

$$y^{(n)}(t) = f(y^{(n-1)}, y^{(n-2)}, \ldots, y, w(t)) + u(t) \tag{1.4}$$

Rewriting (1.4) as a first order differential equation we have the actual dynamical system,

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = y_3$$
$$\vdots$$
$$\dot{y}_{n-1} = y_n \tag{1.5}$$
$$\dot{y}_n = y_{n+1} + u(t)$$
$$\dot{y}_{n+1} = h(y, w(t))$$
$$y = y_1$$

See Guo, [3]. Note that we do not have enough information about the disturbance. Thus, we will take resort to the ESO.

Let us now build the ESO for (1.5) while assuming some information about the disturbance is given. Assume $f(\hat{y}, w)$ to be the given disturbance, then the ESO will be the

following,

$$\dot{\hat{y}}_1 = \hat{y}_2 + l_1(y_1 - \hat{y}_1)$$

$$\vdots$$

$$\dot{\hat{y}}_{n-1} = \hat{y}_n + l_{n-1}(y_1 - \hat{y}_1) \tag{1.6}$$

$$\dot{\hat{y}}_n = \hat{y}_{n+1} + l_n(y_1 - \hat{y}_1) + u(t)$$

$$\dot{\hat{y}}_{n+1} = l_{n+1}(y_1 - \hat{y}_1)$$

where $[\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{n+1}] \in \mathbb{R}^{n+1}$ and $[l_1, l_2, \ldots, l_{n+1}] \in \mathbb{R}^{n+1}$ being the observer gain parameter. See Zheng, [13]. We have just established the ESO form (1.6) for an nth order dynamical system.

1.4 ACTIVE DISTURBANCE REJECTION CONTROL

We have just learned that ESO is an effective method used to estimate the unknown disturbance because of the low amount of information required. However, we still have not controlled the chaotic dynamical system. Now, we need a controller to manage this system. The controller we will be using is called the Active Disturbance Rejection Control (ADRC).

The ADRC was first introduced by J.Han who used the information and underlying ideas of proportional-integral-derivative (PID). See Tian, [7]. This method estimates the internal and external disturbances and treats the system as a simpler model. Thus, without a complete and precise description of the system, we can still assume that the unknown parts of dynamics are part of the disturbance. Therefore, it again brings ESO forward as one of the best methods to estimate the disturbance.

After ADRC was introduced, it became a vastly popular method. Unlike PID, ADRC not only tracks the disturbance using ESO, but it cancels the disturbance as well making the system stabilize at the equilibrium point. See Espe, [15]. Additionally, ADRC was proven

to have a higher accuracy and efficiency than PID.

In this paper, the controller will be denoted as $u_1(t)$ and $u_2(t)$. These are our ADRC controller that will be applied in the system. Recall the Lorenz system, (1.3), with the unknown controllers added in,

$$\dot{x} = p(y - x)$$
$$\dot{y} = Rx - y - xz + w_1 + u_1 \qquad (1.7)$$
$$\dot{z} = xy - bz + w_2 + u_2$$

In the Lorenz system, we controlled the input $u_1(t)$, $u_2(t)$ also called the state feedback control. Unlike the Lorenz system, the $u_1(t)$, $u_2(t)$ of the first order dynamical system of ADRC Controller can be written as the following,

$$u_1(t) = -\hat{y}_2 + k(r_1(t) - \hat{y}_1) \qquad (1.8)$$

where $r_1(t)$ is the reference signal and $k > 0$ is the feedback gain parameter. Refer to Zheng, [12]. The first term $-\hat{y}_2$ in (1.8), will cancel out the disturbance. The second term $k(r_1(t) - \hat{y}_1)$, is the state feedback term to make $\hat{y}_1$ track the reference signal. The reference signal, $u_2(t)$, has a similar solution.

Furthermore, this paper will use ESO and ADRC to stabilize the Lorenz system. It is discussed in this chapter, stabilizing the system without disturbance is not too difficult. Reality, however, include disturbances that result from information loss due to simplification of high-order terms and disturbances that can not be modeled. Therefore, there are two aspects of the system that need to be stabilized. First, the strong result where the disturbance is known. The second case is the weak result where the disturbance is unknown.

1.5   THE MATRIX EXPONENTIAL

In order to derive the equation in the previous page, a brief introduction of the matrix exponential is needed. But, before we go on we should understand when this method is

needed.

Let $\dot{x} = Ax + f$ be given where, $x(0) = x_0$. Then for x_0, f, $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ we have the following equation,

$$x = e^{At}x_0 + \int_0^t e^{A(t-\tau)}f(\tau)d\tau.$$

Now the question is how do we compute e^{At}, the natural exponent raised to a matrix power or matrix exponential.

The matrix exponential is defined as the following

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

This can also be written in series notation,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$$

where A is an $n \times n$ real or complex matrix. As mentioned earlier in the paper, we are working with the variable t in the matrix exponential. Therefore, it can be written as

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{k!}A^kt^k + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k$$

Before continuing, we should note that the eigenvalues of the Hurwitz matrix do not need to be distinct. The eigenvalues could repeat more than once or be complex. There is a theorem that gives a method for constructing the matrix exponential from the solutions of a differential equation. Refer to Leonard, [8].

Theorem 1.6. *Let A be a constant $n \times n$ matrix with characteristic polynomial*

$$p(\lambda) = det(\lambda I - A) = \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0;$$

then

$$e^{At} = x_1(t)I + x_2(t)A + x_3(A)^2 + \cdots + x_n(t)A^{n-1}$$

where the $x_k(t)$, $1 \leq k \leq n$, are the solutions to the nth order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1 x' + c_0 x = 0$$

satisfying the following initial conditions:

$$
\left.\begin{array}{rl}
x_1(0) & = 1 \\
x_1'(0) & = 0 \\
\vdots \\
x_1^{(n-1)}(0) & = 0
\end{array}\right\}
\qquad
\left.\begin{array}{rl}
x_2(0) & = 0 \\
x_2'(0) & = 1 \\
\vdots \\
x_2^{(n-1)}(0) & = 0
\end{array}\right\}
\cdots
\left.\begin{array}{rl}
x_n(0) & = 0 \\
x_n'(0) & = 0 \\
\vdots \\
x_n^{(n-1)}(0) & = 1
\end{array}\right\}
$$

Proof. Let

$$p(\lambda) = det(\lambda I - A) = \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

be a characteristic polynomial, and let A be a constant $n \times n$ matrix. Define

$$\Phi(t) = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1}$$

where the $x_k(t)$, $1 \leq k \leq n$, are the unique solutions to the nth order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1 x' + c_0 x = 0$$

satisfying the initial conditions written in the theorem. Now, let $\Phi(t)$ have an mth order derivative. Then for all $1 \leq m \leq n$,

$$\Phi^{(m)}(t) = x_1^{(m)}(t)I + x_2^{(m)}(t)A + x_3^{(m)}(t)A^2 + \cdots + x_n^{(m)}(t)A^{n-1}.$$

Therefore,

$$
\begin{aligned}
\Phi^{(n)} + c_{n-1}\Phi^{(n-1)} + \cdots + c_1\Phi' + c_0\Phi &= I(x_1^{(n)} + c_{n-1}x_1^{(n-1)} + \cdots + c_1 x_1' + c_0 x_1) \\
&\quad + A(x_1^{(n)} + c_{n-1}x_2^{(n-1)} + \cdots + c_1 x_2' + c_0 x_2) \\
&\quad \vdots \\
&\quad + A^{n-1}(x_n^{(n)} + c_{n-1}x_n^{(n-1)} + \cdots + c_1 x_n' + c_0 x_n) \\
&= 0I + 0A + \cdots + 0A^{n-1} \\
&= 0
\end{aligned}
$$

Therefore,

$$\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) + \cdots + c_1\Phi'(t) + c_0\Phi(t) = 0$$

for all $t \in \mathbb{R}$, and by the initial conditions

$$\Phi(0) = x_1(0)I + x_2(0)A + \cdots + x_n(0)A^{n-1} = I$$

$$\Phi'(0) = x_1'(0)I + x_2'(0)A + \cdots + x_n'(0)A^{n-1} = A$$

$$\vdots$$

$$\Phi^{(n-1)}(0) = x_1^{(n-1)}(0)I + x_2^{(n-1)}(0)A + \cdots + x_n^{(n-1)}(0)A^{n-1} = A^{(n-1)}$$

(1.9)

Thus,

$$\Phi(t) = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1}$$

satisfies the initial value problem

$$\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) + \cdots + c_1\Phi'(t) + c_0\Phi(t) = 0,$$

$\Phi(0) = I$, $\Phi'(0) = A$, $\Phi''(0) = A^2,\ldots,\Phi^{(n-1)}(0) = A^{n-1}$ We concluded that there exists a unique solution that satisfies the initial value problem, $\Phi(t) = e^{At}$. Therefore for all $t \in \mathbb{R}$,

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1}.$$

$\square$

We now know the method on calculating the exponential matrix, e^{At}. This theorem will be used extensively in the stability studies.

