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Blow-Up Solution and Blow-Up Rate of Bose-Einstein Condensates with Rotational Term

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BLOW-UP SOLUTION AND BLOW-UP RATE OF BOSE-EINSTEIN CONDENSATES WITH ROTATIONAL TERM

by

NYLA BASHARAT

(Under the Direction of Shijun Zheng)

ABSTRACT

In this thesis, we discuss the Gross Pitaevskii Equation (GPE) with harmonic potential and with an angular momentum rotational term in space $\mathbb{R}^2$, which describes the model for Bose-Einstein Condensation. Local Well-Posedness of the equation and the conservation identities for mass, energy and angular momentum are presented. Using the virial identities, we derive the condition for blow-up solution in finite time. Then a threshold of L^2 norm of wave function ψ is obtained for global existence, of GPE in term of ground state solution. This method allows us to obtain our main result "Sharp sufficient condition for global existence, of NLS with certain in-homogeneous non-linearity". Furthermore, we estimate the universal upper bound for Blow-up rate in super mass critical regime.

Key Words: Gross Pitaevskii Equation, Harmonic Potential, Bose-Einstein Condensates, Angular Momentum Rotation, Blow-up solution, Ground state, Blow-up rate.

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DEDICATION

First, I give thanks to God for protection and bless that provides energy and passion for continuing my work.

This thesis is dedicated to my Family, for the support they provided me through my entire life. In particular, I would like to thank my mother, my father, my sisters and my brothers. They were always supporting me, prayed for me and encouraging me with their best wishes throughout the time of my research.

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LIST OF SYMBOLS

GPE	Gross-Pitaevskii Equation
NLS	Non Linear Schrödinger Equation
BEC	Bose-Einstein Condensates
$\psi(x,t)$	Complex-Valued Wave function
n	Space Dimension
Δ	Laplace operator
$V(x) = \|x\|^2$	Harmonic Potential in magnetic trap
$\lambda(x)$	Attractive or repulsive interaction for GPE
Ω	Angular Velocity
L_z	Angular Momentum Operator
ψ_0	Intial Data
$\overline{f}$	Complex Cojugate of function f
Σ	Weighted Sobolev Space
$\mathbb{R}$	The set of Real numbers
$\mathbb{C}$	The set of Complex Numbers
$\mathbb{R}^+$	The set of Positive Real Numbers
$\Im(f)$	Imaginary Part of function f
$\Re(f)$	Real part of function f
$E(\psi)$	Energy per particle
$\|\|\psi\|\|_{L^2}$	L^2-norm of function ψ in Σ
$\langle \phi, \Phi \rangle$	Inner product of function ϕ and Φ in Σ
$\int f$	Integration of function f in $\mathbb{R}^2$

CHAPTER 1

Introduction

1.1 History of Bose-Einstein Condensation

Bose Einstein Condensate (BEC), is the fifth state of matter (after solid, liquid, gas, plasma). Bose-Einstein Condensates was first predicted by Satyendra Nath Bose in 1924, when he published a paper and described the statistical nature of light [1]. Using this paper, Albert Einstein predicted a phase transition (Bose-Einstein Condensation). It is a phase transition, like water freezing to ice, but it was related to the quantum mechanical nature of the world. It was thought that this effect is like superconductivity in metals and superfluidity in liquid helium gas, and this phase occur due to these quantum statistical effects.
Both Satyendra and Einstein studied the statistical properties of photons and massive particles with integer spin, which are known as bosons. It was found that this is possible not only for two or more bosons to share the same quantum state, but millions of that bosons actually try to occupy the same state. It was predicted that at a finite temperature, almost all the particles of a bosonic system would occupy the ground state as soon as the quantum wave functions of the particles start to overlap. In a Bose-Einstein condensates, this Phase transition period enable millions of microscopic non-interacting bosons simultaneously to occupy the same quantum state of lowest energy [2].

F. London discover the superfluidity in liquid helium (1938). On the basis of his results he conclude that, this may be one of the first representation of BEC [3]. Experimental efforts to produce BEC in dilute gases was started approximately 32 years ago [4]. In first experiment hydrogen H atoms were used to obtain BEC, But due to the high rate of inelastic collision in H [4], prevented these experiments from success. Then after a long time, In 1998 BEC in H was produced by T. J. Greytak's and D. Kleppner's group[5].

The real break through came in 1995, when the first experiment on BEC was successfully performed by C. E. Wieman's and E. A. Cornell's group with a dilute sample of magnetically trapped rubidium atoms [6]. Then after four months, W. Ketterle's group produce a BEC using sodium atoms [7]. In each of these experiment, the magnetic trap confined the atoms cooling them less than one-millionth of a degree below absolute zero. Also in both experiments the scattering length [8,9] was positive. For their work, C. E. Wieman's and E. A. Cornell were awarded by "Nobel Prize" in Physics (2001).

With the knowledge of dilute bosonic gases [10,11,12], Bose-Einstein Condensation of alkali atoms and hydrogen has been studied in laboratory [13]. In view of potential applications [14], to study about the quantized vortices is an important issue. Many research groups [13,15-17], have produced the results for quantized vortices in trapped Bose-Einstein Condensates (BECs). So far there are two methods to generate quantized vortices from the ground state of BEC:

(a)- A laser beam rotating with an angular velocity on magnetic trap is imposed to produce anisotropic harmonic potential [18-21].

(b)- A narrow, moving Gaussian Potential, representing a far-blue detuned laser added to the stationary magnetic trap [22,23].

To find the blow-up solution for Bose-Einstein Condensate with or without Angular momentum rotational term are not new, lots of work has been done by using different methods e.g [24,25,26] and e.g [27,28,29] respectively. But to find sharp condition on global existence, blow-up solution and to find blow-up rate with rotational term is relatively new.

In this thesis, we drive a sharp sufficient condition for blow-up solution of nonlinear schrödinger equation (NLS) with harmonic potential (known as Gross Pitaevskii

Equation (GPE)) and angular momentum rotational term by using the method [24]. Then a threshold of L^2-norm of wave function is introduced for global existence in term of ground state.we also include the generalized case for focusing constant. Furthermore, we estimate the sharp lower bound and upper bound for Blow-up solution and Blow-up rate respectively.

1.2 Organization of Thesis

We organize the thesis as follows:

In Chapter 2, we formally introduce the GPE with an angular momentum rotational term, then we discuss the Local Well-Posedness (LWP) for the solution of equation. We present the conservation laws such as mass, energy and angular momentum in space $\mathbb{R}^2$.

In Chapter 3, we prove the virial identities, which play important role in proving Blow Up results.

In Chapter 4, First we give a Lemma about finite time blow-up solution. Then using the results of this lemma, we prove the main theorem for "sharp sufficient condition for global existence (Blow-up Solution in finite time)" in term of ground state solution and we also present some graphical results for the ground state.

In Chapter 5, We estimate upper bound on the Blow-up rate of GPE with rotational term.

CHAPTER 2

Gross Pitaevskii Equation

2.1 GPE with Rotational term

In this thesis, we will discuss the Cauchy problem of nonlinear Schrödinger equation with harmonic potential in a rotational frame (also known as Gross-Pitaevskii Equation (GPE), derived by Gross [30] and Pitaevskii [31]) with an angular momentum rotation term in 2-dimensional space represented as

$$\begin{aligned} &i\psi_t = -\Delta\psi + \mathrm{V}(x)\psi - \lambda(x)|\psi|^2\psi - (\Omega \cdot L_z)\psi \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R} \\ &\psi(x,0) = \psi_0(x) \qquad x \in \mathbb{R}^2 \end{aligned} \tag{2.1}$$

where $\psi(x,t)$ represents the wave function, which is a complex-valued function, and describes the properties of BEC in rotational frame at temperature T far below the critical condensation temperature T_c.
Δ is the Laplace Operator on $\mathbb{R}^2$.
Attractive or repulsive interactions for BEC are described by using function $\lambda(x) \in \mathbb{R}^2$ such that there are $\lambda_1 > 0$ and $\lambda_2 > 0$ such that
$(\mathrm{H}_1) : \forall x \in \mathbb{R}^2, \lambda_1 = inf\lambda(x) \leqslant \lambda(x) \leqslant sup\lambda(x) = \lambda_2$
$(\mathrm{H}_2) : \forall x \in \mathbb{R}^2, x \cdot \nabla\lambda(x) \leqslant 0$
And $\mathrm{V}(x) = |x|^2 = x_1^2 + x_2^2$ describe the harmonic potential in magnetic trap.
Finally, the last term $\Omega \cdot L_z = |\Omega|\ L_z$ represents the Angular Momentum Rotational term, where

$$L_z = i(x_2\partial_{x_1} - x_1\partial_{x_2}) \tag{2.2}$$

is the angular momentum operator and Ω is the angular velocity vector and $\Omega \in \mathbb{R}^2$.
For the Cauchy problem (2.1), we define the Hilbert (Sobolev Embedding) Space Σ as follows

$\Sigma = \mathrm{H}^1(\mathbb{R}^2)\bigcap\{\psi : \psi|x| \in \mathrm{L}^2(\mathrm{R}^2)\}$

Inner product in Hilbert space Σ is defined as

$$\langle\phi,\varphi\rangle = \int \nabla\phi\overline{\nabla\varphi} + \phi\overline{\varphi} + |x|^2\phi\overline{\varphi}, \qquad \forall\ \phi,\varphi \in \Sigma$$

We define the norm $||\cdot||_{L^2(\mathbb{R}^2)}$ denoted by$||\cdot||_{L^2}$ in the Space Σ as:

$$||\psi||^2_{L^2} = ||\psi||^2_{L^2} + ||\nabla\psi||^2_{L^2} + ||x\psi||^2_{L^2}, \qquad \forall\ \psi \in \Sigma$$

And the Energy per particle E and Angular Momentum $\langle L_z(t)\rangle$ are well defined in space Σ according to Sobolev embedding theorem(since H_1,H_2 holds true),and are defined by,

$$E(\psi) = \int |\nabla\psi|^2 + |x|^2|\psi|^2 - \frac{\lambda(x)}{2}|\psi|^4 - Re|\Omega|(\overline{\psi}L_z\psi) \tag{2.3}$$

and

$$\langle L_z(t)\rangle = \int \overline{\psi}L_z\psi < +\infty \tag{2.4}$$

respectively. Also this showes that ψ has finite angular momentum.

2.2 Local Well Posedness

The LWP. result and the alternative blowup result for mass-critical and mass-subcritical were proved in [37], and further elaborated for energy-subcritical case that include the endpoint pair $(2, 2n/n-2)$ in [38]. when n=2,it is $(2,\infty)$. Here we present the following theorem from [38].

Theorem 2.1. *Let $1 \le p \le 1 + 4/n$ for $n \geqslant 2$,$r_0 = p+1$ and (q_o, r_0)admissible. We consider (q,r) the endpoint case. Let $\psi_0 \in \Sigma$. Then the integral equation (the*

Duhamel of the NLS) has the unique solution ψ *in* $C((-T^-,T^+),L^2)\cap L^q((-T^-,T^+),L^r)$. *Furthermore, for all* t *in* $(-T^-,T^+)$, $\|\psi(t)\|_2 = \|\psi(0)\|_2$. *If* $T^+ < \infty$ *(resp.* $-T^- > -\infty$*), then we must have the blowup alternative.*

$$\|\psi\|_{L^{q_0}((0,T^+),L^{r_0})} = \infty \quad (resp.\ \|\psi\|_{L^q((-T^-,0),L^r)} = \infty)$$

Here [37] requires $(q_0, r_0 = p+1)$ are non-endpoint admissible pair.

Remark:

Read Theorem 4.7.1 in [37] the above blowup alternative should hold for any admissible pair (q,r) (including the critical case).

Remark 1:

The morale is that in view of Lemma 6.2 in [37] the inequality where [37] only showed for $(\tilde{q},\tilde{r})$ equal to $(q,p+1)$ and $(\infty,2)$: There exists $1 < \sigma \leq \infty$ ($\sigma = 4p/(d+4-pd)$) such that for all ψ,ϕ in $L^q([-T,T],L^{p+1})$

$$\|\Lambda F(\psi) - \Lambda F(\phi)\|_{\tilde{q},\tilde{er}} \leq cT^{p/\sigma}(\|\psi\|^{p-1}_{q,p+1} + \|\phi\|^{p-1}_{q,p+1})\|\psi-\phi\|_{q,p+1} \tag{2.5}$$

where $\Lambda F = \int_0^t e^{-i(t-s)\mathcal{L}}F(s,x)ds$

Note that

1. In the critical case, the exponent of T is zero, we will have to use the $p-1>0$ to show the contraction.

2. In the subcritical case, T has an exponential $p/\sigma > 0$, which will guarantee the contraction.

Remark 2: The blowup alternative not only part of the LWP, but it tells us where to check into about the blowup problem with the scale $\|\cdot\|_{q,r}$.

The proof of the last statement is similar to [38], which is omitted in [37].

Let $T_0 > 0$ be such that the local dispersive estimate holds.

Proof:

First note that T_{max} and T_{min} can be obtained from bootstrap argument if necessary. Following [38], let $u_0 \in \Sigma$ (not necessarily small). Strichartz tells in particular for $q_0 = r_0 = 2 + 4/d$.

$$\|\Psi(t)\psi_0\|_{L^q((-T_0,T_0),L^r)} \leq c\|\psi_0\|_2 \tag{2.6}$$

So for each $\delta > 0$ there exists small $0 < T = T(d,\delta) < T_0$ such that

$$\|\Psi(t)\psi_0\|_{L^{q_0}([-T,T],L^{r_0})} < \delta \tag{2.7}$$

This will allow us to construct a local unique and maximal solution u in $C((-T_{min}, T_{max}), L^2)$. Let $I = [-T, T]$. Define the mapping Φ on the space.

$$E = \{\psi \in L^{q_0}(I, L^{r_0}) : \|\psi\|_{L^{q_0}(I,L^{r_0})} \leq 2\delta\}$$

endowed with the metric $\|\cdot\|_{L^{q_0}(I,L^{r_0})}$:

$$\Phi(\psi) = e^{-it\mathcal{L}}\psi_0 + i\int_0^t \lambda(x)e^{-i(t-s)\mathcal{L}}(|\psi|^{p-1}\psi)(s,x)ds \tag{2.8}$$

Let $F(\psi) = |\psi|^{4/d}u$. For any admissible pair (q,r) in (2.7) we have by (2.7) and inhomogeneous Strichartz.

$$\|\int_0^t \Psi(t-s)F(\psi)\|_{L^q(I,L^r)} \leq c\|\psi\|^{1+4/d}_{L^{q_0}(I,L^{r_0})}$$
$$\|\int_0^t \Psi(t-s)(F(\psi)-F(\phi))\|_{L^q(I,L^r)} \leq c\|\psi-\phi\|_{L^{q_0}(I,L^{r_0})}\cdot$$
$$\left(\|u\psi\|^{4/d}_{L^{q_0}(I,L^{r_0})} + \|\phi\|^{4/d}_{L^{q_0}(I,L^{r_0})}\right)$$

where c does not depend on I. Thus if taking $\delta = \delta(c, d)$ sufficiently small, c being the constants involved in the Strichartz, we see that Φ is a contraction mapping on E because of (2.7) and the above two inequalities. This proves the local existence on $I = [-T, T]$ where $T = T(d, \delta)$ is small so that (2.7) is valid.

Blowup Alternative:

Assume T_{max} is finite and $\|\psi\|_{L^{q_0}([0,T_{max}),L^{r_0})}$ is finite. Let $0 \le t \le t + \rho < T_{max}$. For fix t applying the operator $\Psi(\rho) = e^{-i\rho\mathcal{L}}$ to (2.8) with $\Phi(\psi) = \psi$ and a change of variable give.

$$\psi(t+\rho) = \Psi(t+\rho)\psi_0 + i\int_{-t}^{\rho} \lambda(x)\Psi(\rho - s)F(\psi)(s+t,x)ds \tag{2.9}$$

Subtraction of the above two equations yields:

$$\Psi(\rho)(\psi(t)) = \Psi(\rho)\psi(t,\cdot) = \psi(t+\rho) - i\int_{0}^{\rho} \lambda(x)\Psi(\rho - s)F(\psi)(s+t,x)ds$$

Applying Strichartz to the second term we get with $q_0 = r_0 = 2 + 4/d$.

$$\|\Psi(\cdot)\psi(t,\cdot)\|_{L^{q_0}((0,T_{max}-t),L^{r_0})} \le \|\psi\|_{L^{q_0}((t,T_{max}),L^{r_0})} + c\|\psi\|^{1+4/d}_{L^{q_0}((t,T_{max}),L^{r_0})} \tag{2.10}$$

So

$$\|\Psi(\cdot)\psi(t,\cdot)\|_{L^{q_0}((0,T_{max}-t),L^{r_0})} < \delta/2 \tag{2.11}$$

if $|t - T_{max}| < \eta$ for some small $\eta > 0$.

