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Automorphisms of Graph Curves on K3 Surfaces

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AUTOMORPHISMS OF GRAPH CURVES ON K3 SURFACES

by

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(Under the Direction of Jimmy Dillies)

ABSTRACT

We examine the automorphism group of configurations of rational curves on $K3$ surfaces. We use the properties of finite automorphisms of $\mathbb{P}^1$ to examine what restrictions a given elliptic fibration imposes on the possible finite order non-symplectic automorphisms of the $K3$ surface. We also examine the fixed loci of these automorphisms, and construct an explicit fibration to demonstrate the process.

Key Words: K3 surface, singular fiber, elliptic fibration

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LIST OF SYMBOLS

$\mathbb{R}$	Real Numbers
$\mathbb{C}$	Complex Numbers
$\mathbb{Z}$	Integers
$\mathbb{P}^n$	Complex Projective Space

CHAPTER 1

INTRODUCTION TO $K3$ SURFACES

Over recent decades, much work has been done to study the automorphism group of $K3$ surfaces by authors such as V.V. Nikulin, D. Zhang, M. Artebani, A. Sarti, and many more [1]. When a $K3$ surface X has Picard rank ≥ 5, it admits an elliptic fibration over $\mathbb{P}^1$. The singular fibers of such a fibration are also well studied, and were classified by Kodaira in the 1960's [9][10]. Many of the singular fibers are configurations of intersecting rational curves, which we will call graph curves. We use these results along with properties of $\mathrm{Aut}(\mathbb{P}^1)$ to examine what restrictions a given fibration imposes on the possible finite order symplectic and non-symplectic automorphisms of the $K3$ surface. We also examine the fixed loci of these automorphisms, and construct an explicit fibration to demonstrate the process.

1.1 Definitions

We begin by defining the objects that interest us, beginning with divisors. Throughout this section, assume X is a smooth compact complex variety.

Definition 1.1. A subset H of X is called a *hypersurface* if it is the zero set of a single polynomial $f : X \to \mathbb{C}$. We use the notation

$$H = V(f).$$

If f is irreducible over $\mathbb{C}$, then we say that H is an irreducible hypersurface of X.

Definition 1.2. A *divisor* of X is a finite formal sum

$$D := \sum_i a_i C_i$$

where each $a_i \in \mathbb{Z}$ and each C_i is an irreducible hypersurface of X. We call the union $\cup_i C_i$ the *support* of D. The set of all divisors of X form a free abelian group, denoted $\mathrm{Div}(X)$.

Definition 1.3. Let f be a rational function on X. We define the *principal divisor of* f to be the divisor

$$\operatorname{div}(f) := \sum_{C \subset X} \nu_C(f) C,$$

where the sum runs over all hypersurfaces C in X, and $\nu_C(f)$ is the multiplicity of the zero or pole of f on C. If f is zero on C, then $\nu_C(f) > 0$ and if f is infinite on C, then $\nu_C < 0$. We denote the set of principal divisors by $\operatorname{PDiv}(X)$.

The set $\operatorname{PDiv}(X)$ is a normal subgroup of $\operatorname{Div}(X)$ since

$$\operatorname{div}(f) - \operatorname{div}(g) = \operatorname{div}(f/g).$$

Hence, the quotient $\operatorname{Div}(X)/\operatorname{PDiv}(X)$ is well defined.

Definition 1.4. The quotient group $\operatorname{Pic}(X) := \operatorname{Div}(X)/\operatorname{PDiv}(X)$ is called the *Picard group* of X. Two divisors from the same class in $\operatorname{Pic}(X)$ are called *linearly equivalent.*

Example 1.5. Let $X = \mathbb{P}^n$. Then $\operatorname{Pic}(X) \simeq \mathbb{Z}$. To see this, first define the map $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ by $\sum_i a_i C_i \mapsto \sum_i a_i$. First let f be a polynomial on X. Since f has the same total multiplicity of roots as it does poles, then we have $\deg(\operatorname{div}(f)) = 0$. Now, for any rational polynomial f/g, we have

$$\deg(\operatorname{div}(f/g)) = \deg(\operatorname{div}(f)) - \deg(\operatorname{div}(g)) = 0,$$