CHAPTER 2

ESO ERROR DYNAMICS

In this chapter, the extended state observer (ESO) for the two single-input-single-output (SISO) system will be determined. This system will be based on the modeling of a single-loop thermosyphon system. Recall the ADRC Controlled Lorenz system from Chapter 1 (1.7),

$$\dot{x} = p(y - x)$$

$$\dot{y} = Rx - y - xz + w_1 + u_1$$

$$\dot{z} = xy - bz + w_2 + u_2$$

where x, y, z are the three states, w_1, w_2 are the external disturbances as well as unincluded components of the physical system due to approximation or idealization, and u_1, u_2 are the ADRC controllers.

Let $y_1 = y$, $y_2 = w_1 - xz + Rx \overset{\Delta}{=} f_1$, $z_1 = z$, $z_2 = w_2 + xy \overset{\Delta}{=} f_2$, $\dot{f}_1 = g_1$, and $\dot{f}_2 = g_2$. The design of the two SISO ADRC systems combined to form a multiple-input-multiple-output (MIMO) system is as follows:

$$\dot{y}_1 = -y_1 + y_2 + u_1$$
$$\dot{y}_2 = g_1 \tag{2.1}$$

and

$$\dot{z}_1 = -bz_1 + z_2 + u_2$$
$$\dot{z}_2 = g_2 \tag{2.2}$$

2.1 ESO FOR THE GIVEN MODEL OF THE PLANT

Let us consider the case where the disturbance in chaos dynamical system is known. That is when the ESO has enough information about the disturbance. Once again, our goal is to

stabilize the Lorenz system. First, we will find the ESO of (2.1). Let,

$$\dot{\bar{y}} = A\bar{y} + Eg_1 + Bu_1$$

$$y_1 = C\bar{y}$$

(2.3)

where $\bar{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$, $A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

The ESO for (2.3) is then designed as

$$\dot{\hat{y}}_1 = -\hat{y}_1 + \hat{y}_2 + u_1 + l_1(y_1 - \hat{y}_1)$$

$$\dot{\hat{y}}_2 = g_1(\hat{y}) + l_2(y_1 - \hat{y}_1)$$

(2.4)

where l_1, l_2 are observer gain parameters, and $l_1 > 0$, $l_2 > 0$. In matrix form,

$$\dot{\hat{y}} = A\hat{y} + Bu_1 + LC(\bar{y} - \hat{y})$$

$$\hat{y}_1 = C\hat{y}.$$

(2.5)

The tracking error dynamics is governed by the error system $\dot{\tilde{y}} = A_c\tilde{y} + E[g_1(\bar{y}) - g_1(\hat{y})]$, where $A_c = A - LC$ and $\tilde{y} = \bar{y} - \hat{y}$. It can be obtained by subtracting (2.5) from (2.3),

$$(\bar{y} - \hat{y}) = A(\bar{y} - \hat{y}) + E[g_1(\bar{y}) - g_1(\hat{y})] - LC(\bar{y} - \hat{y}).$$

The close-loop matrix is,

$$A_c = A - LC = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - l_1 & 1 \\ -l_2 & 0 \end{bmatrix}.$$

(2.6)

Now, we need to find l_1, l_2 so that the eigenvalues of A_c are in the left of the complex plane. That is, for this system, we want A_c to be Hurwitz. Let $p(\lambda)$ be the characteristic polynomial of A_c. Then the $p(\lambda)$ of (2.4) is

$$p(\lambda) = (\lambda + \omega_0)^2$$

(2.7)

and ω_0, the observer gain, becomes the only tuning parameter. From (2.6) and (2.7),

$l_1 = 2\omega_0 - 1$ and $l_2 = \omega_0^2$. Now, let $\tilde{y} = \bar{y} - \hat{y}$, $\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$ then the error system between the

original y-system and (2.4) is

$$\dot{\tilde{y}}_1 = -2\omega_0\tilde{y}_1 + \tilde{y}_2$$
$$\dot{\tilde{y}}_2 = -\omega_0^2\tilde{y}_1 + g_1(\bar{y}) - g_1(\hat{y})$$

(2.8)

The equation can be transformed by using change of variable. Let, $\dot{e}_{y_1} = \dot{\tilde{y}}_1$ and $\dot{e}_{y_2} = \frac{\dot{\tilde{y}}_2}{\omega_0}$.

Then,

$$\dot{e}_{y_1} = -2\omega_0 e_{y_1} + e_{y_2}\omega_0$$
$$\dot{e}_{y_2} = -\omega_0 e_{y_1} + \frac{g_1(\bar{y}) - g_1(\hat{y})}{\omega_0}.$$

Let $e_y = \begin{bmatrix} e_{y_1} \\ e_{y_2} \end{bmatrix}$ then (2.8) can be written as

$$\dot{e}_y = \omega_0 A_y e_y + E\left(\frac{g_1(\bar{y}) - g_1(\hat{y})}{\omega_0}\right)$$

(2.9)

where $A_y = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Theorem 2.1. *Suppose g_1 in (2.8) is globally Lipschitz. Then the error system is asymptotically stable if $\omega_0 > c$ where $c > 0$.*

Proof. From (2.9), A_y is Hurwitz. Therefore there exists a unique positive definite matrix P, such that $A_y^T P + P A_y = -I$. Let the Lyapunov function be $v = e_y^T P e_y$. Hence,

$$\dot{v} = \dot{e}_y^T P e_y + e_y^T P \dot{e}_y$$

(2.10)

Let $h = (g_1(\bar{y}) - g_1(\hat{\bar{y}}))/(\omega_0)$ is now,

$$\dot{v} = \left(e_y^T A_y^T \omega_0 + E^T h\right) Pe_y + e_y^T P \left(\omega_0 A_y e_y + Eh\right)$$

$$= e_y^T \omega_0 A_y^T Pe_y + E^T hPe_y + e_y^T \omega_0 PA_y e_y + e_y^T PEh$$

$$= e_y^T \omega_0 \left(A_y^T P + PA_y\right) e_y + 2E^T hPe_y$$

$$= -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2E^T hPe_y$$

$$= -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2hE^T Pe_y$$

$$= -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2\left(\frac{g_1(\bar{y}) - g_1(\hat{\bar{y}})}{\omega_0}\right) E^T Pe_y$$

$$\leq -\omega_0 \left|\left|e_y^T\right|\right|^2 + \frac{2}{\omega_0} \left|g_1(\bar{y}) - g_1(\hat{\bar{y}})\right| \left|E^T Pe_y\right|$$

Since g_1 is globally Lipschitz, there exists a constant $k_g > 0$ such that $|g_1(\bar{y}) - g_1(\hat{\bar{y}})| \leq k_g ||\bar{y} - \hat{\bar{y}}||$. Thus,

$$-\omega_0 \left|\left|e_y^T\right|\right|^2 + \frac{2}{\omega_0} \left|g_1(\bar{y}) - g_1(\hat{\bar{y}})\right| \left|E^T Pe_y\right| \leq -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{||\bar{y} - \hat{\bar{y}}||}{\omega_0} \left|E^T Pe_y\right|$$

By the Cauchy-Schwarz Inequality,

$$-\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{||\bar{y} - \hat{\bar{y}}||}{\omega_0} \left|E^T Pe_y\right| \leq -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{||\bar{y} - \hat{\bar{y}}||}{\omega_0} \left|\left|E^T\right|\right| \left|\left|Pe_y\right|\right|$$

$$\leq -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{||\bar{y} - \hat{\bar{y}}||}{\omega_0} ||P|| \left|\left|e_y\right|\right|$$

$$= -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{\sqrt{(y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2}}{\omega_0} ||P|| \left|\left|e_y\right|\right|$$

