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# Enumerating Graphs Using Integrals From Quantum Field Theory

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**ENUMERATING GRAPHS USING INTEGRALS FROM QUANTUM  
FIELD THEORY**

by

**ANTHONY COGGINS**

(Under the Direction of Jimmy Dillies)

**ABSTRACT**

Enumerating graphs is a relatively new subfield of mathematics. In this thesis, we will discuss a enumerative method that derives from Quantum Field Theory. We begin with the basic ideas of Calculus and extend them into a enumerative method that will allow us to classify graphs embedded on surfaces.

*Key Words:* Map, Hermitian, Surface, Riemann, Gluing

*2010 Mathematics Subject Classification:* 81T18,83C47

ENUMERATING GRAPHS USING INTEGRALS FROM QUANTUM  
FIELD THEORY

by

ANTHONY COGGINS

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## DEDICATION

I would like to thank everyone who has helped me through my undergraduate and graduate degrees over the last 7 years. Without the help of professors and friends alike, I would not have been successful in this endeavor. A very special thanks goes out to the following people: Jimmy and Enka: Without your guidance and mentorship, I would be completely lost through this jumble of mathematical wreckage. Andrew Sills: Your love for mathematics influenced me at a young age, for without this, I would've pursued a very different career. Hua Wang, Sharon Taylor, and Nikki Collins: without the three of you, my degree would've been lost in the administrative mess from all of the hoops I've had to jump through. Lastly my wife, Katelyn. Without your love and support, I would've failed a thousand times over. You've been the constant source of my strength through all of the struggles, and I can truthfully say I would not be here without you.

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## LIST OF SYMBOLS

- $\mathbb{R}$ : Real Numbers
- $\mathbb{C}$ : Complex Numbers
- $\mathbb{Z}$ : Integers
- $\mathbb{N}$ : Natural Numbers
- $\mathcal{H}_N$ : Space of  $N \times N$  Hermitian Matrices

# CHAPTER 1

## COUNTING PATHS OF A PLANAR GRAPH

In this section, we will give a brief overview of graphs and their adjacency matrices. We will then use these matrices to analyze paths through the graph, from one vertex to any other.

### 1.1 Graphs

Our first definition is the word **graph**. A graph is a set of points, or **nodes**, that are connected by line segments, more commonly called **edges**. Figure 1, is a simple example of a graph.

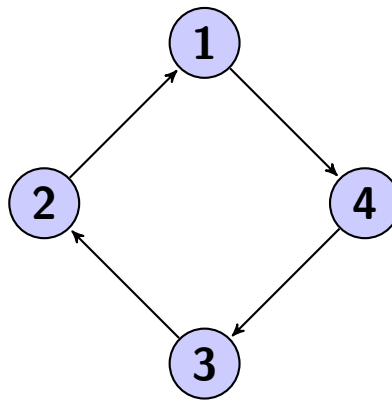


Figure 1: Example of a directed graph.

In the example above, it is important to notice that the edges have arrows. This is what makes the graph **directed**. In this chapter, we would like to answer the following question: How many possible paths are there, say from node 1 to 4? While we simply count to arrive at the answer, there are usually far more complicated graphs, and in that case, what should we do?

## 1.2 Matrices and the Adjacency Matrix

To answer the question of how many possible paths there are in a graph from one node to another, we need to introduce matrices. A **matrix** is an array of numbers, letters, symbols, commonly called **elements**. Matrices are used in almost every subfield of mathematics and physics, the most common being linear algebra. An **adjacency matrix** is a matrix whose elements are the number of paths of length one from each node to any other. More precisely, if  $A_{i,j}$  is the element in row  $i$ , column  $j$ , then  $A_{i,j}$  is the number of paths from  $i$  to  $j$ .

Here we will form an adjacency matrix from the following graph.

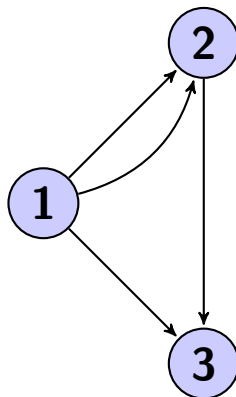


Figure 2: Graph

The first element of the matrix, is the number of paths of length *one* from 1 to 1. So we put a 0 in that spot. For the number of paths of length one between 1 and 2, we put a 2, since there are two paths. Continuing this labeling convention, we obtain the adjacency matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

What we are most interested in though, is how many possible paths are there from node 1 to node 3? Since the graph is directed, we are limited in our choices. Here we should note that if there are any loops, or backwards directed edges, that this question is very easy to answer because we could loop infinitely many times. So what should we do?

### 1.3 Counting paths

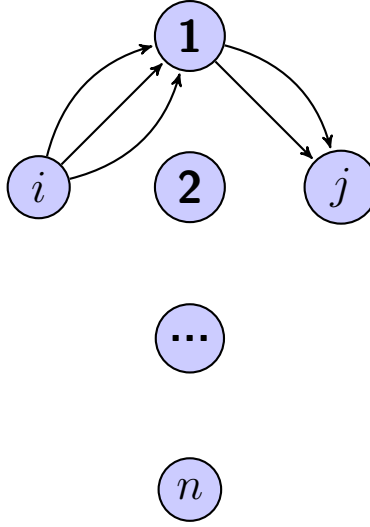
With a graph like the one given in section 2, it is quite manageable to count all the paths from 1 to 3. However, this could easily become a much greater task just by adding, say two or three additional edges or more nodes to the graph between 1 and 3. So let's think about the problem for a moment. If we wanted to find *all* paths from one node to another, we would want all possible paths of length one, length 2, length 3, etc., and add all of those paths together. Let's take a look at what the adjacency matrix for paths of length 2 look like. Again, this can easily be done by hand for a small graph. Here we will introduce a conventional notation,  $A[2]$ , to represent the matrix for paths of length two. In general, we will write  $A[n]$  for the matrix whose elements are paths of length  $n$ , where  $n \in \mathbb{N}$ . So this time, we obtain

$$A[2] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, we can see that  $A[2]$  is actually formed by multiplying  $A$  by  $A$ .

**Lemma 1.3.1.**  $A[n] = A^n$ .

*Proof.* It is easy to see that  $A^1 = A[1]$ , by definition. So let's assume now that our statement holds for some natural number  $k$ , that is,  $A^k = A[k]$ . But to prove that this claim is true for  $k + 1$ , we need to get a picture of what's going on here.

Figure 3: Graph of  $n$  nodes

In Figure 3, we are counting paths from  $i$  to any other node between  $i$  and  $j$ . On the right side, we are counting paths of length one from each node 1 through  $n$  to  $j$ . Here we can multiply the number of paths to get the total number of possible paths, thus

$$\begin{aligned}
 (A[k+1])_{ik} &= \overbrace{(A[k])_{i1}}^{A^k} \cdot \overbrace{(A[1])_{1k}}^A \\
 &\quad + (A[k])_{i2} \cdot (A[1])_{2k} \\
 &\quad \quad \quad + \dots \\
 &= \sum_{\mu=1}^n (A^{n-1})_{i\mu} \cdot (A)_{\mu j} = (A^n)_{ij}
 \end{aligned}$$

This completes the proof. □

#### 1.4 General formula for counting paths

Since we have developed an understanding of how the powers of a nilpotent matrix relate to that of a planar graph, we can now show how to count paths in any

such graph. In the previous section, it is important to note that as the powers of the matrix increase, we begin to see more zero's in the matrix. There is a name for this;

Definition: A **nilpotent** matrix is a matrix  $A$  such that  $A^n = 0$  for some  $n \in \mathbb{N}$ .

We know now almost everything that we need to derive the general formula. Let's take some time to revisit the geometric series. A geometric series is simply a sum of powers, called the common ratio. For example, the sum of  $1 + \frac{1}{2} + \frac{1}{4} + \dots$  converges to 2. This is easy to see using the formula below, given that  $x = \frac{1}{2}$ , which is called our common ratio. In general, a geometric series will converge provided that  $x < 1$ .

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad (1.1)$$

*Proof.* Let

$$s = 1 + x + x^2 + x^3 + \dots$$

$$xs = x + x^2 + x^3 + \dots$$

So taking the difference give us

$$s - xs = 1, \text{ thus } s = \frac{1}{1-x}.$$

□

Notice here that we have something similar to what I mentioned earlier, about the matrices. We are adding up powers of a matrix that tends to the zero matrix. Why does this matrix tend to the zero matrix? As the powers of the matrix increase, i.e, as the length of the path goes up, we are going to run out of paths since there are only a finite number of edges. An example of where this would not occur, is an adjacency matrix of a graph without loops. It is very important that the graph



does not have loops, otherwise the adjacency matrix would not be nilpotent. Once the number of paths exceeds the number of edges, we will have arrived at the zero matrix.

Modifying equation (1.1) for matrices, we obtain

$$A^{(total)} := \sum_{i=1}^{\infty} A^i = (I - A)^{-1} - I, \quad (1.2)$$

where  $I$  is the identity matrix. Here,  $(I - A)^{-1}$  simply means to take the inverse of the matrix,  $(I - A)$ . Let's see the proof of this.

