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Gorenstein Projective (Pre)Covers

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by

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(Under the Direction of Dr. Alina C. Iacob)

Abstract

The existence of the Gorenstein projective precovers is one of the main open problems in Gorenstein Homological algebra. We give sufficient conditions in order for the class of Gorenstein projective complexes to be special precovering in the category of complexes of R-modules $Ch(R)$. More precisely, we prove that if every complex in $Ch(R)$ has a special Gorenstein flat cover, every Gorenstein projective complex is Gorenstein flat, and every Gorenstein flat complex has finite Goenstein projective dimension, then the class of Gorenstein projective complexes, $\mathcal{GP}(\mathcal{C})$, is special precovering in $Ch(R)$.

Keywords: Module, Projective, Injective, Flat, Gorenstein

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Contents

1. Introduction

In 1966 M. Auslander defined the notion of the G-dimension of a finite module over a commutative noetherian local ring. In 1969 Auslander and Bridger extended the definition to two sided noetherian rings. Then in 1995 Enochs and Jenda defined Gorenstein projective modules (whether finitely generated or not) as modules of G-dimension zero, and Gorenstein injective modules over arbitrary rings. Avramov, Buchweitz, Martsinkovshy, and Reiten proved that if the ring R is both right and left noetherian and G is a finite Gorenstein projective module, then Enochs' and Jenda's definition agrees with that of Auslander and Bridger. Another extension of the G-dimension is based on Gorenstein flat modules which were introduced by Enochs, Jenda, and Torrecillas.

Gorenstein Homological algebra is the relative version of homological algebgra that replaces the projective, injective, and flat resolutions with their Gorenstein counterparts. However, there is a big difference between the two areas. Namely, while in classical Homological algebra, every module over any ring has a projective, injective, and flat resolution, it is still not known what is the most general type of ring R such that every left R-module has a Gorenstein projective, injective, and flat left and right resolution. In fact these are the main open problems in Gorenstein homological aglebra.

In this theis we consider the existence of the Gorenstein projective left resolutions. Their existence is known over Gorenstein rings [2]. Then P. Jorgensen [4] proved their existence over commutative noetherian rings with dualizing complexes. In 2011, D. Murfet and Sh. Slarian [5] extended Jorgensen's result to commutative noetherian rings of finite Krull dimesion. Recently D. Murfet and Sh. Salarian's result was extended to right coherent and left n-perfect rings in the work of Estrada-Iacob-Odabsi [3].

We give a sufficient condition for the existence of the special Gorenstein projective precovers in the category of complexes. We show that if the ring R is such that every complex in $Ch(R)$ has a special Gorenstein flat precover, every Gorenstein projective complex is Gorenstein flat and every Gorenstein flat complex has finite Gorenstein projective dimension, then $\mathcal{GP}(\mathcal{C})$ is special precovering in $Ch(R)$. In particular, this is the case for any right coherent and left n-perfect ring.

2. Basic Definitions and Examples

Definition. Let R be a ring (not necessarily commutative nor with 1). A *left R-module* or *left module over* R is a set M together with

(1) a binary operation $+$ on M under which M is an abelian group.
(2) an action of R on M (that is a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies
   (a) $(r+s)m = rm + sm$, for all $r, s \in R$, and for all $m \in M$
   (b) $(rs)m = r(sm)$, for all $r, s \in R$, and for all $m \in M$
   (c) $r(m+n) = rm + rn$, for all $r \in R$, and for all $m, n \in M$.
   If the ring R has a 1 we impose the additional axiom:
   (d) $1m = m$, for all $m \in M$.

The descriptor "left" in the above definition indicates that the ring elements appear on the left; "right" R-modules can be defined analogously. If the ring R is *commutative* and M is a left R-module we can make M into a right R-module by defining $mr = rm$ for $m \in M$ and $r \in R$. If R is not commutative, axiom 2(b) in general will not hold with this definition (so not every left R-module us also a right R-module). Unless explicitly mentioned otherwise the term "module" will always mean "left module." Modules satisfying axiom 2(d) are called *unital* modules and in this paper all rings will be assumed to be associative and all modules will be assumed to be unital (this is to avoid "pathologies" such as having $rm = 0$ for all $r \in R$ and $m \in M$).

Definition. If R and S are rings, then an abelian group M is said to be an (R,S)-*bimodule*, denoted ${}_RM_S$, if M is a left R-module and right S-module and the structures are compatible, that is, $(rx)s = r(xs)$ for all $r \in R$, $s \in S$, and $x \in M$. In particular, any ring R is naturally an (R,R)-bimodule.

Definition. Let R be a ring and let M be an R-module. An R-*submodule* of M is a subgroup N of M which is closed under the action of the ring elements, i.e., $rn \in N$, for all $r \in R$ and $n \in N$.

Submodules of M are therefore just subsets of M which are themselves modules under the restricted operations.

Examples

(1) Let R be any ring. Then $M = R$ is a left R-module, where the action of a ring element on a module element is just the usual multiplication in the ring R (similarly, R is a right R-module over itself). In particular, every field can be considered as a (1-dimensional) vector space over itself. When R is considered as a left module over itself, the submodules of R are precisely the left ideals of R (and if R is considered as a right R-module over itself, its submodules are the right ideals). Thus if R is not commutative it has a left and right module structure over itself and these structures may be different (e.g., the submodules may be different).

(2) Let R be a ring with 1 and let $n \in \mathbb{Z}^+$. Define

$$R^n = \{(a_1, a_2, \ldots, a_n) | a_i \in R \text{ for all } i\}.$$

Make R^n into an R-module by defining addition and multiplication componentwise:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, aa_n + b_n)$$

$$r(a_1, a_2, \ldots, a_n) = (ra_1, ra_2, \ldots, ra_n)$$

where $r \in R$. The module R^n is called the *free module of rank n over R*. An obvious submodule of R^n is given by the i^{th} component, namely the set of n-tuples with arbitrary ring elements in the i^{th} component and zeros in the j^{th} component for all $j \neq i$.

(3) The same abelian group may have the structure of an R-module for a number of different rings R and each of these module structures may carry useful information. Specifically, if M is an R-module and S a subring of R with $1_S = 1_R$, then M is automatically an S-module as well. For instance the field $\mathbb{R}$ is an $\mathbb{R}$-module, a $\mathbb{Q}$-module, and a $\mathbb{Z}$-module.

Example: $\mathbb{Z}$-modules

Let $R = \mathbb{Z}$, let A be *any* abelian group (finite or infinite) and write the operation of A as $+$. Make A into a $\mathbb{Z}$-module as follows: for any $n \in \mathbb{Z}$ and $a \in A$ define

$$na = \begin{cases} a + a + \cdots + a \ (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - \cdots - a \ (n \text{ times}) & \text{if } n < 0 \end{cases}$$

(here 0 is the identity of the additive group A). This definition of an action of the integers on A makes A into a $\mathbb{Z}$-module, and the module axioms show that this is the only possible action of $\mathbb{Z}$ on A making it a unital $\mathbb{Z}$-module. Thus every abelian group is a $\mathbb{Z}$-module. Conversely, if M is any $\mathbb{Z}$-module, a fortiori M is an abelian group, so

$$\mathbb{Z} - \textit{modules are the same as abelian groups.}$$

Furthermore, it is immediate from the definition that

$$\mathbb{Z} - \textit{submodules are the same as subgroups.}$$

Note that for the cyclic group $< a >$ written multiplicatively the additive notation na becomes a^n. Note also that since $\mathbb{Z}$ is commutative these definitions of left and right actions by ring elements give the same module structure.

If A is an abelian group containing an element x of finite order n, then $nx = 0$. Thus, in contrast to vector spaces, a $\mathbb{Z}$-module may have nonzero elements x such that $nx = 0$ for some nonzero ring element n. In particular, if A has order m, then by Lagrange's Theorem $mx = 0$, for all $x \in A$. Note then that A is a module over $\mathbb{Z}/m\mathbb{Z}$.

In particular, if p is prime and A is an abelian group (written additively) such that $px = 0$, for all $x \in A$, then A is a $\mathbb{Z}/p\mathbb{Z}$ module, i.e., can be considered as a vector space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Proposition [1] (*The Submodule Criterion*) Let R be a ring and M an R-module. A subset N of M is a submodule if and only if

(1) $N \neq \varnothing$

(2) $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

3. Quotient Modules and Module Homomorphisms

Definition. Let R be a ring and let M and N be R-modules.

(1) A map $\varphi : M \to N$ is an *R-module homomorphism* if it respects the R-module structures of M and N, i.e.,
   (a) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in M$
   (b) $\varphi(rx) = r\varphi(x)$, for all $x \in M$, and for all $r \in R$.

(2) An R-module homomorphism is an *isomorphism* (*of R-modules*) if it is both injective and surjective. The modules M and N are said to be *isomorphic*, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.