We know that $y_1 - \hat{y}_1 = e_{y1}$ and $y_2 - \hat{y}_2 = \omega_0 e_{y2}$, thus

$$= -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g \frac{\sqrt{(e_{y1})^2 + (\omega_0 e_{y2})^2}}{\omega_0} ||P|| \left|\left|e_y\right|\right|$$

$$< -\omega_0 \left|\left|e_y^T\right|\right|^2 + 2k_g ||P|| \left|\left|e_y\right|\right|^2$$

$$= -\left(\omega_0 - 2k_g ||P||\right) \left|\left|e_y\right|\right|^2$$

Thus, $\omega_0 \geq 1$. Let $c = \max\{1, 2k_g ||P||\}$. If $\omega_0 > c$, then $\dot{v} < 0$. Therefore, $\lim_{t \to \infty} e_y(t) = 0$ for all $\omega_0 > c$. $\qquad \square$

Similar to the *y*-equation, ESO for the *z*-equation is the following,

$$\dot{\hat{z}}_1 = -b\hat{z}_1 + \hat{z}_2 + u_2 + l_3(z_1 - \hat{z}_1)$$

$$\dot{\hat{z}}_2 = g_2 + l_4(z_1 - \hat{z}_1).$$

The close-loop matrix then is,

$$A_c = \begin{bmatrix} -b - l_3 & 1 \\ -l_4 & 0 \end{bmatrix}.$$

By (2.7), $l_3 = 2\omega_0 - b$ and $l_4 = \omega_0^2$. Hence, the error system is

$$\dot{\tilde{z}}_1 = -2\omega_0\tilde{z}_1 + \tilde{z}_2$$

$$\dot{\tilde{z}}_2 = -\omega_0^2\tilde{z}_1 + g_2(\bar{z}) - g_2(\hat{\bar{z}})$$

(2.11)

Let $\dot{e}_{z_1} = \tilde{z}_1, \dot{e}_{z_2} = \tilde{z}_2/\omega_0$, and $e_z = \begin{bmatrix} e_{z_1} \\ e_{z_2} \end{bmatrix}$ then, (2.11) can be simplified to

$$\dot{e}_z = \omega_0 A_z e_z + E\left(\frac{g_2(\bar{z}) - g_2(\hat{\bar{z}})}{\omega_0}\right)$$

where $A_z = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Theorem 2.2. *Suppose g_2 in (2.11) is globally Lipschitz. Then, there exists a constant $c_z > 0$, such that the error system is asymptotically stable.*

Proof. The proof is similar to Theorem 2.1. Thus, $lim_{t\to\infty}e_z(t) = 0$ for $\omega_0 > c_z$. □

It has been proven that the error system of the ESO is asymptotically stable when the model of the plant is given. That is, $\lim_{t\to\infty}e_y(t) = 0$ and $\lim_{t\to\infty}e_z(t) = 0$. In here, we observed the accuracy of the ESO tracking the disturbances in the *y* and *z* states. We will now analyze the stability of the error system of the ESO when the dynamics of the plant is largely unknown. That is, the disturbances in the system is unknown.

2.2 ESO FOR THE DYNAMICS OF THE PLANT UNKNOWN

In the real world scenarios, the plant dynamics are mostly unknown. See Guo, [2]. In this case, the ESO takes the form of

$$
\begin{aligned}
\dot{\hat{y}}_1 &= -\hat{y}_1 + \hat{y}_2 + u_1 + l_1(y_1 - \hat{y}_1) \\
\dot{\hat{y}}_2 &= l_2(y_1 - \hat{y}_1).
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\dot{\hat{z}}_1 &= -b\hat{z}_1 + \hat{z}_2 + u_1 + l_3(z_1 - \hat{z}_1) \\
\dot{\hat{z}}_2 &= l_4(z_1 - \hat{z}_1).
\end{aligned}
\tag{2.13}
$$

The only difference from the form of known disturbance is the existence of g_1, g_2. As it was defined earlier, $l_1 = 2\omega_0 - 1$, $l_2 = \omega_0^2$, $l_3 = 2\omega_0 - b$, and $l_4 = \omega_0^2$. Then the observer estimation error for (2.12) and (2.13) could be written as,

$$
\begin{aligned}
\dot{\tilde{y}}_1 &= -2\omega_0\tilde{y}_1 + \tilde{y}_2 \\
\dot{\tilde{y}}_2 &= -\omega_0^2\tilde{y}_1 + g_1
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\dot{\tilde{z}}_1 &= -2\omega_0\tilde{z}_1 + \tilde{z}_2 \\
\dot{\tilde{z}}_2 &= -\omega_0^2\tilde{z}_1 + g_2
\end{aligned}
\tag{2.15}
$$

where $\tilde{y} = \bar{y} - \hat{y}$, $\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$, and $\tilde{z} = \bar{z} - \hat{z}$, $\tilde{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}$, respectively.

The equations (2.14) can be transformed through a use of change of variables into, $\dot{e}_{y_1} = \dot{\tilde{y}}_1$ and $\dot{e}_{y_2} = \dot{\tilde{y}}_2/\omega_0$. Similarly let $\dot{e}_{z_1} = \dot{\tilde{z}}_1$ and $\dot{e}_{z_2} = \dot{\tilde{z}}_2/\omega_0$. Thus,

$$
\begin{aligned}
\dot{e}_{y_1} &= -2\omega_0 e_{y_1} + e_{y_2}\omega_0 \\
\dot{e}_{y_2} &= -\omega_0 e_{y_1} + \frac{g_1}{\omega_0}
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_{z_1} &= -2\omega_0 e_{z_1} + e_{z_2}\omega_0 \\
\dot{e}_{z_2} &= -\omega_0 e_{z_1} + \frac{g_2}{\omega_0}
\end{aligned}
$$

Let $e_y = \begin{bmatrix} e_{y_1} \\ e_{y_2} \end{bmatrix}$ and $e_z = \begin{bmatrix} e_{z_1} \\ e_{z_2} \end{bmatrix}$ then,

$$\dot{e}_y = \omega_0 A e_y + E\left(\frac{g_1}{\omega_0}\right)$$

$$\dot{e}_z = \omega_0 A e_z + E\left(\frac{g_2}{\omega_0}\right)$$

where $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Up to now the solution looks similar to the ESO part in Chapter 2, however, instead of using Lyapunov function we will continue the process with the use of the matrix exponential method,

$$e_y = e^{\omega_0 A t} e_y(0) + \int_0^t e^{\omega_0 A(t-\tau)} E\frac{g_1}{\omega_0} d\tau \tag{2.16}$$

$$e_z = e^{\omega_0 A t} e_z(0) + \int_0^t e^{\omega_0 A(t-\tau)} E\frac{g_2}{\omega_0} d\tau \tag{2.17}$$

Theorem 2.3. *Suppose g_1 in (2.14) is bounded. Then, the error system (2.14) is stable if the observer gain ω_0 is sufficiently large.*

Proof. Let g_1 in (2.14) be bounded. By using Theorem 1.4 from the matrix exponential method, we can compute e^{At}. The matrix A for the equation (2.16) is 2×2, therefore, $p(\lambda) = \lambda^2 + 2\lambda + 1$. Since $\lambda = -1$, therefore, $x(t) = c_1 e^{-t} + c_2 t e^{-t}$. Note that $e^{At} = x_1 I + x_2 A$ and the initial condition is the following,

$$\left.\begin{array}{l} x_1(0) = 1 \\ x_1' = 0 \end{array}\right\} \qquad \left.\begin{array}{l} x_2(0) = 0 \\ x_2' = 1 \end{array}\right\}$$

Then by calculus, $x_1(t) = e^{-t} + t e^{-t}$ and $x_2(t) = t e^{-t}$. As a result,

$$e^{At} = \begin{bmatrix} e^{-t} - t e^{-t} & t e^{-t} \\ -t e^{-t} & e^{-t} + t e^{-t} \end{bmatrix} \tag{2.18}$$

Now we need to solve for (2.16) by replacing e^{At} by (2.18).

$$e^{\omega_0 At} = \begin{bmatrix} e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t} & \omega_0 t e^{-\omega_0 t} \\ -\omega_0 t e^{-\omega_0 t} & e^{-\omega_0 t} + \omega_0 t e^{-\omega_0 t} \end{bmatrix} \tag{2.19}$$

Note that for the known disturbance we showed that $\lim_{t \to \infty} e_y = 0$ when $\omega_0 > c$. Now, for the unknown disturbance we cannot show that $\lim_{t \to \infty} e_y = 0$, instead we can construct, $||e_y|| < \varepsilon(1/\omega_0)$ where ε depends on $1/\omega_0$. This shows that the bigger the ω_0 the smaller the error.