*Proof.* Let  $A$  be a nilpotent,  $n \times n$  matrix. If we add  $I$  to each side of (1.2), we obtain

$$I + A + A^2 + A^3 + \dots = (I - A)^{-1}$$

If we let  $s$  equal the left hand side of the above equation and multiply by  $A$ , we get

$$As = A + A^2 + A^3 + \dots.$$

Taking the difference,

$$s - As = I$$

$$s(I - A) = I$$

$$s = (I - A)^{-1}$$

If we subtract  $I$  from both sides, we complete the proof. □

In our example, plugging the matrix  $A$  into the general formula above gives us the following matrix:

$$A^{(total)} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$A^{(total)}$  is the matrix that answers the question we asked long ago. How many possible paths are there from 1 to 3? If we look at the 3rd element on the 1st row, we can see that the answer is 3.

## CHAPTER 2

### FEYNMAN INTEGRALS

#### 2.1 Feynman Diagrams

Feynman Diagrams are a way to represent the mathematical expressions describing interactions of subatomic particles. The diagrams are named for American physicist, Richard P. Feynman, who first introduced them in 1948 [4]. The Feynman diagrams allow for a simple visualization of very complex subatomic particle processes. When one wants to calculate the scattering cross-section in particle physics, the interactions can be described by a free form field for the incoming and outgoing particles, as well as including the interaction Hamiltonian for how the particles deflect. One can write down the scattering amplitude by summing over all the possible interactions of the particles. Instead of writing long, complex expressions for these interactions, Feynman suggested that one can simply sum over Feynman diagrams, where at each interaction vertex, both energy and momentum are conserved, but the length of the energy-momentum four vector is not equal to the mass. The calculations of the probability amplitudes for particle interactions to occur can be very tedious and complex (no pun intended). The integrals, however, do have a nice structure, and can be represented by the diagrams. Within the canonical formulation for Quantum Field Theory, a Feynman diagram represents a term in Wick's Expansion of a perturbative S-matrix.

Feynman wrote up a prescription if you will, of how to calculate the probability amplitude for a particle interaction (i.e diagram) from a Lagrangian. These are the Feynman rules. Each internal line corresponds to a factor of the virtual particle's propagator, each vertex where lines meet gives a factor derived from an interaction term in the Lagrangian, and incoming and outgoing lines carry an energy, momentum, and spin. Aside from their mathematical value, these diagrams grant deep insight

into the nature of particle interactions. Particles will interact in every way possible, so finding the interaction probability amplitude can be a tricky task. Doing these calculations without care can often yield infinite probability amplitudes, because short distance interactions require careful limiting, the so-called renormalization. After renormalization though, many of the ‘infinities’ can be eliminated, and the math very closely matches that of the data from physics experiments.

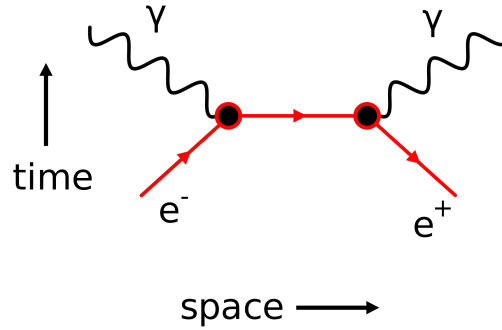


Figure 1: Electron-Positron Annihilation

The Figure 1 represents electron-positron annihilation. Time is upwards and space moves to the right, as shown below in the figure. The arrow points backwards on the positron because Feynman said the anti-particles move backwards through time.

As we will see in the coming sections, the probability density of these interactions are often given by something in the form of

$$\exp\{-(\text{quadratic function}) - (\text{interaction term})\}$$

which is a Gaussian form. However, we will need to extend the space from  $\mathbb{R}^N$  to  $\mathcal{H}^N$ , which will be introduced later. Let us now move to discussing several Gaussian forms on  $\mathbb{R}$  and extending those to  $\mathbb{R}^N$ .

## 2.2 Gaussian Integrals

In this section, we will focus on the combinatorial interpretation of integrals over a gaussian measure, and prove some identities related to such integrals. On the real line  $\mathbb{R}$ , the standard Gaussian measure  $\mu$  with the measure given by

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \quad x \in \mathbb{R}$$

This measure has several properties, all of which we will prove.

1. It is *normalized*, that is, the measure over the real line is 1:

$$\int_{\mathbb{R}} d\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1$$

2. The *mean* is 0:

$$\int_{\mathbb{R}} x d\mu(x) = 0$$

3. The *variance* is 1:

$$\int_{\mathbb{R}} x^2 d\mu(x) = 1$$

4. The *characteristic function* (or Fourier transform) is

$$\phi(t) = \int_{\mathbb{R}} e^{-itx} d\mu(x) = e^{-\frac{t^2}{2}}$$

5. The integral

$$\int_{\mathbb{R}} p(x) e^{-\frac{x^2}{2}} dx$$

converges for any polynomial  $p \in \Pi_n(x)$ , where  $\Pi_n(x)$  is the space of  $n^{\text{th}}$  degree polynomials with real coefficients.

6. For any positive real number  $b$ , we have

$$\int_{\mathbb{R}} e^{-\frac{bx^2}{2}} dx = \frac{1}{\sqrt{b}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx.$$

*Proof.* 1. 
$$\begin{aligned} \left( \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \right)^2 &= \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} = \int_0^{\infty} \int_0^{2\pi} r e^{-\frac{r^2}{2}} d\phi dr \\ &= 2\pi \int_0^{\infty} r e^{-\frac{r^2}{2}} dr = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} d\left(\frac{1}{2}r^2\right) = 2\pi \end{aligned}$$

2. Since the integrand  $x e^{-\frac{x^2}{2}}$  is an odd function being integrated over symmetric limits, the value is 0.

3. This integral will be verified using the gamma function along with a change of variable.

For those not familiar with the gamma function, it is defined as follows:

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx = (n-1)!$$

for  $n \in \mathbb{Z}$ . The integral we want to verify is defined as follows:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$

To make our lives easy, we will execute a change of variable,  $u = x^2/2$ . This change of variable yields the following integral

$$\frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2ue^{-u} \frac{du}{\sqrt{2u}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} u^{1/2} e^{-u} du = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = 1$$

4. For this integral, we will use an integral table, namely

$$\mathcal{F}_x[e^{-ax^2}](t) = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 t^2}{a}}$$

A direct application of the formula will give the desired result.

5. For this integral, we will rely on a result that has not yet been proved, so let us prove this first. We need to prove

$$\int_{\mathbb{R}} x^{2n} d\mu(x) = (2n - 1)!!$$

where  $(2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1$ .

To start this proof, we will simply rewrite the integral using a change of variable,  $u = x^2/2$  again. This give us

$$\frac{2}{\sqrt{2\pi i}} \int_0^\infty (2u)^n e^{-u} \frac{du}{\sqrt{2u}} = \frac{2^n}{\sqrt{\pi}} \int_0^\infty u^{n-\frac{1}{2}} e^{-u} du = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$

The above gamma function simplifies to  $\frac{(2n-1)!!}{2^n} \sqrt{\pi}$ . Thus our integral reduces to  $(2n - 1)!!$ .

Going back to the original problem, we want to show that

$$\int_{\mathbb{R}} p(x) e^{-\frac{x^2}{2}} dx$$

converges for any polynomial  $p$ . Any polynomial (with  $c_i \in \mathbb{R} \forall i \in \mathbb{N}$ ) can be written as a linear combination of it's power basis. So, let  $p(x) = \sum_{i=0}^n c_i x^i$ .

So now our integral becomes

$$\int_{\mathbb{R}} \sum_{i=0}^n c_i x^i e^{-\frac{x^2}{2}} dx = \sum_{i=0}^n \int_{\mathbb{R}} c_i x^i e^{-\frac{x^2}{2}} dx$$

(by linearity). Each term of the sum either has an odd integrand, or an even integrand. If the integrand is odd, we know this goes to 0 by property (2). If the integrand is even, we know it converges by the proof completed above. Therefore,

$$\sum_{i=0}^n \int_{\mathbb{R}} c_i x^i e^{-\frac{x^2}{2}} dx < \infty \quad \forall n \in \mathbb{N}.$$

6. For this integral, we will use a change of variable again. As one might suspect at this point, the change of variable is simply  $u = \frac{bx^2}{2}$ . After some basic algebra, this results in the following integral:

$$2 \int_0^\infty e^{-u} \frac{1}{b} \sqrt{\frac{2u}{b}} du = \frac{2}{\sqrt{2b}} \int_0^\infty e^{-u} u^{-1/2} du$$

In this case, the integral above is simply  $\Gamma\left(\frac{1}{2}\right)$ , so the integral simplifies to be  $\sqrt{\frac{2\pi}{b}}$ , which is what we needed to show.

□

There are a few other notable integrals worth verifying here, that are of particular use to us. We will move on to describing Gaussian measures in  $\mathbb{R}^N$ . Consider a point  $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ . By  $(x, y)$ , we will define the ordinary scalar product in  $\mathbb{R}^N$ , that is,  $(x, y) = x_1 y_1 + \dots + x_N y_N$ .