showing that $\operatorname{PDiv}(X) \subset \ker(\deg)$. Now, suppose that $\alpha = \sum_{i=1}^m a_i C_i \in \ker(\deg)$. That is, $\sum a_i = 0$. Without loss of generality, assume that $a_i > 0$ for $i = 1, \ldots, k$ and $a_i < 0$ for $i = k+1, \ldots, m$. Let f_i be the irreducible polynomials such that $C_i = V(f_i)$ for $i = 1, \ldots, m$. We have

$$\alpha = \operatorname{div}\left(\frac{f_1^{a_1} \cdots f_k^{a_k}}{f_{k+1}^{a_{k+1}} \cdots f_m^{a_m}} \right),$$

which shows that $\ker(\deg) = \operatorname{PDiv}(\mathrm{X})$. Since deg is clearly surjective, then

$$\operatorname{Pic}(X) = \frac{\operatorname{Div}(X)}{\operatorname{PDiv}(X)} = \frac{\operatorname{Div}(X)}{\ker(\deg)} \cong \mathbb{Z}.$$

Hence, $\operatorname{Pic}(X)$ is generated by the class of a single hyperplane $[H]$. That is, $\operatorname{Pic}(X) = \mathbb{Z}[H]$. □

Remark 1.6. For a $K3$ surface X, the Picard group $\operatorname{Pic}(X)$ is isomorphic to $\operatorname{NS}(X)$, the so called Nerón-Severi group of X.

While a divisor D may not globally be a principal divisor, there exists an open covering $\{U_i\}$ of X and rational functions f_i such that $D|_{U_i} = \operatorname{div}(f_i|_{U_i})$ for all i. Let $C \subset X$ be a curve not contained in the support of D. We denote by $D|_C$, the restriction given locally by $\operatorname{div}(f_i|_{U_i \cap C})$.

Definition 1.7. Suppose X is an n dimension variety. Let ω be a meromorphic top form on X. Let $\{U_i\}$ be an open covering of X with local coordinates $\{z_{i,1}, \ldots, z_{i,n}\}$. Then locally we can write $\omega|_{U_i} = f_i \; dz_{i,1} \wedge \ldots \wedge dz_{i,n}$. The collection of principal divisors $\operatorname{div}(f_i)$ defines a divisor of X that we call the *canonical divisor* of X. We denote the canonical divisor of X by K_X.

Proposition 1.8. *The class of K_X in* $\operatorname{Pic}(X)$ *is independent of choice of meromorphic form.*

Proof. Let ω and ω' be meromorphic top forms on X, with f_i and f_i' being the maps associated to the cover $\{U_i\}$. Suppose K_X and K_X' are the divisors determined by ω and ω', respectively. On any overlap $U_i \cap U_j$, we have $f_i = J_{ij} f_j$ and $f_i' = J_{ij} f_j'$ where J_{ij} is the jacobian of the change of coordinates. This shows that f_i/f_i' is the same over all X. We have $K_X - K_X' = \operatorname{div}(f_i/f_i')$. □

Example 1.9. Let $X = \mathbb{P}^1$. For homogeneous coordinates $[z_1 : z_2]$, define an open covering $\{U_i\}$ where $U_i = \{[z_1 : z_2] : z_i \neq 0\}$ for $i = 1, 2$. We have the local coordinates $Z_1 = z_2/z_1$, $Z_2 = 1/Z_1$. Define the one form ω where on U_1, we have $\omega|_{U_1} = Z_2 \; dZ_1$. Here we have a pole at $[1 : 0]$. Note that $Z_2 \; dZ_1 = -Z_1 \; dZ_2$ on

$U_1 \cap U_2$, so $\omega|_{U_2} = -Z_1\, dZ_2$, which has a pole at $[0:1]$. Thus, $K_X = -([1:0]) - ([0:1]) = -2[pt] \in \mathrm{Pic}(X)$ where $[pt]$ is the class of a point in $\mathrm{Pic}(X)$. □

Definition 1.10. A $K3$ surface is a surface X such that

$$K_X \equiv 0 \quad \text{and} \quad \pi_1(X) = 0.$$

where $\pi_1(X)$ denotes the fundamental group of X.

Remark 1.11. The condition $\pi_1(X) = 0$ excludes Abelian surfaces.

Example 1.12.

- A double cover of $\mathbb{P}^2$ branched along a smooth sextic is a $K3$ surface.
- A non-singular degree 4 surface in $\mathbb{P}^3$ is a $K3$ surface (see example 1.15).

The Hodge diamond of a $K3$ surface X is a diagram containing the dimensions of each of the spaces $h^{p,q}(X)$. These spaces are computed in [8] and the hodge diamond is shown in figure 1.1.