By the definition of infinite norm which is $||x||_\infty = max\{|x_i|\}_{i=1}^n$, we get

$$||e_y||_\infty = \left|\left| e^{\omega_0 At} e_y(0) + \int_0^t e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} d\tau \right|\right|_\infty$$

By the triangle inequality we get the following,

$$||e_y||_\infty \leq ||e^{\omega_0 At} e_y(0)||_\infty + \left|\left| \int_0^t e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} d\tau \right|\right|_\infty \tag{2.20}$$

Consider the first term in the right hand side of (2.20). From (2.19), we obtained

$$e^{\omega_0 At} e_y(0) = \begin{bmatrix} e^{-\omega_0 t}(1 - \omega_0 t) e_1(0) + \omega_0 t e^{-\omega_0 t} e_2(0) \\ -\omega_0 t e^{-\omega_0 t} e_1(0) + e^{-\omega_0 t}(1 + \omega_0 t) e_2(0) \end{bmatrix}$$

Hence,

$$||e^{\omega_0 At} e_y(0)||_\infty = max\{|e^{-\omega_0 t}(1 - \omega_0 t) e_1(0) + \omega_0 t e^{-\omega_0 t} e_2(0)|, \tag{2.21}$$

$$|-\omega_0 t e^{-\omega_0 t} e_1(0) + e^{-\omega_0 t}(1 + \omega_0 t) e_2(0)|\}$$

Let us look at the first term in the right hand side of (2.21). By the triangle inequality,

$$|e^{-\omega_0 t}(1 - \omega_0 t) e_1(0) + \omega_0 t e^{-\omega_0 t} e_2(0)| \leq |e^{-\omega_0 t}(1 - \omega_0 t)||e_1(0)| + |\omega_0 t e^{-\omega_0 t}||e_2(0)|$$

Now, let $m = max\{|e_1(0)|, |e_2(0)|\}$. Thus,

$$|e^{-\omega_0 t}(1 - \omega_0 t)||e_1(0)| + |\omega_0 t e^{-\omega_0 t}||e_2(0)| \leq m|e^{-\omega_0 t}(1 - \omega_0 t)| + m|\omega_0 t e^{-\omega_0 t}|$$

Since $e^{-\omega_0 t}(1 - \omega_0 t)$ and $\omega_0 t e^{-\omega_0 t}$ are both positive we can simplify to,

$$m|e^{-\omega_0 t}(1 - \omega_0 t)| + m|\omega_0 t e^{-\omega_0 t}| = m(2\omega_0 t - 1)e^{-\omega_0 t}$$

Similarly the second term in the right hand side of (2.21) follows,

$$|-\omega_0 t e^{-\omega_0 t} e_1(0) + e^{-\omega_0 t}(1 + \omega_0 t)e_2(0)| \le m(2\omega_0 t + 1)e^{-\omega_0 t}$$

By the definition of infinity norm, we get

$$||e^{\omega_0 At} e_y(0)||_\infty \le m\frac{(2\omega_0 t + 1)}{e^{\omega_0 t}} \tag{2.22}$$

From (2.22), we can use L'Hopital's rule and get,

$$\lim_{t \to \infty} \frac{2\omega_0 t + 1}{e^{w_0 t}} = \lim_{t \to \infty} \frac{2\omega_0}{\omega_0 e^{w_0 t}} = \lim_{t \to \infty} \frac{2}{e^{w_0 t}} = 0$$

Note that $\lim_{t \to \infty} f(t) = 0$ for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that if $t > N$, then $|f(t) - 0| < \varepsilon$.

Thus, there exists $1/(m\omega_0^2)$ and $T_1 > 0$, such that

$$\frac{2\omega_0 t + 1}{e^{\omega_0 t}} < \frac{1}{m\omega_0^2}$$

when $t > T_1$ Therefore,

$$||e^{\omega_0 At} e_y(0)||_\infty < \frac{1}{\omega_0^2}. \tag{2.23}$$

We have just concluded the first term of (2.20). Now, let us look at the second term.

$$\left\|\int_0^t e^{\omega A(t-\tau)} E \frac{g_1}{\omega_0} d\tau \right\|_\infty \tag{2.24}$$

By looking at this equation, $\tau < t$. Without the integration, (2.24) can be written as,

$$e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} = \begin{bmatrix} g_1(t - \tau)e^{-\omega_0(t-\tau)} \\ \frac{g_1(1 + \omega_0(t-\tau))e^{-\omega_0(t-\tau)}}{\omega_0} \end{bmatrix} \tag{2.25}$$

Hence in infinity norm,

$$\left\| e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} \right\|_\infty = \max \left\{ \left| g_1(t-\tau)e^{-\omega_0(t-\tau)} \right|, \left| \frac{g_1(1+\omega_0(t-\tau))e^{-\omega_0(t-\tau)}}{\omega_0} \right| \right\} \quad (2.26)$$

The first term in the right hand side of (2.26), $(t-\tau)e^{-\omega_0(t-\tau)}$ is positive, therefore,

$$\left| g_1(t-\tau)e^{-\omega_0(t-\tau)} \right| = |g_1|(t-\tau)e^{-\omega_0(t-\tau)}$$

Let $|g_1| \leq M_1$, then we have

$$|g_1|(t-\tau)e^{-\omega_0(t-\tau)} \leq M_1(t-\tau)e^{-\omega_0(t-\tau)} \quad (2.27)$$

By the Cauchy-Schwarz inequality (2.27) is now,

$$\int_0^t \left| g_1(t-\tau)e^{-\omega_0(t-\tau)} \right| d\tau \leq M_1 \int_0^t (t-\tau)e^{-\omega_0(t-\tau)} d\tau$$

Using integration by parts, we obtain

$$= \frac{M_1}{\omega_0^2}(1-e^{-\omega_0 t}) - \frac{M_1}{\omega_0}te^{-\omega_0 t}$$

Similarly for the second term in the right hand side of (2.26),

$$\left| \frac{g_1(1+\omega_0(t-\tau))e^{-\omega_0(t-\tau)}}{\omega_0} \right| \leq \frac{M_1}{\omega_0}(1+\omega_0(t-\tau))e^{-\omega_0(t-\tau)}$$

$$= \frac{2M_1}{\omega_0^2}(1-e^{-\omega_0 t}) - \frac{M_1}{\omega_0}te^{-\omega_0 t}$$

Let $q(t) = (2M_1/\omega_0^2)(1-e^{-\omega_0 t}) - (M_1/\omega_0)te^{-\omega_0 t}$. By l'hopital's rule,

$$\lim_{t \to \infty} q(t) = \frac{2M_1}{\omega_0^2}$$

Therefore, for all $\varepsilon > 0$, there exists $T_2 > 0$, such that $\left| q(t) - 2M_1/\omega_0^2 \right| < \varepsilon$ when $t > T_2$.

Let $\varepsilon = M_1/\omega_0^2$ then,

$$q(t) < \frac{3M_1}{\omega_0^2}.$$

Therefore,

$$\left\|\int_0^t e^{\omega A(t-\tau)} E \frac{g_1}{\omega_0} d\tau\right\|_\infty < \frac{3M_1}{\omega_0^2} \qquad (2.28)$$

By (2.23) and (2.28) we have concluded,

$$||e_y||_\infty \leq ||e^{\omega_0 At} e_y(0)||_\infty + \left\|\int_0^t e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} d\tau\right\|_\infty$$

There exist $T = max\{T_1, T_2\}$ where $t > T$, such that

$$||e^{\omega_0 At} e_y(0)||_\infty + \left\|\int_0^t e^{\omega_0 A(t-\tau)} E \frac{g_1}{\omega_0} d\tau\right\|_\infty \leq \frac{1}{\omega_0^2} + \frac{3M_1}{\omega_0^2}$$

Therefore,

$$||e_y||_\infty < \frac{1 + 3M_1}{\omega_0^2} \qquad (2.29)$$

$\square$

Theorem 2.4. *Suppose g_2 in (2.15) is bounded. Then, the error system (2.15) is stable if the observer gain ω_0 is sufficiently large.*

Proof. The proof is similar to Theorem 2.3. $\square$

We have just discovered that the error system of the ESO can be controlled by increasing the observer gain parameter. In practical application, increasing the observer gain is not the prefered method. This is due to the high cost required to pump the state observer.