For a positively defined symmetric  $N \times N$ -matrix  $A$  the measure is defined by the *density*

$$d\mu(x) = \text{Const} \times e^{-\frac{1}{2}(Ax, x)} dv(x) \tag{2.1}$$

where  $dv(x) = dx_1 \dots dx_N$  is the volume form on  $\mathbb{R}^N$ . For this measure to be normalized, we must take the constant to be

$$\text{Const} = (2\pi)^{-N/2}(\det A)^{1/2}. \quad (2.2)$$

**Lemma 2.2.1.** *Let  $A$  be a symmetric matrix. Then*

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(A\underline{x}, \underline{x})} d^N x = (2\pi)^{N/2}(\det A)^{1/2}$$

where  $(A\underline{x}, \underline{x}) = \underline{x}^T A \underline{x}$  is the usual inner-product notation.

*Proof.* For simplicity in the notation, all  $x$ 's are understood to be  $N$ -dimensional column vectors. This proof will require diagonalization of  $A$ . Let  $D = O^T A O$ , where  $O$  is an orthogonal matrix, and  $D$  is the diagonal matrix consisting of the eigenvalues of  $A$ . This give us the following:

$$\begin{aligned} (Ax, x) &= x^T Ax = x^T (OO^T)A(OO^T)x \\ &= (x^T O)(O^T A O)(O^T x) \\ &= (x^T O)D(O^T x). \end{aligned}$$

By using the orthogonal transformation  $\underline{y} = O^T \underline{x}$  and denoting the eigenvalues of  $A$  by  $\lambda_j$  for  $1 \leq j \leq N$ , we can write the integral as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T Ax} d^N x &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j y_j^2} d^N y = \prod_{j=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_j y_j^2} dy_j \\ &= \prod_{j=1}^N \left( \frac{2\pi}{\lambda_j} \right)^{\frac{1}{2}} \\ &= \left( \frac{(2\pi)^N}{\prod_{j=1}^N \lambda_j} \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{(2\pi)^N}{\det(A)}} \end{aligned}$$

□



**Lemma 2.2.2.**  $\int_{\mathbb{R}} e^{-\frac{ax^2}{2}+bx} dx = e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}}$

*Proof.* First we will complete the square in the exponent of the integrand.

$$\begin{aligned} -\frac{ax^2}{2} + bx &= -\frac{a}{2} \left( x^2 - \frac{2b}{a}x \right) \\ &= -\frac{a}{2} \left( x^2 - \frac{2b}{a}x + \frac{b^2}{a^2} - \frac{b^2}{a^2} \right) \\ &= -\frac{a}{2} \left( \left( x - \frac{b}{a} \right)^2 - \frac{b^2}{a^2} \right) \\ &= -\frac{a}{2} \left( x - \frac{b}{a} \right)^2 + \frac{b^2}{2a} \end{aligned}$$

Now our integral becomes

$$\int_{\mathbb{R}} e^{-\frac{a}{2}\left(x-\frac{b}{a}\right)^2} e^{\frac{b^2}{2a}} dx$$

Setting  $u = x - \frac{b}{a}$ , we can write

$$e^{\frac{b^2}{2a}} \int_{\mathbb{R}} e^{-\frac{a}{2}u^2} du$$

An application of the standard gaussian to the integral will give the desired result of  $e^{\frac{b^2}{2a}} \sqrt{\frac{2\pi}{a}}$ . □

**Lemma 2.2.3.**  $\int_{\mathbb{R}^N} e^{-\frac{1}{2}(x^T Ax)+b^T x} d^N x = \sqrt{\frac{(2\pi)^N}{\det(A)}} e^{\frac{1}{2}b^T A^{-1}b}$

*Proof.* The idea used to prove this result is identical to the previous integral we proved in Lemma 1. By using the same diagonalization, we arrive at the integral:

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(x^T Ax)+b^T x} = \int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j y_j^2 + \frac{1}{2}\sum_{j=1}^N \tilde{b}_j y_j} d^N y$$

where  $\tilde{b} = b^T O$ . Now we can rewrite the integral above as a product of integrals over  $\mathbb{R}$ .

$$\prod_{j=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_j y_j^2 + \tilde{b}_j y_j} dy_j \quad (2.3)$$

Now if we let  $a = \lambda_j$  and  $b = \tilde{b}_j$ , we can use Lemma 2 to simplify (2.4).

$$\begin{aligned} \prod_{j=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_j y_j^2 + \tilde{b}_j y_j} dy_j &= \prod_{j=1}^N \sqrt{\frac{2\pi}{\lambda_j}} e^{\frac{\tilde{b}_j^2}{2\lambda_j}} \\ &= \sqrt{\frac{(2\pi)^N}{\det(A)}} \prod_{j=1}^N e^{\frac{1}{2}\frac{\tilde{b}_j^2}{\lambda_j}} \\ &= \sqrt{\frac{(2\pi)^N}{\det(A)}} e^{b^T A^{-1} b} \end{aligned}$$

Note that  $\frac{\tilde{b}_1^2}{\lambda_1} + \frac{\tilde{b}_2^2}{\lambda_2} + \dots + \frac{\tilde{b}_N^2}{\lambda_N} = \sum_{j=1}^N \tilde{b}_j^2 \lambda_j^{-1} = b^T A^{-1} b$ . This completes the proof.  $\square$

In probability, the inverse matrix  $C = (c_{ij}) = A^{-1}$  is called the **covariance matrix**, and we have  $\langle x_i \rangle = 0$ ,  $\langle x_i x_j \rangle = c_{ij}$ . These properties follow from the fact that they are satisfied in the case  $A$  and  $C$  are diagonal because of the properties of the Gaussian measure on the line. A Gaussian measure is called *standard* if both  $A$  and  $C$  are the identity matrices.

For the remainder of this paper, we will regularly use a notation borrowed from physics (don't worry, we asked). For any measure  $\mu$  on  $X$ , and for any function  $f : X \rightarrow \mathbb{R}$  or  $f : X \rightarrow \mathbb{C}$ , we will denote  $\langle f \rangle$  the *mean*, or the *average* value, of  $f$  with respect to the measure  $\mu$ :

$$\langle f \rangle = \int_X f(x) dx.$$

The measure  $\mu$  and the space  $X$  will always be clear from the context. For example, the above formulas can be rewritten as

$$\langle 1 \rangle = 1, \quad \langle x \rangle = 0, \quad \langle x^2 \rangle = 1, \quad \langle e^{itx} \rangle = e^{-\frac{t^2}{2}}.$$

### 2.3 M-Point Functions

The so-called **correlation functions** contain a lot of information about physical systems. In particular, they can be used to compute scattering amplitudes of the Feynman diagrams mentioned earlier.

Let us recall the integral from the last section, and give it a name.

$$Z_b = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(x^T Ax) + b^T x} d^N x = \sqrt{\frac{(2\pi)^N}{\det(A)}} e^{\frac{1}{2}b^T A^{-1}b}$$

**Definition** Consider the multiset of indices  $i_1, \dots, i_m$ . We define the **m-point function** as

$$\langle x_{i_1} \cdots x_{i_m} \rangle = \frac{1}{Z_0} \int_{\mathbb{R}^N} e^{\frac{1}{2}x^T Ax} x_{i_1} \cdots x_{i_m} d^N x$$

The combinatorial nature of the above integrals comes from the fact that the m-point functions can be obtained through differentiation.

**Lemma 2.3.1.**  $\langle x_i \rangle = \frac{1}{Z_0} \frac{\partial Z_b}{\partial b_i} \Big|_{b=0}$

*Proof.*

$$\begin{aligned} \frac{\partial Z_b}{\partial b_i} &= \frac{\partial}{\partial b_i} \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T Ax + b^T x} d^N x \\ &= \int_{\mathbb{R}^N} \frac{\partial}{\partial b_i} e^{-\frac{1}{2}x^T Ax + b^T x} d^N x \\ &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T Ax} \frac{\partial}{\partial b_i} e^{b^T x} d^N x \\ &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T Ax} x_i e^{b^T x} d^N x \Big|_{b=0} \\ &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T Ax} x_i d^N x. \end{aligned}$$

Multiplying by  $\frac{1}{Z_0}$ , we get the desired result. Here, the result  $\langle x_i \rangle$  is called the one-point function.  $\square$

**Lemma 2.3.2.**  $\langle x_{i_1} \cdots x_{i_m} \rangle = \frac{1}{Z_0} \left( \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_m}} Z_b \right) |_{b=0}$

*Proof.* Since we have proven the case for the 1-point function and the partial derivatives are linearly independent operations, the result follows from repeated application of the derivative (as shown below):

$$\begin{aligned} \frac{\partial^2}{\partial b_{i_1} \partial b_{i_2}} Z_b &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T A x} x_{i_1} x_{i_2} e^{b^T x} d^N x |_{b=0} \\ &= \int_{\mathbb{R}^N} e^{-\frac{1}{2}x^T A x} x_{i_1} x_{i_2} d^N x \end{aligned}$$

Continuing up to the  $m^{\text{th}}$  order, the result follows.  $\square$

## 2.4 The Wick Formula

A large part of the workings of Quantum Mechanics consists of computing integrals with respect to a Gaussian measure. The goal of this section is to develop a technique of integrating polynomials. Knowing that  $\langle x_i \rangle = 0$ ,  $i = 1, \dots, n$  and  $\langle x_i x_j \rangle = c_{ij}$ ,  $i, j = 1, \dots, n$ , we can easily compute the integral of any polynomial in  $x_1, \dots, x_n$  of quadratic degree. For higher degrees, we need to develop some more advanced techniques, which will be the focus of this section.