(3) If $\varphi : M \to N$ is an R-module homomorphism, let the *kernel* of φ, denoted $\mathrm{Ker}(\varphi)$, be defined by $\mathrm{Ker}(\varphi) = \{m \in M \mid \varphi(m) = 0\}$, let the image of φ, denoted $\mathrm{Im}(\varphi)$, be defined by $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$), and let the *cokernel* of φ, denoted $\mathrm{Coker}(\varphi)$, be defined by $\mathrm{Coker}\ \varphi = N/\mathrm{Im}\,\varphi$.

(4) Let M and N be R-modules and define $\mathrm{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M to N.

Note that any R-module homomorphism is also a homomorphism of the additive groups, but not every group homomorphism need be a module homomorphism (because condition (b) may not be satisfied).

Examples

(1) If R is a ring and $M = R$ is a module over itself, then R-module homomorphisms (even from R to itself) need not be ring homomorphisms and ring homomorphisms need not be R-module homomrphism. For example, when $R = \mathbb{Z}$ the $\mathbb{Z}$-module homomorphism $x \mapsto 2x$ is not a ring homomoprhism (1 does not map to 1). When $R = F[x]$ the ring homorphism $\varphi : f(x) \mapsto f(x^2)$ is not an $F[x]$-module homorphism (if it were, we would have $x^2 = \varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = x$).

(2) Let R be a ring, let $n \in \mathbb{Z}^+$, and let $M = R^n$. Then we have that for each $i \in \{1, 2, \ldots, n\}$ the projection map

$$\pi_i : R^n \to R \quad \text{by} \quad \pi_i(x_1, \ldots, x_n) = x_i$$

is a surjective R-module homomorphism with kernel equal to the submodule of n-tuples which have a zero in position i.

(3) For the ring $R = \mathbb{Z}$ the action of ring elements (integers) on any $\mathbb{Z}$-module amounts to just adding and subtracting within the (additive) abelian group structure of the module so that in this case condition (b)of a homomorphism is implied by condition (a). For example, $\varphi(2x) = \varphi(x+x) = \varphi(x)+\varphi(x) = 2\varphi(x)$, etc. It follows that

$\mathbb{Z}$–module homomorphisms are the same as abelian group homomorphisms.

Proposition [1] Let M, N, and L be R-modules

(1) A map $\varphi : M \to N$ is an R-module homomorphism if and only if $\varphi(rx + y) = r\varphi(x) + \varphi(y)$ for all $x, y \in M$ and for all $r \in R$.

(2) Let φ, ψ be elements of $\mathrm{Hom}_R(M, N)$. Define $\varphi + \psi$ by

$$(\varphi + \psi)(m) = \varphi(m) + \psi(m) \quad \text{for all } m \in M.$$

Then $\varphi + \psi \in \mathrm{Hom}_R(M, N)$ and with this operation $\mathrm{Hom}_R(M, N)$ is an abelian group. If R is a commutative ring then for $r \in R$ define $r\varphi$ by

$$(r\varphi)(m) = r(\varphi(m)) \quad \text{for all } m \in M.$$

Then $r\varphi \in \mathrm{Hom}_R(M, N)$ and with this action of the commutative ring R the abelian group $\mathrm{Hom}_R(M, N)$ is an R-module.

Proposition [1] Let R be a ring, let M be an R-module, and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x + N) = (rx) + N, \quad \text{for all } r \in R, x + N \in M/N.$$

The natural projection map $\pi : M \to M/N$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

Theorem [1] (Isomorphism Theorem)

(1) Let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then $\ker \varphi$ is a submodule of M and $M/\ker\varphi \cong \varphi(M)$.

Proposition [2] If M is an R-module, then the map $\varphi : \mathrm{Hom}_R(R, M) \to M$ defined by $\varphi(f) = f(1)$ is an R-module isomorphism.

4. Generation of Modules, Direct Sums, and Free Modules

Definition. Let M be an R-module and let $N_1, \ldots, N_n$ be submodules of M.

(1) The *sum* of $N_1, \ldots, N_n$ is the set of all finite sums of elements from the sets $N_i : \{a_1 + a_2 + \cdots + a_n | a_i \in N_i \quad \text{for all } i\}$. Denote this sum by $N_1 + \cdots + N_n$.

(2) For any subset A of M let

$$RA = \{r_1 a_1 + r_2 a_2 + \ldots + r_m a_m | r_1, \ldots, r_m \in R, a_1, \ldots, a_m \in A, m \in \mathbb{Z}^+\}$$

(where by convention $RA = \{0\}$ if $A = \varnothing$). If A is the finite set $\{a_1, \ldots, a_n\}$ we shall write $Ra_a + \cdots Ra_n$ for RA. Call RA the *submodule of M generated by A*. If N is a submodule of M (possibly $N = M$) and $N = RA$, for some subset A of M, we call A a *set of generators* or *generating set* for N, and we say N is *generated by A*.

(3) A submodule N of M (possibly $N = M$) is *finitely generated* if there is some finite subset A of M such that $N = RA$, that is, if N is generated by some finite subset.

(4) A submodule N of M (possibly $N = M$) is *cyclic* if there exists an element $a \in M$ such that $N = Ra$, that is, if N is generated by one element:

$$N = RA = \{ra | r \in R\}.$$

Definition. Let $M_1, \ldots, M_k$ be a collection of R-modules. The collection of k-tuples $(m_1, \ldots, m_k)$ where $m_i \in M_i$ with addition and action of R defined componentwise is called the *direct product* of $M_1, \ldots, M_k$ and is denoted $\prod_{i \in I} M_i = M_1 \times \cdots \times M_k$.

Proposition [2] Let M be an R-module, $(N_i)_{i \in I}$ a collection of R-modules, and $e_j : N_j \to \oplus_I N_i$ be the jth embedding. Then the map

$$\varphi : \operatorname{Hom}_R\left(\bigoplus_I N_i, M\right) \to \prod_I \operatorname{Hom}_R(N_i, M)$$

defined by $\varphi(f) = (f \circ e_i)_I$ is an isomorphism.

A similar proof gives the following.

Proposition [2] Let M be an R-module, $(N_i)_{i \in I}$ a collection of R-modules, and $\pi_j : \prod_I N_i \to N_j$ for each j be the projection map. Then the map

$$\varphi : \operatorname{Hom}_R\left(M, \prod_I N_i\right) \to \prod_I \operatorname{Hom}_R(M, N_i)$$

defined by $\varphi(f) = (\pi_i \circ f)_I$ is an isomorphism.

Definition. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exists unique nonzero elements $r_1, \dots, r_k$ of R and unique $a_1, \dots, a_k$ in A such that $x = r_1 a_1 + \cdots + r_k a_k$, for some $k \in \mathbb{Z}^+$. Equivalently, an R-module F is said to be *free* if it is a direct sum of copies of R, that is, if $M = \oplus_{i \in I} M_i$ where $M_i = R$ for all i.

Proposition Every R-module is a quotient of a free R-module.

Proof. Let M be an R-module and $\{x_i : i \in I\}$ be a set of generators of M. Then $R^{(I)} = \oplus_{i \in I} R_i$, where $R_i = R$ for all i, is a free R-module. Define a map $\varphi : R^{(I)} \to M$ by $\varphi((r_i)_{i \in I}) = \sum_{i \in I} r_i x_i$. Then φ is surjective and so $M \cong R^{(I)}/\mathrm{Ker}\varphi$ by the first Isomorphism Theorem.

□

Corollary An R-module is finitely generated if and only if it is a quotient of R^n for some integer $n > 0$.

5. Tensor Products of Modules

Definition. Let M be a right R-module, N be a left R-module, and G an Abelian group. Then a map $\sigma : M \times N \to G$ is said to be *balanced* if it is additive in both variables (*biadditive*), that is,

$$\begin{aligned} \sigma(x + x', y) &= \sigma(x, y) + \sigma(x', y) \\ \sigma(x, y + y') &= \sigma(x, y) + \sigma(x, y') \\ \sigma(xr, y) &= \sigma(x, ry) \end{aligned}$$

for all $x, x' \in M, y, y' \in N$, and $r \in R$. Note that the term *bilinear* is used when R is commutative and when we add the condition $\sigma(x, ry) = r\sigma(x, y)$ for all x, y, and r.

Definition. A balanced map $\sigma : M \times N \to G$ is said to be *universal* or we say σ solves the "*universal mapping problem*" for G, if for every Abelian group G' and balanced map $\sigma' : M \times N \to G'$, there exists a unique map $h : G \to G'$ such that $\sigma' = h\sigma$.