In summary, it has been proven that when the plant model is given, the error system of the ESO is asymptotically stable; and when the plant dynamics are largely unknown, the error system of the ESO is bounded and stable for sufficiently large ω_o. In the next chapter, the stability of the ADRC will be analyzed.

CHAPTER 3

STABILITY ANALYSIS OF ADRC

In Chapter 1, we briefly introduced the ADRC system. Recall the ADRC controlled Lorenz system from (1.7),

$$\dot{x} = p(y - x)$$

$$\dot{y} = Rx - y - xz + w_1 + u_1$$

$$\dot{z} = xy - bz + w_2 + u_2$$

where w_1, w_2 are the external disturbances. In here, u_1, u_2 are the so-called ADRC controllers. Earlier in Chapter 2, we saw that the ESO tracked the state variables y, z and the disturbances f_1, f_2, where $f_1 = Rx - xz + w_1$ and $f_2 = xy + w_2$, respectively. Now, we will determine the ADRC controllers to stabilize the given Lorenz system. Let us first solve for y-equation. In general, for $r_1 = r_1(t)$, u_1 is written as

$$u_1 = \omega_c(r_1 - \hat{y}_1) - \hat{y}_2 + \dot{r}_1 \tag{3.1}$$

where r is the reference signal and ω_c is the controller gain parameter. The equation (3.1) cannot be used in the ESO system, however, it can be used in the original equation. Recall the y-equation from (1.7) with the substitution of (3.1),

$$\dot{y} = -y + f_1 + \omega_c(r - \hat{y}_1) - \hat{y}_2 + \dot{r}_1$$

where $f_1 = Rx - xz + w_1$. We proved in Theorem 2.1 that $y_2 \to \hat{y}_2$ and $y_2 = f_1$, thus

$$(y \dot{-} r) = -y + (f_1 - \hat{y}_2) + \omega_c(r_1 - \hat{y}_1). \tag{3.2}$$

Before continuing, modification of (3.1) is needed in order to cancel out the remaining y term in (3.2). That is, to stabilize the system, the equation for the ADRC controller should be modified so that it can cancel out the disturbance. Thus, equation for u_1 is now,

$$u_1 = \omega_c(r_1 - \hat{y}_1) - \hat{y}_2 + \dot{r}_1 + r_1. \tag{3.3}$$

Hence,

$$(y \dot{-} r_1) = (r_1 - y) + (f_1 - \hat{y}_2) + \omega_c(r_1 - y) + \omega_c(y - \hat{y}_1) \tag{3.4}$$

Let $e_r = y - r_1$, then (3.4) can be written as

$$\dot{e}_r = -e_r(\omega_c + 1) + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1).$$

Let $k = \omega_c + 1$, then we obtain

$$\dot{e}_r = -ke_r + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1). \tag{3.5}$$

Similar to the y-equation, for $r_2 = r_2(t)$, u_2 is written as

$$u_2 = \tilde{\omega}_c(r_2 - \hat{z}_1) - \hat{z}_2 + \dot{r}_2 + br_2 \tag{3.6}$$

where r_2 is the reference signal and $\tilde{\omega}_c$ is the controller gain parameter. Recall (1.7) with the substitution of (3.6), then we obtain

$$(z \dot{-} r_2) = (br_2 - bz) + (f_2 - \hat{z}_2) + \tilde{\omega}_c(r_2 - z) + \tilde{\omega}_c(z - \hat{z}_1). \tag{3.7}$$

Let $e_{rz} = z - r_2$ and $k_z = \tilde{\omega}_c + b$, then (3.7) can be written as

$$\dot{e}_{rz} = -k_z e_{rz} + (f_2 - \hat{z}_2) + \tilde{\omega}_c(z - \hat{z}_1)$$

3.1 ADRC FOR THE GIVEN MODEL OF THE PLANT

Consider the case where the model of the plant is given. Note that in Chapter 2, we have determined that the error system of the ESO is stable. Now our goal is to build the ADRC Controller in such way so that it can track the disturbances. Before the stability of the ADRC is analyzed, the following lemma is constructed.

Lemma 3.1. *Let $\dot{\eta} = -\mu\eta + g$ for all $\mu > 0$ and $\lim_{t \to \infty} |g(t)| = 0$. Then, $\lim_{t \to \infty} |\eta(t)| = 0$.*

Proof. Since $\lim_{t\to\infty}|g(t)|=0$, $\dot{\eta}=-\mu\eta+g$ can be written as

$$\eta(t)=e^{-\mu t}\eta(0)+\int_0^t e^{-\mu(t-\tau)}g(\tau)d(\tau) \tag{3.8}$$

Because $\lim_{t\to\infty}e^{-\mu t}=0$, therefore, for all $\varepsilon>0$, there exists T_1 such that $|e^{-\mu t}e_r(0)|<\varepsilon/3$ when $t>T_1$. Let $B=2k/9$. Since $lim_{t\to\infty}g(t)=0$, thus, there exists T_2 such that $|g(t)|<\beta\varepsilon$ when $t>T_2$. Hence we have,

$$\begin{aligned}
|\eta| &\leq |e^{-\mu t}\eta(0)|+\left|\int_0^{T_2}e^{-\mu(t-\tau)}g(\tau)d\tau+\int_{T_2}^t e^{-\mu(t-\tau)}g(\tau)d\tau\right| \\
&\leq |e^{-\mu t}\eta(0)|+\left|\int_0^{T_2}e^{-\mu(t-\tau)}g(\tau)d\tau\right|+\left|\int_{T_2}^t e^{-\mu(t-\tau)}g(\tau)d\tau\right|
\end{aligned} \tag{3.9}$$

Let us now look at the second term in the right hand side of (3.9).

$$\left|\int_0^{T_2}e^{-\mu(t-\tau)}g(\tau)d\tau\right|\leq \int_0^{T_2}e^{-\mu(t-\tau)}|g(\tau)|d\tau$$

Let $|g|<M_g$, then

$$\begin{aligned}
\int_0^{T_2}e^{-\mu(t-\tau)}|g(\tau)|d\tau &< M_g\int_0^{T_2}e^{-\mu(t-\tau)}d\tau \\
&= \frac{M_g}{\mu}(e^{-\mu(t-T_2)}-e^{-\mu t})
\end{aligned}$$

Since $\lim_{t\to\infty}e^{-\mu(t-T_2)}=0$ and $\lim_{t\to\infty}e^{-\mu t}=0$, therefore, there exists T_3, such that

$$\frac{M_g}{\mu}(e^{-\mu(t-T_2)}-e^{-\mu t})<\frac{\varepsilon}{3} \tag{3.10}$$

where $t>T_3$. Similar to the second term in (3.9), looking at the third term we obtain,

$$\begin{aligned}
\left|\int_{T_2}^t e^{-\mu(t-\tau)}g(\tau)d\tau\right| &\leq \int_{T_2}^t e^{-\mu(t-\tau)}|g(\tau)|d\tau \\
&< \beta\varepsilon\int_{T_2}^t e^{-\mu(t-\tau)}d\tau
\end{aligned}$$

Simplifying and substituting β we get,

$$\beta\varepsilon\int_{T_2}^t e^{-\mu(t-\tau)}d\tau=\frac{2\varepsilon}{9}(1-e^{-\mu(t-T_2)})$$

Since $\lim_{t \to \infty}(1 - e^{-\mu(t-T_2)}) = 1$, therefore, there exist T_4, such that $1 - e^{-\mu(t-T_2)} < 3/2$ when $t < T_4$. Thus,

$$\left| \int_{T_2}^{t} e^{-\mu(t-\tau)} g(\tau) d\tau \right| < \frac{\varepsilon}{3}. \tag{3.11}$$

when $t > T_4$. Let $T = \max\{T_1, T_2, T_3, T_4\}$ then, $|\eta| < \varepsilon$ for all $t > T$. $\square$

Using Lemma 3.1 we can construct two theorems, first for the y-equation and second for the z-equation.