**Lemma 2.4.1.** *If  $f(x)$  is a monomial of odd degree, then  $\langle f \rangle = 0$ .*

This is easily checked because  $f$  is an odd function being integrated over symmetric limits. Now, we will introduce a useful formula, the **Wick Formula**, which reduces the integration of *any* degree polynomial to 2.

**Theorem 2.4.1.** (*Wick's Lemma*) *Let  $f_1, f_2, \dots, f_{2n}$  be a set of (not necessarily distinct) linear functions of  $x_1, \dots, x_k$ . Then*

$$\langle f_1 f_2 \dots f_{2n} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \dots \langle f_{p_n} f_{q_n} \rangle$$

where the sum is taken over all permutations of  $p_1 q_1 p_2 \dots q_n$  of the set of indices  $1, 2, \dots, 2n$  such that  $p_1 < p_2 < \dots < p_n$ ,  $p_1 < q_1, \dots, p_n < q_n$ .

The number of summands on the right side is equal to  $(2n - 1)!!$ . A partition of the set  $1, 2, \dots, 2n$  into couples  $(p_i, q_i)$  satisfying the condition of the Wick Formula is a **Wick coupling**.

To see this formula in action, let's take a simple example that we know the answer to, and compute it using the Wick Formula.

**Example** Let us apply the Wick Formula to compute the following integral:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^6 e^{-\frac{x^2}{2}} dx$$

We have  $x^6 = f_1 f_2 f_3 f_4 f_5 f_6$ , where each  $f_i = x$  (for  $1 \leq i \leq 6$ ). Therefore,

$$\begin{aligned} \langle f_1 f_2 f_3 f_4 f_5 f_6 \rangle &= (\langle f_1 f_2 \rangle \langle f_3 f_4 \rangle \langle f_5 f_6 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle \langle f_5 f_6 \rangle \\ &+ \dots + \langle f_1 f_6 \rangle \langle f_2 f_3 \rangle \langle f_4 f_5 \rangle) + (\langle f_2 f_3 \rangle \langle f_1 f_4 \rangle \langle f_5 f_6 \rangle \\ &+ \dots + \langle f_2 f_6 \rangle \langle f_1 f_3 \rangle \langle f_4 f_5 \rangle) + \dots + \langle f_5 f_6 \rangle \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle. \end{aligned}$$

Each of these products are equal to 1, so we have a total of 15. According the formula proved earlier, this is precisely what we expect to get.

A different way to state the Wick Formula, perhaps more useful, is as follows:

$$\frac{\partial}{\partial b^{i_1}} \dots \frac{\partial}{\partial b^{i_m}} e^{\frac{1}{2} b^T A^{-1} b} \Big|_{b=0} = \sum A_{i_{p_1}, i_{p_2}}^{-1} \dots A_{i_{p_{m-1}}, i_{p_m}}^{-1}$$

where the sum is taken over all pairings  $(i_{p_1}, i_{p_2}), \dots, (i_{p_{m-1}}, i_{p_m})$  of  $i_1, \dots, i_m$ .

To begin analyzing the statement, it is much easier to write  $e^{\frac{1}{2}b^T A^{-1}b}$  with  $b^T A^{-1}b = \sum A_{i,j}^{-1}b^i b^j$ , where the sum runs from  $i, j = 1$  to  $d$ . Rewriting this in terms of the power series for the exponential function, we arrive at the following expression:

$$e^{\frac{1}{2}b^T A^{-1}b} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left( \sum A_{i,j}^{-1} b^i b^j \right)^n$$

However, as discussed in the lemma, only even terms of this series will appear, meaning the general term will have the form of

$$\frac{1}{2^n n!} \left( \sum A_{i,j}^{-1} b^i b^j \right)^n.$$

Note that this is a homogeneous polynomial in  $b_i$  of degree  $2n$ . If we try to differentiate this polynomial  $k$ -times and evaluate it at 0, we will get 0 unless  $k = 2n$ . So now we have what we are most concerned with. Our next step is to analyze how  $2n$  differentiations will ‘look’ on  $\frac{1}{2^n n!} \left( \sum A_{i,j}^{-1} b^i b^j \right)^n$

Since the coefficients are of no concern to us, we can ignore them for now. The differentiation we are going to do most here is the following:

$$\frac{\partial}{\partial b^k} \left( \frac{1}{2} \sum_{i,j=1}^d A_{i,j}^{-1} b^i b^j \right) = \sum_{i=1}^d A_{i,k}^{-1} b^i$$

Now let us compute a few terms of this polynomial to get an idea of what it looks like. From this point on, we will abbreviate  $\frac{\partial}{\partial b^i}$  as  $\partial_i$ .

For the case of  $n = 1$ , we have

$$\partial_2 \partial_1 \left( \frac{1}{2} \sum_{i,j=1}^d A_{i,j}^{-1} b^i b^j \right) = \partial_2 \left( \sum_j A_{1,j}^{-1} b^j \right) = A_{1,2}^{-1}.$$

Likewise,  $\partial_1 \partial_1 \left( \frac{1}{2} \sum_{i,j=1}^d A_{i,j}^{-1} b^i b^j \right) = A_{1,1}^{-1}$ . Note that (1, 2) and (2, 1) count as the same pairing here, so we do not double count it.

For the case of  $n = 2$ , we have

$$\begin{aligned}
\partial_4 \partial_3 \partial_2 \partial_1 \frac{1}{2!2^2} \left( \sum_{i,j=1}^d A_{i,j}^{-1} b^i b^j \right)^2 &= \partial_4 \partial_3 \partial_2 \left( \frac{1}{2} \sum A_{1,j}^{-1} b^i b^j \right) \left( \sum A_{1,j}^{-1} b^j \right) \\
&= \partial_4 \partial_3 \left[ \left( \sum A_{2,j}^{-1} b^j \right) (A_{1,j}^{-1} b^j) + (1/2) \left( \sum A_{i,j}^{-1} b^i b^j \right) A_{1,2}^{-1} \right] \\
&= \partial_4 \left[ A_{2,3}^{-1} \left( \sum A_{1,j}^{-1} b^j \right) + \left( \sum A_{2,j}^{-1} \right) A_{1,3}^{-1} + \left( \sum A_{3,j}^{-1} \right) A_{1,2}^{-1} \right] \\
&= A_{2,3}^{-1} A_{1,4}^{-1} + A_{2,4}^{-1} A_{1,3}^{-1} + A_{3,4}^{-1} A_{1,2}^{-1}
\end{aligned}$$

Similarly, one can use this same method to show that  $\partial_4 \partial_3 \partial_1 \partial_1$  gives  $2A_{1,4}^{-1} A_{1,3}^{-1} + A_{3,4}^{-1} A_{1,1}^{-1}$ , and so on for any other pairing of the indices. It is easy to see that this process can get very tedious, which is why we need the Wick Formula to help us, where it becomes a problem of pairing indices as opposed to computing multiple partial derivatives.

Now we will prove the Wick Lemma.

*Proof.* Here, we can rewrite the expression for the Wick Lemma as an expansion of exponential functions [1].

$$\left\langle \exp \left( t \sum_{i=1}^N b^i x^i \right) \right\rangle = \exp \left( \frac{t^2}{2} \sum_{i,j=1}^N A_{i,j}^{-1} b^i b^j \right)$$

From here, expanding both sides in powers of  $t$ , the odd powers will vanish (due to the symmetry). The term on order  $2n$  will yield

$$\begin{aligned}
\frac{1}{(2n)!} \sum b^{i_1} b^{i_2} \dots b^{i_{2n}} \langle x^{i_1} x^{i_2} \dots x^{i_{2n}} \rangle &= \frac{1}{2^n n!} \left( \sum_{i,j} A_{i,j}^{-1} b^i b^j \right)^n \\
&= \frac{1}{2^n n!} \sum_P b^{i_1} b^{i_2} \dots b^{i_{2n}} \frac{1}{(2n)!} \sum_P A_{i_{p_1}, i_{p_2}}^{-1} \dots A_{i_{p_{2n-1}}, i_{p_{2n}}}^{-1}
\end{aligned}$$

Note here that  $P$  means a permutation of the indexing. Now if we identify the coefficients of  $b^{i_1} b^{i_2} \dots b^{i_{2n}}$ , this completes the proof.  $\square$

## 2.5 Graphs of the M-point Functions

The functions computed in the previous section can be represented graphically. More explicitly, we'll associate a 4 point correlation function  $\langle x^1, x^2, x^2, x^4, \rangle$  to the graphs with vertices labeled 1, 2, 3, 4 and where the edges correspond to the ordering of the indices.

In the previous section, we have shown how to compute these  $m$ -point functions. Let us, without calculation, write down some and show how to draw their corresponding graphs.

$$\langle v^1, v^2 \rangle = A_{1,2}^{-1}$$

$$\langle v^1, v^1 \rangle = A_{1,1}^{-1}$$

$$\langle v^1, v^2, v^3, v^4 \rangle = A_{2,3}^{-1}A_{1,4}^{-1} + A_{2,4}^{-1}A_{1,3}^{-1} + A_{3,4}^{-1}A_{1,2}^{-1}$$

$$\langle v^1, v^1, v^3, v^4 \rangle = 2A_{1,4}^{-1}A_{1,3}^{-1} + A_{3,4}^{-1}A_{1,1}^{-1}$$

$$\langle v^1, v^1, v^1, v^4 \rangle = 3A_{1,4}^{-1}A_{1,1}^{-1}$$

$$\langle v^1, v^1, v^4, v^4 \rangle = 2A_{1,4}^{-1}A_{1,4}^{-1} + A_{4,4}^{-1}A_{1,1}^{-1}$$

$$\langle v^1, v^1, v^1, v^1 \rangle = 3A_{1,1}^{-1}A_{1,1}^{-1}$$

Now, let us represent each of these functions as a graph. There are 2 different ways to write the graphs down. The first way, is to simply consider your vertices as the 'powers' of the  $v$ 's listed in the function. Each  $A_{i,j}^{-1}$  becomes an edge from  $i$  to  $j$ .