Definition. A *tensor product* of a right R-module M and a left R-module N is an Abelian group T together with a universal balanced map $\sigma : M \times N \to T$.
If $\sigma : M \times N \to T$ and $\sigma' : M \times N \to T'$ are both universal balanced maps, then we can complete the diagram

$$\begin{tikzcd} & T \arrow[d, "h"] \\ M \times N \arrow[ur, "\sigma"] \arrow[r, "\sigma'"] \arrow[dr, "\sigma"'] & T' \arrow[d, "f"] \\ & T \end{tikzcd}$$

such that the diagram commutes. Then we have that $fh = \mathrm{id}_T$. Similarly, we have $hf = \mathrm{id}_{T'}$ and thus h is an isomorphism. Thus tensor products are unique up to isomorphism. We will thus speak of the tensor product of M_R and $_RN$, and will denote it by $M \otimes_R N$ or simpliy $M \otimes N$ if the ring R is understood.

Theorem The tensor product of M_R and $_RN$ exists.

Proof. Let F be the free abelian group with base $M \times N$, that is,

$$F = \left\{ \sum_i m_i(x_i, y_i) : m_i \in \mathbb{Z}, (x_i, y_i) \in M \times N \right\}.$$

Let S be the subgroup of F generated by elements of F of the form

$$(x+x',y)-(x,y)-(x',y),(x,y+y')-(x,y)-(x,y'),(rx,y)-(x,ry)$$

where $x, x' \in M, y, y' \in N$, and $r \in R$. Define a map $\sigma : M \times N \to N/S$ by $\sigma(x,y) = (x,y)+S$. Then σ is clearly balanced. Now let $\sigma' : M \times N \to G'$ be a balanced map into an Abelian group G'. However, since F is free on $M \times N$, there is a unique homomorphism $h' : F \to G'$ that extends σ', that is, $h'(x,y) = \sigma'(x,y)$. But clearly $S \subset \mathrm{Ker}h'$ since σ' is balanced. Then we get a unique induced map $h : F/S \to G'$ such that $\sigma' = h\sigma$. Thus $F/S = M \otimes_R N$. $\square$

Remark We see from the proof above that F/S is generated as an Abelian group by cosets $(x,y)+S$. We denote $(x,y)+S$ by $x \otimes y$. Then $M \otimes_R N$ is generated as an Abelian group by the elements $x \otimes y$. Since $-(x \otimes y) = (-x) \otimes y$, the elements of $M \otimes_R N$ are of the form $\sum x_i \otimes y_i$. Furthermore, if $x, x' \in M, y, y' \in N$, and $r \in R$, then

$$\begin{aligned}(x+x')\otimes y &= x\otimes y + x'\otimes y,\\ x\otimes(y+y') &= x\otimes y + x\otimes y',\\ (xr)\otimes y &= x\otimes(ry).\end{aligned}$$

Examples

(1) Let M be a right R-module, N be a left R-module, and $m \in M$. Then in the tensor product $M \otimes_R N$ we have

$$m\otimes 0 = m\otimes(0+0) = m\otimes 0 + m\otimes 0.$$

Thus $m \otimes 0 = 0$. Similarly, we have that $0 \otimes n = 0$ for any $n \in N$.

(2) We have that $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$. To see this we first notice that $x3 = x$ for any $x \in \mathbb{Z}/2\mathbb{Z}$. Then we have that $x \otimes y = x3 \otimes y = x \otimes 3y$. However, $3y = 0$ for any $y \in \mathbb{Z}/3\mathbb{Z}$. Thus $x \otimes y = x \otimes 3y = x \otimes 0 = 0$ by the previous example.

We establish some properties of the tensor product.

Proposition [2] $M_R \otimes R \cong M$ for every right R-module M, and $R \otimes_R N \cong N$ for every left R-module N.

Theorem [2] (Associativity of the Tensor Product) Suppose M is a right R-module, N is an (R, T)-bimodule, and L is a left T-module. Then there is a unique isomorphism

$$(M \otimes_R N) \otimes_T L \cong M \otimes_R (N \otimes_T L)$$

of abelian groups such that $(m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)$. If M is an (S, R)-bimodule, then this is an isomorphism of S-modules.

Theorem [2] (Tensor Product of Direct Sums) Let $(M_i)_I$ be a collection of right R-modules and N be a left R-module. Then

$$\left(\bigoplus_I M_i\right) \otimes_R N \cong \bigoplus_I (M_i \otimes N).$$

Similarly we have the following result:

Theorem [2] Let $(N_i)_I$ be a collection of left R-modules and let N be a right R-module. Then

$$M \otimes_R (\bigoplus N_i) \cong \bigoplus_I (M \otimes N_i).$$

Theorem [1] Suppose that R is a commutative ring. Then for and R-modules M and N, we have

$$M \otimes_R N \cong N \otimes_R M.$$

Proposition [1] (Extension of Scalars for Free Modules) The module obtained from the free R-module $N \cong R^n$ by extension of scalars from R to S is the free S-module S^n, i.e.,

$$S \otimes_R R^n \cong S^n$$

Proposition [1] Let $f : M \to M'$ and $g : N \to N'$ be homomorphisms of right and left R-modules respectively. Then there is a unique homomorphism $h : M \otimes_R N \to M' \otimes_R N'$ such that $h(x \otimes y) = f(x) \otimes g(y)$.

Remark The map $h : M \otimes_R N \to M' \otimes_R N'$ in the above propostion is denoted by $f \otimes g$. Now suppose $f' : M' \to M"$ and $g' : N' \to N"$ are homomorphisms of right and left R-modules respectively. Then we get a map $f' \otimes g' : M' \otimes N' \to M" \otimes N"$ and it is clear that $(f' \otimes g') \circ (f \otimes g) = f'f \otimes g'g$ by evaluating the maps on a generator $x \otimes y \in M \otimes N$. We also note that $\mathrm{id}_M \otimes \mathrm{id}_N : M \otimes N \to M \otimes N$ is clearly the identity on $M \otimes N$, and if $f : M \to N$ and $g : N \to N'$ are isomorphisms, then $f \otimes g$ is an isomorphism with $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$.

Now let I be a right ideal of R and M be an R-module. Then IM, the set of all finite sums of the form $\sum_{i=1}^{n} r_i x_i$, where $r_i \in I$ and $x_i \in M$, is a subgroup of M.

Corollary Let I be a right ideal of R and M be a left R-module. Then

$$(R/I) \otimes_R M \cong M/IM.$$

Definition. The intersection of all maximal left ideals of a ring R is called the *Jacobson radical* of R and is denoted $\mathrm{rad}(R)$. an R-module M is said to be *simple* if it is isomorphic to $R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of R, or equivalently, has no submodules except for the trivial module 0 and the ring R itself. Thus it is easy to see that $\mathrm{rad}(R) = \{r \in R : rM = 0$ for every simple left R-module $M\}$. Si $\mathrm{rad}(R)$is a two-sided ideal of R. Moreover, $\mathrm{rad}(R)$ consists precisely of elements $r \in R$ such that $1 - sr$ is invertible for all $s \in R$. But then $1 - sr \notin \mathfrak{m}$ for each maximal left ideal $\mathfrak{m}$ of R and every $s \in R$. Hence $1 - sr$ is invertible for if not then the left-ideal $R(1 - sr)$ would be contained in some maximal left ideal $\mathfrak{m}$. Conversely, if $r \notin \mathrm{rad}(R)$, then $r \notin \mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$. But then $Rr + \mathfrak{m} = R$ and so there is an $s \in R$ such that $1 - sr \in \mathfrak{m}$. That is, $1 - sr$ is not invetible. In particular, if $r \in \mathrm{rad}(R)$, then $1 - r$ is not invertible.

Proposition [2] (Nakayama's Lemma) Let M be an R-module and I be a submodule of the additive group of R such that either

(a) I is nilpotent (that is $I^n = 0$ for some $n \geq 1$)

(b) $I \subset \mathrm{rad}(R)$ and M is finitely generated.

Then $IM = M$ implies that $M = 0$.

Corollary [2] Let M be an R-module, N be a submodule of M, and I a subgroup of the additive group of R such that either

(a) I is nilpotent

(b) $I \subset \mathrm{rad}(R)$ and M is finitely generated.

Then $IM + N = M$ implies that $M = N$.

Proposition [2] If M is a nonzero finitely generated R-module and $I \subset \mathrm{rad}(R)$ is a right ideal, then $(R/I) \otimes_R M \neq 0$.

6. Exact Sequences

Definition.

(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* (at Y) if $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

(2) A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an *exact sequence* if it is exact at every X_n between a pair of homomorphisms.

Proposition Let A, B, and C be R-modules over some ring R. Then

(1) The sequence $0 \to A \xrightarrow{\psi} B$ is exact (at A) if and only if ψ is injective.

(2) The sequence $B \xrightarrow{\varphi} C \to 0$ is exact (at C) if and only if φ is surjective.

Proof. The (uniquely defined) homomorphism $0 \to A$ has image 0 in A. This will be the kernel of ψ if and only if ψ is injective. Similarly, the (uniquely defined) zero homomorphism $C \to 0$ is all of C, which is the image of φ if and only if φ is surjective.

$\square$

Corollary The sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is exact if and only if ψ is injective, φ is surjective, and $\mathrm{Im}(\psi)$=$\mathrm{Ker}(\varphi)$.