Note that

the L.H.S alone does NOT suggest this property since $\Psi(\cdot)\psi(t)$ is a solution of LS with initial data $\psi(t)$, so the Strichartz can only tell $\|\Psi(\cdot)\psi(t)\|_{L^q((0,T_0,L^r)} \le c\|\psi(t)\|_2$ and hence $\|\Psi(\cdot)\psi(t)\|_{L^q((0,t_1,L^r)} < \delta/2$ for t_1 small. However, this does Not tell exactly if t_1 can be just chosen as $T_{max} - t$. Therefore, we can consider the time interval

$[t, T_{max} + t']$ with $t' = t'(t, \delta) > 0$, on which

$$\|\Psi(\cdot)u(t)\|_{L^{q_0}((0,T_{max}+t'-t),L^{r_0})} < \delta \tag{2.12}$$

Applying the map Φ on the space:

$E_{[t,T_{max}+t']} := \{\psi : \|\psi\|_{L^{q_0}([t,T_{max}+t'),L^{r_0})} < 2\delta\}$

We obtain the same way as in the proof of the LWP, that the existence of a solution that is defined on an interval beyond. If (q, r) is an admissible pair other than (q_0, r_0) and $r > r_0 = q_0$, then for $T < T_{max}$ we have by Hölder (let $a = \frac{2(r-r_0)}{r-2}, b = q = \frac{4r}{d(r-2)}$),

$$\|\psi\|_{L^{q_0}((0,T),L^{r_0})} = (\int_0^T (\int_x |\psi(t,x)|^{r_0=a+b} dx) dt)^{1/r_0}$$

$$\leq \left(\int_0^T (\int_x |\psi(t,x)|^2 dx)^{a/2} (\int_x |\psi(t,x)|^{bp'} dx)^{1/p'} dt\right)^{1/r_0} \qquad p = 2/a$$

$$\leq \sup_{t\in(0,T)} \|\psi\|_{L^2_x}^{a/r_0} \left(\int_0^T (\int_x |\psi(t,x)|^{bp'} dx)^{1/p'} dt\right)^{1/r_0}$$

$$= \sup_{t\in(0,T)} \|\psi\|_{L^2_x}^{a/r_0} \cdot \left(\int_0^T (\int_x |\psi(t,x)|^{r} dx)^{q/r} dt\right)^{1/r_0}$$

$$= \sup_{t\in(0,T)} \|\psi\|_{L^2_x}^{a/r_0} \cdot \|\psi(t,x)\|_{L^q(0,T),L^q)}^{1-a/r_0}$$

where, it is easy to check that $1/r_0 = (1 - a/r_0)\frac{1}{q}$, $1/p' = q/r = 4/d(r-2)$, $0 < a < 2$, $1/p + 1/p' = 1$.

Now we immediately see that $\|\psi\|_{L^q((0,T_{max}),L^r)}$ is infinite, from

$$\|\psi\|_{L^{q_0}((0,T),L^{r_0})} \leq \|\psi_0\|_{L^2_x}^{a/r_0} \cdot \|\psi\|_{L^q((0,T),L^r)}^{1-a/r_0} \quad \forall T < T_{max}$$

This proves the blowup alternative for any pair (q, r) with $r \geq r_0$.

2.3 Laws of Conservation

Since by using the above Local Well-Posedness theorem, we can also prove the following proposition.

Proposition 2.2. *Let $\psi_0 \in \Sigma(\mathbb{R}^2)$ and ψ is a unique solution of Cauchy problem (2.1), then ψ satisfies the following laws of conservation for all $t \geqslant 0$.*

1- Mass conservation Law

$$||\psi||_{L^2} = ||\psi_o||_{L^2} \tag{2.13}$$

2-Energy Conservation Law

$$E(\psi) = \int |\nabla\psi|^2 + |x|^2|\psi|^2 - \frac{\lambda(x)}{2}|\psi|^4 - \Re|\Omega| \langle L_z(t)\rangle = E(\psi_0) \tag{2.14}$$

3-Angular Momentum Conservation Law

$$\langle L_z(t)\rangle = \int \overline{\psi} L_z \psi = \langle L_z(0)\rangle \tag{2.15}$$

Moreover, $\langle L_z(t)\rangle = \Re \langle L_z(t)\rangle$

Proof:

Proof of 1:

First we prove the mass conservation law. For this prove, we take the derivative of L.H.S of equation (2.13) w.r.t time t. Then we have

$$\begin{aligned}
\frac{d}{dt}(||\psi||_{L^2})^2 &= \frac{d}{dt}\int \psi\overline{\psi} \\
&= \int (\psi\overline{\psi_t} + \psi_t\overline{\psi} \\
&= 2\Re \int \overline{\psi}\psi_t \\
&= -2\Im \int \overline{\psi}(i\psi_t)
\end{aligned}$$

Now using equation (2.1) we have

$$\frac{d}{dt}(||\psi||_{L^2})^2 = -2\Im\int(-\overline{\psi}\Delta\psi + |x|^2|\psi|^2 - \lambda(x)|\psi|^4 + |\Omega|\overline{\psi}L_z\psi)$$

applying the divergence theorem on the first term of R.H.S we get

$$\frac{d}{dt}|(|\psi||_{L^2})^2 = 2\Im\int|\nabla\psi|^2 - 2\Im\int|x|^2|\psi|^2 + 2\Im\int\lambda(x)|\psi|^4 + 2\Im|\Omega|\int\overline{\psi}L_z\psi$$

since $L_z = i|\Omega|(x_2\partial_{x_1} - x_1\partial_{x_2})$

Therefore,

$$\frac{d}{dt}(||\psi||_{L^2})^2 = 0 - 0 + 0 - 2\Re|\Omega|\int\overline{\psi}(x_2\partial x_1\psi - x_1\partial x_2\psi) \tag{2.16}$$

Let

$A = |\Omega|(\int\overline{\psi}(x_2\partial_{x_1}\psi - x_1\partial_{x_2}\psi))$

using integration by parts we get

$$A = |\Omega|(\int(-x_2\partial_{x_1}\overline{\psi}\psi + x_1\partial_{x_2}\overline{\psi})$$

$$= -|\Omega|(\int\psi(x_2\partial_{x_1}\overline{\psi} - x_1\partial_{x_2}\overline{\psi}))$$

$$A = -\overline{A}$$

$$2\Re A = 0$$

therefore, from equation (2.16)

$$\frac{d}{dt}(||\psi||_{L^2})^2 = 0$$

$$\frac{d}{dt}||\psi||_{L^2} = 0$$

which proves, that Identity (2.13) is true for all $t \geqslant 0$.

Proof of 2:

To prove the law of conservation of energy first we prove the non-rotating part of energy, then we prove the angular momentum conservation law, after combining these two results we get the required Identity (2.14).

Conservation law for non-rotating part of energy, defined as:

$$E_1(\psi) = \int (|\nabla\psi|^2 + |x|^2|\psi|^2 - \frac{\lambda(x)}{2}|\psi|^4) = E_1(\psi_0) \tag{2.17}$$

To prove the above identity we will consider the non rotating part of equation,(2.1)

$$i\psi_t + \Delta\psi - |x|^2\psi + \lambda(x)|\psi|^2\psi = 0$$

Now multiplying the above equation by $\bar{\psi}_t$. We get

$$i\psi_t\bar{\psi}_t + \Delta\psi\bar{\psi}_t - |x|^2\psi\bar{\psi}_t + \lambda(x)|\psi|^2\psi\bar{\psi}_t = 0$$

taking the conjugate of above equation,

$$-i\bar{\psi}_t\psi_t + \Delta\bar{\psi}\psi_t - |x|^2\bar{\psi}\psi_t + \lambda(x)|\psi|^2\bar{\psi}\psi_t = 0$$

adding the above two equations we have

$$\Delta\bar{\psi}\psi_t + \Delta\psi\bar{\psi}_t - |x|^2(\bar{\psi}\psi_t + \psi\bar{\psi}_t) + \lambda(x)|\psi|^2(\bar{\psi}\psi_t + \psi\bar{\psi}_t) = 0$$

We can also write the above equation as

$$div_x(\nabla\psi\bar{\psi}_t) - (\nabla\psi\nabla\bar{\psi}_t) + \Delta\psi\psi_t + \Delta\psi\psi_t - |x|^2\partial_t(\psi\bar{\psi}) + \frac{\lambda(x)}{2}|\psi|^2|\psi|^2) = 0$$

Integrating to get, using Green's formula and asymptotic $\nabla\psi(t,x) \to 0$ as $x \to \infty$

$$\frac{d}{dt}\left(-\int \nabla\psi\nabla\bar{\psi}dx - \int |x|^2|\psi|^2dx + \frac{1}{2}\int \lambda(x)|\psi|^4dx\right) = 0$$

Finally, from L.H.S of equation (2.17) we have

$$\frac{d}{dt}(E_1(\psi)) = 0$$

Hence it is proved that the non-rotating part of energy is conserved.

Proof of 3:

To prove the law of angular momentum conservation. We will consider the following equation:

$$\langle L_z(t)\rangle = \int \bar{\psi} L_z \psi \tag{2.18}$$

Taking the derivative on both sides w.r.t time "t".

$$\begin{aligned}\frac{d}{dt}(\langle L_z(t)\rangle) &= \frac{d}{dt}\left(\int \bar{\psi} L_z \psi\right)\\ &= \int (\bar{\psi}_t L_z \psi + \bar{\psi} L_Z \psi_t)\end{aligned}$$

Since $L_z = i(x_2\partial_{x_1} - x_1\partial x_2)$. Therefore,

$$\frac{d}{dt}(\langle L_z(t)\rangle) = -\int \overline{i\psi_t}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi + \int \bar{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})(i\psi_t)$$

applying integration by parts on second term, we have

$$\begin{aligned}\frac{d}{dt}(\langle L_z(t)\rangle) &= -\int \overline{i\psi_t}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi - \int i\psi_t(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}\\ &= -\int \big((-\Delta\bar{\psi} + |x|^2\bar{\psi} - \lambda(x)|\psi|^2\bar{\psi} - |\Omega|\overline{L_z\psi})(x_2\partial_{x_1}\psi - x_1\partial_{x_2}\psi)\big)\\ &\quad - \int \big((-\Delta\psi + |x|^2\psi - \lambda(x)|\psi|^2\psi - |\Omega|L_z\psi)(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi})\big)\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}(\langle L_z(t)\rangle) = &\int 2\Re(\Delta\psi(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi}) - \int |x|^2(\bar{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi + \psi(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}) \\
&+ \int \lambda(x)|\psi|^2(\bar{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi + \psi(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}) \\
&+ |\Omega| \int \left[i(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi - i(x_2\partial_{x_1} - x_1\partial_{x_2})\psi(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}\right] \quad = \\
&+ \int \lambda(x)|\psi|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2 +
\end{aligned}
$$

After simplification we have

$$
\begin{aligned}
\frac{d}{dt}(\langle L_z(t)\rangle) = &\int 2\Re(\Delta\psi(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi}) - \int |x|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2 \\
&+ \int \lambda(x)|\psi|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2 + 0
\end{aligned}
$$

Let the above expression be,

$$
\frac{d}{dt}(\langle L_z(t)\rangle) = 2\Re(I_1) - I_2 + I_3 \tag{2.19}
$$

Then we compute I_1, I_2, I_3 separately and recollect their results in (2.20).

Computing I_1,

$$
\begin{aligned}
I_1 &= \int \Delta\psi(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi}) \\
&= \int \left(\partial^2_{x_1}\psi(x_2\partial_{x_1}\bar{\psi}) - \partial^2_{x_2}\psi(x_1\partial_{x_2}\bar{\psi}) - \partial^2_{x_1}\psi(x_1\partial_{x_2}\bar{\psi}) - \partial^2_{x_2}\psi(x_2\partial_{x_1}\bar{\psi})\right)
\end{aligned}
$$

Applying integration by parts for all term on R.H.S, we have

$$
\begin{aligned}
I_1 = &-\int x_2\partial_{x_1}\psi\partial^2_{x_1}\bar{\psi} + \int x_1\partial_{x_2}\psi\partial^2_{x_2}\bar{\psi} + \int \partial_{x_2}\bar{\psi}\partial_{x_1}\psi + \int x_1\partial_{x_1}\psi\partial^2_{x_2x_1}\bar{\psi} \\
&- \int \partial_{x_1}\bar{\psi}\partial_{x_2}\psi - \int x_2\partial_{x_2}\psi\partial^2_{x_1x_2}\bar{\psi} \\
= &-\int \Delta\bar{\psi}(x_2\partial_{x_1} - x_1\partial_{x_2})\psi + 2(i)Im\int \partial_{x_1}\psi\partial_{x_2}\bar{\psi} + \int (x_1\partial_{x_1}\psi\partial^2_{x_2x_1}\bar{\psi} - x_2\partial_{x_2}\psi\partial^2_{x_1x_2}\bar{\psi})
\end{aligned}
$$

Now taking the $\Re$ on both sides

$$\begin{aligned}\Re I_1 &= -\Re\overline{I_1} + 0 + \Re I_{1,1}\\ \Re I_1 + \Re\overline{I_1} &= \Re I_{1,1}\\ 2\Re I_1 &= \Re I_{1,1}\end{aligned}$$

where

$$I_{1,1} = \int (x_1\partial_{x_1}\psi\partial^2_{x_2x_1}\bar{\psi} - x_2\partial_{x_2}\psi\partial^2_{x_1x_2}\bar{\psi})$$

After applying integration by parts on $I_{1,1}$ we get

$$\begin{aligned}I_{1,1} &= -\int \left(\partial_{x_2}(x_1\partial_{x_1}\psi)\partial_{x_1}\bar{\psi} + \partial_{x_1}(x_2\partial_{x_2}\psi)\partial_{x_2}\bar{\psi}\right)\\ &= -\int (x_1\partial_{x_1}\bar{\psi}\partial^2_{x_1x_2}\psi - x_2\partial_{x_2}\bar{\psi}\partial^2_{x_1x_2}\psi)\\ I_{1,1} &= -\overline{I_{1,1}}\end{aligned}$$

Therefore,

$$\Re(I_{1,1}) = 0$$

Finally, we have

$$2\Re(I_1) = 0$$

For I_2, we have

$$
\begin{aligned}
I_2 \quad &= \int |x|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2 \\
&= -\int \partial_{x_1}(|x|^2x_2)|\psi|^2 + \int \partial_{x_2}(|x|^2x_1)|\psi|^2 \\
&= -\int 2x_1x_2|\psi|^2 + \int 2x_2x_1|\psi|^2
\end{aligned}
$$

$$I_2 = 0$$

Now for I_3,

$$
\begin{aligned}
I_3 &= \int \lambda(x)|\psi|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2 \\
&= -\int x_2\partial_{x_1}(\lambda(x)|\psi|^2)|\psi|^2 + \int x_1\partial_{x_2}(\lambda(x)|\psi|^2)|\psi|^2 \\
&= -\int x_2\partial_{x_1}\lambda(x)|\psi|^4 + \int x_1\partial_{x_2}\lambda(x)|\psi|^4 - \int \left[\lambda(x)|\psi|^2(x_2\partial_{x_1} - x_1\partial_{x_2})|\psi|^2\right] \\
&= -\int (x_2\partial_{x_1}\lambda(x) - x_1\partial_{x_2}\lambda(x))|\psi|^4 - I_3 \\
2I_3 &= -\int (x_2\partial_{x_1}\lambda(x) - x_1\partial_{x_2}\lambda(x))|\psi|^4 \\
&= -\int |\psi|^4\left((x_2\partial_{x_1} - x_1\partial_{x_2})\,\lambda(x)\right)
\end{aligned}
$$

Since $\lambda(x)$ is radial. Then we have $\lambda(x) = \lambda(r) \quad where \quad r = r(x) = |x|^{\frac{1}{2}}$. Then $\partial_{x_1}\lambda(r) = \lambda'(r)(\frac{x_1}{r})$ and $\partial_{x_2}\lambda(r) = \lambda'(r)(\frac{x_2}{r})$

Putting these values in above expression we get

$I_3 = 0$

Combining the results of I_1, I_2, I_3 in (2.20), we get

$$\frac{d}{dt}(\langle L_z(t)\rangle) = 0$$

which yields the Identity (2.15).

Recollecting the (2.17) and (2.15), we get the desired identity (2.14) for conservation of energy with rotational term.