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Figure 1.1: The Hodge diamond of a $K3$ surface

Proposition 1.13. *(see [8]) Let C be a smooth curve on a surface X. Then*

$$K_C = (K_X + C)\,|_C\,.$$

The formula above is called the adjunction formula.

Example 1.14. Let $X = \mathbb{P}^2$ and $C = \mathbb{P}^1$. We can use adjunction to show that $K_X = -3[pt]$. Since C is linearly equivalent to a line L in $\mathbb{P}^2$, then by proposition 1.13 and example 1.9, we have

$$-2[pt] = (K_X + L)|_L = K_X|_L + [pt].$$

Thus, $K_X|_L = -3[pt]$, which implies $K_X = -3[L]$ where $[L]$ is the class of a line in $\text{Pic}(X)$. This process can be applied inductively to deduce that

$$K_{\mathbb{P}^n} = -(n+1)[H]$$

where $[H]$ is the class of a codimension 1 hypersurface in $\text{Pic}(\mathbb{P}^n)$. $\square$

Example 1.15. Let $X = \mathbb{P}^3$ and C be a non-singular degree 4 surface in X. We can use adjunction to show that $K_C \equiv 0$. By example 1.14, $K_X = -4[H]$. Also, in $\text{Pic}(X)$, we have $[C] = 4[H]$. Hence,

$$K_C = (K_X + C)|_C \equiv (-4[H] + 4[H])|_C = 0.$$

1.2 The Picard Lattice

Definition 1.16. Let D be a divisor of X and let $C \subset X$ be an irreducible curve not contained in the support of D. Define the intersection $D \cdot C$ to be the integer $D \cdot C := \deg(D|_C)$, where deg is the map described in example 1.5.

Lemma 1.17. *If D and D' are linearly equivalent divisors of X, and $C \subset X$ is a curve not contained in the support of D or D', then $D \cdot C = D' \cdot C$.*

Proof. Suppose $D = D' + \operatorname{div}(f)$. Then

$$\begin{aligned} D \cdot C &= (D' + \operatorname{div}(f)) \cdot C \\ &= \deg((D' + \operatorname{div}(f))|_C) \\ &= \deg(D'|_C) + \deg(\operatorname{div}(f)|_C) \\ &= \deg(D'|_C) + \deg(\operatorname{div}(f|_C) \\ &= \deg(D'|_C). \end{aligned}$$

□

Corollary 1.18. *The intersection $D \cdot C$ is well defined even if the curve C is contained in the support of D.*

Proof. Suppose C is contained in the support of D. Write $D := \sum_i a_i C_i$ as in definition 1.2. Since C is irreducible, then $C = C_j$ for some j and $a_j \neq 0$. Let f be a rational function such that $\nu_C(f) = a_j$. Then D is linearly equivalent to $D - \operatorname{div}(f)$ and C is not contained in the support of $D - \operatorname{div}(f)$. Define

$$D \cdot C := (D - \operatorname{div}(f)) \cdot C.$$

□

For divisors D and $\sum_i a_i C_i$ we can express their intersection number as

$$D \cdot \sum_i a_i C_i := \sum_i a_i (D \cdot C_i).$$

Lemma 1.19. *For divisors D and D' on X, we have*

i. $D \cdot (aD') = a(D \cdot D')$ *for all* $a \in \mathbb{Z}$,

ii. $D \cdot D' = D' \cdot D$.

Proposition 1.20. *The intersection number defines an integer valued symmetric bilinear form on* $\operatorname{Pic}(X)$.

Proof. This follows from lemma 1.17 and lemma 1.19. □

Definition 1.21. Let G be a finitely generated free abelian group, and let $q : G \times G \to \mathbb{Z}$ be a quadratic form. The pair (G, q) is called a *lattice*. The lattice (G, q) is called *even* if $q(x) \in 2\mathbb{Z}$ for all $x \in G$. Let V be the real vector space $G \otimes_{\mathbb{Z}} \mathbb{R}$ and let Q be the extension of q to V. The *rank* of (G, q) is defined to be $\dim V$ and the *signature* of (G, q) is the pair of positive integers (b_+, b_-) where b_+ is the number of positive eigenvalues of Q and b_- is the number of negative eigenvalues of Q. The Gram matrix of a lattice is a matrix representation of the form Q. If the determinant of the Gram matrix of G is ± 1, then we say that (G, q) is *unimodular*.