Definition. The exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is called a *short exact sequence*. In this case, $\mathrm{Coker}(\psi) = B/\mathrm{Im}(\psi) \cong C$.

Remark Let M be an R-module. Then M is a quotient of a free R-module, say F_0 by Proposition *. Then we have a short exact sequence

$$0 \longrightarrow K_1 \longrightarrow F_0 \xrightarrow{\partial_0} M \longrightarrow 0 \tag{1.1}$$

where $M \cong F_0/K_1$ from above.
But K_1 is quotient of a free module, say F_1. Then we have an exact sequence

$$0 \longrightarrow K_2 \longrightarrow F_1 \xrightarrow{\partial_1} K_1 \longrightarrow 0. \tag{1.2}$$

Now combine (1.1) and (1.2) to get

$$0 \longrightarrow K_2 \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0. \tag{1.3}$$

Note that $\mathrm{Im}(\partial_1) = K_1 = \mathrm{Ker}(\partial_0)$.
Now repeat to get an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Where each F_i is a free R-module. This is called a *free resolution* of M.

Proposition [2] The following statements hold:

(1) If $0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$ is an exact sequence of R-modules, then for each R-module M the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, N') \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}_R(M, N) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}_R(M, N'')$$

is also exact.

(2) If $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules, then for each R-module N the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \xrightarrow{\mathrm{Hom}(g,N)} \mathrm{Hom}_R(M, N) \xrightarrow{\mathrm{Hom}(f,N)} \mathrm{Hom}_R(M', N)$$

is also exact.

Proposition [2] If $N' \xrightarrow{f} N \xrightarrow{g} N" \to 0$ is an exact sequence of left R-modules, then for each right R-module M, the sequence $M \otimes N' \xrightarrow{\mathrm{id}_M \otimes f} M \otimes N \xrightarrow{\mathrm{id}_M \otimes g} M \otimes N" \to 0$ is also exact.

Proposition [2] (Snake Lemma) Suppose

$$\begin{array}{ccccccc} & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{f'} & N & \xrightarrow{g'} & N' & & \end{array}$$

is a commutative diagram (that is, $f'\sigma' = \sigma f$ and $g'\sigma = \sigma'' g$) of R-modules with exact rows. Then there is an exact sequence

$$\mathrm{Ker}(\sigma') \xrightarrow{\overline{f}} \mathrm{Ker}(\sigma) \to \mathrm{Ker}(\sigma'') \xrightarrow{d} \mathrm{Coker}(\sigma') \to \mathrm{Coker}(\sigma) \xrightarrow{\overline{g}'} \mathrm{Coker}(\sigma'').$$

Furthermore, if f is injective, then $\overline{f}$ is also injective and if g' is surjective, then $\overline{g}'$ is also surjective. The map d is called a *connecting homomorphism.*

Definition. An exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ of R-modules is said to be *split exact*, or we say the sequence *splits*, if $\mathrm{Im}(f)$ is a direct summand of M.

Proposition [2] Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules. Then the following are equivalent:

(1) The sequence is split exact

(2) There exists an R-module homomorphism $f' : M \to M'$ such that $f' \circ f = \mathrm{id}_{M'}$

(3) There exists an R-module homomorphism $g'' : M'' \to M$ such that $g \circ g'' = \mathrm{id}_{M''}$

Definition. Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules.

(1) A *homomorphism of short exact sequences* is a triple α, β, γ of module homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

The homomorphism is an *isomorphism of short exact sequences* if α, β, and γ are all isomomorphisms, in which case the extensions B and B' are said to be *isomorphic extensions.*

(2) The two extensions are called *equivalent* if $A = A', C = C'$, and there is an isomorphism between them as in (1) that is the identity maps on A and C (i.e., α and γ are the identity). In this case the corresponding extensions B and B' are said to be *equivalent* extensions.

Proposition [1] (The Short Five Lemma) Let α, β, and γ be a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

(1) If α and γ are injective, then so is β.

(2) If α and γ are surjective, then so is β.

(3) If α and γ are isomorphisms, then so is β (and the two sequences are isomorphic).

7. Categories and Functors

Definition. A *category* C consists of the following:

(1) a class of objects, denoted $\mathrm{Ob}(\mathsf{C})$

(2) for any pair $A, B \in \mathrm{Ob}((\mathrm{C}))$, a set denoted $\mathrm{Hom}_{\mathsf{C}}(A,B)$ with the property that $\mathrm{Hom}_{\mathsf{C}}(A,B) \cap \mathrm{Hom}_{\mathsf{C}}(A',B') = \varnothing$ whenever $(A,B) \neq (A',B')$. The set $\mathrm{Hom}_{\mathsf{C}}(A,B)$ is called the *set of morphisms* from A to B. If $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ we write $f : A \to B$ and say f is a *morphism* of C from A to B.

(3) a *composition* $\mathrm{Hom}_{\mathsf{C}}(B,C) \times \mathrm{Hom}_{\mathsf{C}}(A,B) \to \mathrm{Hom}_{\mathsf{C}}(A,C)$ for all objects A, B, and C, and is denoted $(g,f) \mapsto gf$ (or $g \circ f$), satisfying the following properties:

   (i) for each $A \in \mathrm{Ob}(\mathsf{C})$, there is an *identity morphism*, denoted $\mathrm{id}_A \in \mathrm{Hom}_{\mathsf{C}}(A,A)$ such that $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$.

   (ii) $h(gf) = (hg)f$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$, $g \in \mathrm{Hom}_{\mathsf{C}}(B,C)$, and $h \in \mathrm{Hom}_{\mathsf{C}}(C,D)$

In this thesis we will be considering the category ${}_R\mathbf{Mod}$ whose objects are left R-modules and whose morphisms are R-module homomorphisms.

Now let $\mathrm{Mor}(\mathsf{C})$ denote the set of all morphisms of C. Then

$$\mathrm{Mor}(\mathsf{C}) = \bigcup_{A,B \in \mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(A,B).$$

If $f : A \to B$ is a morphism in C, then f is said to be an *isomorphism* if there is a morphism $g : B \to A$ in C such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$. Clearly, g is unique if it exists, and is denoted by f^{-1}. The morphism f is said to be a *monomorphism* if for every pair of morphisms $g, h : C \to A$, in C, we have that $fg = fh$ implies that $g = h$. The morphism f is said to be a *epimorphism* if for every pair of morphisms $g, h : B \to C$, in C, we have that $gf = hf$ implies that $g = h$.

Definition. If C and C' are categories, then C' is said to be a *subcategory* of C if

(1) $\mathrm{Ob}(\mathsf{C}') \subset \mathrm{Ob}(\mathsf{C}), \mathrm{Mor}(\mathsf{C}') \subset \mathrm{Mor}(\mathsf{C})$, and $\mathrm{Hom}_{\mathsf{C}'}(A',B') = \mathrm{Hom}_{\mathsf{C}}(A',B') \cap \mathrm{Mor}(\mathsf{C}')$

(2) For any $A' \in \mathrm{Ob}(\mathsf{C}')$, the identity morphism on A' in C and C' are the same, and if $f' \in \mathrm{Hom}_{\mathsf{C}'}(A',B')$, and $g' \in \mathrm{Hom}_{\mathsf{C}'}(B',C')$, then the map $g' \circ f'$ is the same in C' as it is in C.

Definition. A subcategory C' of C is said to be a *full subcategory* if $\mathrm{Hom}_{\mathsf{C}'}(A,B) = \mathrm{Hom}_{\mathsf{C}}(A,B)$ for all $A, B \in \mathrm{Ob}(\mathsf{C}')$.

Definition. If C and D are categories, then we say that we have a *functor* $F : \mathsf{C} \to \mathsf{D}$ if we have

(1) a function $F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$

(2) functions $F : \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(A), F(B))$ such that

 (i) if $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$, then $F(gf) = F(g)F(f)$

 (ii) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for each $A \in \mathrm{Ob}(\mathsf{C})$.

A functor is sometimes called a *covariant functor*. A function $\mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$ is said to be *functorial* if it agrees with a functor from C to D.

Example

(1) Let M be a left R-module. Define $F :_R\mathbf{Mod} \to \mathbf{Ab}$ by $F(N) = \mathrm{Hom}_R(M, N)$ (where $\mathbf{Ab}$ is the category of abelian groups whose morphisms are group homomorphisms). For $f \in \mathrm{Hom}(N', N)$ define $F(f) : \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N)$ by $F(f)(h) = fh$. Then F is a covariant functor and is denoted by $\mathrm{Hom}(M, —)$.

Definition. We say that we have a *contravariant functor* $F : \mathsf{C} \to \mathsf{D}$ if we have

(1) a function $F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$

(2) functions $F : \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(B), F(A))$ such that

 (a) if $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$, then $F(gf) = F(f)F(g)$

 (b) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for each $A \in \mathrm{Ob}(\mathsf{C})$.