Figure 2 is the graph corresponding to the  $A_{1,2}^{-1}A_{3,4}^{-1}$  term in the 4-point function  $\langle v^1, v^2, v^3, v^4 \rangle$ . Another method for writing down these graphs, is to simply consider each listing of a point in the  $m$ -point function has a half edge from that point. Then the graphs can be formed by all the possible connections of the half edges.



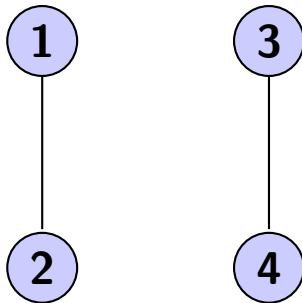


Figure 2: A graph of  $A_{1,2}^{-1}A_{3,4}^{-1}$

## 2.6 Calculations with the ‘Feynman Rules’

In this section, we will discuss in detail how to compute integrals according to the Feynman rules. The integrals in question, are similar to ones we have previously discussed, so we will approach them with the developed framework. Our integral of interest is

$$Z_U = \int_{\mathbb{R}^N} e^{-\frac{1}{2}v^T A v + \hbar U(\mathbf{v})} d^N v.$$

If  $U(\mathbf{v})$  is a polynomial of the coordinate functions  $v^1, \dots, v^N$ , then we can use the methodology developed in the previous section and treat this as an  $m$ -point function.

This integral can be rewritten using the formal power series for the exponential function as follows

$$\begin{aligned} Z_U &= \int e^{-\frac{1}{2}v^T A v} e^{\hbar U(\mathbf{v})} d^N v \\ &= \int e^{-\frac{1}{2}v^T A v} \sum_n \frac{1}{n!} (\hbar U(\mathbf{v}))^n d^N v \end{aligned}$$

We can consider this as an  $m$ -point function, and evaluate it by writing it

$$Z_U = Z_0 e^{\hbar U\left(\frac{\partial}{\partial \mathbf{b}}\right)} e^{\frac{1}{2}b^T A^{-1}b} \Big|_{b=0}$$

Now we will compute an example. Consider the potential  $U(\mathbf{v}) = \sum_{i,j,k} u_{i,j,k} v^i v^j v^k$ .

We want to analyze

$$\begin{aligned} Z_U &= \int \exp\left(-\frac{1}{2}v^T A v + \hbar \sum_{i,j,k} u_{i,j,k} v^i v^j v^k\right) d^N v \\ &= Z_0 \exp\left(\hbar \sum_{i,j,k} u_{i,j,k} \partial_i \partial_j \partial_k\right) \exp\left(\frac{1}{2}b^T A^{-1} b\right) \Big|_{b=0}. \end{aligned}$$

Using the same abbreviation as last time, we will compute the terms of degree 2 in  $\hbar$ . Since there are 3 variables and we are concerned with order 2, there will be 6 derivatives whose sum is

$$\sum_{i,j,k} \sum_{i',j',k'} u_{i,j,k} u_{i',j',k'} \partial_i \partial_j \partial_k \partial_{i'} \partial_{j'} \partial_{k'} \exp\left(\frac{1}{2}b^T A^{-1} b\right) \Big|_{b=0}$$

By Wick's Formula, we can rewrite the sum as

$$\sum_{i,j,k} \sum_{i',j',k'} \sum_{i_1,i_2} A_{i_1,i_2}^{-1} A_{i_3,i_4}^{-1} A_{i_5,i_6}^{-1} u_{i,j,k} u_{i',j',k'},$$

where the inner sum is taken over all pairings  $(i_1, i_2), (i_3, i_4), (i_5, i_6)$  of the indices  $i, j, k, i', j', k'$ .

As we saw in the previous section, there will be 2 vertices, one for  $u_{i,j,k}$  and one for  $u_{i',j',k'}$ . Without working out all of the details, we know there will be 6 edges, so there are only two distinct graphs to be formed from this set. One is the ‘dumbbell’ and the other is the ‘theta’ graph, as shown here.

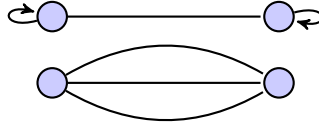


Figure 3: A dumbbell and theta graph

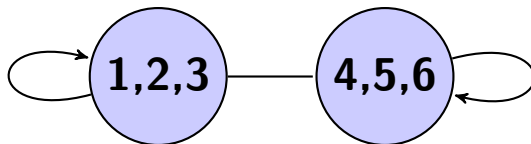


Figure 4: One possible dumbbell graph

One of the 8 labellings that result to the product  $A_{1,3}^{-1}A_{2,5}^{-1}A_{4,6}^{-1}u_{1,2,3}u_{4,5,6}$  is seen in Figure 4. Each edge can be permuted, which is where the other 7 graphs are fromed from.

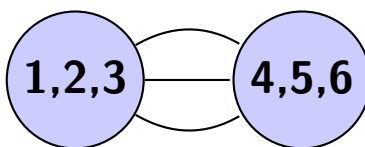


Figure 5: One possible theta graph

Figure 5 is one of the graphs that appear in the product  $u_{1,2,3}u_{4,5,6}A_{1,4}^{-1}A_{2,5}^{-1}A_{3,6}^{-1}$ . The other 5 graphs, and their reflections, can again be formed by permuting the edges.

Graphs that have 3 vertices at an edge are called **trivalent**. The ‘dumbbell’ graph has an **automorphism** group of order 8, and the ‘theta’ graph has an automorphism group of order 12. The order of the automorphism group is the number of ways to permute the edges of the graph. With this in mind, the coefficient of the  $\hbar^2$  can be written as

$$\sum_G \frac{1}{|\text{Aut}G|} \sum_{\text{labellings}} \prod_v u_{\text{vertex label}} \prod_e A_{\text{edge label}}^{-1}$$

where  $|\text{Aut}G|$  is the order of the automorphism group. The sum  $\sum_G$  is taken over the set of topologically distinct trivalent graphs with two vertices, the products are taken over the set of all of the vertices  $v$ , and all of the edges  $e$ .

Generally speaking, the Feynam rules for computing the coefficient of  $\hbar^{2n}$  in the

expansion of  $Z_U$  are done in exactly this way, except the sum is taken over all trivalent graphs with  $2n$  vertices and  $3n$  edges.

## 2.7 Correlation Functions

As stated previously in the beginning of the chapter, the manner in which path integrals are used in QFT is roughly to compute the probability of something going from point  $v_1$  to point  $v_2$ , as an integral over all possible ways to get from  $v_1$  to  $v_2$ . In the model developed in this chapter, each pathway is represented by a vector  $v$  in  $\mathbb{R}^N$  and the probability measure is

$$\frac{1}{Z_U} \exp\left(-\frac{1}{2}v^T A v + \hbar U(v)\right) v^1 v^2 d^N v$$

So the integral, which is called the 2-point function, is given by

$$\langle v^1, v^2 \rangle = \frac{1}{Z_U} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}v^T A v + \hbar U(v)\right) v^1 v^2 d^N v.$$

We will now call this the correlation function. Let us continue with our example of the cubic potential  $U(v) = \sum_{i,j,k} u_{i,j,k} v^i v^j v^k$ . By our previous calculations, we know that this will become

$$\langle v^1, v^2 \rangle = \frac{1}{Z_U} \partial_1 \partial_2 \exp\left(\hbar \sum_{i,j,k} u_{i,j,k} \partial_i \partial_j \partial_k\right) \left(\exp\left(\frac{1}{2}b^T A^{-1} b\right)\right) \Big|_{b=0}.$$

By using Wick's Formula and the graph interpretation of the pairings, this will be

$$\sum_G \frac{\hbar^n}{|\text{Aut}G|} \sum_{\text{labellings}} \prod_v u_{\text{vertex label}} \prod_e A_{\text{edge label}}^{-1},$$

where the sum is taken over all graphs  $G$  with two-single valent vertices labeled 1 and 2 and  $n$  trivalent vertices.

The  $k$ -point correlation functions are similarly defined. The methods developed in this section can be applied to them for calculations of the graphs and amplitudes. Figure 6 is a graph that appears in the calculation of the  $\hbar^2$  coefficient of  $\langle v^1, v^2, v^3, v^4 \rangle$ .