Example 1.22. The following are some well known lattices.

- The U lattice has rank 2, is unimodular, and has signature $(1, 1)$. Its Gram matrix is
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
- The E_8 lattice has rank 8, is unimodular, even, and has signature $(0, 8)$. Its Gram matrix is
$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Corollary 1.23. *The group* Pic(X) *together with the intersection number defines a lattice called the* Picard Lattice *of X.*

1.3 The K3 Lattice

Recall that an isometry of lattices is an isomorphism ϕ of groups such that $\phi(x)\cdot\phi(y) = x\cdot y$. It is well known that the $K3$ lattice, Λ_{K3} is isometric to $E_8^2 \oplus U^3$. For an in depth justification of this fact, see [8]. The argument follows from the fact that the lattice $H^2(X, \mathbb{C})$ with intersection is even, unimodular, and has signature $(b_+, b_-) = (3, 19)$. Then refer to the theorem by Milnor [11].

Theorem 1.24. *Let Λ be an indefinite unimodular lattice. If Λ is even, then $\Lambda \simeq E_8(\pm 1)^m \oplus U^n$ for some integers m and n.*

Thus, the Picard lattice of a $K3$ surface is a sub-lattice of $E_8^2 \oplus U^3$.

CHAPTER 2

ACTION ON ELLIPTICALLY FIBERED K3 SURFACES

2.1 Preliminaries

Now we look at how configurations of rational curves arise from $K3$ surfaces[1]. Let X be a $K3$ surface with an elliptic fibration $\pi : X \to \mathbb{P}^1$. By this we mean that the generic fiber $\pi^{-1}(t)$ is a smooth elliptic curve. If the rank of $\mathrm{Pic}(X) \geq 5$, then such a fibration exists [7]. Recall that a section s of the fibration is a map $s : \mathbb{P}^1 \to X$, such that $\pi \circ s = \mathrm{id}$. We will often refer to the image a section s by simply s. If the fibration admits a section, then we can express the fiber at t using the Weierstrass equation $y^2 = x^3 + f(t)x + g(t)$. Recall that this expression defines a non-singular elliptic curve if and only if the discriminant $\Delta = 4f^3 + 27g^2$ is non-zero. As one might expect, the most interesting fibers are those at which $\Delta = 0$, the singular fibers. Depending on the vanishing orders of Δ, singular fibers have 11 different configurations. These make up the famous Kodaira's list of singular fibers (see appendix A.1). Many of these curves are indeed configurations of rational curves. So, it will be useful to remind ourselves of some properties of $\mathrm{Aut}(\mathbb{P}^1)$.

For an automorphism ϕ of $\mathbb{P}^1$, there is a transformation $A \in GL_2(\mathbb{C})$ such that the following diagram commutes for the coordinate maps X_1, X_2.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{A} & \mathbb{C}^2 \\ {\scriptstyle X_i}\downarrow & & \downarrow{\scriptstyle X_i} \\ \mathbb{P}^1 & \xrightarrow[\phi]{} & \mathbb{P}^1 \end{array}$$

Up to a change of coordinates, either

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

[1] Not all rational curves on $K3$ surfaces come from elliptic fibrations, but we will focus on the rational curves that do for this work.

with non-zero eigenvalues. If we consider only finite automorphisms of $\mathbb{P}^1$, say $\phi^n = 1$, then the only possibility is

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1^n = \lambda_2^n$, or equivalently, λ_1/λ_2 is an nth root of unity. Notice that, with these coordinates, we immediately have two fixed points. Namely, ϕ fixes 0 and ∞. Now, suppose that there is another fixed point $[x : y]$ not equal to 0 or ∞. That is, suppose we have non-zero $x, y \in \mathbb{C}$ such that $[\lambda_1 x : \lambda_2 y] = [x : y]$. This implies $\lambda_1 = \lambda_2$, which means $\phi = \mathrm{id}$. We have proved the following useful fact.