Example

(1) Let M be a left R-module. Define $F :_R\mathbf{Mod} \to \mathbf{Ab}$ by $F(N) = \mathrm{Hom}_R(N, M)$. For $f \in \mathrm{Hom}(N', N)$ define $F(f) : \mathrm{Hom}(N, M) \to \mathrm{Hom}(N', M)$ by $F(f)(h) = hf$. Then F is a contravariant functor and is denoted by $\mathrm{Hom}(—, M)$.

Definition. If C and D are abelian categories, then a covariant functor $F : \mathsf{C} \to \mathsf{D}$ is said to be *left exact* if for every short exact sequence $0 \to A \to B \to C \to 0$ in C the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in D and F is said to be *right exact* if $F(A) \to F(B) \to F(C) \to 0$ is exact. If the functor F is contravariant, then it is *left exact* if $0 \to F(C) \to F(B) \to F(A)$ is exact and *right exact* if $F(C) \to F(B) \to F(A) \to 0$ is exact. A functor F is said to be an *exact* functor if it is both left and right exact.

Definition. Let C be an abelian category. Then a *pushout* of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

in C is an object D together with morphisms $h : B \to D$ and $k : C \to D$ such that $kg = hf$ and if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"] & D' \end{tikzcd}$$

is any commutative diagram in C, then there is a unique morphism $D \to D'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \arrow[ddr, "g'"] & \\ C \arrow[r, "k"] \arrow[drr, "f'"'] & D \arrow[dr] & \\ & & D' \end{tikzcd}$$

is commutative. The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

in the above is called a *pushout diagram.*

Dually, a *pullback diagram* is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

such that if

$$\begin{array}{ccc} P' & \xrightarrow{h'} & A \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is any commutative diagram in C, then there is a unique morphism $\sigma : P' \to P$ such that $h\sigma = h'$ and $k\sigma = k'$. In this case, P with morphisms h, k is called a *pullback* of morphisms $f : A \to C$ and $g : B \to C$.

8. Complexes of Modules

Definition. A *(chain) complex* $\mathbf{C}$ of R-modules is a sequence

$$\mathbf{C} : \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots$$

of R-modules and R-module homomorphisms such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$. The complex $\mathbf{C}$ is denoted $((\mathbf{C}_n), (\partial_n))$. If F is a covariant additive functor into some category of modules, then the sequence

$$F(\mathbf{C}) : \cdots \to F(C_2) \xrightarrow{F(\partial_2)} F(C_1) \xrightarrow{F(\partial_1)} F(C_0) \xrightarrow{F(\partial_0)} F(C_{-1}) \xrightarrow{F(\partial_{-1})} F(C_{-2}) \to \cdots$$

is also a chain complex. Similarly if F is a contravariant additive functor then the sequence

$$F(\mathbf{C}) : \cdots \to F(C_{-2}) \xrightarrow{F(\partial_{-1})} F(C_{-1}) \xrightarrow{F(\partial_0)} F(C_0) \xrightarrow{F(\partial_1)} F(C_1) \xrightarrow{F(\partial_2)} F(C_2) \to \cdots$$

is also a chain complex.

Definition. Let $\mathbf{C} = ((C_n), (\partial_n))$ and $\mathbf{C'} = ((C'_n), (\partial'_n))$ be two complexes of R-modules. Then a *map (or chain map)* $f : \mathbf{C} \to \mathbf{C'}$ is a sequence of maps $f_n : \mathbf{C}_n \to \mathbf{C'}_n$ such that the diagram

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \end{tikzcd}$$

is commutative for each $n \in \mathbb{Z}$. The map f is denoted by (f_n).

Remark: We note that if $g = (g_n) : \mathbf{C} \to \mathbf{C'}$ is another map, then $(f_n + g_n) : \mathbf{C} \to \mathbf{C'}$ is also a map. If $h : \mathbf{C'} \to \mathbf{C''}$ is another map of complexes, then we can define a map $hf : \mathbf{C} \to \mathbf{C''}$ by $hf = (h_n \circ f_n)$. Then we get a category, the category of complexes of R-modules which is denoted by $Ch(R)$. We note that $Ch(R)$ is an Abelian category.

Definition. If $\mathbf{C} = ((C_n), (\partial_n))$ is a complex, then Im $\partial_{n+1} \subset$ Ker ∂_n. Then the nth *homology module* of $\mathbf{C}$ is defined to be Ker ∂_n/Im ∂_{n+1} and is denoted by $H_n(\mathbf{C})$. So $H_n(\mathbf{C}) = 0$ if and only if $\mathbf{C}$ is exact at C_n. We note that Ker ∂_n and Im ∂_{n+1} are usually denoted by $Z_n(\mathbf{C})$ and $B_n(\mathbf{C})$ and their elements are called *n-cycles* and *n-boundaries* respectively.

Now suppose that $f : \mathbf{C} \to \mathbf{C'}$ is a chain map. Then we have the following commutative diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1}
\end{tikzcd}$$

If $x \in \mathrm{Ker}\partial_n$, then $\partial'_n(f_n(x)) = f_{n-1}(\partial_n(x)) = 0$ and so $f_n(x) \in \mathrm{Ker}\partial'_n$. Hence we get an induced map $\mathrm{Ker}\partial_n \to \mathrm{Ker}\partial'_n$. Futhermore, suppose that $x \in \mathrm{Im}\partial_{n+1}$. Then $x = \partial_{n+1}(y)$, where $y \in C_{n+1}$. So we have $\partial'_{n+1}(f_{n+1}(y)) = f_n(\partial_{n+1}(y)) = f_n(x)$. That is $f_n(x) \in \mathrm{Im}\partial'_{n+1}$. So we consider the composition $\mathrm{Ker}\partial_n \to \mathrm{Ker}\partial'_n \to \mathrm{Ker}\partial'_n/\mathrm{Im}\partial'_{n+1} = H_n(\mathbf{C'})$. This composition maps $\mathrm{Im}\partial_{n+1}$ onto zero by the above. So we get an induced map

$$H_n(\mathbf{C}) = \mathrm{Ker}\ \partial_n/\mathrm{Im}\ \partial_{n+1} \to \mathrm{Ker}\ \partial'_n/\mathrm{Im}\ \partial'_{n+1} = H_n(\mathbf{C'})$$

given by $x + \mathrm{Im}\ \partial_{n+1} \mapsto f_n(x) + \mathrm{Im}\ \partial'_{n+1}$. This map is denoted by $H_n(f)$.

Remark We note that if $g : \mathbf{C'} \to \mathbf{C''}$ is another chain map, then $H_n(g) : H_n(\mathbf{C'}) \to H_n(\mathbf{C''})$ maps $x' + \mathrm{Im}\ \partial'_{n+1}$ onto $g_n(x') + \mathrm{Im}\ \partial''_{n+1}$. Hence $H_n(g) \circ H_n(f) = H_n(g \circ f)$. Also $H_n(\mathrm{id}_C) = \mathrm{id}_{H_n(C)}$ and if $f_1, f_2 : \mathbf{C} \to \mathbf{C'}$ are chain maps, then $H_n(f_1 + f_2) = H_n(f_1) + H_n(f_2)$. This gives the following result.

Theorem [2] The functor H_n :Ch(R)$\to_R$ **Mod** defined by $H_n(\mathbf{C}) = \mathrm{Ker}\ \partial_n/\mathrm{Im}\ \partial_{n+1}$ is an additive covariant functor.

Definition. A complex $\mathbf{C'} = ((C'_n), (\partial'_n))$ is said to be a *subcomplex* of the complex $\mathbf{C} = ((C_n), (\partial_n))$ if $C'_n \subset C_n$ and ∂_n agrees with ∂'_n on C'_n. In this case, we can form a complex $((C_n/C'_n), (\bar{\partial_n}))$ where $\bar{\partial_n} : C_n/C'_n \to C_{n-1}/C'_{n-1}$ is the induced map given by $\bar{\partial_n}(x + C'_n) = \partial_n(x) + C'_{n-1}$. This complex is called the *quotient complex* and is denoted by $\mathbf{C}/\mathbf{C'}$.

Definition. If $f : \mathbf{C} \to \mathbf{C'}$ and $g : \mathbf{C'} \to \mathbf{C''}$ are chain maps, then we say that $\mathbf{C} \xrightarrow{f} \mathbf{C'} \xrightarrow{g} \mathbf{C''}$ is an *exact sequence* if it is exact for each $n \in \mathbb{Z}$.

Definition. Let $(\mathbf{A}, \alpha)$ and $(\mathbf{B}, \beta)$ be two complexes. If $f : \mathbf{A} \to \mathbf{B}$ is a chain map, then the *mapping cone* of f is the complex denoted by $\mathbf{cone}(f)$ whose degree n term is given by $A_{n-1} \oplus B_n$ and whose differential $\partial_n : A_{n-1} \oplus B_n \to A_{n-2} \oplus B_{n-1}$ is given by $(a_{n-1}, b_n) \mapsto (-\alpha(a_{n-1}), \beta(b_n) - f_{n-1}(a_{n-1}))$.