CHAPTER 3

Virial Identity

To prove the blow up results, we introduce a relevant quantity which plays a vital role in the analysis of blow-up phenomena, called the variance and defined as

$$J(t) = \int |x|^2 |\psi|^2 \tag{3.1}$$

Moreover,the first and second order derivatives of $J(t)$ in term of ψ, the solution of Cauchy problem (2.1) is an important tool to describe the blow-up results. Now we prove the following proposition.

Proposition 3.1. *If ψ be the solution of Cauchy problem (2.1) in $C^1([0,T),\Sigma)$ and let ψ_0 is in Σ.Then the variance $J(t)$ satisfies the following identities:*

$$J'(t) = -4\Im \int x\psi \cdot \nabla\overline{\psi} \tag{3.2}$$

and

$$J''(t) = 8E(\psi_0) + 8\Re \langle L_z(0)\rangle - 16J(t) + \frac{1}{2}\int x \cdot \nabla\lambda(x)|\psi|^4 \tag{3.3}$$

Proof:

Consider the variance

$$J(t) = \int |x|^2 |\psi|^2$$

Now differentiating $J(t)$ w.r.t time t we have

$$\begin{aligned} J'(t) &= \int |x|^2(\psi_t\bar{\psi} + \psi\bar{\psi}_t) \\ &= 2\Re(i)\int |x|^2\psi(\overline{i\psi_t}) \end{aligned}$$

Using (2.1) in above expression we get

$$J'(t) = -2\Im\int |x|^2\psi(-\Delta\bar{\psi} + |x|^2\bar{\psi} - \lambda(x)|\psi|^2\bar{\psi} - |\Omega|\bar{L_z}\bar{\psi})$$
$$= -2\Im\int |x|^2\psi\Delta\bar{\psi} + 2\Im\int |x|^4|\psi|^2 - 2\Im\int \lambda(x)|\psi|^4 - 2\Im(i)\int |\Omega||x|^2\psi(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}$$
$$= -2\Im\int \nabla(|x|^2\psi)\cdot\nabla\bar{\psi} - 2\Re|\Omega|\int |x|^2\psi(x_2\partial_{x_1} - x_1\partial_{x_2})\bar{\psi}$$
$$= -4\Im\int x\psi\nabla\bar{\psi} - 2Re|\Omega|(A)$$

Consider A,

$$A = \int |x|^2\psi(x_2\partial_{x_1}\bar{\psi}) - \int |x|^2\psi(x_1\partial_{x_2}\bar{\psi})$$
$$= -\int \partial_{x_1}(|x|^2x_2\psi)\bar{\psi} + \int \partial_{x_2}(|x|^2x_1\psi)\bar{\psi}$$
$$= -\int 2x_1x_2\psi\bar{\psi} - \int x_2|x|^2\partial_{x_1}\psi\bar{\psi} + \int 2x_1x_2\psi\bar{\psi} + \int x_1|x|^2\partial_{x_2}\psi\bar{\psi}$$
$$= -\int |x|^2\bar{\psi}(x_2\partial_{x-1} - x_1\partial_{x_2})\psi$$
$$A = -\bar{A}$$
$$A + \bar{A} = 0$$
$$2\Re(A) = 0$$

Therefor we get the proof of identity (3.2).

$$J'(t) = -4\Im\int x\psi\nabla\bar{\psi}$$

Now we will prove the second identity (3.3).

Differentiating $J'(t)$ w.r.t time t we have,

$$J''(t) = -4\Im\int x(\psi_t\cdot\nabla\bar{\psi} + \psi\cdot\nabla\bar{\psi}_t)$$

Applying the divergence theorem on both terms of R.H.S

$$
\begin{aligned}
J''(t) &= -4\Im\left[\int \nabla\cdot(x\psi_t)\bar{\psi} - \int \nabla\cdot(x\psi)\bar{\psi}_t\right] \\
&= -4\Im\left[-2\int x\psi_t\bar{\psi} - \int x\nabla\psi_t\bar{\psi} - 2\int \psi\bar{\psi}_t - \int x\nabla\psi\bar{\psi}_t\right] \\
&= -4\Im\left[-2\Re\int \psi_t\bar{\psi} - \int x\bar{\psi}\nabla\psi_t - \int x\nabla\psi\bar{\psi}_t\right]
\end{aligned}
$$

Since the first term vanish and using integration by parts on second term of R.H.S

$$
\begin{aligned}
J''(t) &= -4\Im\left[+\int \nabla(x\bar{\psi})\psi_t - \int x\nabla\psi\bar{\psi}_t\right] \\
&= -4\Im\left[\int (2\bar{\psi}\psi_t + x\nabla\bar{\psi}\psi_t - x\nabla\psi\bar{\psi}_t)\right] \\
&= -8\Im\int \bar{\psi}\psi_t + 4\Im\left[2i\Im\int x\nabla\psi\bar{\psi}_t\right] \\
&= -8\Im\int \bar{\psi}\psi_t + 8\Im\int x\nabla\psi\bar{\psi}_t \\
&= 4\left[-2\Im\int \bar{\psi}\psi_t + 2\Im\int x\nabla\psi\bar{\psi}_t\right]
\end{aligned}
$$

Let the above identity is,

$$
J''(t) = 4\left[C + D\right] \tag{3.4}
$$

where

$$
\begin{aligned}
C &= -2\Im\int \bar{\psi}\psi_t \\
&= 2\Im(i)\int \bar{\psi}(i\psi_t)
\end{aligned}
$$

Now using (2.1) in above identity we have,

$$
\begin{aligned}
C &= 2\Re\int \bar{\psi}(-\Delta\psi + |x|^2\psi - \lambda(x)|\psi|^2\psi - |\Omega|L_z\psi) \\
&= 2\Re\left[-\int \bar{\psi}\Delta\psi + |x|^2|\psi|^2 - \lambda(x)|\psi|^4 - |\Omega|\bar{\psi}L_z\psi\right]
\end{aligned}
$$

applying integration by parts on first term

$$C = 2\Re\left[+\int \nabla\bar{\psi}\cdot\nabla\psi\right] + 2\int |x|^2|\psi|^2 - 2\int \lambda(x)|\psi|^4 - 2|\Omega|\int \bar{\psi}L_z\psi$$

$$C = 2\int |\nabla\psi|^2 + 2\int |x|^2|\psi|^2 - 2\int \lambda(x)|\psi|^4 - 2\Re|\Omega|\int \bar{\psi}L_z\psi \tag{3.5}$$

Now we consider,

$$\begin{aligned} D &= 2\Im\int x\nabla\psi\bar{\psi}_t \\ &= 2\Im(i)\int x\nabla\psi\overline{(i\psi_t)} \\ &= 2\Re\int x\cdot\nabla\psi(-\overline{\Delta\psi} + |x|^2\bar{\psi} - \lambda(x)|\psi|^2\bar{\psi} - |\Omega|\overline{L_z\psi}) \\ &= -2\Re\int x\cdot\nabla\psi\Delta\bar{\psi} + 2\Re\int x\cdot\nabla\psi|x|^2\bar{\psi} - 2\Re\int x\cdot\nabla\psi\lambda(x)|\psi|^2\bar{\psi} - 2\Re|\Omega|\int x\cdot\nabla\psi\overline{L_z\psi} \end{aligned}$$

Let

$$D = -2\Re I_1 + 2\Re I_2 - 2ReI_3 - |\Omega|\left[2\Re I_4\right] \tag{3.6}$$

We solve I_1, I_2, I_3, I_4 separately then combine these results to get D in the equation (3.6).

Consider

$$I_1 = \int x\cdot\nabla\psi\Delta\bar{\psi}$$

We can write L.H.S in the following form

$$\begin{aligned} I_1 &= \sum_k\sum_j\int x_k\partial_k\psi\partial_j\cdot\partial_j\bar{\psi} \qquad where \quad k,j = 1,2 \\ &= -\sum_k\sum_j\int \partial_j(x_k\partial_k\psi)\cdot\partial_j\bar{\psi} \\ &= -\sum_k\sum_j\int \left(\partial_j(x_k)\partial_k\psi\cdot\partial_j\bar{\psi} + x_k\partial_j(\partial_k\psi)\cdot\partial_j\bar{\psi}\right) \end{aligned}$$

$$\begin{aligned}
I_1 &= -\sum_k \sum_j \int \delta_{jk} \partial_k \psi \cdot \partial_j \bar{\psi} - \sum_k \sum_j \int x_k \partial_k (\partial_j \psi) \partial_j \bar{\psi} \\
&= -\sum_k \int \partial_k \psi \cdot \partial_k \bar{\psi} - \sum_k \sum_j \int x_k \partial_k (\partial_j \psi) \partial_j \bar{\psi} \\
&= -\int |\nabla \psi|^2 - I_{1,1}
\end{aligned}$$

Now we solve the second term of above equation.

$$I_{1,1} = -\sum_k \sum_j \int x_k \partial_k (\partial_j \psi) \partial_j \bar{\psi}$$

applying integration by parts we have

$$\begin{aligned}
I_{1,1} &= -\sum_j \sum_k \int \partial_k (x_k \partial_j \bar{\psi}) \partial_j \psi \\
&= -\sum_j \int 2 \partial_j \psi \partial_j \bar{\psi} - \sum_j \sum_k \int x_k \partial_k (\partial_j \bar{\psi}) \partial_j \psi \\
&= -2 \int |\nabla \psi|^2 - \overline{I_{!,1}} \\
2\Re(I_{1,1}) &= -2 \int |\nabla \psi|^2
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
-2\Re(I_1) &= -2\Re \left[-\int |\nabla \psi|^2 \right] + 2\Re I_{1,1} \\
&= +2 \int |\nabla \psi|^2 - 2 \int |\nabla \psi|^2
\end{aligned}$$

Finally, we get the following result for first term of (3.6)

$$-2\Re(I_1) = 0 \tag{3.7}$$

Now consider I_2.

$$\begin{aligned}
I_2 &= \int x \cdot \nabla\psi |x|^2 \bar{\psi} \\
&= -\int \nabla(x|x|^2\bar{\psi})\psi \\
&= -\int \left(2|x|^2\bar{\psi}\psi + x \cdot 2x\bar{\psi}\psi + x|x|^2\nabla\bar{\psi}\psi\right) \\
&= -2\int |x|^2|\psi|^2 - 2\int |x|^2|\psi|^2 - \overline{I_2}
\end{aligned}$$

Therefore, we get the result for second term of (3.6).

$$2\Re(I_2) = -4\int |x|^2|\psi|^2 \tag{3.8}$$

We will solve for I_3.

$$\begin{aligned}
I_3 &= \int x\lambda(x)|\psi|^2\bar{\psi} \cdot \nabla\psi \\
&= -\int \nabla \cdot (x\lambda(x)|\psi|^2\bar{\psi})\psi \\
&= -\int 2\lambda(x)|\psi|^2\bar{\psi}\psi - \int x\nabla\lambda(x)|\psi|^2\bar{\psi}\psi - \int x\lambda(x)\nabla(|\psi|^2)\bar{\psi}\psi - \int x\lambda(x)|\psi|^2\psi\nabla\bar{\psi} \\
&= -2\int \lambda(x)|\psi|^4 - \int x\nabla\lambda(x)|\psi|^4 - \int x\lambda(x)|\psi|^2(\nabla\psi\bar{\psi} + \psi\nabla\bar{\psi}) - \overline{I_3} \\
&= -2\int \lambda(x)|\psi|^4 - \int x\nabla\lambda(x)|\psi|^4 - \int x\lambda(x)|\psi|^2\nabla\psi\bar{\psi} - \int x\lambda(x)|\psi|^2\psi\nabla\bar{\psi} - \overline{I_3} \\
I_3 + \overline{I_3} &= -2\int \lambda(x)|\psi|^4 - \int x\nabla\lambda(x)|\psi|^4 - I_3 - \overline{I_3} \\
2(I_3 + \overline{I_3}) &= -2\int \lambda(x)|\psi|^4 - \int x\nabla\lambda(x)|\psi|^4 \\
I_3 + \overline{I_3} &= -\int \lambda(x)|\psi|^4 - \frac{1}{2}\int x\nabla\lambda(x)|\psi|^4 \\
2\Re(I_3) &= -\int \lambda(x)|\psi|^4 - \frac{1}{2}\int x\nabla\lambda(x)|\psi|^4
\end{aligned}$$

So we have following result for third term of (3.6).

$$-2\Re I_3 = \int \lambda(x)|\psi|^4 + \frac{1}{2}\int x\nabla\lambda(x)|\psi|^4 \tag{3.9}$$

Consider I_4.

$$I_4 = \int x \cdot \nabla\psi \overline{L_z\psi}$$

From (2.3) putting the value of L_z.

$$\begin{aligned}
I_4 &= -i\int x \cdot \nabla\psi(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi}) \\
&= -i\int \left[(x_1\partial_{x_1}\psi + x_2\partial_{x_2}\psi)(x_2\partial_{x_1}\bar{\psi} - x_1\partial_{x_2}\bar{\psi})\right] \\
&= -i\left(\int x_1x_2\partial_{x_1}\bar{\psi}\partial_{x_1}\psi - \int {x_1}^2\partial_{x_1}\psi\partial_{x_2}\bar{\psi} + {x_2}^2\partial_{x_2}\psi\partial_{x_1}\bar{\psi} - x_1x_2\partial_{x_2}\psi\partial_{x_2}\bar{\psi}\right) \\
&= -i\left[\int x_1x_2|\partial_{x_1}\psi|^2 - \int x_1x_2|\partial_{x_2}\psi|^2 + \int({x_2}^2\partial_{x_2}\psi\partial_{x_1}\bar{\psi} - \int {x_1}^2\partial_{x_1}\psi\partial_{x_2}\bar{\psi})\right] \\
2\Re(I_4) &= -2\Re(i)\left[\int x_1x_2|\partial_{x_1}\psi|^2\right] - 2\Re(i)\left[\int x_1x_2|\partial_{x_2}\psi|^2\right] + 2\Re\left[I_{4,1}\right] \\
2\Re(I_4) &= -0 - 0 + 2\Re\left[I_{4,1}\right]
\end{aligned}$$

Now we solve for $I_{4,1}$.

$$\begin{aligned}
I_{4,1} &= -i\int({x_2}^2\partial x_2\psi\partial_{x_1}\bar{\psi} - \int {x_1}^2\partial_{x_1}\psi\partial_{x_2}\bar{\psi}) \\
&= -i\left(-\int \partial x_1({x_2}^2\partial_{x_2}\psi)\bar{\psi} + \int \partial_{x_2}({x_1}^2\partial_{x_1}\psi)\bar{\psi}\right) \\
&= -i\left(-\int {x_2}^2\partial^2_{x_1x_2}\psi\bar{\psi} + \int {x_1}^2\partial^2_{x_1x_2}\psi\bar{\psi}\right) \\
&= -i\left(\int \partial x_2({x_2}^2\bar{\psi})\partial_{x_1}\psi - \int \partial_{x_1}({x_2}^2\bar{\psi})\partial_{x_2}\psi\right) \\
&= -i\left(\int 2x_2\bar{\psi}\partial_{x_1}\psi + \int {x_2}^2\partial_{x_2}\bar{\psi}\partial_{x_1}\psi - \int 2x_1\bar{\psi}\partial_{x_2}\psi - \int {x_1}^2\partial_{x_1}\bar{\psi}\partial_{x_2}\psi\right) \\
&= -i\left(2\int \bar{\psi}(x_2\partial_{x_1}\psi - x_1\partial_{x_2}\psi\right) - \overline{I}_{4,1}
\end{aligned}$$

by using (2.3),

$$2\Re(I_{4,1}) = -2\int \bar{\psi}L_z\psi$$

from results of $I_{4,1}$ we obtain the result for fourth term of (3.6)

$$-2\Re(I_4) = 2\Re\int \bar{\psi} L_z \psi \tag{3.10}$$

Combining all the results of (3.7), (3.8), (3.9) and (3.10) in (3.6) we get,

$$D = -4\int |x|^2|\psi|^2 + \int \lambda(x)|\psi|^4 + \frac{1}{2}\int x\cdot\nabla\lambda(x)|\psi|^4 + 2\Re\int \bar{\psi} L_z\psi$$

Finally, recollecting the results of C and D in (3.4) we have,

$$J''(t) = 8\left[\int (|\nabla\psi|^2 + |x|^2|\psi|^2 - \frac{\lambda(x)}{2}|\psi|^4\right] - 16\int |x|^2|\psi|^2 + \frac{1}{2}\int x\cdot\nabla\lambda(x)|\psi|^4$$

By the law of conservation of energy and momentum (2.14), (2.15) respectively and the moment (variance) (3.1), we complete the proof of identity (3.3).

$$J''(t) = 8E(\psi_0) + 8\Re|\Omega|\left\langle L_z(0)\right\rangle - 16J(t) + \frac{1}{2}\int x\cdot\nabla\lambda(x)|\psi|^4$$

CHAPTER 4

Sharp Condition For Global Existence

To prove the sharp condition first we will consider the following equation,

$$-\Delta\psi + \psi - |\psi|^2\psi = 0, \qquad \psi \in H^1(\mathbb{R}^2) \tag{4.1}$$

Now we are concerned with the relation between the global existence and the ground state solution of equation (4.1). Many authors has been studied the existence of ground state solution for (4.1) (see [32], [33]).