Theorem 3.2. *The ADRC system is $\dot{e}_r = -ke_r + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1)$. Let $g_1 = \hat{f}_1$, and g_1 is known as well as globally lipschitz, then the system is asymptotically stable.*

Proof. Let $e_r = y - r_1$ and $h = f_1 - \hat{y}_2 + \omega_c(y - \hat{y}_1)$. According to Theorem 2.1, $\lim_{t \to \infty} \|h(t)\| = 0$ if g_1 is globally Lipschitz. Then by (3.8) from Lemma 3.1, $\dot{e}_r = -ke_r + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1)$ can be written as

$$e_r(t) = e^{-kt} e_r(0) + \int_{0}^{t} e^{-k(t-\tau)} h(\tau) d(\tau). \tag{3.12}$$

Since A is Hurwitz from (2.9), according to Theorem 2.1 and Lemma 3.1, it can be concluded that there exist constants $\omega_0 > 0$ and $\omega_c > 0$ such that $\lim_{t \to \infty} e_r(t) = 0$. In other words, the system is asymptotically stable. $\square$

Theorem 3.3. *The ADRC system is $\dot{e}_{rz} = -k_z e_{rz} + (f_2 - \hat{z}_2) + \tilde{\omega}_c(z - \hat{z}_1)$. Let $g_2 = \hat{f}_2$, and g_2 is known as well as globally Lipschitz, then the system is asymptotically stable.*

Proof. The proof is similar to Theorem 3.2. $\square$

We have just stabilized the ADRC Controlled Lorenz system for the y-, and z-equations. In the next section, we will determine the convergence of the ADRC with plant dynamics largely unknown.

3.2   ADRC FOR THE DYNAMICS OF THE PLANT UNKNOWN

Now we consider the case where the dynamics of the plant are largely unknown. Since we have proven the stability of the error system, we now can construct two theorems that prove the stability of the ADRC Controlled Lorenz system for the y-, and z-equations.

Theorem 3.4. *The ADRC system is* $\dot{e}_r = -ke_r + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1)$. *Let* $g_1 = \dot{f}_1$, *and* $|g_1| \leq M_1$. *Then, there exists* $\varepsilon_\Sigma > 0$ *when* $t > T$. *Then, the system is stable.*

Proof. Recall from the previous section, the ADRC system is given as,

$$\dot{e}_r = -ke_r + (f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1)$$

where $k = \omega_c + 1$. Similar to the (3.8) from the known disturbance, the $e_r(t)$ for this ADRC system is,

$$e_r(t) = e^{-kt}e_r(0) + \int_0^t e^{-k(t-\tau)}[(f_1 - \hat{y}_2) + \omega_c(y - \hat{y}_1)]d\tau \tag{3.13}$$

Since g_1 is bounded, (3.13) is now,

$$|e_r(t)| \leq |e^{-kt}e_r(0)| + \int_0^t e^{-k(t-\tau)}(|f_1 - \hat{y}_2| + \omega_c|y - \hat{y}_1|)d\tau \tag{3.14}$$

According to Theorem 2.3, there exists a finite time T_1 such that

$$|f_1 - \hat{y}_2| \leq \frac{1 + 3M_1}{\omega_0^2}$$

and

$$|y - \hat{y}_1| \leq \frac{1 + 3M_1}{\omega_0^2}$$

for all $t > T_1$. Therefore, by the property of definite integral (3.14) is now,

$$|e_r(t)| \leq |e^{-kt}e_r(0)| + \int_0^{T_1} e^{-k(t-\tau)}(|f_1 - \hat{y}_2| + \omega_c|y - \hat{y}_1|)d\tau$$
$$+ \int_{T_1}^t e^{-k(t-\tau)}(|f_1 - \hat{y}_2| + \omega_c|y - \hat{y}_1|)d\tau$$

Then $|f_1 - \hat{y}_2| + \omega_c|y - \hat{y}_1|$ is continuous over $[0, T_1]$. Thus, there exists $M_2 > 0$, such that $|f_1 - \hat{y}_2| + \omega_c|y - \hat{y}_1| \leq M_2$. Hence

$$
\begin{aligned}
|e_r(t)| &\leq |e^{-kt}e_r(0)| + M_2 \int_0^{T_1} e^{-k(t-\tau)}d\tau + (1 + \omega_c)\left(\frac{1 + 3M_1}{\omega_0^2}\right)\int_{T_1}^t e^{-k(t-\tau)}d\tau \\
&= |e^{-kt}e_r(0)| + \frac{M_2}{k}[e^{-k(t-\tau)} - e^{-kt}] + \frac{1 + 3M_1}{\omega_0^2}[1 - e^{-k(t-T_1)}]
\end{aligned}
\tag{3.15}
$$

Note that $\lim_{t\to\infty} e^{-kt} = 0$ as well as $\lim_{t\to\infty} e^{-k(t-T_1)} - e^{-kt} = 0$ and $\lim_{t\to\infty} e^{-k(t-T_1)} = 0$. From $\lim_{t\to\infty} e^{-kt} = 0$, there exists $1/(k|e_r(0)|) > 0$ and a finite time $T_2 > 0$ such that

$$
e^{-kt} < \frac{1}{k|e_r(0)|}
$$

for all $t > T_2$. Therefore,

$$
|e^{-kt}e_r(0)| < \frac{1}{k}
\tag{3.16}
$$

for all $t > T_2$. From $\lim_{t\to\infty} e^{-k(t-T_1)} - e^{-kt} = 0$ there exists $1/M_2 > 0$ and a finite time $T_3 > 0$ such that,

$$
e^{-k(t-T_1)} - e^{-kt} < \frac{1}{M_2}
$$

for all $t > T_3$. Therefore,

$$
\frac{M_2}{k}[e^{-k(t-T_1)} - e^{-kt}] < \frac{1}{k}
\tag{3.17}
$$

for all $t > T_3$. From $\lim_{t\to\infty} 1 - e^{-k(t-T_1)} = 1$ there exists a finite time $T_4 > 0$ such that

$$
1 - e^{-k(t-T_1)} < \frac{3}{2}.
$$

Therefore,

$$
\frac{1 + 3M_1}{\omega_0^2}[1 - e^{-k(t-T_1)}] < \frac{3(1 + 3M_1)}{2\omega_0^2}.
\tag{3.18}
$$

Now, let $T = \max\{T_1, T_2, T_3, T_4\}$, then we obtain

$$
\begin{aligned}
|e_r(t)| &\leq \frac{1}{k} + \frac{1}{k} + \frac{3(1 + 3M_1)}{2\omega_0^2} \\
&= \frac{2}{k} + \frac{3(1 + 3M_1)}{2\omega_0^2}
\end{aligned}
\tag{3.19}
$$

Denote (3.19) as ε_Σ. Thus the system is stable. □

Similarly, we can construct the theorem for the z-equation.

Theorem 3.5. *The ADRC system is $\dot{e}_{rz} = -k_z e_{rz} + (f_2 - \hat{z}_2) + \tilde{\omega}_c(z - \hat{z}_1)$. Let $g_2 = \dot{f}_2$, and $|g_2| \leq M_2$. Then, there exists $\varepsilon_\Sigma > 0$ when $t > T$. Then, the system is stable.*

Proof. The proof is similar to Theorem 3.4. □

From Theorem 3.4 and Theorem 3.5, the system is bounded by the reciprocal of k and ω_0, where k is the feedback control gains parameter and ω_0 is the observer gains parameter. That is, to stabilize the system, we can increase both the k and ω_0.

In summary, it has been determined that with the plant model given, the ADRC system is asymptotically stable; and when the plant dynamics are unknown, the error system is stable when observer gains and feedback control gains are sufficiently large. In the next section, the stability of the ADRC controlled Lorenz system will be analyzed.

CHAPTER 4

STABILITY OF THE ADRC CONTROLLED LORENZ SYSTEM AT THE ORIGIN

In previous section, we have proven that the given ADRC system is stable. Now, we need to show that the ADRC controlled Lorenz system is stable at the given equilibrium point. In the following results, we have chosen $r_1(t) = 0$, $r_2(t) = 0$. Thus, we will stabilize the Lorenz system at the origin, which is one of the equilibrium points. Two theorems are then constructed; first, when the model of the plant is known and the second with the dynamics of the plant largely unknown.