Proposition [2] Let

$$\begin{tikzcd}
\cdots \arrow[r] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"] & A_{-1} \arrow[r, "\alpha_{-1}"] \arrow[d, "f_{-1}"] & A_{-2} \arrow[r] \arrow[d, "f_{-2}"] & \cdots \\
\cdots \arrow[r] & B_2 \arrow[r, "\beta_2"] & B_1 \arrow[r, "\beta_1"] & B_0 \arrow[r, "\beta_0"] & B_{-1} \arrow[r, "\beta_{-1}"] & B_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

be a commutative diagram where the rows are complexes. Form the complex

$$\cdots \longrightarrow A_1 \oplus B_2 \longrightarrow A_0 \oplus B_1 \longrightarrow A_{-1} \oplus B_0 \longrightarrow \cdots$$

where the map $\varphi_n : A_n \oplus B_{n+1} \to A_{n-1} \oplus B_n$ is given by $(a_{n-1}, b_n) \mapsto (-\alpha(a_{n-1}), \beta(b_n) - f_{n-1}(a_{n-1}))$. Then this complex is exact at $A_{n-1} \oplus B_n$ if the complex (A, α) is exact at A_{n-1} and the complex (B, β) is exact at B_n.

9. Projective, Injective, and Flat Modules

Definition. An R-module P is said to be *projective* if given an exact sequence $A \xrightarrow{\psi} B \to 0$ of R-modules and an R-homomorphism $f : P \to B$, there exists an R-homomorphism $g : P \to A$ such that $f = \psi \circ g$, that is, such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ A \arrow[r, "\psi"'] & B \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram. Thus every free module is projective and hence every R-module M has a *projective resolution*, that is, an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective.

Theorem [2] The following are equivalent for an R-module P:

(1) P is projective

(2) $\mathrm{Hom}(P, —)$ is right exact

(3) Every exact sequence $0 \to A \to B \to P \to 0$ is split exact

(4) P is a direct summand of a free R-module

Definition. An R-module F is said to be *flat* if given any exact sequence $0 \to A \to B$ of right R-modules, the tensored sequence $0 \to A \otimes_R F \to B \otimes_R F$ is exact.

Proposition [2] The direct sum $\bigoplus_{i \in I} F_i$ is flat if and only if each F_i is flat.
Corollary Every projective module is flat.

Definition. It now follows that every R-module has a *flat resolution*, that is, an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ with each F_i flat.

Remark Suppose that M is a right R-module. Then for every left R-module N the tensor product $M \otimes_R N$ is an abelian group and the functor $M \otimes_R —$ is covariant and right exact, i.e., for any short exact sequence of left R-modules

$$M \otimes A \to M \otimes B \to M \otimes C \to 0$$

is an exact sequence of abelian groups. This sequence can extended at the left to a long exact sequence as follows: Let

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \xrightarrow{d_0} N \to 0$$

be a projective resolution of N and apply the covariant functor $M \otimes_R —$ to obtain

$$\cdots \to M \otimes P_n \xrightarrow{1 \otimes d_n} M \otimes P_{n-1} \to \cdots \to M \otimes P_0 \xrightarrow{1 \otimes d_0} M \otimes N \to 0.$$

Definition. Let M be a right R-module and let N be a left R-module. For any projective resolution of N by left R-modules as in the remark above let $1 \otimes \partial_n : M \otimes P_n \to M \otimes P_{n-1}$ for $n \geq 1$. Define $\mathrm{Tor}_i^R(M, N) = \mathrm{Ker}(1 \otimes d_i)/\mathrm{Im}(1 \otimes d_{i+1})$.

Theorem [2] The following are equivalent for an R-module F:

(1) F is flat

(2) $— \otimes_R F$ is left exact

(3) $\mathrm{Tor}_i^R(M, F) = 0$ for all right R-modules M and for all $i \geq 1$

(4) $\mathrm{Tor}_1^R(M, F) = 0$ for all right R-modules M

(5) $\mathrm{Tor}_1^R(M, F) = 0$ for all finitely generated right R-modules M

Theorem [2] Let R and S be commutative rings and let $f : R \to S$ be a ring homomorphism that makes S into a flat left R-module. If M and N are R-modules, then

$$\mathrm{Tor}_i^R(M, N) \otimes_R S \cong \mathrm{Tor}_i^R(M \otimes_R S, N \otimes_R S)$$

Definition. An R-module F is said to be *faithfully flat* if $0 \to A \to B$ is an exact sequence of right R-modules if and only if $0 \to A \otimes_R F \to B \otimes_R F$ is exact. It is easy to see that every free R-module is faithfully flat.

Lemma [2] The following are equivalent for a left R-module F:

(1) F is faithfully flat

(2) F is flat and for any right R-module N we have $N \otimes F = 0$ implies $N = 0$

(3) F is flat and $\mathfrak{m}F \neq F$ for every maximal right ideal $\mathfrak{m}$ of R.

Definition. An R-module E is said to be *injective* if given R-modules $A \subset B$ and a homomorphism $f : A \to E$, there exists a homomorphism $g : B \to E$ such that $g|_A = f$, that is, such that

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

is a commutative diagram.

Theorem [2] The following are equivalent for an R-module E:

(1) E is injective

(2) $\mathrm{Hom}(—, E)$ is right exact

(3) E is a direct summand of every R-modules containing E

Theorem [2] (Baer's Criterion) An R-module E is injective if and only if for all ideals I of R, every homomorphism $f : I \to E$ can be extended to R.

Definition. Let I be a left R-module. Then I is said to be *divisible* if for any $r \in R$ that is a non zero divisor and for any $y \in I$, then there exists $x \in I$ such that $y = rx$.

Theorem [2]Let R be a principal ideal domain. Then an R-module M is injective if and only of it is divisible.

Corollary Every abelian group can be embedded in an injective abelian group.

Proposition [2] If $f : R \to S$ is a ring homomorphism and if E is an injective left R-module, then $\mathrm{Hom}_R(S, E)$ is an injective left S-module.

Theorem [2]Every R-module can be embedded in an injective R-module.

Remark Let M be any R-module and

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \xrightarrow{d_0} M \to 0$$

be a projective resolution of M. Then apply the contravariant functor $\mathrm{Hom}_R(—, N)$ to the above resolution to obtain

$$0 \to \mathrm{Hom}_R(M, N) \xrightarrow{d_0} \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1} \mathrm{Hom}_R(P_1, N) \to \cdots$$

This sequence is not necessarily exact, however it is a cochain complex.

Definition. Let M and N be R-modules. For any projective resoltion of M as in the remark above let $d_n : \mathrm{Hom}_R(P_{n-1}, N) \to \mathrm{Hom}_R(P_n, N)$. Define $\mathrm{Ext}^i_R = \mathrm{Ker}(d_{i+1})/\mathrm{Im}(d_i)$.

Theorem [2] The following are equivalent for an R-module E:

(1) E is injective

(2) $\mathrm{Ext}^i(M, E) = 0$ for all R-modules M and for all $i \geq 1$

(3) $\mathrm{Ext}^1(M, E) = 0$ for all R-modules M

(4) $\mathrm{Ext}^i(R/I, E) = 0$ for all ideals I of R and for all $i \geq 1$

(5) $\mathrm{Ext}^1(R/I, E) = 0$ for all ideals I of R

Corollary [2]A product of R-modules $\prod_{i \in I} E_i$ is injective if and only if each E_i is injective.

Definition. If M is a submodule of an injective R-module E, then $M \subset E$ is called an *injective extension* of M. It then follows that every R-module has an injective extension.

Definition. Let $A \subset B$ be R-modules. Then B is said to be an *essential extension* of A if for each submodule N of B, we have that $N \cap A = 0$ implies that $N = 0$. In this case, A is said to be an *essential submodule* of B.

Definition. An injective module E which is an essential extension of an R-module M is said to be an *injective envelope* of M.

Theorem [2]Every R-module has an injective envelope which is unique up to isomorphism.

Remark We can construct an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots$ with each E^i injective using injective envelopes. This sequence is called a *minimal injective resolution* of M.

Notation An injective envelope of an R-module M is denoted by $E(M)$. We see that if $M \subset E$ with E injective, then E contains an injective envelope of M (just extend the identity $M \to E$ to $E(M) \to E$).

Definition. An R-module M is said to have *injective dimension* at most n, denoted inj dim $\leq n$, if there is an injective resolution $0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0$. If n is the least, then we set inj $\dim(M) = n$. The *projective dimension* and *flat dimension* of an R-module are defined similarly using projective and flat resolutions respectively. These are denoted proj $\dim(M)$ and flat $\dim(M)$ respectively.