From [34] and [35], we can assume that ground state $Q(x)$ is the unique solution of (4.1) where, $x \in H^1(R^2)$ Then from [34] and [35], we have following lemma.

Lemma 4.1. *For equation (4.1), there exists a unique positive radially symmetric solution $Q(x)$ called the ground state solution depending only on x. Also $\frac{1}{2}\int Q^2$ is the minimum of the functional,*

$$I(\psi) = \frac{\left(\int |\nabla\psi|^2\right)\left(\int |\psi|^2\right)}{\left(\int |\psi|^4\right)}, \quad \psi \in \Sigma$$

Remark:

Using the above lemma 4.1, we can get the following Gagliardo-Nirenberg Inequality,

$$\int |\psi|^4 \leqslant \frac{2\left(\int |\nabla\psi|^2\right)\left(\int |\psi|^2\right)}{\int Q^2} \tag{4.2}$$

From [34] and [36], we introduce the following inequality and will prove it.

Lemma 4.2. *Let $\psi \in \Sigma$, then we have the following inequality:*

$$\int |\psi|^2 \leqslant \left(\int |\nabla\psi|^2\right)^{\frac{1}{2}} \left(\int |x|^2|\psi|^2\right)^{\frac{1}{2}} \tag{4.3}$$

Proof:

First we consider the following identity and prove it,

$$-2\int |\psi|^2 = 2\Re\int \bar{\psi} x \cdot \nabla\psi \tag{4.4}$$

To prove this we consider the R.H.S. Let

$$\begin{aligned} B &= \int \bar{\psi} x \cdot \nabla \psi \\ &= -\int \nabla(\bar{\psi} x) \cdot \psi \\ &= -\int \left(\nabla \bar{\psi} x + \bar{\psi} \nabla x\right) \cdot \psi \\ &= -\int \psi x \nabla \bar{\psi} - \int 2\psi \cdot \bar{\psi} \\ &= -\overline{B} - 2\int |\psi|^2 \\ 2\Re(B) &= -2\int |\psi|^2 \end{aligned}$$

Hence the identity (4.4) is proved.

Now we Consider the absolute value of term $|2Re \int \bar{\psi} x \nabla \psi|$ then after applying Cauchy-Schwarz inequality, we have,

$$|2\Re \int \bar{\psi} x \cdot \nabla \psi| \leqslant 2\left(\int |\nabla \psi|^2\right)^{\frac{1}{2}} \left(\int |x\bar{\psi}|^2\right)^{\frac{1}{2}}$$

$$|2\Re \int \bar{\psi} x \cdot \nabla \psi| \leqslant 2\left(\int |\nabla \psi|^2\right)^{\frac{1}{2}} \left(\int |x|^2 |\psi|^2\right)^{\frac{1}{2}}$$

Using the identity (4.4) we get,

$$|-2\int |\psi|^2| \leqslant 2\left(\int |\nabla \psi|^2\right)^{\frac{1}{2}} \left(\int |x|^2 |\psi|^2\right)^{\frac{1}{2}}$$

$$\int |\psi|^2 \leqslant \left(\int |\nabla \psi|^2\right)^{\frac{1}{2}} \left(\int |x|^2 |\psi|^2\right)^{\frac{1}{2}}$$

Therefore, we get the desire inequality (4.3).

Now we proof following lemma from [24,27], which plays an important role in proving our main results.

4.1 Blow up in finite time

Lemma 4.3. *Let* $p \in [1+\frac{4}{n}, 1+\frac{4}{n-2}]$, $n = 2$. *Suppose* $\psi_0 \neq 0 \in H^1(\mathbb{R}^2)$ *satisfying the following condition:*

$$J(0) = \int |x|^2|\psi_0|^2 \geqslant E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle$$

and

$$J'(0) = -4\Im \int x\psi_0 \nabla\psi_0 \leqslant 0$$

Then the corresponding solution $\psi(x,t)$ *of equation (2.1) blows up in finite time.*

Proof:

To prove this lemma we will consider the following two cases :

- Case 1 : If $E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle \ \geqslant 0$

From the proposition 3.1, we have the following identity :

$$J''(t) - 16J(t) = 8E(\psi_0) + 8\Re|\Omega| \langle L_z(0)\rangle + \frac{1}{2}\int x\cdot\nabla\lambda(x)|\psi|^4 \tag{4.5}$$

which is the second order differential equation and for the solution of this equation we will find

$$J(t) = J_{p_1}(t) + J_{p_2}(t)$$

For J_{p_1} we consider the following equation

$$J''(t) + J(t) = 8(E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle)$$

Since L.H.S of above equation is constant. Therefore the solution of this equation is

$$J_{p_1}(t) = \frac{1}{16} \times 8(E(\psi_0) + \Re|\Omega| \langle L_z(0) \rangle)$$

Finally we have

$$J_{p_1}(t) = \frac{1}{2}(E(\psi_0) + \Re|\Omega| \langle L_z(0) \rangle) \tag{4.6}$$

Now we consider the following equation to find $J_{p_2}(t)$,

$$J''(t) - 16J(t) = \frac{1}{2} \int x \nabla \lambda(x) |\psi|^4$$

Let the above equation can be written as,

$$J''(t) - 16J(t) = f(t)$$

where $f(t) = \frac{1}{2} \int x \cdot \nabla \lambda(x) |\psi(t,x)|^4$.

Now we can write the above equation in the Matrix form as

$$\frac{d}{dt}\begin{pmatrix} J(t) \\ J'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix} \begin{pmatrix} J(t) \\ J'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Then applying Duhamel's Principle we have,

$$\begin{pmatrix} J(t) \\ J'(t) \end{pmatrix} = e^{At} \begin{pmatrix} J(0) \\ J'(0) \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \tag{4.7}$$

Where $A = \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix}$

Since we know that, the expansion series of e^{At} is:

$e^{At} = I + At + (At)^2/2! + (At)^3/3! = (At)^4/4! + (At)^5/5! + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

after simplifying the above expansion we obtain

$$e^{At} = \cos 4t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\sin 4t \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix}$$

Similarly we get

$$e^{A(t-s)} = \cos 4(t-s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{4}\sin 4(t-s) \begin{pmatrix} 0 & 1 \\ -16 & 0 \end{pmatrix}$$

Putting the values of e^{At} and $e^{A(t-s)}$ in (4.7), we obtain the desire solution of $J_{p_2}(t) as follow$

$$J_{p_2}(t) = J(0)\cos 4t + J'(0)\frac{1}{4}\sin 4t + \frac{1}{4}\int_0^t \sin 4(t-s) f(s) ds \tag{4.8}$$

After combining the results of (4.6) and (4.8), we get the following solution for equation (4.5).

$J(t) = J(0)\cos 4t + \frac{1}{4}\sin 4t + \frac{1}{2}(E(\psi_0) + Re\langle L_z(0)\rangle) + \frac{1}{4}\int_0^t \sin 4(t-s) f(s) ds$

Let the above solution can be written as,

$$J(t) = r\sin(4t+\beta) + \frac{1}{2}E(\psi_0) + \frac{1}{2}\Re|\Omega|\langle L_z(0)\rangle + \frac{1}{4}\int_0^t \sin 4(t-s) f(s) \tag{4.9}$$

Where $r \geqslant 0$ and $\beta \in [0, 2\pi]$ are constants determined by $J(0)$, $J'(0)$ and $E(\psi_0) + Re|\Omega|\langle L_z(0)\rangle$.

Also

$$r^2 = \left(J(0) - \frac{1}{2}E(\psi_0) - \frac{1}{2}Re|\Omega|\langle L_z(0)\rangle\right)^2 + \frac{1}{16}\left(J'(0)\right)^2 \tag{4.10}$$

Since H_2 is true then for any $t \in [0, \frac{\pi}{4}]$, we have $f(t) \leqslant 0$. Therefore, from equation (4.10) we have

$$0 \leqslant J(t) \leqslant r\sin(4t+\beta) + \frac{1}{2}E(\psi_0) + \frac{1}{2}\Re|\Omega| \langle L_z(0)\rangle \tag{4.11}$$

If we have $J(0) \geqslant E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle \ \geqslant 0$ and $J'(0) = 4\cos\beta \leqslant 0$, then we have $\beta \in [\frac{\pi}{2}, \pi)$. Now from equation (4.10) and equation(4.11), we can claim that there exits $T \in [\frac{\pi}{8}, \frac{\pi}{4})$, such that

$$\lim_{t\to T} J(t) = 0$$

From lemma 4.2, we have the following results.

$$\lim_{t\to T} \int |\nabla\psi|^2 = +\infty$$

Which shows that, the solution $\psi(t,x)$ of equation (2.1) blows up in finite time.

- Case 2: If $E(\psi_0) + \Re|\Omega| \langle L_z\rangle(0) \ < 0$

Since H_2 is true and $x \cdot \nabla\lambda(x) \leqslant 0$, $\forall\ x \in (\mathbb{R}^2)$ then from equation (3.3) we have,

$$J''(t) \leqslant 8E(\psi_0) + 8\Re|\Omega| \langle L_z\rangle(0)$$

By using an analytical identity we get,

$$J(t) = J(0) + J'(0)t + \int_0^t J''(s)(t-s)ds$$

Then using the above result of $J''(t)$ we have,

$$J(t) \leqslant J(0) + J'(0)t + 8E(\psi_0) + 8\Re|\Omega| \langle L_z \rangle (0)$$

Since we have $E(\psi_0) + Re|\Omega| \langle L_z \rangle (0) \; < 0$, $J(0) \geqslant 0$ and $J'(0) \leqslant 0$, then there exits a T with $0 < T < +\infty$ such that,

$$\lim_{t \to T} J(t) = 0$$

Therefore, again from lemma 4.2, we have the following results.

$$\lim_{t \to T} \int |\nabla \psi|^2 = +\infty$$

Which shows that the solution $\psi(t, x)$ of (2.1) blows up in finite time. Hence lemma 4.3 is proved.

Now we consider the following equation:

$$-\Delta\psi + \psi - \lambda_2 |\psi|^2 \psi = 0, \;\; \psi \in H^1(\mathbb{R}^2) \tag{4.12}$$

Then from [24] and by using scaling argument, we have the following remark:

Remark:

Equation (4.12) has a unique positive radially symmetric solution Q_{λ_2} called the ground state solution. Also we have $||Q_{\lambda_2}|| = (\lambda_2)^{-\frac{1}{2}} ||Q||_{L^2}$ and $Q_{\lambda_2}(x)$ satisfies that,

$$2\int |\nabla Q_{\lambda_2}|^2 = \lambda_2 \int |Q_{\lambda_2}|^4 \tag{4.13}$$

Which is also known as Pohozaev identity.

Now with the help of lemma 4.3, we will prove our main theorem about the sharp condition of global existence of (2.1). Also we will show that how small or large is the initial data for which we get global solution or blow-up result.

4.2 Main Result

Theorem 4.4. *Assume that Q_{λ_2} be the positive radially symmetric solution of equation (4.12). If the initial data $\psi_0 \in \Sigma$ satisfying these two condition:*

condition(a). $J(0) = \int |x|^2|\psi_0|^2 \geqslant E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle$

condition(b). $J'(0) = -4\Im \int x\psi_0\nabla\psi_0 \leqslant 0$

and $(H_1)$and (H_2) are true, then we have following results

1- If initial data and ground state solution of (4.12) has relation $||\psi_0||_{L^2} < ||Q_{\lambda_2}||_{L^2}$. Then the solution of (2.1), $\psi(t,x)$ exists globally in time t.

2- If initial data is $\psi(0,x) = k\,\alpha\, Q_{\lambda_2}(\alpha x)$, where α be any positive real number, k be any real number and satisfies the condition, $|k| \geqslant (\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}} \geqslant 1$. Then $||\psi_0||_{L^2} \geqslant ||Q_{\lambda_2}||_{L^2}$ and the corresponding solution of equation (2.1), $\psi(t,x)$ blows up in finite time.

Proof (1):

Let $\psi(t,x) \in C([0,T),\Sigma)$ be the solution of equation (2.1), where $[0,T)$ represents the maximal existence of time.

From conservation of energy and angular momentum, we have,

$$E(\psi_0) = \int |\nabla\psi|^2 + \int |x|^2|\psi|^2 - \frac{1}{2}\int \lambda(x)|\psi|^4 - \Re|\Omega| \langle L_z(0)\rangle$$

Then

$$\frac{1}{2}\int \lambda(x)|\psi|^4 = \int |\nabla\psi|^2 + \int |x|^2|\psi|^2 - E(\psi_0) - \Re|\Omega| \langle L_z(0)\rangle$$

Now from the lemma 4.1, we have the following Gagliardo-Nirenberg inequality.

$$\int |\psi|^4 \leqslant 2\frac{\left(\int |\nabla\psi|^2\right)\left(\int |\psi|^2\right)}{\int Q^2}$$

from (H_1) we have,

$$\frac{1}{2}\int \lambda(x)|\psi|^4 \leqslant \lambda_2\frac{\left(\int |\nabla\psi|^2\right)\left(\int |\psi|^2\right)}{\int Q^2}$$

After combining the above two results we get

$$\int |\nabla\psi|^2 + \int |x|^2|\psi_0|^2 - E(\psi_0) - \Re|\Omega| \langle L_z(0)\rangle \leqslant \lambda_2\frac{\left(\int |\nabla\psi|^2\right)\left(\int |\psi_0|^2\right)}{\int Q^2}$$

Then after simplification we have,

$$\int \left[1 - \lambda_2\left(\frac{\int |\psi_0|^2}{\int Q^2}\right)\right] |\nabla\psi|^2 + \int |x|^2|\psi|^2 \leqslant E(\psi_0) + \Re|\Omega| \langle L_z(0)\rangle$$

Since we have $||Q_{\lambda_2}||_{L^2} = (\lambda_2)^{\frac{-1}{2}}||Q||_{L^2}$. Therefore, the above inequality can be written as

$$\int\left[1-\left(\frac{\int|\psi_0|^2}{\int Q_{\lambda_2}^2}\right)\right]|\nabla\psi|^2+\int|x|^2|\psi|^2\leqslant E(\psi_0)+\Re|\Omega|\left\langle L_z(0)\right\rangle$$

Since we have $||\psi_0||_{L2}<||Q_{\lambda_2}||_{L^2}$. So we obtain that, $\int|x|^2|\psi|^2$ and $\int|\nabla\psi|^2$ are bounded for any $t\in[0,T)$. Then from theorem 2.1 of LWP for any $T<+\infty$ we get the required result, that the corresponding solution $\psi(t,x)$ of (2.1) globally exists in time $t\in[0,+\infty)$.