Proposition 2.1. *Let $\phi \in \mathrm{Aut}(\mathbb{P}^1)$ be an automorphism of finite order n. Then ϕ either has exactly two fixed points, or is the identity.*

We can also examine how ϕ acts near fixed points. Near zero, we have

$$X = \frac{x}{y} \mapsto \frac{\lambda_1 x}{\lambda_2 y} = \frac{\lambda_1}{\lambda_2} X,$$

and near ∞ we have

$$Y = \frac{y}{x} \mapsto \frac{\lambda_2 y}{\lambda_1 x} = \frac{\lambda_2}{\lambda_1} Y = \left(\frac{\lambda_1}{\lambda_2}\right)^{-1} Y.$$

As stated above, λ_1/λ_2 is an nth root of unity. So, we could say that near zero, $X \mapsto \zeta_n^k X$, and near infinity, $Y \mapsto \zeta_n^{-k} Y$ for some $k \in \mathbb{Z}$.

Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism of order $n < \infty$. Since $K_X = 0$, then there exists a nowhere vanishing holomorphic volume form ω, and since $h^{2,0} = 1$, then $\sigma^* \omega = \lambda \omega$ for some $\lambda \in \mathbb{C}$ satisfying $\lambda^n = 1$.

Definition 2.2. If $\lambda = 1$, then $\sigma^* \omega = \omega$ and we say that σ is a *symplectic* automorphism. Otherwise, $\lambda = \zeta_n^k \neq 1$ and we say that σ is *non-symplectic*. If, in addition, $\gcd(n, k) = 1$, then σ is called *purely non-symplectic*.

Let x be an intersection point of two stable curves under σ. Locally, we have the action $\sigma_* \in \mathrm{Aut}(T_x X)$. If σ is non-symplectic, then $\det \sigma_* = \zeta_n$ and, up to a change of coordinates, σ_* is given by

$$\sigma_* = \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{1-k} \end{pmatrix} \tag{2.1}$$

for some integer k. If σ is symplectic, then $\det \sigma_* = 1$ and, up to a change of coordinates, σ_* is given by

$$\sigma_* = \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix} \tag{2.2}$$

for some integer k. Since the two curves in question are stable under σ then in both cases the eigendirections of σ_* are along the intersecting curves.

The curves we will be considering are rational curves. Hence, we now apply the properties of $\mathrm{Aut}(\mathbb{P}^1)$ above. First, by proposition 2.1, if an automorphism of $\mathbb{P}^1$ has three distinct fixed points, then it is the identity map. This allows us to say that if a stable curve intersects three other stable curves, then it is a fixed curve. Suppose a stable curve ℓ intersects two other stable curves. Then we have two fixed points on ℓ, which can be identified with 0 and ∞ by using a proper change of coordinates. By looking at the action σ_* near one fixed point 0 described in (2.1) and (2.2), we can assume that $X \mapsto \zeta_n^k X$ near 0. By the above discussion, we then know that $Y \mapsto \zeta_n^{-k}$ near the other intersection point (at infinity). Using this process along the curves and the determinant restriction at intersection points, we can determine the action of σ on all curves adjacent to ℓ.

We demonstrate this process in figure 2.1. Notice that for this to be consistent, we must have $1 - k \equiv -3 - k \pmod{n}$, which implies $4 \equiv 0 \pmod{n}$. We explore more restrictions such as this in section 2.2.

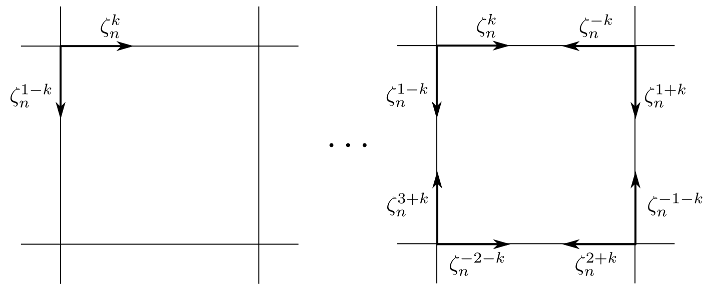

Figure 2.1: Computing a symplectic action on I_4.

2.2 Graph Curves

As mentioned above, many of the singular fibers in Kodaira's list are configurations of rational curves. In particular, the $I_v, I_v^*, II^*, III^*, IV$, and IV^* fibers are called graph curves.

Definition 2.3. A *graph curve* Γ is a connected collection of rational curves.

Note that this definition is a relaxed version of what may be used in other work. These types of fibers offer some restrictions on the order of automorphisms that we may have.

As mentioned in [5], the rational curves in a $K3$ surface are "rigid" in X, meaning that we can characterize automorphisms of X by the action on the rational curves in X.