Remark We note that flat $\dim(M) \leq$ proj $\dim(M)$ and equality holds if R is left Noetherian and M is finitely generated.

10. Covers and Envelopes

Definiton. Let R be a ring and let $\mathcal{F}$ be a class of R-modules. Then for an R-module M, a morphism $\varphi : C \to M$ where $C \in \mathcal{F}$ is called an $\mathcal{F}$-*cover* of M if

(1) any diagram with $C' \in \mathcal{F}$

$$\begin{tikzcd} C' \arrow[d, dashed, "g"] \arrow[dr, "f"] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

can be completed to a commutative diagram (that is $f = \varphi \circ g$)

(2) the diagram

$$\begin{tikzcd} C \arrow[d, dashed, "g"] \arrow[dr, "\varphi"] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

can be completed only by automorphisms g of C.

So if an $\mathcal{F}$-cover exists, then it is unique up to isomorphism. If $\varphi : C \to M$ satisfies (1) but maybe not (2), then it is called an $\mathcal{F}$-*precover* of M. For example, if $\mathcal{F}$ is the class of projective modules, an $\mathcal{F}$-cover (precover) is called a *projective cover (precover)*. We say that a class $\mathcal{F}$ is *(pre)covering* if every R-module has an $\mathcal{F}$-(pre)cover.

Proposition [2] Let M be an R-module. Then the $\mathcal{F}$-cover of M, if it exists, is a direct summand of any $\mathcal{F}$-precover of M.

Theorem [2]For any ring R and any R-module M, if M has a flat pre-cover, then it also has a flat cover.

Theorem [2]Let $\mathcal{P}roj$ be the class of projective R-modules. Then the following are equivalent:

(1) Every flat R-module is projective

(2) Every projective pre-cover is a flat pre-cover

(3) $\mathcal{P}roj$ is covering (that is, R is left perfect)

Theorem [2] If R is a local ring, then every finitely generated R-module has a projective cover.

Remark Rings for which every finitely generated module has a projective cover are said to be *semiperfect*. So we see that local rings are semiperfect and every perfect ring is semiperfect.

Theorem [2] Let $\mathcal{E}$ be the class of injective left R-modules. Then the following are equivalent:

(1) R is left Noetherian

(2) $\mathcal{E}$ is pre-covering

(3) $\mathcal{E}$ is covering

Lemma [2]Let R be a commutative Noetherian ring. If M is a finitely generated R-module and $\mathfrak{p}$ is a prime ideal of R with $\mathrm{Hom}(E(R/\mathfrak{p}), M) \neq 0$, then $\mathfrak{p}$ is a maximal ideal.

Theorem[2] Let R be a commutative Noetherian ring. Then the injective cover of a finitely generated R-module is a direct sum of finitely many copies of $E(R/\mathfrak{m})$ for finitely many maximal ideals $\mathfrak{m}$.

Definition. Let R be a ring and let $\mathcal{F}$ be a class of R-modules. Then for an R-module M, a morphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is called an $\mathcal{F}$-*envelope* of M if

(1) any diagram with $F' \in \mathcal{F}$

$$\begin{tikzcd} & F' \\ M \arrow[ur, "f"] \arrow[r, "\varphi"'] & F \arrow[u, dashed, "g"'] \end{tikzcd}$$

can be completed to a commutative diagram (that is $f = g \circ \varphi$)

(2) the diagram

$$\begin{tikzcd} & F \\ M \arrow[ur, "\varphi"] \arrow[r, "\varphi"'] & F \arrow[u, dashed, "g"'] \end{tikzcd}$$

can be completed only by automorphisms g of F.

So if envelopes exist, they are unique up to isomorphism. It is easy to check that if $\mathcal{F}$ is the class of injective modules, then we get the usual injective envelopes. Similarly, we get pure injective envelopes if $\mathcal{F}$ is the class of pure injectives. We note that if the class $\mathcal{F}$ contains injectives, then $\mathcal{F}$-preenvelopes are monomorphisms. If every R-module has an $\mathcal{F}$-(pre)envelope, we say that $\mathcal{F}$ is *(pre)enveloping*. For example, we know that the class of injective R-modules is enveloping

Proposition [2] Let M be an R-module, then the $\mathcal{F}$-envelope of M, if it exists, is a direct summand of any $\mathcal{F}$-pre-envelope of M.

Definition. Given a class $\mathcal{C}$ of R-modules, we let ${}^\perp\mathcal{C}$ be the class of R-modules F such that $\mathrm{Ext}^1(F, C) = 0$ for all $C \in \mathcal{C}$. We let $C^\perp$ be the class of modules G such that $\mathrm{Ext}^1(C, G) = 0$ for all $C \in \mathcal{C}$. The classes ${}^\perp C$ and $C^\perp$ are called *orthogonal classes* of $\mathcal{C}$.

Remark We note that for any $\mathcal{C}$, we have that $\mathcal{C} \subset {}^\perp(\mathcal{C}^\perp)$ and $\mathcal{C} \subset ({}^\perp\mathcal{C})^\perp$. Also $\mathcal{C}_1 \subset \mathcal{C}_2$ implies ${}^\perp\mathcal{C}_2 \subset^\perp \mathcal{C}_1$ and $\mathcal{C}_2^\perp \subset \mathcal{C}_1^\perp$. From this it follows that $({}^\perp(\mathcal{C}^\perp))^\perp = \mathcal{C}^\perp$ and ${}^\perp(({}^\perp\mathcal{C})^\perp) =^\perp \mathcal{C}$ for all $\mathcal{C}$.

Definition. A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion theory* (for the category of R-modules) if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$. A class $\mathcal{D}$ is said to *generate* the cotorsion theory if ${}^\perp\mathcal{D} = \mathcal{F}$ (and so $\mathcal{D} \subset \mathcal{C}$) and a class $\mathcal{G}$ is said to *cogenerate* $(\mathcal{F}, \mathcal{C})$ if $\mathcal{G}^\perp = \mathcal{F}$ (and so $\mathcal{G} \subset \mathcal{F}$).

Example The pairs $(\mathcal{M}, \mathcal{I}nj)$ and $(\mathcal{P}roj, \mathcal{M})$ are cotorsion theories where $\mathcal{M}$ denotes the class of left R-modules and $\mathcal{I}nj$ and $\mathcal{P}roj$ denote the classes of injective and projective modules respectively. The cotorsion theory $(\mathcal{M}, \mathcal{I}nj)$ is cogenerated by the set of modules R/I where I is a left ideal, and is generated by the class of injective modules.
Lemma [2] If $\mathcal{F}$ is the class of flat R-modules and if $\mathcal{F}^\perp = \mathcal{C}$, then $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory.

Definition. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to have *enough injectives* if for every module M there is an exact sequence $0 \to M \to C \to F \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say that $(\mathcal{F}, \mathcal{C})$ has *enough projectives* if for every module M there is an exact sequence
$0 \to C \to F \to M \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$.

Definition. Given a class $\mathcal{F}$, a module M is said to have a *special $\mathcal{F}$-precover* if there is an exact sequence $0 \to C \to F \to M \to 0$ with $F \in \mathcal{F}$ and $C \in \mathcal{F}^\perp$. A module M is said to have a *special $\mathcal{F}$-preenvelope* if there is an exact sequence $0 \to M \to D \to F \to 0$ with $F \in \mathcal{F}$ and $D \in^\perp \mathcal{F}$. So if a cotorsion theory $(\mathcal{F}, \mathcal{C})$ has enough injectives and projectives, every module M has a special $\mathcal{F}$-precover and a speical $\mathcal{C}$-preenvelope.

11. Gorenstein Modules

Definition. A module N is said to be *Gorenstein injective* if there exists a $\mathrm{Hom}(\mathcal{I}nj, —)$ exact exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective modules such that $N = \mathrm{Ker}(E^0 \to E^1)$. We note that in the above definition, the complex $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ is a complete $\mathcal{I}nj$-reolution of N. Moreover, if N is a Gorenstein injective R-module, then $\mathrm{Ext}^i(E, N) = 0$ for all $i \geq 1$ and all injective R-modules E, or equivalently, every right $\mathcal{I}nj$-resolution of N is a left $\mathcal{I}nj$-resolution.

Proposition [2] The injective dimension of a Gorenstein injective R-module is either zero or infinite.

Definition. A module M is said to be *Gorenstein projective* if there is a $\mathrm{Hom}(—, \mathcal{P}roj)$ exact exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules such that $M = \mathrm{Ker}(P^0 \to P^1)$.

Remark The complex $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is a complete $\mathcal{P}roj$-resolution of M. We note that if M is Gorenstein projective, then $\mathrm{Ext}^i(M, P) = 0$ for all $i \geq 1$ and all projective R-modules P and so by induction, $\mathrm{Ext}^i(M, L) = 0$ for all $i \geq 1$ and all R-modules L of finite projective dimension. In particular, every left $\mathcal{P}roj$-resolution of M is $\mathrm{Hom}(—, \mathcal{P}roj)$ exact.