Proof(2):

Now we consider the following initial data

$$\psi(0,x)=\psi_0=k\;\alpha\;Q_{\lambda_2}(\alpha x)$$

Where α is any arbitrary positive value and k is any real number satisfying the condition, $|k|\geqslant(\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}}\geqslant 1$.
Then we have,

$$||\psi_0||_{L^2}=|k||\alpha|||Q_{\lambda_2}||_{L^2}\geqslant\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{2}}||Q_{\lambda_2}||_{L^2}\geqslant||Q_{\lambda_2}||_{L^2}$$

Now from conservation of energy and angular momentum we have

$$E(\psi_0)+|\Omega|\Re\left\langle L_z(0)\right\rangle=\int|\nabla\psi(0,x)|^2+\int|x|^2|\psi(0,x)|^2-\int\frac{\lambda(x)}{2}|\psi(0,x)|^4$$

Since $\lambda_1\leqslant\lambda(x)\leqslant\lambda_2$. Then

$$E(\psi_0) + |\Omega|\Re\langle L_z(0)\rangle \leqslant \int |\nabla\psi(0,x)|^2 + \int |x|^2|\psi(0,x)|^2 - \int \frac{\lambda_1}{2}|\psi(0,x)|^4$$

Now using the value of $\psi(0,x) = \psi_0 = k\ \alpha\ Q_{\lambda_2}(\alpha x)$ we have

$$E(\psi_0)+|\Omega|\Re\langle L_z(0)\rangle \leqslant |k|^2|\alpha|^2 \int |\nabla Q_{\lambda_2}(\alpha x)||^2+|k|^2|\alpha|^2 \int |x|^2|Q_{\lambda_2}(\alpha x)|^2-\frac{\lambda_1}{2}k^4\alpha^4 \int |Q_{\lambda_2}(\alpha x)|^4$$

Using the equation (4.13) we obtain

$$E(\psi_0)+|\Omega|\Re\langle L_z(0)\rangle \leqslant |k|^2|\alpha|^2 \int |\nabla Q_{\lambda_2}(x)+\int |x|^2|k\alpha Q_{\lambda_2}(x)|^2-\frac{\lambda_1}{2}|k|^4\alpha^4\frac{1}{\alpha^2}\frac{2}{\lambda_2} \int |\nabla Q_{\lambda_2}(x)|^2$$

$$E(\psi_0) + |\Omega|\Re\langle L_z(0)\rangle \leqslant \left(1 - \frac{\lambda_1}{\lambda_2}|k|^2\right)|k|^2\alpha^2 \int |\nabla Q_{\lambda_2}(x) + \int |x|^2|k\alpha Q_{\lambda_2}(x)|^2$$

Since $|k|^2 \geqslant \frac{\lambda_2}{\lambda_1} \geqslant 1$. Therefore,

$$\begin{aligned} E(\psi_0) + |\Omega|\Re\langle L_z(0)\rangle &\leqslant \left(1 - \frac{\lambda_1}{\lambda_2}|k|^2\right)|k|^2\alpha^2 \int |\nabla Q_{\lambda_2}(x) + \int |x|^2|k\alpha Q_{\lambda_2}(x)|^2 \\ &\leqslant \int |x|^2|\psi_0|^2 \\ &\leqslant J(0) \end{aligned}$$

and

$$J'(0) = -4\Im \int \psi_0 \nabla\overline{\psi}_0$$

Then using $\psi(0,x) = \psi_0 = k\ \alpha\ Q_{\lambda_2}(\alpha x)$ we have,

$$J'(0) = -4|k|^2\alpha^2\Im \int Q_{\lambda_2}\nabla Q\lambda_2 = 0 \leqslant 0$$

Thus from condition (a), condition (b) and above results for $J(0)$ and $J'(0)$, we obtain the required result that, solution $\psi(t,x)$ of equation (2.1) blows up in finite time. Hence the theorem 4.4 is proved.

4.3 Graphical Results for Ground State

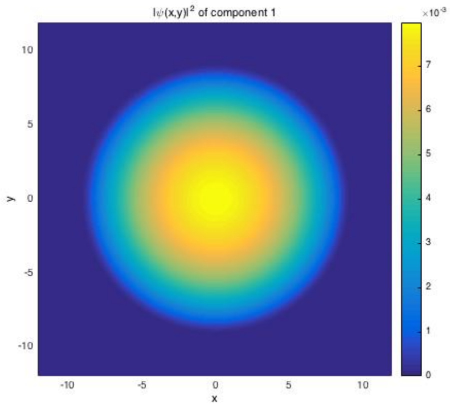
In this section we will look at the graphs for the solution of ground state of GPE (2.1) at various value of angular velocity Ω , two type of the initial data ψ_0 and strong nonlinearity. All simulation will be run in in two dimensional space.
For all these simulations we use the 'BESP' method, the "MATLAB" code used to solve the ground state of GPE with rotation. Initially, this code was used but special thanks is given to Xavier Antoine and Romain Duboscq for GPELab [39], an open source MATLAB package that greatly sped up the simulation process.
For the matter of our perspective we use the cubic nonlinear GPE with rotational term described in equation (2.1).

4.3.1 Intial data ψ_0 as Gaussian Approximation

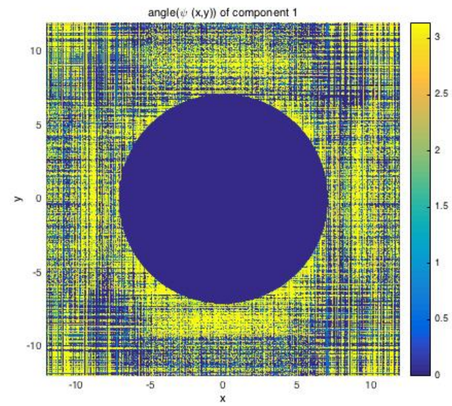
First we will run all the simulation for 2d defocusing case $\lambda > 0$ with the following gaussian approximation initial condition:

$$\psi_o(x) = \frac{1}{\sqrt{\pi}} e^{\frac{-({x_1}^2+{x_2}^2)}{2}} = 1$$

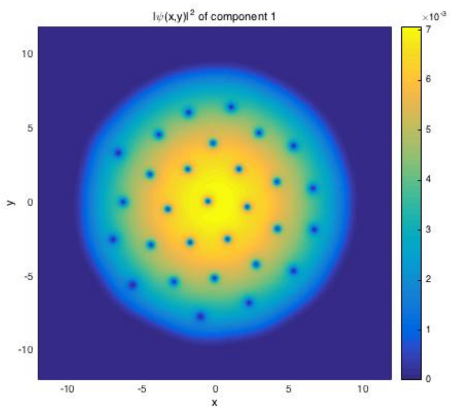
with step size $\Delta t = 0.001$, $x \in [-12, 12]^2$ where $x \in \mathbb{R}^2$, the cubic nonlinearity $\lambda = 5000$ and the different values of Ω. We obtain the following graphs, Figure 4.1 and Figure 4.2.



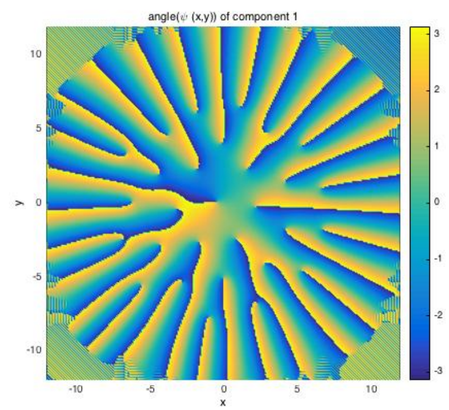
(a) Modulus of ground state for $\Omega = 0$



(b) Phase of ground state for $\Omega = 0$

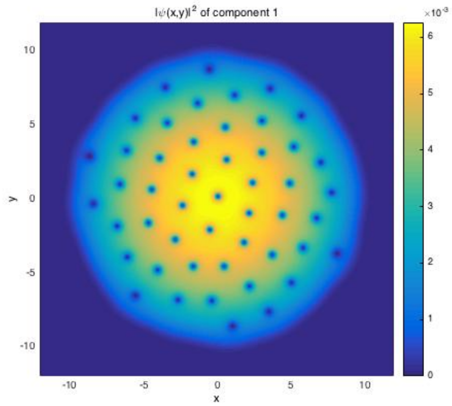


(c) Modulus of ground state for $\Omega = 0.5$

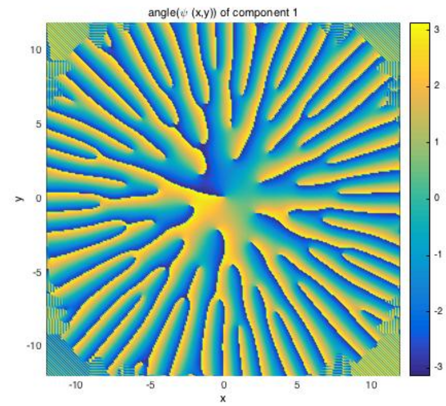


(d) Phase of ground state for $\Omega = 0.5$

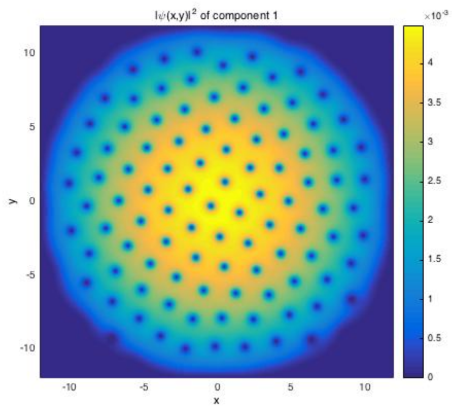
Figure 4.1: Plots of Ground State density function $\psi(x, y)$ in 2D with Gaussian initial data and different values of Ω.



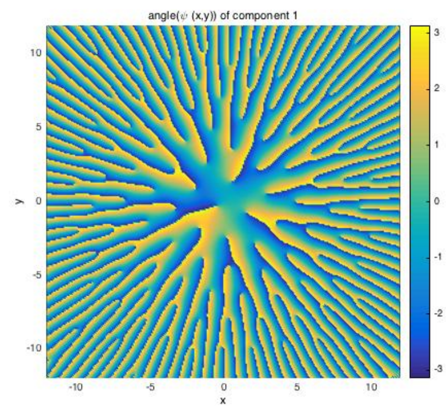
(a) Modulus of ground state for $\Omega = 0.65$



(b) Phase of ground state for $\Omega = 0.65$



(c) Modulus of ground state for $\Omega = 0.85$



(d) Phase of ground state for $\Omega = 0.85$

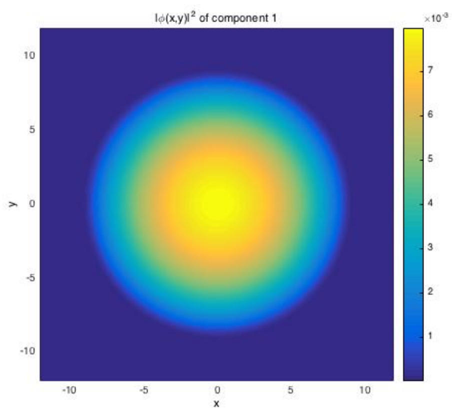
Figure 4.2: Plots of Ground State density function $\psi(x, y)$ in 2D with Gaussian initial data and different values of Ω.

4.3.2 Initial data ψ_0 as Thomas-Fermi approximation

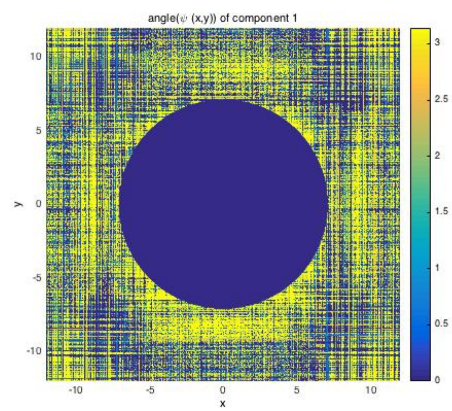
Now we will run all the simulation for 2D defocusing case $\lambda > 0$ with the following Thomas-Fermi (TF) approximation initial condition:

$$\psi_0{}^{TF}(x) = \begin{cases} \sqrt{\mu_\lambda{}^{TF} - V(x)/\lambda} & if \quad \lambda^{TF} > V(x) \\ 0 & otherwise \end{cases}$$

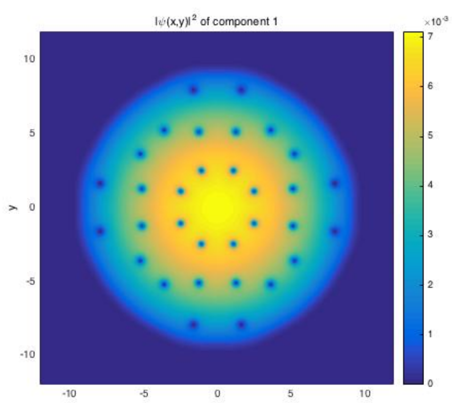
where eigenvalue approximation $\mu_\lambda{}^{TF} = \frac{(4\lambda/\pi)^{1/2}}{2}$, for the value of λ^{TF} and more detail about Thomas-Fermi initial data see [40], with step size $\Delta t = 0.001$, $x \in [-12, 12]^2$, the cubic nonlinearity $\lambda = 5000$, $\psi_0^{TF}(x) = 2$ and the different values of Ω. We obtain the following graphs, Figure4.3 and Figure 4.4.



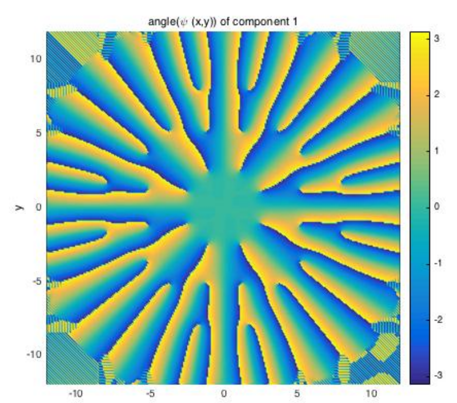
(a) Modulus of ground state for $\Omega = 0$



(b) Phase of ground state for $\Omega = 0$

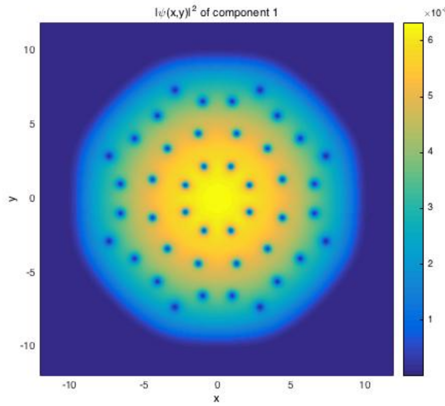


(c) Modulus of ground state for $\Omega = 0.5$

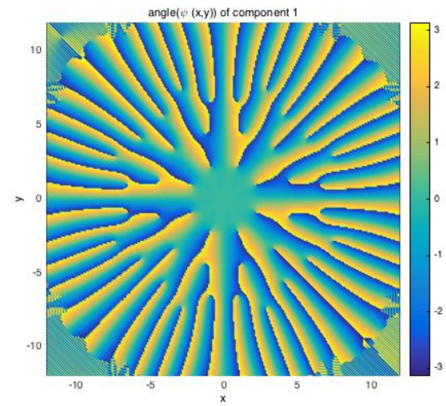


(d) Phase of ground state for $\Omega = 0.5$

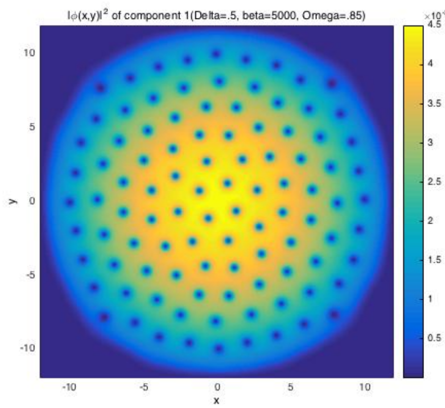
Figure 4.3: Plots of Ground State density function $\psi(x, y)$ in 2D with Thomas-Fermi initial data and different values of Ω.



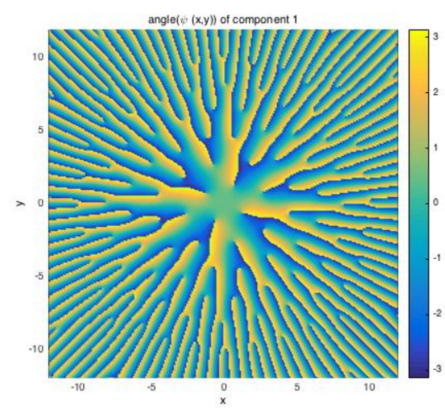
(a) Modulus of ground state for $\Omega = 0.65$



(b) Phase of ground state for $\Omega = 0.65$



(c) Modulus of ground state for $\Omega = 0.85$



(d) Phase of ground state for $\Omega = 0.85$

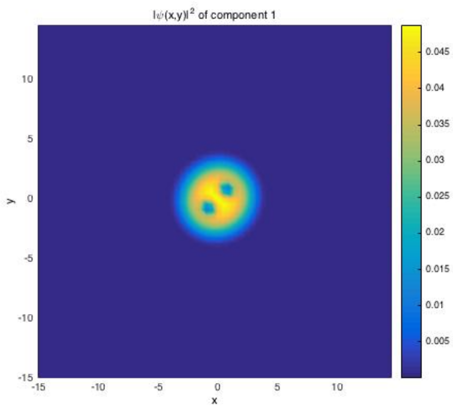
Figure 4.4: Plots of Ground State density function $\psi(x, y)$ in 2D with Thomas-Fermi initial data and different values of Ω.

From all above Figures we see that if the initial data is gaussian then we also see the vortex at the center.