Theorem 4.1. *Assume g_1 and g_2 are globally Lipschitz and the ESOs of the y, z subsystems are asymptotically stable. Then, the ADRC controlled Lorenz system is asymptotically stable at the origin.*

Proof. According to Theorem 3 and Theorem 7, $\lim_{t \to \infty} y(t) = 0$ and $\lim_{t \to \infty} z(t) = 0$. Then by Lemma 1, $\lim_{t \to \infty} x(t) = 0$. Therefore, the ADRC controlled Lorenz system is asymptotically stable. $\qquad\square$

Theorem 4.2. *Assume g_1 and g_2 are bounded and the ESOs of the y, z subsystems are stable. Then, the ADRC controlled Lorenz system is stable for sufficiently large observer gains and feedback control gains.*

Proof. According to Theorem 4 and Theorem 8 for sufficiently large observer gains and feedback control gains, the y, z subsystems are stable. For all $\varepsilon > 0$, let $|y(t)| < \varepsilon/3$, where $t > T_1$. From our original Lorenz system, $\dot{x}$ can be rewritten as,

$$\dot{x} = -px + py.$$

Thus,

$$x = e^{-pt}x(0) + p\int_0^t e^{-p(t-\tau)}y(\tau)d\tau$$
$$= e^{-pt}x(0) + p\int_0^{T_1} e^{-p(t-\tau)}y(\tau)d\tau + p\int_{T_1}^t e^{-p(t-\tau)}y(\tau)d\tau$$

Therefore,

$$|x| \le e^{-pt}|x(0)| + p \int_0^{T_1} e^{-p(t-\tau)}|y(\tau)|d\tau + p \int_{T_1}^t e^{-p(t-\tau)}|y(\tau)|d\tau$$

Let $|y(\tau)| < M$ over $[0, T_1]$. Hence,

$$
\begin{aligned}
|x| &\le e^{-pt}|x(0)| + M(e^{-p(t-T_1)} - e^{-pt}) + \frac{\varepsilon}{3}(1 - e^{-pt}) \\
&= e^{-pt}\left(|x(0)| - M - \frac{\varepsilon}{3}\right) + Me^{-p(t-T_1)} + \frac{\varepsilon}{3} \\
&\le e^{-pt}|x(0)| + Me^{-p(t-T_1)} + \frac{\varepsilon}{3}.
\end{aligned}
\tag{4.1}
$$

Then, there exists T_2 such that $e^{-pt}|x(0)| < \varepsilon/3$ and T_3 such that $Me^{-p(t-T_1)} < \varepsilon/3$. The inequality (4.1) is now,

$$|x| \le \varepsilon. \tag{4.2}$$

$\square$

CHAPTER 5

NUMERICAL RESULTS

We have just established the stability of the ADRC Controlled Lorenz system. Now, the system will be simulated visually in MATLAB. Our goal here is to verify the analytical results in a numerical representation. We will consider two cases. First, we will show the stability of the system when $r_1 = 0$, $r_2 = 0$; next, when at least one of the reference signal is not zero. For example, when reference signal for y-equation is $r_1 = 0$, then the reference signal for z-equation is $r_2(t) \neq 0$. Note that we are only working with a chaotic system, thus, the Rayleigh number should be large enough as mentioned in Chapter 1.

We will begin by setting the parameters to be $R = 50$, $b = 8/3$, and $p = 10$ where R is the Rayleigh number, b is the geometric factor, and p is the Prandtl number, respectively. From the analytical results, we have proved that, when the observer bandwidth is big enough, the ESO system is stable. Thus, let $\omega_0 = 60$ and $k = 20$ where ω_0 is the observer bandwidth and k is the feedback gain. Moreover, the ADRC Controller is set to activate when $t = 20$.

Let us first consider the case where the reference signal for the y- and z-equations are $r_1 = 0$ and $r_2 = 0$. For the disturbances, we will choose sinusoids since they are bounded and close to practical situation. Thus, the disturbances are chosen as follows

$$
\begin{aligned}
w_1(t) &= 10\cos(4t) \\
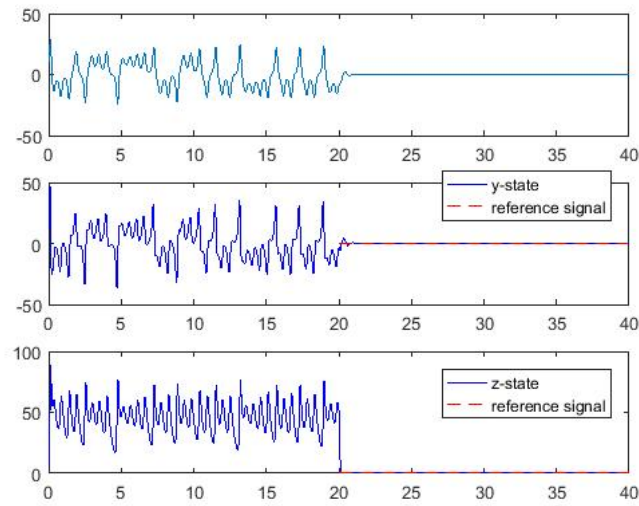w_2(t) &= \sin(3t)^2.
\end{aligned}
\tag{5.1}
$$

Figure 5.1: ADRC Controlled Lorenz system when $r_1 = 0$, $r_2 = 0$.

From Figure 5.1, we can see the chaotic system from $t = 0$ to $t = 20$. However, when the ADRC Controllers track the disturbances, the system is close to the reference signals. Now, let us look at the ESO for the y- and z-equation.
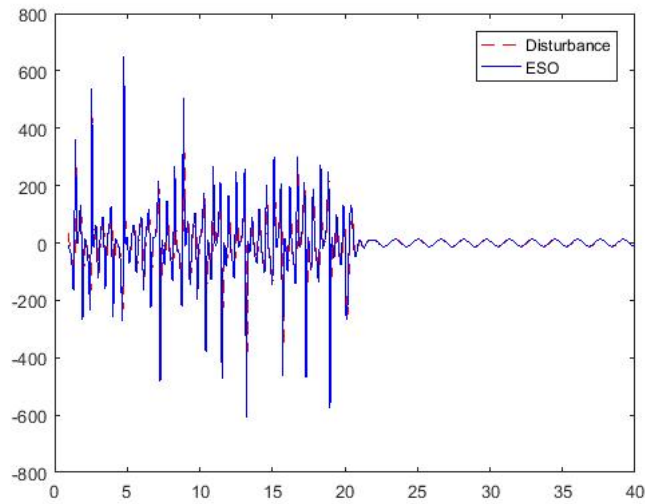

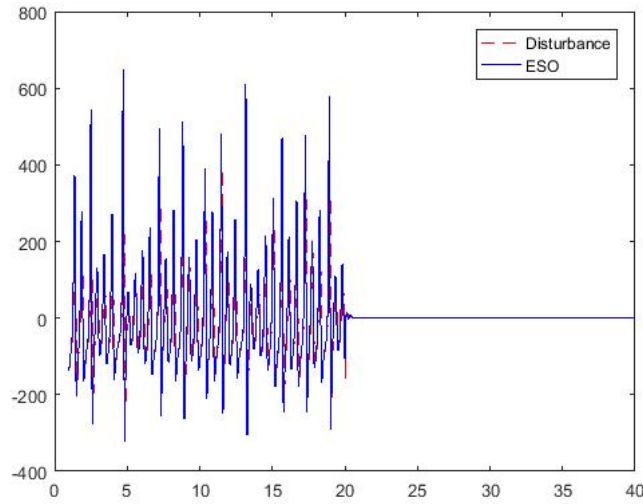Figure 5.2: ESO for the y-state when $r_1 = 0$, $r_2 = 0$.

Figure 5.3: ESO for the z-state when $r_1 = 0$, $r_2 = 0$.

From Figure 5.2, when the ESO tracks the disturbance, the system behaves more like a sinusoid. On the other hand, Figure 5.3 shows the system near the origin when the ESO becomes active. Earlier in the analytical result, we discovered that the accuracy of the ESO tracking the system highly depends on the observer bandwidth. Thus, changing the w_0 can give a slightly different result.