Proposition[2] The projective dimension of a Gorenstein projective module is either zero or infinite.

Definition. A module M is said to be *Gorenstein flat* if there exists an $\mathcal{I}nj \otimes —$ exact exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat modules such that $M = \mathrm{Ker}(F^0 \to F^1)$.

12. Main Results

Proposition 1. If **G** is a complex of finite Gorenstein projective dimension, then **G** has a special Gorenstein projective precover.

Proof. Let **G** be a complex of finite Gorenstein projective dimension, that is
Gor. proj dim $= d < \infty$. Then there exists an exact sequence of complexes

$$0 \longrightarrow C \longrightarrow P_{d-1} \dashrightarrow P_0 \longrightarrow G \longrightarrow 0$$

with each P_i being a projective complex and $C \in \mathcal{GP}(\mathcal{C})$. Since $C \in \mathcal{GP}(\mathcal{C})$ there is an exact and $\mathrm{Hom}(—, \mathrm{Proj})$ exact sequence

$$0 \longrightarrow C \longrightarrow T_{d-1} \dashrightarrow T_0 \longrightarrow D \longrightarrow 0$$

with each T_j being a projective complex and D being a Gorenstein projective complex. The fact that each P_i is projective allows the construction of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & T_{d-1} \arrow[r, dashed] \arrow[d] & T_0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & P_{d-1} \arrow[r, dashed] & P_0 \arrow[r] & G \arrow[r] & 0.
\end{tikzcd}$$

So we have a map of exact complexes and so the mapping cone is also exact. This gives an exact sequence

$$0 \longrightarrow T_{d-1} \longrightarrow P_{d-1} \oplus T_{d-2} \dashrightarrow P_1 \oplus T_0 \longrightarrow P_0 \oplus D \xrightarrow{\delta} G \longrightarrow 0.$$

The exact sequence

with all P_i and T_j being projective shows that proj dim $\mathrm{Ker}(\delta) < \infty$. Thus
$\mathrm{Ker}(\delta) \in \mathcal{GP}(\mathcal{C})^{\perp}$. So we have an exact sequence $0 \to \mathrm{Ker}\ \delta \mapsto P_0 \oplus D \xrightarrow{\delta} G \to 0$ with $P_0 \oplus D \in \mathcal{GP}(\mathcal{C})$ and $\mathrm{Ker}\ \delta \in \mathcal{GP}(\mathcal{C})^{\perp}$. Thus $P_0 \oplus D \xrightarrow{\delta} G$ is a special $\mathcal{GP}(\mathcal{C})$-precover of G. □

Theorem Let R be such that

(i) every complex in $\mathbf{Ch}(R)$ has a special Gorenstein flat precover
(ii) every Gorenstein projective complex is Gorenstein flat
(iii) every Gorenstein flat complex has finite Gorenstein projective dimension.

Then $\mathcal{GP}(\mathcal{C})$ is special precovering in $\mathbf{Ch}(R)$.

Proof. Let X be any complex in $\mathbf{Ch}(R)$. Then by assumption we have that there exists an exact sequence $0 \to Y \to G \to X \to 0$ where G is a Gorenstein flat complex and $Y \in \mathcal{GF}(\mathcal{C})^{\perp}$. Since $\mathrm{Gpd}(G) < \infty$, by the previous proposition, G has a special Gorenstein projective precover. So there exists an exact sequence $0 \to L \to P \to G \to 0$ with $P \in \mathcal{GP}(\mathcal{C})$ and $L \in \mathcal{GP}(\mathcal{C})^{\perp}$ (since $\mathrm{pd}(L) < \infty$). Form the pullback diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r, "\alpha"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Since $\mathcal{GP}(\mathcal{C}) \subseteq \mathcal{GF}(\mathcal{C})$ we have that $\mathcal{GF}(\mathcal{C})^{\perp} \subseteq \mathcal{GP}(\mathcal{C})^{\perp}$. So both L and Y are in $\mathcal{GP}(\mathcal{C})^{\perp}$. Then the exact sequence $0 \to L \to M \to Y \to 0$ gives $M \in \mathcal{GP}(\mathcal{C})^{\perp}$. The exact sequence $0 \to M \to P \xrightarrow{\alpha} X \to 0$ with P Gorenstein projective and $M \in \mathcal{GP}(\mathcal{C})^{\perp}$gives that α is a special $\mathcal{GP}$-precover of X.

□

Corollary If the ring R is right coherent and left n-perfect, then the class of Gorenstein projective complexes is special precovering in $\mathbf{Ch}(R)$.

Proof. It is known that if R is a right coherent and left n-perfect ring, then every Gorenstein flat R-module M has Gorenstein projective dimension less than or equal to n (Estrada-Odabasi). It is also known that when R is a coherent ring, a complex G is Gorenstein flat if and only if it is a complex of Gorenstein flat modules [8]. We show that $Gpd(G) \leq n$ for any Gorenstein flat complex G. Consider a paritial projective resolution of G:

$$0 \to C \to P_{n-1} \to \cdots \to P_1 \to P_0 \to G \to 0$$

Then for each j we have an exact sequence of modules:

$$0 \to C_j \to P_{n-1,j} \to \cdots \to P_{1,j} \to P_{0,j} \to G_j \to 0$$

Since each $P_{i,j}$ is projective and $Gpd(G_j) \leq n$, it follows that C_j is a Gorenstein projective module for each j. Thus C is a Gorenstein projective complex. Since over a coherent ring $\mathcal{G}$ flat complex means a complex of Gorenstein flat modules, and $\mathcal{GP}|$ complex means a complex of Gorenstein projective modules (the second statement holds over any ring), it follows that we have $\mathcal{GP}(\mathcal{C}) \subseteq \mathcal{GP}(\mathcal{F})$ (every Gorenstein projective complex is Gorenstein flat) whenever R is right coherent and left n-perfect.

$\square$

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13. Apendix: Noetherian and Coherent Rings

Definition. An R-module M is said to be *Noetherian* if every ascending chain of submodules of M terminates.

Remark It is easy to see that an R-module M is Noetherian if and only if every nonempty set of submodules of M has a maximal element. To see this suppose there is a nonempty set of submodules of M that has no maximal element. Let M_1 be an element of this set. Then M_1 is not maximal. So there is an element M_2 is the set such that $M_1 \subsetneq M_2$. Repeating the arugment we get a chain of submodules $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$ of M that never terminates. The converse is clear.

Proposition [2] An R-module M is Noetherian if and only if every submodule of M is finitely generated.

Definition. A ring R is said to be *left Noetherian* if it is Noetherian as a left module over itself. By Noetherian we will always mean left Noetherian.

Corollary A ring R is Noetherian if and only if every left ideal of R is finitely generated.

Lemma [2] Let $0 \to M' \to M \to M" \to 0$ be an exact sequence with $M' \subset M$ and $M" = M/M'$ with the usual maps. Suppose S_1, S_2 are submodules of M such that $S_1 \subset S_2$and $S_1 \cap M' = S_2 \cap M'$. If $(M' + S_1)/M' = (M' + S_2)/M'$, then $S_1 = S_2$.

Proposition[2] Let $0 \to M' \to M \to M" \to 0$ be an exact sequence of R-modules. Then M is Noetherian if and only if M' and $M"$ are Noetherian.

Corollary A finite direct sum of Noetherian R-modules is also Noetherian.

Corollary A finitely generated module over a Noetherian ring is Noetherian. In particular, if R is Noetherian, then an R-module M is Noetherian if and only if M is finitely generated.

Corollary A ring R is Noetherian if and only if every submodule of a finitely generated R-module is finitely generated.

Definition. An R-module M is said to be *finitely presented* if there is an exact sequence $F_1 \to F_0 \to M \to 0$ where F_0 and F_1 are finitely generated free R-modules.

Remark It is easy to see that an R-module M is finitely presented if and only if there is an exact sequence $0 \to K \to F \to M \to 0$ where K and F are finitely generated R-modules and F is free. In particular, every finitely presented R-module is finitely generated and the converse holds if R is left Noetherian.

Definition. A ring R is said to be *right coherent* if every finitely generated right ideal of R is finitely presented. It follows from the above remark that every right Noetherian is right coherent.

Theorem [2] The following are equivalent for a ring R:

(1) R is right coherent

(2) Every product of flat left R-modules is flat

(3) $\bigoplus_{i \in I} R_i$ is a flat left R-module for any set I

(4) Every finitely generated submodule of a finitely presented right R-module is finitely presented.

Remark Suppose M is a finitely presented right R-module. Then there is an exact sequence $0 \to K \to F_0 \to M \to 0$ with K and F_0 finitely generated and F_0 free. If R is right coherent, then K is finitely presented by the previous theorem. Thus continuing in this manner, we see that if R is right coherent then every finitely presented right R-module M has a free resolution $\cdots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated and free.