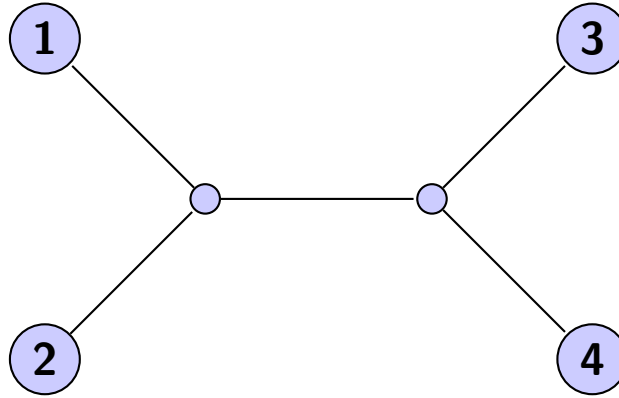


Figure 6: ‘Feynman’ Graph

## CHAPTER 3

### EMBEDDING GRAPHS ON SURFACES

In this chapter, we will care about graphs embedded on topological surfaces. That is, graphs will come with a preferred orientation of the edges at each vertex.

#### 3.1 Surfaces

For the remainder of this paper, we will consider **compact, oriented** 2-dimensional **manifolds** without **boundary**. First, let's discuss some preliminary definitions that will help us get a better understanding of what a surface is. A **topological space**  $X$  is called *locally Euclidean* if there exists a non-negative integer  $n$  such that every point  $x \in X$ , there is a neighborhood that is homeomorphic to the Euclidean space  $E^n$ , or equivalently, the real space  $\mathbb{R}^n$ . An example of this would be to define a topology on  $\mathbb{R}$ . The standard topology for this set generated by open intervals.

A homeomorphism is simply a continuous map that has a continuous inverse. A topological manifold, or simply manifold, is a locally Euclidean **Hausdorff** space. Hausdorff spaces, or  $T_2$  spaces, are spaces in which distinct points have disjoint neighborhoods. This is commonly called a *separated* space. A trivial example of a Hausdorff space would be the real numbers. Now that we know what a manifold is, we can put further restrictions on it. We will define a compact surface to be one in which there are a finite number of triangles in a **triangulation** of the surface. A triangulation is simply dividing a surface into triangles with the restriction that each triangle side is entirely shared by two adjacent triangles.

Finally, a surface is called **orientable** if a figure cannot be moved along the surface to where it started, so that it looks like its own mirror image. From this point on, we will call these mathematical objects surfaces.

The **genus** of a surface,  $g \geq 0$ , is the number of 'holes' in it, or the number of

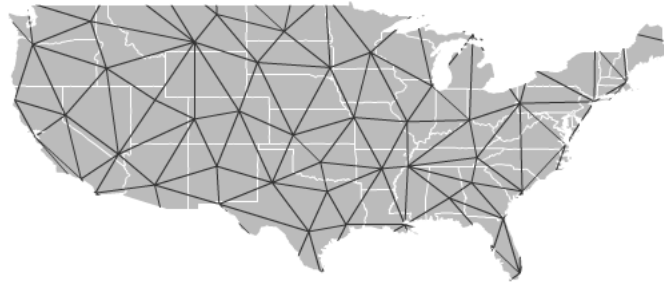


Figure 1: A triangulation of the contiguous United States

'handles'.

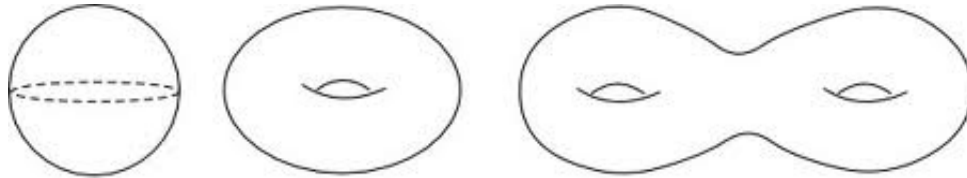


Figure 2: Surfaces of genus  $g = 0, 1, 2$ .

Mathematicians classify surfaces according to their genus. For a given genus  $g$ , there is only one surface, up to homeomorphism. This means that any surface with genus 0 looks like a sphere, and any surface with genus 1 looks like a torus, so on and so forth.

### 3.2 Maps

A **map** is a graph *drawn* on a surface. The differences between a graph and a map are very subtle. The reason why these differences are subtle, is because whenever we deal with graphs, we draw them ourselves. Simply drawing a graph is enough to endow it with some mathematical structure. Here is what we mean:

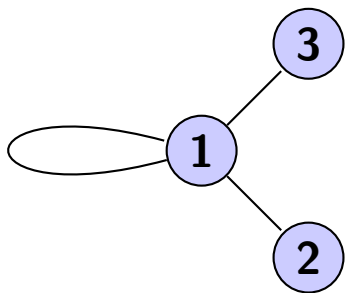
A map is a graph **embedded** into a surface in such a way that the edges do not intersect, and 'cutting' the surface along the edges of the graph will give us a disjoint

union of sets which are homeomorphic to an open disk. These disjoint sets are called **faces** of the map. This ‘cutting’ condition implies connectedness of the graph. This simply means that there is a path from every vertex to another along edges. Note that a graph itself does not have faces, but only edges and vertices. The **degree** of a vertex is the number of edges attached to it, and the degree of a face is the number of boundary edges. Edges adjacent to a face ‘from both sides’ are counted twice.

The Euler characteristic of a map is given by  $\chi = V - E + F = 2 - 2g$ , where  $g$  is the genus of the surface,  $V$  is the number of vertices,  $E$  is the number of edges, and  $F$  is the number of faces. Something that is interesting to note here, is that the Euler characteristic of a map depends only on the genus of the surface. For example, the Euler characteristic of *any* map drawn on a sphere is precisely 2.

Note that sometimes it is necessary to consider non-connected maps. In such a scenario, the map is drawn on several surfaces, particularly, one for each component of the graph. When this happens, the Euler characteristic is additive over the surfaces, meaning that the Euler characteristic for the map is the sum of the Euler characteristics of each component. This happens because the number of edges, vertices, and faces are additive, even though the genus of the surface is not.

Let’s see some examples. In the images below, we see the same graph, but two different maps. We will consider these drawn on the surface of a sphere.



The upper map in Figure 3 has two faces, one of degree 5 (the outer face), and one of degree 1. The outer face has degree 5 because the edges  $(1) - (3)$  and  $(1) - (2)$



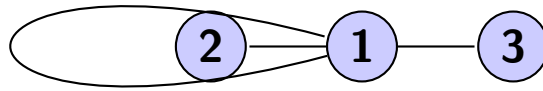


Figure 3: One graph, but two maps

are counted twice. The lower map in the same figure has two faces of degree 3, where in either case, the edges  $(1) - (2)$  and  $(1) - (3)$  are the counted twice.

### 3.3 A Combinatorial Description of Maps

Now let us examine some other properties of graphs. For instance, what additional properties does a graph need to be a map? If we consider a small neighborhood of a vertex in a graph, it will consists of the ‘ends‘ of the edges, or half-edges; if an edge is a loop, then the neighborhood of the vertex will contain both half-edges of the loop. In a graph, the ‘set of half-edges‘ is a set, without any structure though. To get a better description of maps, we will introduce a **cyclic order** on this set.

If a graph is already drawn on a surface, the cyclic order of the half-edges is determined by the orientation of the surface. The key fact of this information is sufficient information to reconstruct the map. This leads us to a proposition: The cyclic order of the half-edges of a graph (chosen randomly), uniequely determines the embedding of the graph on the surface. We will take this proof for granted, and instead, give a geometric construction which will explain how we can reconstruct a map from the cyclic ordrders.

We begin by consider the edge of a graph to be a ‘two-way‘ street. For example, if there are two vertices with an edge between them, we will replace it by two directed edges, as shown in Figure 4.

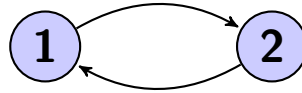


Figure 4: A two-way street

Now we will represent each vertex as a crossroad made up like a ‘roundabout’, as seen in Figure 5. This geometric notion of a roundabout corresponds quite nicely to that of the formal notion of the cyclic order of a vertex. Let us move through the city following only one rule: *At each crossroad, we have to turn right.*

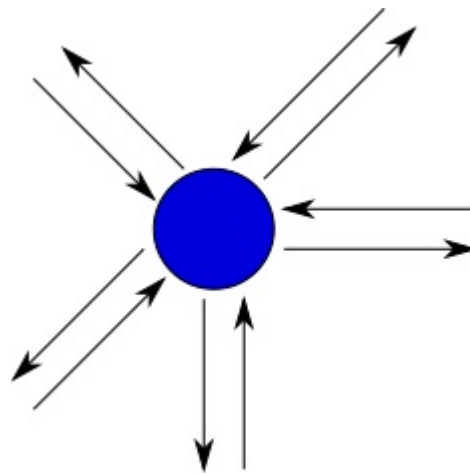


Figure 5: A Roundabout Crossroad

Then all of the possible routes from a set of disjoint cycles.

By going right, we see that this cycle forms a face (Figure 6). Note here that the streets at each crossroad are oriented counter-clockwise, but the boundary of the faces are oriented clockwise. This construction is commonly called a **ribbon graph**.

Now let’s consider a dual approach to this construction, going in reverse. We can start with a given number of faces, whose boundaries are oriented clockwise, provided that the total number of edges is even. From here, we will take the edges pair-wise, and glue them together in an arbitrary (but connected) manner following this rule: If two edges are to be glued, they must face in opposite directions, as shown in Figure 7.