Instead of stabilizing the Lorenz system with disturbances to the origin, we can also tackle it differently. Let's say that we want the system to look more periodic. Hence, we examine two cases; first, when $r_1(t) \neq 0$, $r_2(t) \neq 0$, second, when only one of the reference signals of either the y or z state is zero. This allows us to observe how the states interfere with each other.

Let us consider the case where the reference signals for both y- and z-equations are sinusoids. Thus, we will let

$$r_1(t) = 5\cos(3t)$$

$$r_2(t) = 5\sin(4t)^2$$

for (5.1). Then, when ADRC Controller begins to track the disturbances, the system looks
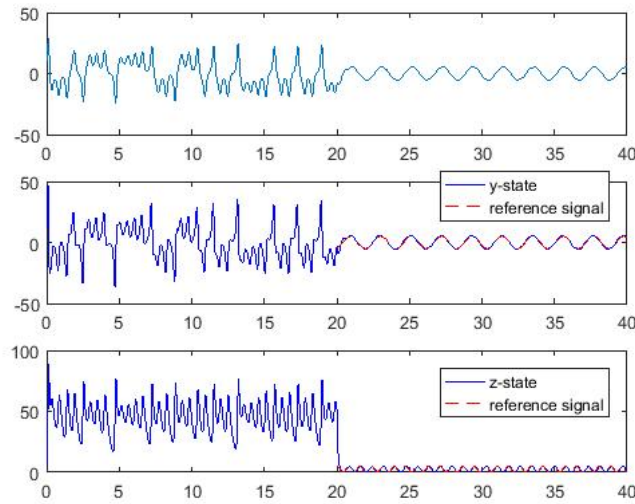
more like the periodic wave shown in Figure 5.4.

Figure 5.4: ADRC Controlled Lorenz system when $r_1(t) \neq 0$, $r_2(t) \neq 0$.

Finally, let us consider the case where the reference signal for one of the states is zero while the other one is not. For example, let $r_1(t) = 0$ and $r_2(t) = 5\sin(4t)^2$ for (5.1). In this case, we want the y state to be stabilized near the origin and z state to be near the given $r_2(t)$. As we expected, the result looks like the Figure 5.5.
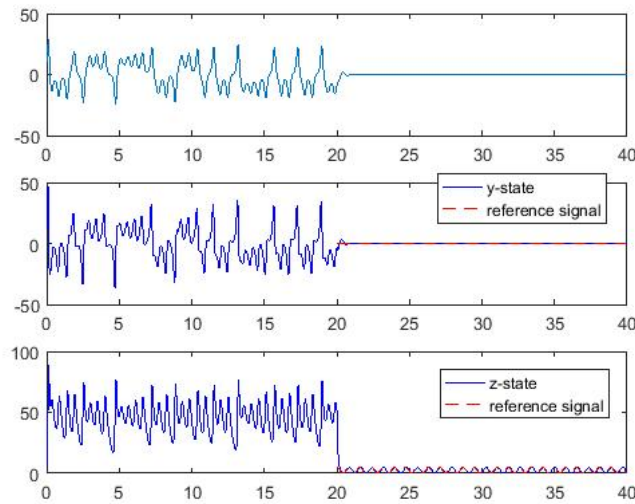
Figure 5.5: ADRC Controlled Lorenz system when $r_1(t) = 0$ and $r_2(t) = 5\sin(4t)^2$.

Instead of having the reference signal a sinusoid, we can let it be a constant number. For example, let $u_1 = 0$ and $u_2 = 8$. Thus, the result looks like the Figure 5.6.
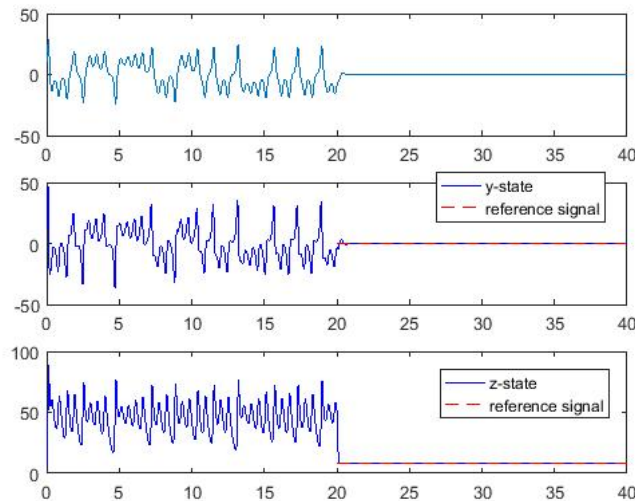

Figure 5.6: ADRC Controlled Lorenz system when $u_1 = 0$ and $u_2 = 8$.

We have just shown two cases to test the stability of the ADRC Controlled Lorenz system. However, instead of playing with the reference signal, we can also modify the

disturbance as described earlier. For example, let

$$w_1(t) = z \sin(10t)$$

$$w_2(t) = zy \sin(3t)^2$$

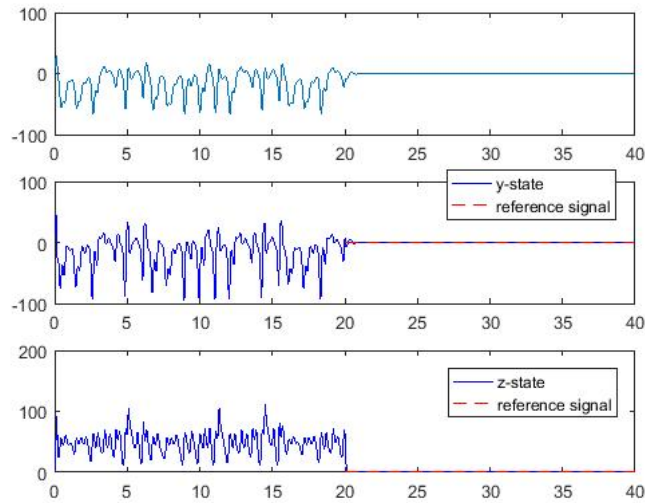with $r_1 = 0$, $r_2 = 0$. Then the result looks like the following Figure 5.7.

Figure 5.7: ADRC Controlled Lorenz system with $w_1(t) = z \sin(10t)$ and $w_2(t) = zy \sin(3t)^2$ when $r_1 = 0$, $r_2 = 0$.

CHAPTER 6

CONCLUSION

In reality, there are dynamics or disturbances that cannot be modeled and must be omitted during the modeling process. Thus, we used a robust controller called ADRC to stabilize the nonlinear dynamical system with disturbances. Not only have we used the ADRC but also the ESO to track the three states and disturbances in the system. As a result, we have successfully stabilized a system in two cases. The system is asymptotically stabilized when the model of the plant is given. When the disturbance is totally unknown, the state trajectories are bounded by the feedback control gains and the observer gains. We we have found that increasing the two gains will stabilize the system with unknown disturbances.

In this paper, we have designed a MIMO ADRC system from two SISO ADRC systems. See Wenchao, [14]. The two disturbances are w_1, w_2 in y- and z-equations, respectively. However, with the same system we used, we can build a MIMO system instead of a combination of two SISO systems. Recall (1.7),

$$\dot{x} = p(y - x)$$

$$\dot{y} = Rx - y - xz + w_1 + u_1$$

$$\dot{z} = xy - bz + w_2 + u_2$$

where x, y, z are three states, w_1, w_2 are the disturbances, and u_1, u_2 are the ADRC controllers. Instead $f_1 = Rx - xz + w_1$, we will let $f_1 = -xz + w_1 = x_4$. Moreover, $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_5 = w_2 + xy = f_2$. Now, the (1.7) can be written as

$$\dot{x}_1 = p(x_2 - x_1)$$

$$\dot{x}_2 = Rx_1 - x_2 + x_4 + u_1$$

$$\dot{x}_3 = -bx_3 + x_5 + u_2$$

$$\dot{x}_4 = \dot{f}_1$$

$$\dot{x}_5 = \dot{f}_2$$

Now the system is associated with 5×5 matrix. Thus, the characteristic polynomial for this matrix is $p(\lambda) = (\lambda + \omega_0)^5$. There will be ten observer gains we need to find and the system equations are nonlinear which is challenging to solve.

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