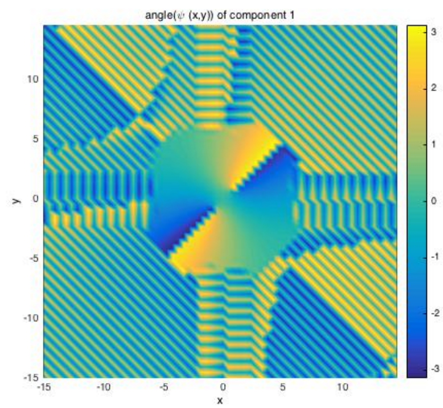
For stationary state (ground state) with strong nonlinearity ($\lambda = 5000$), we can also use the Thomas-Fermi approximation.

Also for Thomas-Fermi approximation, we see more symmetric pattern for vortices as compare to gaussian approximation.

Following Figure 4.5 represents the modulus of ground sate and its phase with small nonlinearity and Thomas-Fermi initial data.



(a) Modulus of ground state for $\Omega = 0.6$



(b) Phase of ground state for $\Omega = 0.6$

Figure 4.5: Plots of Ground State density function $\psi(x, y)$ in 2D with Thomas-Fermi initial data and small nonlinearity i.e $\lambda = 100$.

CHAPTER 5

Universal Upper Bound On Blow Up Rate

Theorem 5.1. *Let $\psi(t,x) \in C([0,T), H^1)$ be a radially symmetric finite time blow-up solution of GPE with rotational term as described by equation (2.1). Let $(H_1), (H_2)$ are true and*

$$n \geqslant 2, \qquad 1+\frac{4}{n} < p < 5$$

$$s_c = \frac{n}{2} - \frac{2}{p-1} \qquad \textit{with} \qquad 0 < s_c < 1$$

Then there exists following universal upper bound in mass critical regime,

$$\int_t^T (T-\tau)\,||\nabla\psi(\tau)||_{L^2}^2 d\tau \leqslant C(\psi_0)(T-t)^{\frac{2\gamma}{1+\gamma}} \tag{5.1}$$

where $\quad \gamma = \frac{5-p}{(p-1)(n-1)}$

Proof:

Step 1:

To prove the above theorem, first we will prove the localized virial identity associated with the equation (2.1), defined as

$$i\psi_t = -\Delta\psi + |x|^2\psi - \lambda(x)|\psi|^{p-1}\psi - |\Omega| L_z\psi \qquad (x,t) \in R^n \times R$$
$$\psi(x,0) = \psi_0(x), \qquad n = 2, \qquad 1+\frac{4}{n} < p < 5$$

$\lambda(x)$ is a positive radial function,

Ω is positive constants,

$L_z = i(x_2\partial_{x_1} - x_1\partial_{x_2})$.

Let $\psi_0 \in H^1$ with radial symmetry and assume that the corresponding solution $\psi \in C([0,T), H^1)$ be a radially symmetric finite time blow up solution at $0 < T < \infty$.

Pick a time $t_0 < T$ and a radius $0 < R = R(t_0) \ll 1$ to be chosen. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ be a radially symmetric cutoff function, defined as:

$\chi = u_R = R^2 u(\frac{x}{R})$ where $u(x) = \frac{|x|^2}{2}$ for $|x| \leqslant 2$ and $u(x) = 0$ for $|x| \geqslant 3$ Now we define

$$\frac{1}{2}\int \chi|\psi|^2 = \frac{1}{2}\int \chi\psi\overline{\psi}.$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \chi|\psi|^2 = \frac{1}{2}\int \chi\left(\psi_t\overline{\psi} + \psi\overline{\psi}_t\right) &= \Re\left(\int \chi\overline{\psi}\psi_t\right) \\
&= \Re\left(\int \chi\overline{\psi}\left(i\Delta\psi - i|x|^2\psi + i\lambda(x)|\psi|^{p-1}\psi + i\Omega L_z\psi\right)\right) \\
&= \Re\left(\int \chi\overline{\psi}i\Delta\psi\right) + \Re\left(\int \chi\overline{\psi}(-i|x|^2\psi)\right) \\
&\quad + \Re\left(\int \chi\overline{\psi}i\lambda(x)|\psi|^{p-1}\psi\right) + \Re\left(\int \chi\overline{\psi}i\Omega L_z\psi\right) \\
&:= (I_1 + I_2 + I_3 + I_4).
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 = \Re\left(i\int \chi\overline{\psi}\Delta\psi\right) &= \Re\left(-i\int \nabla(\chi\overline{\psi})\cdot\nabla\psi\right) = \Re\left(-i\int \overline{\psi}\nabla\chi\cdot\nabla\psi\right) + \Re\left(-i\int \chi\nabla\overline{\psi}\cdot\nabla\psi\right) \\
= \Re\left(i\int \nabla\cdot(\overline{\psi}\nabla\chi)\psi\right) &= \Re\left(i\int \nabla\overline{\psi}\cdot\nabla\chi\psi\right) + \Re\left(i\int \overline{\psi}\Delta\chi\psi\right) = \Im\int \nabla\chi\cdot\nabla\psi\overline{\psi}.
\end{aligned}$$

For I_2, we have

$$I_2 = \Re\left(-i\int \chi|x|^2|\psi|^2\right) = 0.$$

For I_3, we have

$$I_3 = \Re\left(i\int \chi\lambda(x)|\psi|^{p+1}\right) = 0.$$

For I_4, note that L_z is self adjoint, so

$$I_4 = \Re\left(i\Omega\int \chi\overline{\psi}L_z\psi\right) = \Re\left(i\Omega\langle L_z\psi, \chi\psi\rangle\right) = \Re\left(i\Omega\langle \psi, L_z(\chi\psi)\rangle\right) = \Re\left(i\Omega\langle \psi, \psi L_z\chi + \chi L_z\psi\rangle\right)$$
$$= \Re\left(i\Omega\langle \psi, \chi L_z\psi\rangle\right) = \Re\left(i\Omega\int \psi\chi\overline{L_z\psi}\right) = -I_4.$$

This implies that $I_4 = 0$.

Collecting I_1, I_2, I_3 and I_4, we have

$$\frac{1}{2}\frac{d}{dt}\int \chi|\psi|^2 = \Im\int \nabla\chi\cdot\nabla\psi\overline{\psi}. \tag{5.2}$$

Now we consider,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\overline{\psi}\right) &= \frac{1}{2}\Im\left(\int \nabla\chi\cdot\nabla\psi_t\overline{\psi} + \int \nabla\chi\cdot\nabla\psi\overline{\psi}_t\right)\\
&= \frac{1}{2}\Im\left(-\int \nabla\cdot(\overline{\psi}\nabla\chi)\psi_t + \int \nabla\chi\cdot\nabla\psi\overline{\psi}_t\right)\\
&= \frac{1}{2}\Im\left(-\int \nabla\overline{\psi}\cdot\nabla\chi\psi_t - \int \overline{\psi}\Delta\chi\psi_t + \int \nabla\chi\cdot\nabla\psi\overline{\psi}_t\right)\\
&= -\frac{1}{2}\Im\left(\int \Delta\chi\overline{\psi}\psi_t\right) - \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi\psi_t\right)\\
&:= -\frac{1}{2}S - T.
\end{aligned}$$

Now we compute S,

$$\begin{aligned}
S &= \Im\left(\int \Delta\chi\overline{\psi}\left(i\Delta\psi - i|x|^2\psi + i\lambda(x)|\psi|^{p-1}\psi + i\Omega L_z\psi\right)\right)\\
&= \Im\left(\int \Delta\chi\overline{\psi}i\Delta\psi\right) + \Im\left(\int \Delta\chi\overline{\psi}\left(-i|x|^2\psi\right)\right) + \Im\left(\int \Delta\chi\overline{\psi}i\lambda(x)|\psi|^{p-1}\psi\right) + \Im\left(\int \Delta\chi\overline{\psi}i\Omega L_z\psi\right)\\
&:= S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

For S_1,

$$\begin{aligned}
S_1 &= \Im\left(i\int \Delta\chi\overline{\psi}\Delta\psi\right) = \Im\int (i\psi(\Delta(\psi\Delta\chi))) \\
&= \Im\left(i\int \psi\Delta\overline{\psi}\Delta\chi\right) + \Im\left(2i\int \psi\nabla\overline{\psi}\cdot\nabla\Delta\chi\right) + \Im\left(i\int \psi\overline{\psi}\Delta^2\chi\right) \\
&= \Im\left(i\int \psi\Delta\overline{\psi}\Delta\chi\right) + \Im\left(-2i\int \nabla\cdot(\psi\nabla\overline{\psi})\Delta\chi\right) + \int \Delta^2\chi|\psi|^2 \\
&= \Im\left(i\int \psi\Delta\overline{\psi}\Delta\chi\right) + \Im\left(-2i\int \nabla\psi\cdot\nabla\overline{\psi}\Delta\chi\right) + \Im\left(-2i\int \psi\Delta\overline{\psi}\Delta\chi\right) + \int \Delta^2\chi|\psi|^2 \\
&= -S_1 - 2\int \Delta\chi|\nabla\psi|^2 + \int \Delta^2\chi|\psi|^2.
\end{aligned}$$

This shows that

$$S_1 = -\int \Delta\chi|\nabla\psi|^2 + \frac{1}{2}\int \Delta^2\chi|\psi|^2.$$

For S_2, one has

$$S_2 = \Im\left(-i\int \Delta\chi|x|^2|\psi|^2\right) = -\int \Delta\chi|x|^2|\psi|^2.$$

For S_3, one has

$$S_3 = \Im\left(i\int \lambda(x)\Delta\chi|\psi|^{p+1}\right) = \int \lambda(x)\Delta\chi|\psi|^{p+1}.$$

For S_4, we will first show that $\int \Delta\chi\overline{\psi}L_z\psi$ is real. Indeed, we note that

$$\int \Delta\chi\overline{\psi}L_z\psi = \langle L_z\psi, \psi\Delta\chi\rangle = \langle \psi, L_z(\psi\Delta\chi)\rangle = \langle \psi, L_z\psi\Delta\chi + \psi L_z\Delta\chi\rangle.$$

Since χ is radial, we know $\Delta\chi = \chi''(r) + \frac{n-1}{r}\chi'(r)$ is also radial, so $L_z\Delta\chi = 0$. Thus

$$\int \Delta\chi\overline{\psi}L_z\psi = \langle \psi, L_z\psi\Delta\chi\rangle = \int \psi\overline{L_z\psi}\Delta\chi,$$

and this means $\int \Delta\chi\overline{\psi}L_z\psi$ is real. Then we have

$$S_4 = \Im\left(i\Omega\int \Delta\chi\overline{\psi}L_z\psi\right) = \Omega\int \Delta\chi\overline{\psi}L_z\psi.$$

Collecting S_1, S_2, S_3, and S_4, we have

$$S = -\int \Delta\chi|\nabla\psi|^2 + \frac{1}{2}\int \Delta^2\chi|\psi|^2 - \int \Delta\chi|x|^2|\psi|^2 + \int \lambda(x)\Delta\chi|\psi|^{p+1} + \Omega\int \Delta\chi\overline{\psi}L_z\psi.$$

To compute T, we have

$$\begin{aligned}
T &= \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi\left(i\Delta\psi - i|x|^2\psi + i\lambda(x)|\psi|^{p-1}\psi + i\Omega L_z\psi\right)\right)\\
&= \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi i\Delta\psi\right) + \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi(-i|x|^2\psi)\right)\\
&\quad + \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi i\lambda(x)|\psi|^{p-1}\psi\right) + \Im\left(\int \nabla\overline{\psi}\cdot\nabla\chi i\Omega L_z\psi\right)\\
&:= T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

For T_1, we have

$$\begin{aligned}
T_1 &= \Im\left(i\int \nabla\overline{\psi}\cdot\nabla\chi\Delta\psi\right)\\
&= \Im\left(i\sum_k\sum_j\int \overline{\psi}_j\chi_j\psi_{kk}\right) \qquad k, j = 1, 2\\
&= \Im\left(-i\sum_k\sum_j\int \left(\overline{\psi}_j\chi_j\right)_k\psi_k\right)\\
&= \Im\left(-i\sum_k\sum_j\int \overline{\psi}_{jk}\chi_j\psi_k\right) + \Im\left(-i\sum_k\sum_j\int \overline{\psi}_j\chi_{jk}\psi_k\right)\\
&:= T_{1,1} + T_{1,2}
\end{aligned}$$

For $T_{1,1}$, we have

$$\begin{aligned}
T_{1,1} &= \Im\left(-i\sum_k\sum_j\int \overline{\psi}_{jk}\chi_j\psi_k\right) = \Im\left(i\sum_k\sum_j\int \overline{\psi}_k\left(\chi_j\psi_k\right)_j\right)\\
&= \Im\left(i\sum_k\sum_j\int \overline{\psi}_k\chi_{jj}\psi_k\right) + \Im\left(i\sum_k\sum_j\int \overline{\psi}_k\chi_j\psi_{kj}\right)\\
&= \int \Delta\chi|\nabla\psi|^2 - T_{1,1}.
\end{aligned}$$

This shows that

$$T_{1,1} = \frac{1}{2}\int \Delta\chi|\nabla\psi|^2.$$

For $T_{1,2}$, recall that χ is radial, so the derivatives are

$$\chi_j = \chi'(r)\ \frac{x_j}{r}, \chi_{jk} = \chi''(r)\frac{x_j x_k}{r^2} + \chi'(r)\frac{\delta_{j,k}}{r} + \chi'(r)\left(-\frac{x_j}{r^2}\right)\frac{x_k}{r},$$

where $\delta_{j,k}$ is the Kronecker delta. Then

$$\begin{aligned} T_{1,2} &= \Im\left(-i\sum_k\sum_j\int\overline{\psi}_j\left(\chi''(r)\frac{x_j x_k}{r^2} + \chi'(r)\frac{\delta_{j,k}}{r} + \chi'(r)\left(-\frac{x_j}{r^2}\right)\frac{x_k}{r}\right)\psi_k\right) \\ &= -\int\frac{\chi''}{r^2}|x\cdot\nabla\psi|^2 - \int\frac{\chi'}{r}|\nabla u|^2 + \int\frac{\chi'}{r^3}|x\cdot\nabla\psi|^2. \end{aligned}$$

Recall that ψ is radial, so

$$\nabla\psi = \psi'(r)\frac{x}{r}, x\cdot\nabla\psi = r\psi', \text{and}|\nabla\psi| = |\psi'(r)|.$$

Bringing these back into $T_{1,2}$ we have

$$T_{1,2} = -\int\chi''|\nabla\psi|^2.$$

Combining $T_{1,1}$ and $T_{1,2}$ we get

$$T_1 = \frac{1}{2}\int\Delta\chi|\nabla\psi|^2 - \int\chi''|\nabla\psi|^2.$$

For T_2, one has

$$\begin{aligned} T_2 &= \Im\left(-i\int\nabla\overline{\psi}\cdot\nabla\chi|x|^2\psi\right) = \Im\left(i\int\nabla\cdot\left(\nabla\chi|x|^2\psi\right)\overline{\psi}\right) \\ &= \Im\left(i\int\Delta\chi|x|^2\psi\overline{\psi}\right) + \Im\left(i\int\nabla\chi\cdot 2x\psi\overline{\psi}\right) + \Im\left(i\int\nabla\chi\cdot|x|^2\nabla\psi\overline{\psi}\right) \\ &= \int\Delta\chi|x|^2|\psi|^2 + 2\int x\cdot\nabla\chi|\psi|^2 - T_2. \end{aligned}$$

This shows that

$$T_2 = \frac{1}{2}\int\Delta\chi|x|^2|\psi|^2 + \int x\cdot\nabla\chi|\psi|^2.$$

For T_3, one has

$$
\begin{aligned}
T_3 &= \Im\left(i\int \nabla\overline{\psi}\cdot\nabla\chi\lambda(x)\psi^{\frac{p+1}{2}}\overline{\psi}^{\frac{p-1}{2}}\right) = \Im\left(-i\int \nabla\cdot\left(\nabla\chi\lambda(x)\psi^{\frac{p+1}{2}}\overline{\psi}^{\frac{p-1}{2}}\right)\overline{\psi}\right)\\
&= \Im\left(-i\int \Delta\chi\lambda(x)\psi^{\frac{p+1}{2}}\overline{\psi}^{\frac{p-1}{2}}\overline{\psi}\right) + \Im\left(-i\int \nabla\chi\cdot\nabla\lambda(x)\psi^{\frac{p+1}{2}}\overline{\psi}^{\frac{p-1}{2}}\overline{\psi}\right)\\
&\quad + \Im\left(-i\int \nabla\chi\cdot\lambda(x)\frac{p+1}{2}\psi^{\frac{p-1}{2}}\nabla\psi\overline{\psi}^{\frac{p-1}{2}}\overline{\psi}\right) + \Im\left(-i\int \nabla\chi\cdot\lambda(x)\psi^{\frac{p+1}{2}}\frac{p-1}{2}\overline{\psi}^{\frac{p-3}{2}}\nabla\overline{\psi\psi}\right)\\
&= -\int \lambda(x)\Delta\chi|\psi|^{p+1} - \int \nabla\chi\cdot\nabla\lambda(x)|\psi|^{p+1} - \frac{p+1}{2}T_3 - \frac{p-1}{2}T_3.
\end{aligned}
$$