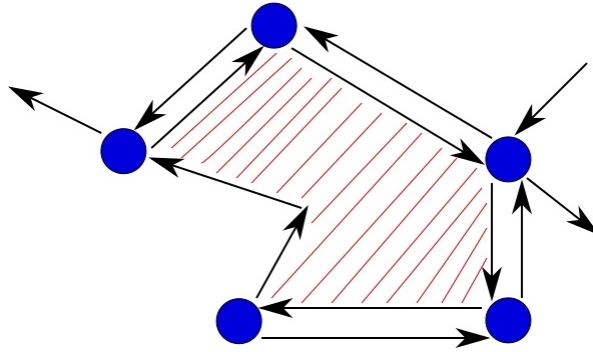


Figure 6: A face

Then the cyclic order of a vertex is given by the rule: the next edge is the one that we may access by the interior of the face.

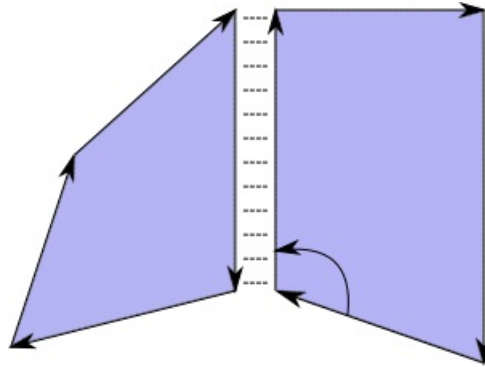


Figure 7: Gluing Faces

The number of cyclic orders of edges of a graph at a given vertex  $v$  and of degree  $d$  is  $(d_v - 1)!$ . So the total number of orderings of any graph is  $\prod_v (d_v - 1)!$ , where the product is taken over all vertices of the graph.

### 3.4 Building Surfaces from Polygons

We all know from elementary geometry the construction of a torus where we start with a rectangle and identify its opposite sides. This construction is actually

more general and is a standard way to build arbitrary topological surfaces. Take a  $n$ -gon,  $n$  being even, and identify its sides two by two. If the sides are identified so that their orientation is opposite (their orientation coming from rotating cyclically around the polygon), the resulting surface will be oriented.

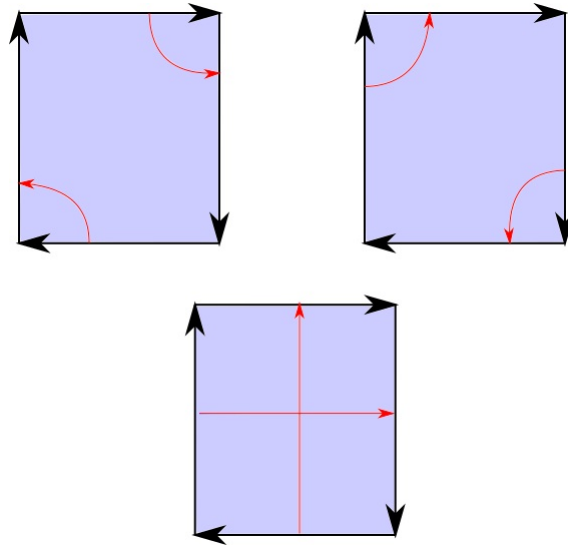


Figure 8: 3 Gluings of Square

In Figure 8, the edges to be adjoined are marked by a line connecting them. The first and second ways of gluing the square give a sphere, the third way a torus. Similarly, there exists a total of 15 glueings of the hexagon. Five of the glueings give a sphere, while the other 10 yield a torus.

The identification of the sides gives actually more than an oriented surface: the identified sides of the polygon embed as map on resulting surface. The topology of the resulting surface and the embedded graph are actually closely related. Indeed, the topology of a compact surface is essentially determined by its Euler characteristic, which in turn can be computed from the associated map.

While it is generally difficult to predict *a priori* the genus of a surface after gluing, we have the following result:

**Proposition 3.4.1.** *Consider a surface obtained by gluing the sides of a polygon.*

*The following statements are equivalent:*

- (i) The resulting a surface is a sphere;*
- (ii) The graph formed by the identified sides is a tree;*
- (iii) Chords connecting pairs of identified sides do not intersect.*

*Proof.* Before we begin, recall that the Euler characteristic

$$\chi = 2 - 2g$$

can be computed as  $V - E + F$ , where  $V$ ,  $E$  and  $F$  are respectively the number of vertices, edges and faces of a (suitable)<sup>1</sup> embedded graph.

Note that in our case,  $F$  is always one, since we deal strictly with one face maps by construction.

- $(i) \implies (ii)$

This is almost immediate by definition. If the resultant of gluing the polygon is a sphere, then the genus is precisely 0. This means that  $1 = V - E$ . Since the edges and vertices differ by one, and the graph must be simply connected, then the graph must be a tree.

- $(ii) \implies (iii)$

By contradiction. Assume two chords intersect. By stretching them in the polygon, we obtain two segments which intersect exactly once. Now, on a sphere two circles cannot intersect with multiplicity 1, a contradiction.

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<sup>1</sup>That is, any two edges intersect at a vertex, there is a vertex at the end of every edge and the complement of the graph is homeomorphic to a disjoint union of discs.

- (iii)  $\implies$  (i)

By induction on the number of sides of the polygon. The case of the square can be handled by hand. For an  $n$ -gon, if no two chords intersect, there must be two adjoining sides which are identified. We can identify directly by pushing them 'inside' the  $n$ -gon and obtain a  $n - 2$  gon.

□

Given the above proposition, we can actually do some counting:

**Theorem 3.4.1.** *Let  $n = 2k$ , there are  $\frac{1}{k+1} \binom{n}{k}$  ways of gluing opposite sides so to get a sphere.*

*Proof.* Since we get a sphere when identified edges are joined with non-intersecting arcs, what we are actually counting are **Dyck** words, i.e. words in  $x$  and  $y$  (e.g.  $xyxy$ ) where no initial segment has more  $x$ 's than  $y$ 's. (the  $x$ 's represent here outgoing arcs and the  $y$ 's incoming arcs) It is well known, see [10] e.g., that the number of Dyck words is the quantity above. □

**Remark 3.4.1.** *The number  $C_k = \frac{1}{k+1} \binom{2k}{k}$  is the  $k$ -th **Catalan** number<sup>2</sup>. See [10] for a nice introduction to the ubiquitous of Catalan numbers.*

If we call  $\epsilon_g(k)$  the number of ways to identify the sides of a polygon with  $2k$  sides to obtain a surface of genus  $g$ , the above result can be rephrased as

$$\epsilon_0(k) = C_k.$$

It is in general much more difficult to compute  $\epsilon_g(k)$  using classical counting methods. Nevertheless, we know that for a fixed  $k$  we can estimate the total sum of the  $\epsilon_g$ 's.

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<sup>2</sup>Catalan numbers are named after Belgian mathematician Eugène Catalan who studied them while investigating the *Tower of Hanoi* game. These numbers appear earlier in the work of Euler.

**Lemma 3.4.1.**  $\sum_{g=0}^{\lfloor k/2 \rfloor} \epsilon_g(k) = (2k - 1)!!.$

*Proof.* From the above discussion it is sufficient to count the number of ways sides of  $2k$ -gon can be identified pairwise.

The first side can be identified with any other of the  $2k - 1$  remaining sides. Moving cyclically around the polygon, next free side can be identified with with any of the  $2k - 3$  remaining sides. Et caetera.  $\square$

To determine the numbers  $\epsilon_g(k)$ , we will need to resort to our Quantum Field integrals.

### 3.5 A Gaussian Measure on the Space of Hermitian Matrices

A matrix  $H = (h_{ij})$  is called **Hermitian** when it satisfies  $H^\dagger = H - \dagger$  denoting the conjugate transpose. We denote the space of  $N \times N$  Hermitian matrices by  $\mathcal{H}_N$ . It is an  $N^2$ -dimensional real vector space whose simplest parametrization is

$$H \mapsto (x_{11} = h_{11}, \dots, x_{NN} = h_{NN}; x_{i<j} = \mathcal{R}(h_{ij}); y_{i<j} = \mathcal{I}(h_{ij}))$$

Here, the  $\mathcal{R}$  denotes the real part of  $a + bi$ , and  $\mathcal{I}$  denotes the imaginary part.

The ordinary volume form (or measure) on  $\mathcal{H}_N = \mathbb{R}^{N^2}$  is

$$dv(H) = \prod_{i=1}^N dx_{ii} \prod_{i<j} dx_{ij} dy_{ij}$$

While  $\mathcal{H}_N \cong \mathbb{R}^{N^2}$  is not different from any other vector space of the same dimension, we would like the measure to reflect the matrix properties of its elements.

The space of Hermitian matrices  $\mathcal{H}_N$  comes with a well behaved quadratic form:

$$tr(HH^\dagger).$$

In terms of the above coordinates,  $x_{ij}$ , this quadratic form can be represented by the matrix  $B = \text{diag}(1, \dots, 1; 2, \dots, 2; 2, \dots, 2)$ .