This shows that

$$
T_3 = -\frac{1}{p+1}\int \lambda(x)\Delta\chi|\psi|^{p+1} - \frac{1}{p+1}\int \nabla\chi\cdot\nabla\lambda(x)|\psi|^{p+1}.
$$

For T_4, before computing the term, we define $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. Then it is easy to check the following:

$$
\begin{aligned}
&\nabla^\perp\cdot\nabla = \nabla\cdot\nabla^\perp = 0,\\
&\nabla(L_z\psi) = i\nabla^\perp\psi + L_z(\nabla\psi).
\end{aligned}
$$

Note that

$$
\begin{aligned}
T_4 &= \Im\left(i\Omega\int \nabla\overline{\psi}\cdot\nabla\chi L_z\psi\right) = \Im\left(i\Omega\langle L_z\psi, \nabla\psi\cdot\nabla\chi\rangle\right) = \Im\left(i\Omega\langle\psi, L_z\left(\nabla\psi\cdot\nabla\chi\right)\rangle\right)\\
&= \Im\left(i\Omega\langle\psi, L_z\left(\nabla\psi\right)\cdot\nabla\chi\rangle\right) + \Im\left(i\Omega\langle\psi, \nabla\psi\cdot L_z\left(\nabla\chi\right)\rangle\right)\\
&= \Im\left(i\Omega\langle\psi, \left(\nabla(L_z\psi) - i\nabla^\perp\psi\right)\cdot\nabla\chi\rangle\right) + \Im\left(i\Omega\langle\psi, \nabla\psi\cdot\left(\nabla(L_z\chi) - i\nabla^\perp\chi\right)\rangle\right)\\
&= \Im\left(i\Omega\langle\psi, \nabla(L_z\psi)\cdot\nabla\chi\rangle\right) + \Im\left(i\Omega\langle\psi, -i\nabla^\perp\psi\cdot\nabla\chi\rangle\right) + \Im\left(i\Omega\langle\psi, \nabla\psi\cdot\left(-i\nabla^\perp\chi\right)\rangle\right)\\
&:= T_{4,1} + T_{4,2} + T_{4,3}.
\end{aligned}
$$

For $T_{4,1}$, one has

$$
\begin{aligned}
T_{4,1} &= \Im\left(i\Omega\int \psi\nabla\overline{L_z\psi}\cdot\nabla\chi\right) = \Im\left(-i\Omega\int \nabla\cdot\left(\psi\nabla\chi\right)\overline{L_z\psi}\right)\\
&= \Im\left(-i\Omega\int \nabla\psi\cdot\nabla\chi\overline{L_z\psi}\right) + \Im\left(-i\Omega\int \psi\Delta\chi\overline{L_z\psi}\right)\\
&= -T_4 - \Omega\int \psi\Delta\chi\overline{L_z\psi}.
\end{aligned}
$$

The last equality is due to $\int \psi \Delta\chi \overline{L_z \psi}$ being real (see previous computation).

For $T_{4,2}$, one has

$$\begin{aligned} T_{4,2} &= \Im\left(i\Omega \int \psi i \nabla^{\perp}\overline{\psi} \cdot \nabla\chi\right) = \Im\left(-\Omega \int \psi \nabla^{\perp}\overline{\psi} \cdot \nabla\chi\right) = \Im\left(\Omega \int \nabla \cdot \left(\psi \nabla^{\perp}\overline{\psi}\right)\chi\right) \\ &= \Im\left(\Omega \int \nabla\psi \cdot \nabla^{\perp}\overline{\psi}\chi\right) + \Im\left(\Omega \int \psi \nabla \cdot \nabla^{\perp}\overline{\psi}\chi\right) = \Im\left(\Omega \int \nabla\psi \cdot \nabla^{\perp}\overline{\psi}\chi\right). \end{aligned}$$

For $T_{4,3}$, one has

$$\begin{aligned} T_{4,3} &= \Im\left(i\Omega \int \psi \nabla\overline{\psi} \cdot i\nabla^{\perp}\chi\right) = \Im\left(-\Omega \int \psi \nabla\overline{\psi} \cdot \nabla^{\perp}\chi\right) = \Im\left(\Omega \int \nabla^{\perp} \cdot \left(\psi \nabla\overline{\psi}\right)\chi\right) \\ &= \Im\left(\Omega \int \nabla^{\perp}\psi \cdot \nabla\overline{\psi}\chi\right) + \Im\left(\Omega \int \psi \nabla^{\perp} \cdot \nabla\overline{\psi}\chi\right) = \Im\left(\Omega \int \nabla^{\perp}\psi \cdot \nabla\overline{\psi}\chi\right). \end{aligned}$$

Combining $T_{4,1}$, $T_{4,2}$, and $T_{4,3}$ one has

$$\begin{aligned} T_4 &= -T_4 - \Omega \int \psi \Delta\chi \overline{L_z \psi} + \Im\left(\Omega \int \nabla\psi \cdot \nabla^{\perp}\overline{\psi}\chi\right) + \Im\left(\Omega \int \nabla^{\perp}\psi \cdot \nabla\overline{\psi}\chi\right) \\ &= -T_4 - \Omega \int \psi \Delta\chi \overline{L_z \psi}. \end{aligned}$$

This shows that

$$T_4 = -\frac{\Omega}{2} \int \psi \Delta\chi \overline{L_z \psi}.$$

Collecting T_1, T_2, T_3, and T_4, one has

$$\begin{aligned} T = &\frac{1}{2} \int \Delta\chi |\nabla\psi|^2 - \int \chi'' |\nabla\psi|^2 + \frac{1}{2} \int \Delta\chi |x|^2 |u|^2 + \int x \cdot \nabla\chi |\psi|^2 \\ &- \frac{1}{p+1} \int \lambda(x) \Delta\chi |\psi|^{p+1} - \frac{1}{p+1} \int \nabla\chi \cdot \nabla\lambda(x) |\psi|^{p+1} - \frac{\Omega}{2} \int \psi \Delta\chi \overline{L_z \psi}. \end{aligned}$$

Finally, we obtain the localized virial identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\bar{\psi}\right) &= -\frac{1}{2}\left(-\int \Delta\chi|\nabla\psi|^2+\frac{1}{2}\int \Delta^2\chi|\psi|^2-\int \Delta\chi|x|^2|\psi|^2+\int \lambda(x)\Delta\chi|\psi|^{p+1}\right)\\ &\quad -\frac{\Omega}{2}\int \Delta\chi\overline{\psi}L_z\psi-\frac{1}{2}\int \Delta\chi|\nabla\psi|^2+\int \chi''|\nabla\psi|^2-\frac{1}{2}\int \Delta\chi|x|^2|\psi|^2\\ &\quad -\int x\cdot\nabla\chi|\psi|^2+\frac{1}{p+1}\int \lambda(x)\Delta\chi|\psi|^{p+1}+\frac{1}{p+1}\int \nabla\chi\cdot\nabla\lambda(x)|\psi|^{p+1}\\ &\quad +\frac{\Omega}{2}\int \psi\Delta\chi\overline{L_z\psi}.\\ &= \int \chi''|\nabla\psi|^2-\frac{1}{4}\int \Delta^2\chi|\psi|^2-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int \lambda(x)\Delta\chi|\psi|^{p+1}\\ &\quad -\int x\cdot\nabla\chi|\psi|^2+\frac{1}{p+1}\int \nabla\chi\cdot\nabla\lambda(x)|\psi|^{p+1}.\end{aligned}$$

Now applying with $\chi = u_R = R^2u(\frac{x}{R})$ where $u(x) = \frac{|x|^2}{2}$ for $|x| \leqslant 2$ and $u(x) = 0$ for $|x| \geqslant 3$, we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\bar{\psi}\right) &= \int u''(\frac{x}{R})|\nabla\psi|^2-\frac{1}{4R^2}\int \Delta^2u(\frac{x}{R})|\psi|^2-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int \lambda(x)\Delta u(\frac{x}{R})|\psi|^{p+1}\\ &\quad -R\int x\cdot\nabla u(\frac{x}{R})|\psi|^2+\frac{R}{p+1}\int \nabla u(\frac{x}{R})\cdot\nabla\lambda(x)|\psi|^{p+1}\\ &\leqslant \int |\nabla\psi|^2+\frac{C}{R^2}\int_{2R\leqslant|x|\leqslant 3R}|\psi|^2-n(\frac{1}{2}-\frac{1}{p-1})\int_{2R\leqslant|x|\leqslant 3R}\lambda(x)|\psi|^{p+1}\\ &\quad +C\int_{|x|>R}\lambda(x)|\psi|^{p+1}-\int |x|^2|\psi|^2+\int x\cdot\nabla\lambda(x)|\psi|^{p+1}\end{aligned}$$

Since $x\cdot\nabla\lambda(x) \leqslant 0$ for all $x\in\mathbb{R}^2$ and $\int |x|^2|\psi|^2 > 0$ then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\bar{\psi}\right) &\leqslant \int \nabla\psi|^2-n(\frac{1}{2}-\frac{1}{p-1})\int_{2R\leqslant|x|\leqslant 3R}\lambda(x)|\psi|^{p+1}\\ &\quad +C\left[\frac{1}{R^2}\int_{2R\leqslant|x|\leqslant 3R}|\psi|^2+C\int_{|x|>R}\lambda(x)|\psi|^{p+1}\right]\end{aligned}\tag{5.3}$$

Now from the conservation law of energy we have:

$$E(\psi_0) = \int |\nabla\psi|^2+\int |x|^2|\psi|^2-\frac{2}{p+1}\int \lambda(x)|\psi|^{p+1}-Re|\Omega|\left\langle L_z(0)\right\rangle$$

Then from above equation:

$$\frac{2}{p+1}\int \lambda(x)|\psi|^{p+1} = \int |\nabla\psi|^2 + \int |x|^2|\psi|^2 - E(\psi_0) - Re|\Omega|\left\langle L_z(0)\right\rangle$$

Also,

$$\int \lambda(x)|\psi|^{p+1} = \frac{n(p-1)}{2}\int |x|^2|\psi|^2 - \frac{p+1}{2}E(\psi_0) - \frac{p+1}{2}\Re|\Omega|\left\langle L_z(0)\right\rangle$$

From which we have

$$\begin{aligned}\int |\nabla\psi|^2 - n(\frac{1}{2} - \frac{1}{p-1})\int \lambda(x)|\psi|^{p+1} = {} & \frac{n(p-1)}{2}E(\psi_0) + \frac{n(p-1)}{2}Re|\Omega|\left\langle L_z(0)\right\rangle \\ & - \frac{n(p-1)}{2}\int |x|^2|\psi|^2 - \frac{2s_c}{n-2s_c}\int |\nabla\psi|^2\end{aligned}$$

where $s_c = \frac{n}{2} - \frac{2}{p-1}$

Now using the above equation in (5.3), we get

$$\begin{aligned}\frac{2s_c}{n-2s_c}\int |\nabla\psi|^2 + \frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\bar{\psi}\right) \lesssim {} & \frac{n(p-1)}{2}E(\psi_0) + \frac{n(p-1)}{2}Re|\Omega|\left\langle L_z(0)\right\rangle \\ & + C\left[\frac{1}{R^2}\int_{2R\leqslant|x|\leqslant 3R}|\psi|^2 + \int_{|x|>R}\lambda(x)|\psi|^{p+1}\right] \\ & - \frac{n(p-1)}{2}\int |x|^2|\psi|^2 \\ \leqslant {} & \frac{n(p-1)}{2}\left(E(\psi_0) + Re|\Omega|\left\langle L_z(0)\right\rangle\right) \\ & + C\left[\frac{1}{R^2}\int_{2R\leqslant|x|\leqslant 3R}|\psi|^2 + \int_{|x|>R}\lambda(x)|\psi|^{p+1}\right]\end{aligned}$$

Now by using conservation of mass, energy , we have

$$\frac{2s_c}{n-s_c}\int |\nabla\psi|^2 + \frac{1}{2}\frac{d}{dt}\left(\Im\int \nabla\chi\cdot\nabla\psi\bar{\psi}\right) \leqslant C(\psi_0)\left[1 + \frac{1}{R^2} + \int_{|x|\geqslant R}|\psi|^{p+1}\right] \tag{5.4}$$

Step 2:

In this step we will find the upper bound for the outer nonlinear term in (5.4).

For this first we will consider the following Radial Gagliardo-Nirenberg interpolation estimate.

$$\|\psi\|_{L^\infty(r\geqslant R)} \leqslant \frac{\|\nabla\psi\|_{L^2}^{\frac{1}{2}}\|\psi\|_{L^2}^{\frac{1}{2}}}{R^{\frac{n-1}{2}}}$$

Now we consider:

$$\int_{|x|\geqslant R} |\psi|^{p+1} \leqslant \int_{|x|\geqslant R} |\psi|^{p-1}|\psi|^2 = \|\psi\|_{L^\infty(r\geqslant R)}^{p-1} \int |\psi|^2$$

Now using the above Gagliardo-Nirenberg inequality we have

$$\begin{aligned}\int_{|x|\geqslant R} |\psi|^{p+1} &\leqslant \frac{\|\nabla\psi\|_{L^2}^{\frac{p-1}{2}}\|\psi\|_{L^2}^{\frac{p-1}{2}}}{R^{\frac{(n-1)(p-1)}{2}}} \int |\psi|^2 \\ &\leqslant \frac{C(\psi_0)}{R^{\frac{(n-1)(p-1)}{2}}}\|\nabla\psi\|_{L^2}^{\frac{p-1}{2}} \\ &\leqslant \left(\frac{4}{p-1}\delta\frac{2s_c}{n-2s_c}\right)^{\frac{p-1}{4}} \|\nabla\psi\|_{L^2}^{\frac{p-1}{2}} \frac{C(\psi_0)}{R^{\frac{(n-1)(p-1)}{2}}}\end{aligned}$$

By Young's Inequality we know that

$$a.b \leqslant \frac{a^m}{m} + \frac{b^l}{l}$$

such that

$$\frac{1}{m} + \frac{1}{l} = 1$$

Now let we assume that $a = \left(\frac{4}{p-1}\delta\frac{2s_c}{n-2s_c}\right)^{\frac{p-1}{4}} \|\nabla\psi\|_{L^2}^{\frac{p-1}{2}}$ and $b = \frac{C(\psi_0)}{R^{\frac{(n-1)(p-1)}{2}}}$ and $m = \frac{4}{p-1}$ and $l = \frac{4}{5-p}$ Then we have

$$\begin{aligned}\int_{|x|\geqslant R} |\psi|^{p+1} &\leqslant \left(\delta\frac{2s_c}{n-2s_c}\right)\|\nabla\psi\|_{L^2}^2 + \frac{C(\psi_0)}{R^{\frac{2(n-1)(p-1)}{5-p}}} \\ &= \left(\delta\frac{2s_c}{n-2s_c}\right)\int |\nabla\psi|^2 + \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}\end{aligned}$$

Now for $p < 5$, $\gamma = \frac{5-p}{(n-1)(p-1)}$ and for $\delta = \frac{1}{2} > 0$ small enough using $R \ll 1$ and $0 < \gamma < 1$ we get from (5.4)

$$\frac{s_c}{n - s_c} \int |\nabla\psi|^2 + \frac{d}{dt}\left(\Im \int \nabla\chi \cdot \nabla\psi\bar{\psi}\right) \leqslant \frac{C(\psi_0, p)}{R^{\frac{2}{\gamma}}} \tag{5.5}$$

Step 3:

Now we integrate (5.5) twice in time on $[t_0, t_2]$ by using Fubini in time:

$$\int_{t_0}^{t_2}\int_{t_0}^{t} \frac{d}{d\tau}\left(\Im \int \nabla\chi \cdot \nabla\psi\bar{\psi}\right) d\tau dt + \frac{s_c}{n - s_c}\int_{t_0}^{t_2}\int_{t_0}^{t}\int |\nabla\psi|^2 dx d\tau dt \leqslant \frac{C(\psi_0, p)}{R^{\frac{2}{\gamma}}}\int_{t_0}^{t_2}\int_{t_0}^{t} d\tau dt$$

$$\int_{t_0}^{t_2}\left([\Im \int \nabla\chi \cdot \nabla\psi\bar{\psi}](t)\right) dt - \left(\Im\left[\int \nabla\chi \cdot \nabla\psi\bar{\psi}\right](t_0)\right)\int_{t_0}^{t_2} dt + \int_{t_0}^{t_2}\int_{t}^{t_2}(\int |\nabla\psi(\tau)|^2) dt d\tau$$
$$\leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}\int_{t_0}^{t_2}(t - t_0) dt$$