*Proof.* We write down the case the two dimensional case, the other cases being similar.

$$\begin{aligned} \text{tr}(HH^\dagger) &= \text{tr} \left( \begin{pmatrix} x_{11} & x_{12} + iy_{12} \\ x_{12} - iy_{12} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} - iy_{12} \\ x_{12} + iy_{12} & x_{22} \end{pmatrix} \right) \\ &= x_{11}^2 + x_{22}^2 + 2x_{12}^2 + 2y_{12}^2 \quad (3.1) \end{aligned}$$

□

From Lemma 2.2.1 in Chapter 2, this means that

$$\int_{\mathcal{H}_N} e^{\text{tr}(HH^\dagger)} dv(H) = (2\pi)^{-\frac{N^2}{2}} \det(B)^{\frac{1}{2}} = (2\pi)^{-\frac{N^2}{2}} 2^{\frac{N^2-N}{2}}$$

i.e., we need to multiply our measure 3.5 by a factor of  $(2\pi)^{-\frac{N^2}{2}} 2^{\frac{N^2-N}{2}}$  to normalize it. We will call  $d\mu(H)$  the normalized Gaussian measure :

$$d\mu(H) = (2\pi)^{-\frac{N^2}{2}} 2^{\frac{N^2-N}{2}} e^{\text{tr}(HH^\dagger)} dv(H)$$

Let us compute a few 2-point functions with this measure. Recall that  $h_{ij}$  denote the entry of a generic Hermitian matrix  $H$  and that it is a linear map in the  $x_{ij}$  coordinates.

**Lemma 3.5.1.** *The two point function in  $h$  takes the following values<sup>3</sup>*

$$\langle h_{ij} h_{kl} \rangle = \delta_{il} \delta_{jk}$$

*Proof.* The proof is a straight application of the Gaussian integrals seen in the previous chapter. □

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<sup>3</sup>The symbol  $\delta_{ij}$  is called the Kronecker symbol and it is equal to 1 when  $i = j$  and 0 otherwise.



### 3.6 Geometric interpretation of a Gaussian integral

Consider a  $2k$  gone with vertices labelled cyclically as  $i_1, \dots, i_{2k}$ . An edge of the polygon is defined by a pair of consecutive indices such as  $i_5i_6$ . Let us associate to each edge a linear map (or matrix entry) :  $i_5i_6 \leftrightarrow h_{i_5i_6}$

A pairwise identification of the sides of the polygon can be encoded unequivocally by a 2 point function. For example,  $\langle h_{i_5i_6} h_{i_1i_2} \rangle$  represents the identification of edges  $i_5i_6$  and  $i_1i_2$ .

**Example 3.6.1.** Here are two examples of surfaces encoded by products of 2 point functions.

1. Consider the octagon as in Figure 9. The gluing is encoded by the product

$$\langle h_{i_1i_2} h_{i_4i_5} \rangle \langle h_{i_2i_3} h_{i_5i_6} \rangle \langle h_{i_3i_4} h_{i_8i_1} \rangle \langle h_{i_6i_7} h_{i_7i_8} \rangle$$

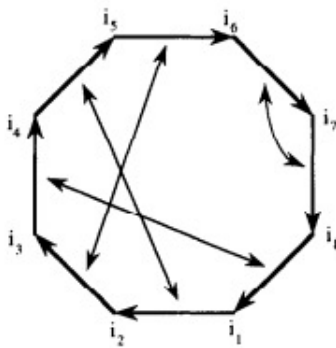


Figure 9: Identifications of the side of an octagon

2. The gluing of a square into a torus is represented by the product  $\langle h_{i_1i_2} h_{i_3i_4} \rangle \langle h_{i_2i_3} h_{i_4i_1} \rangle$ .

**Proposition 3.6.1.** The  $2k$  point function

$$\langle \text{tr}(H^{2k}) \rangle = \int_{\mathcal{H}_N} \text{tr}(H^{2k}) d\mu \quad (3.2)$$

can be expanded topologically as

$$\langle \text{tr}(H^{2k}) \rangle = N^{k+1} \sum_{g=0}^{\lfloor k/2 \rfloor} \epsilon_g(k) \left( \frac{1}{N^2} \right)^g. \quad (3.3)$$

*Proof.* The matrix  $\text{tr}(H^{2k})$  is polynomial of degree  $2k$  whose terms are of the form  $h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} \dots h_{i_{2k} i_1}$ . Hence, by Wick's Lemma, the  $2k$  point function can be expanded as a sum of  $(2k - 1)!!$  Wick couplings (products of 2-point functions). By the above discussion, the non-zero couplings correspond to genus  $g$  surfaces. Using Euler's formula for trees, i.e. one faced maps, we see that  $2g = k + 1 - V$ . Moreover, a Wick coupling with  $V$  vertices has  $N^V$  different combinations corresponding to equivalent gluing. This means that the contributions of each such polygon gluing is  $N^{k+1-2g}$  : it is graded by genus.  $\square$

**Example 3.6.2.** *As an appetizer, let us consider two simple applications*

1. *Case  $k = 1$ . We start with the trivial example of a 2-gon. Its two sides can only be identified in a unique way to generate a sphere. Expanding Equation 3.3 for  $k = 1$  we get,*

$$\langle \text{tr}(H^2) \rangle = N^2 \epsilon_0(1)$$

*Let  $N = 1$ , i.e. let us work on the set of one dimensional hermitian matrices, that is the set of real numbers. We have seen in Chapter 2 that  $\langle (x^2) \rangle = 1$  and thus that  $\epsilon_0(1) = 1$  as we expected.*

2. *Case  $N = 1$ . Let  $k$  be arbitrary and  $N = 1$ . We will recover Lemma 3.4.1. Indeed, Equation 3.3 for  $N = 1$  becomes*

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \epsilon_g(k) = \langle x^{2k} \rangle = (2k - 1)!!$$

*where the second equality was derived in Chapter 2.*

**Example 3.6.3.**

Let us now study the case  $k = 2$ . This is the example we discussed extensively in the beginning where we identify the sides of a square. The surface we obtain after gluing is either a sphere or a torus.

For  $k = 2$ , Equation 3.3 becomes

$$\langle \text{tr}(H^4) \rangle = N^3 \left( \epsilon_0(2) + \epsilon_1(2) \frac{1}{N^2} \right)$$

Again, let us focus for a moment on  $N = 1$ . From Chapter 2 we know that  $\langle x^4 \rangle = 3$  and we see that  $\epsilon_0(2) + \epsilon_1(2) = 3$ . This is exactly the observation made in Figure 8.

Note that this equality alone does not determine  $\epsilon_0(2)$  and  $\epsilon_1(2)$ . We need a second relation which we will obtain from considering the case of  $2 \times 2$  Hermitian matrices. Equation 3.2 becomes

$$(2\pi)^{-\frac{N^2}{2}} 2^{\frac{N^2-N}{2}} \int_{\mathbb{R}^{N^2}} \text{tr}(H^{2k}) e^{-\frac{1}{2} \text{tr}(H^2)} dx^{N^2} \quad (3.4)$$

Consider a Hermitian matrix  $H \in \mathcal{H}_2$ , it can be written as  $H = \begin{pmatrix} a & b + ci \\ b - ci & d \end{pmatrix}$  where  $a, \dots, d$  are real. The traces of its square and fourth power are respectively:

$$\text{tr}(H^2) = a^2 + 2(b^2 + c^2) + d^2$$

and

$$\text{tr}(H^4) = a^4 + d^4 + 4(a^2 + d^2)(b^2 + c^2) + 4ad(b^2 + c^2) + 2(b^2 + c^2)^2.$$

Now, using the above parametrization, Equation 3.4 becomes

$$\frac{1}{2\pi^2} \iiint_{\mathbb{R}^4} (a^4 + d^4 + 4(a^2 + ad + d^2)(b^2 + c^2) + 2(b^2 + c^2)^2) e^{-\frac{1}{2}(a^2+2(b^2+c^2)+d^2)} da db dc dd \quad (3.5)$$

To compute this integral we pass to polar coordinates in the  $(a, d)$  plane as well as in the  $(b, c)$  plane:

$$\begin{cases} a = s \cos(\phi) \\ b = r \cos(\theta) \\ c = r \sin(\theta) \\ d = s \sin(\phi) \end{cases} .$$

Our integral now becomes

$$\frac{1}{2\pi^2} \int_{s=0}^{\infty} \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} (s^4 - 2s^4 \cos^2(\phi) \sin^2(\phi) + 4s^2 r^2 + 4s^2 r^2 \sin(\phi) \cos(\phi) + 2r^4) e^{-\frac{1}{2}(s^2+2r^2)} r s d\phi d\theta dr ds \quad (3.6)$$

The integral with respect to  $\theta$  is trivial and, with respect to  $\phi$  it boils down to two elementary integrals; we obtain thus:

$$2 \int_{s=0}^{\infty} \int_{r=0}^{\infty} \left( \frac{3}{4} s^4 + 4s^2 r^2 + 2r^4 \right) e^{-\frac{1}{2}(s^2+2r^2)} r s dr ds \quad (3.7)$$

We can now leisurely split the integral and solve three elementary Gaussian integrals to get as final result

18.

In our situation, the right hand side of Equation 3.3 is  $8\epsilon_0(2) + 2\epsilon_1(2)$  and we get thus the system

$$\begin{cases} \epsilon_0(2) + \epsilon_1(2) = 3 \\ 8\epsilon_0(2) + 2\epsilon_1(2) = 18 \end{cases}$$

Whose solution is obviously  $\epsilon_0(2) = 2$  and  $\epsilon_1(2) = 1$  as could be seen from *Figure 8*. Mille viae ducunt homines per saecula Romam.

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