Now using (5.2)

$$\int_{t_0}^{t_2}\frac{d}{dt}\left(\int \chi|\psi(t)|^2\right) dt + \int_{t_0}^{t_2}\int_{t}^{t_2}(\int |\nabla\psi(\tau)|^2) dt d\tau$$
$$\leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}(t_2 - t_0)^2 + (t_2 - t_0)\left|\Im\left(\int \nabla\chi \cdot \nabla\psi\bar{\psi}\right)(t_0)\right|$$

As we have $\chi = u_R = R^2 u(\frac{x}{R})$. Then

$$\int u_R|\psi(t_2)|^2 - \int u_R|\psi(t_0)|^2 + \int_{t_0}^{t_2}(t_2 - t)|\|\nabla\psi(t)\|_{L^2}^2 dt$$
$$\leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}(t_2 - t_0)^2 + (t_2 - t_0)\left|\Im\left(\int \nabla u_R \cdot \nabla\psi\bar{\psi}\right)(t_0)\right|$$

Finally, we have

$$\int u_R|\psi(t_2)|^2 + \int_{t_0}^{t_2} (t_2 - t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}(t_2 - t_0)^2 + CR(t_2 - t_0)\left|\left(\int \nabla\psi\bar{\psi}\right)(t_0)\right| + \int u_R|\psi(t_0)|^2$$

Now applying Cauchy-Schwarz Inequality, $\left|\int fg\right| \leqslant \left(\int |f|^2\right)^{\frac{1}{2}}\left(\int |g|^2\right)^{\frac{1}{2}}$ on second term of R.H.S in above Inequality. Then we have

$$\int u_R|\psi(t_2)|^2 + \int_{t_0}^{t_2} (t_2 - t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}(t_2 - t_0)^2 + R(t_2 - t_0)\|\nabla\psi(t_0)\|_{L^2}\|\psi_0\|_{L^2} + R^2\|\psi_0\|_{L^2}^2$$

From conservation of mass:

$$\begin{aligned}\int_{t_0}^{t_2} (t_2 - t)\|\nabla\psi(t)\|_{L^2}^2 dt &\leqslant \frac{C(\psi_0)}{R^{\frac{2}{\gamma}}}(t_2 - t_0)^2 + C(\psi_0)R(t_2 - t_0)\|\nabla\psi(t_0)\|_{L^2} + C(\psi_0)R^2 \\ &\leqslant C(\psi_0)\left(\frac{(t_2 - t_0)^2}{R^{\frac{2}{\gamma}}} + R(t_2 - t_0)\|\nabla\psi(t_0)\|_{L^2} + R^2\right)\end{aligned} \tag{5.6}$$

Now we assume that $t_2 \to T = T_{max}$. We conclude that the integral in (5.6) on the left hand side converges and

$$\int_{t_0}^{T} (T - t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant C(\psi_0)\left(\frac{(T - t_0)^2}{R^{\frac{2}{\gamma}}} + R(T - t_0)\|\nabla\psi(t_0)\|_{L^2} + R^2\right) \tag{5.7}$$

Now we optimize R by choosing:

$$R^2 = \frac{(T - t_0)^2}{R^{\frac{2}{\gamma}}} \qquad ie \qquad R(t_0) = (T - t_0)^{\frac{\gamma}{1+\gamma}} \ll 1$$

Then (5.7) yields:

$$\int_{t_0}^{T} (T - t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant C(\psi_0)\left((T - t_0)^{\frac{2\gamma}{1+\gamma}} + (T - t_0)^{\frac{\gamma}{1+\gamma}}(T - t_0)\|\nabla\psi(t_0)\|_{L^2} + (T - t_0)^{\frac{2\gamma}{1+\gamma}}\right)$$

After combining first and third term, and applying the Young's inequality ($a.b \leqslant \frac{a^2}{2} + \frac{b^2}{2}$) on second term, let $a = (T - t_0)^{\frac{\gamma}{1+\gamma}}$ and

$b = (T-t_0)\|\nabla\psi(t_0)\|_{L^2}$, then we have

$$\int_{t_0}^{T} (T-t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant C(\psi_0)\left((T-t_0)^{\frac{2\gamma}{1+\gamma}} + (T-t_0)^2\|\nabla\psi(t_0)\|_{L^2}^2 + (T-t_0)^{\frac{2\gamma}{1+\gamma}}\right)$$

Again after combining first and third term we have:

$$\int_{t_0}^{T} (T-t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant C(\psi_0)(T-t_0)^{\frac{2\gamma}{1+\gamma}} + (T-t_0)^2\|\nabla\psi(t_0)\|_{L^2}^2 \tag{5.8}$$

Now to solve the above inequality we assume that, let

$$g(t_0) = \int_{t_0}^{T} (T-t)\|\nabla\psi(t)\|_{L^2}^2 dt, \tag{5.9}$$

Then we can write (5.8) as:

$$g(t) \leqslant C(T-t)^{\frac{2\gamma}{1+\gamma}} - (T-t)g'(t) \tag{5.10}$$

By using the quotient rule of derivatives and using (5.10):

$$\begin{aligned}(\frac{g}{T-t})' &= \frac{1}{(T-t)^2}\left((T-t)g' + g\right)\\ &\leqslant \frac{C(\psi_0)}{(T-t)^{2-\frac{2\gamma}{1+\gamma}}}\end{aligned}$$

Now after integrating the above expression in time t we have:

$$(\frac{g}{T-t}) \leqslant \frac{C(\psi_0)}{(T-t)^{1-\frac{2\alpha}{1+\alpha}}}$$

Finally, we have

$$g(t) \leqslant C(\psi_0)(T-t)^{\frac{2\gamma}{1+\gamma}}$$

From (5.9) and for t close enough to T, yield (5.1)

$$\int_{t_0}^{T} (T-t)\|\nabla\psi(t)\|_{L^2}^2 dt \leqslant C(\psi_0)(T-t)^{\frac{2\gamma}{1+\gamma}}$$

Hence theorem 5.1 is proved.

CHAPTER 6

Conclusion

In this work we have studied the sharp condition for GWP and universal upper bound on blow up rate of Bose-Einstein Condensate. For this we used GPE with angular momentum rotational term with certain inhomogeneous nonlinearity (2.1) for rotating BEC in $\mathbb{R}^2$ with initial data. Many methods has been proposed to find GWP with Constant value of nonlinearity, with and without the rotational term, but to establish the sharp condition on GWP in term of ground state and upper bound on blow up rate with inhomogeneous nonlinearity and rotational term in 2D is new.

In the first part, we studied the question related to relation between GWP and ground state solution. we presented the local well-posedness of the solution ψ of equation (2.1) proved in [37] and [38]. we proved the conservation laws of mass, energy and angular momentum with external trapped potential, radially symmetric. Then we introduced the virial identity and proved its first and second order derivatives by using the divergence theorem and method of integration by parts.

we also derive a new condition on virial identity and its 1st order derivative for blow up solution in finite time by applying the newly developed method. Then we utilized the characterization of ground state solution Q_{λ_2} of equation:

$$-\Delta\psi + \psi - \lambda_2|\psi|^2\psi = 0, \ \psi \in H^1(\mathbb{R}^2)$$

Where λ_2 is the suprimum of inhomogeneous nonlinearity.

For initial data $\psi_0 \in \sum$ (Sobolev embedding space), we established the following threshold of the L^2 norm of initial data in terms of ground state solution:

$$||\psi_0||_{L^2} < ||Q_{\lambda_2}||_{L^2}$$

And infer that the solution of (2.1), $\psi(t, x)$ exists globally in time t.

For the prove of blow up solution, introduce following initial data in term of ground state after scaling:

$$\psi(0, x) = k\, \alpha\, Q_{\lambda_2}(\alpha x)$$

with condition, $\alpha > 0$ and $|k| \geqslant (\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}} \geqslant 1$. Clearly the above introduced $\psi_0 \in \sum$ and $||\psi_0||_{L^2} \geqslant ||Q_{\lambda_2}||_{L^2}$, we get the expected results that how large or small my initial data should be for the blow up solution of equation (2.1).

In the second part, we studied the Universal upper bound on Blow up rate of BEC in mass critical regime . For this we again used the technique of introducing the localize virial identity for radially symmetric data with a radially symmetric cutoff function. Then using the conservation laws and applying radial Gagliardo-Nirenberg interpolation estimate we find an upper bound for sum of $\int |\nabla\psi|^2$ and 2nd order derivative of virial identity in terms of $C(\psi_0)$ and radius R.Then using the Fubini in time for double integral and optimize R to get grownwall type inequality. After solving this inequality and applying the rules of differentiation, we get our required result for universal upper bound on blow up speed in mass critical regime.

Finally, it is worth noting that the method which we use to find GWP tell us, that exactly how large or small our initial data should be for this blow up solution. Moreover we also infer that universal upper bound is sharp.

In future I am interested to elaborate my result of blow up solution for n-dimensional space. I am also interested in finding the Universal upper bound on Blow-up rate if data is not radial.

REFERENCES

[1] Satyendra Nath Bose, Warmegleichgewicht im strahlungsfeld bei anwesenheit von materie, Paul Zeitschrift fur Physik ,A Hadrons and Nuclei, 27 (1924), no.1, 384-393.

[2] Allan Griffin, David W Snoke, and Sandro Stringari, ‘" Bose-Einstein condensation", *Cambridge University Press*, (1996).

[3] F London, "The-phenomenon of liquid helium and the Bose-Einstein degeneracy", *Nature* 141(1938), no. 3571, 643-644

[4] I.F. Silvera and J.T.M. Walraven, J. Appl. Phys. 52, 2304 (1981); R. Sprik, J.T.M. Walraven, and I.F. Silvera, Phys. Rev. Lett. 51 479 (1983); R. Sprik, J.T.M.Walraven, and I.F. Silvera, Phys. Rev. B 32 5668 (1985).

[5] D.G. Fried, T.C. Killian, L. Willmann, D. Landhuis, S.C. Moss, D. Kleppner, and T.J. Greytak, Phys. Rev. Lett. 81, 3811 (1998).

[6] M. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science 269, 198 (1995).

[7] K.B. Davis, M-O. Mewes, M.R. Andrews. N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

[8] C.C. Bradley, C.A. Sackett, R.G. Hulet, "Bose-Einstein condensation of lithium: observation of limited condensate number", Phys. Rev. Lett. 78 (1997) 985?989.

[9] M. Wadati, T. Tsurumi," Critical number of atoms for the magnetically trapped Bose-Einstein condensate with negative s-wave scattering length", Phys. Lett. A 247 (1998) 287?293.

[10] M. H. Anderson, J. R. Ensher, M. R. Matthewa, C. E. Wieman, and E. A. Cornell, "Ob- servation of Bose-Einstein condensation in a dilute atomic vapor", Science, 269 (1995), pp. 198-201.

[11] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, "Theory of Bose-Einstein conden- sation in trapped gases", Rev. Mod. Phys., 71 (1999), pp. 463-512.

[12] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, "Bose-Einstein condensation in a gas of sodium atoms", Phys. Rev. Lett., 75 (1995), pp. 3969-3973.

[13] C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and W. Ketterle, "Vortex nucleation in a stirred Bose-Einstein condensate", Phys. Rev. Lett., 87 (2001), 210-402.

[14] L. P. Pitaevskii and S. Stringari, "Bose-Einstein Condensation", Clarendon Press, Oxford, UK, 2003.

[15] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wiemann, and E. A. Cornell, "Vortices in a Bose-Einstein condensate", Phys. Rev. Lett., 83 (1999), pp. 2498- 2501.

[16] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, "Vortex formation in a stirred Bose-Einstein condensate", Phys. Rev. Lett., 84 (2000), pp. 806-809.

[17] K. W. Madison, F. Chevy, V. Bretin, and J. Dalibard, "Stationary states of a rotating Bose-Einstein condensate: Routes to vortex nucleation", Phys. Rev. Lett., 86 (2001), pp. 4443-4446.

[18] Y. Castin and R. Dum, "Bose-Einstein condensates with vortices in rotating traps", Eur. Phys. J. DAt. Mol. Opt. Phys., 7(1999), pp. 399-412.

[19] K. Kasamatsu, M. Tsubota, and M. Ueda," Nonlinear dynamics of vortex lattice formation in a rotating Bose-Einstein condensate", Phys. Rev. A, 67(2003), 033610.

[20] A. Aftalion and Q. Du, "Vortices in a rotating Bose-Einstein condensate: Critical angular velocities and energy diagrams in the Thomas-Fermi regime", Phys. Rev. A, 64 (2001), 063603.

[21] W. Bao, H. Wang, and P. A. Markowich, "Ground, symmetric and central vortex states in rotating Bose-Einstein condensates", Commun. Math. Sci., 3 (2005), pp. 57-88.

[22] B. Jackson, J. F. McCann, and C. S. Adams, "Vortex formation in dilute inhomogeneous Bose-Einstein condensates", Phys. Rev. Lett., 80 (1998), pp. 3903-3906.

[23] W. Bao, D. Jaksch, and P. A. Markowich, ”Numerical solution of the Gross-Pitaevskii equation for Bose-Einstein condensation”, J. Comput. Phys., 187 (2003), pp. 318-342.

[24] Qian Liu, Yuquin Zhou, Jian Zhang and Weinian Zhang, ”Sharp condition of global existence for nonlinear schrödinger equation with a hormonic potential”, J. Applioed Mathematics and Computation,177 (2006), pp 482-487.

[25] Luca Fanelli, Eugenio Montefusco, ”On the blow up threshold for weakly coupled nonlinear Schrödinger equations”, arXiv:0705.0491, V1 Math. Ap ,(2007), pp. 1-14.

[26] Guanggan Chen, Jian Zheng, ”Remarks on global existence for the supercritical nonlinear schrödinger equation with harmonic potential,” J. Math. Anal. Appl. 320 (2006), pp. 591-598.

[27] Zhongxue Lu, and Zuhan Liu, ” Sharp Threshold of Boss-Einstein Condensates with an angular momentum rotational term ”, J. Appl. Math and Informatics, 29 (2011), no. 3 - 4, pp. 901 - 908.

[28] Chengchun Hao, Ling Hsiao and Hai-Liang Li, ”Global well-posednesss for the Gross Pitaevskii equation with an angular momentum rotational term”, Math. Meth. Appl. Sci. 31 (2008), pp 655-664.

[29] Paolo Antonelli, Daniel Marahrens and Christof Sparber, ” On Cauchy problem for nonlinear schrödinger equation with rotational term”, arXiv:1009.4160 ,V1 Math. Ap (2010), pp. 1-13.

[30] Gross EP, ”Structure of quantized vortices in boson systems”, Nuovo Cimento, 20 (1961), pp.454-477.

[31] Pitaevskii LP, ”Vortex line in an imperfect Bose gas”, Zhurnal Eksperimentalnoi Teoreticheskoi, 40 (1961), pp.646-651.

[32] M. S. Berger. ”On the existence andstructure of stationary states for nonlinear Klein-Gordon equations”, J. Funct. Anal. 9 (1972), 249-261.

[33] W. A. Strauss, ”Existence of solitary waves in higher dimensions, Commun. Math. Phys., 55 (1977) 149-162.

[34] M. I. Weinstein, ”Nonlinear Schrödinger equations and sharp interpolation estimate”, Commun. Mth. Phys. 87 (1983) 567-576.

[35] M. K. Kwong, ”Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch, Rat, Mech. Anal. 105 (1989) 243-266.

[36] Jian Zhang, ”Stability of standing waves for nonlinear Schrödinger with unbounded potentials”. Z. Angew. Math. Phys. 51 (2000) 498-503.

[37] De. Bouard, ”Nonlinear Schrödinger equations with magnetic fields”. Diff. Integral Equations. bf 4 (1991) no. 1, 73-88.

[38] S. Zheng, ”Fractional regularity for Nonlinear Schrödinger equations with magnetic fields”. Contemp. Math Volume 581 (2012) 1-11.

[39] GPELab, http://gpelab.math.cnrs.fr/downloads.htm

[40] GPElab, http://gpelab.math.cnrs.fr/downloads/GPELab-UserGuide.pdf