Singular fibers are also divisors of X, so let us focus on the subgroup $H(n)$ of $\mathrm{Aut}(X)$ given by

$$H(n) = \{\sigma \in \mathrm{Aut}(X) : \sigma^*|_{\mathrm{Pic}(X)} = \mathrm{id}, \sigma^n = \mathrm{id}, \sigma \text{ is non-symplectic}\}$$

for $n \in \mathbb{Z}$. Let Γ be a graph curve that is a singular fiber of X. Let $\sigma \in H(n)$. Then σ induces an action $\sigma|_\Gamma$ on Γ and we know several facts about this action. We can apply this knowledge to induce an action on the Gram graph of Γ. The Gram graph of Γ is the simple graph with vertices representing rational curves of Γ, such that two vertices are connected by an edge if and only if the curves that they represent intersect. The induced action on the Gram graph of Γ will give us a simple way to compute invariants of a fibration.

2.3 Non-Symplectic Action on Gram Graphs

We begin by constructing a convenient way to represent the action of $\sigma \in H(n)$ on Γ using the Gram graph of Γ. As seen in figure 2.1, locally σ acts by multiplication of an nth root of unity. To simplify notation, we use the isomorphism $\{\zeta_n^i\} \to \mathbb{Z}_n$ given by $\zeta_n^i \mapsto i$.

We will now construct an action of $\mathbb{Z}_n$ on a Gram graph G which has at least one vertex of degree > 2. We will see that this condition does not cause any problems. Since each edge of G represents a fixed point of σ that has two eigenvalues, we need a way to temporarily decide which value to place on the edge. The choice will not matter in the end since, as discussed in section 2.1, we can determine the action on all of G based on how σ acts at one fixed point. We only need to find some restrictions that keep our model consistent with the observations in section 2.1.

Let G' be the graph G with an arbitrary direction and a weight of 0 (mod n) given to each edge.

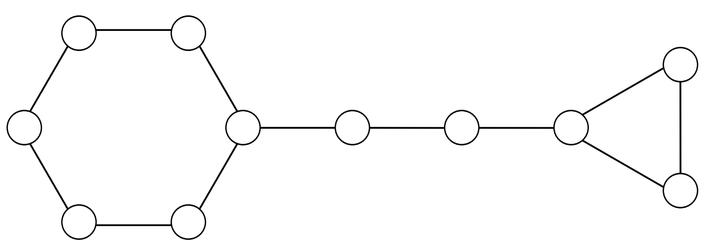

Figure 2.2: Example of a Gram graph G

Recall that vertices of our graphs represent rational curves, and edges represent intersection points. To see what these directions and weights represent, see figure 2.4,

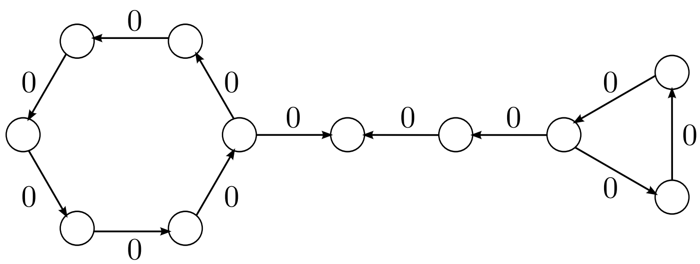

Figure 2.3: Example of G'

which on the left is two different representations of the same action on the right.

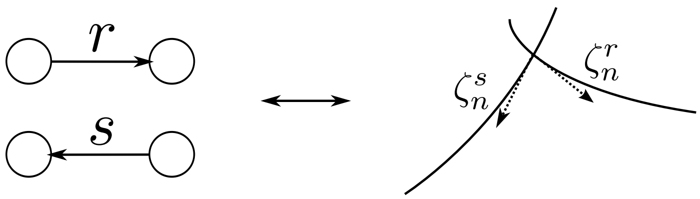

Figure 2.4: Translation between directed graph and graph curve

Let $k \in \mathbb{Z}_n$. In order to define the action of k on G that represents $\sigma^k(\Gamma)$, we first define the action of k on G'. Starting at any vertex v with degree at least 3, place weights from $\mathbb{Z}_n$ as follows. Following a path starting at v, add $w(j)$ to the weight on the j-th edge in the path where

$$w(j) = \begin{cases} j \cdot k & \text{if the direction of the edge is the same as the path taken} \\ k - j \cdot k & \text{if not} \end{cases}$$

See figure 2.5 for an example of how these weights are added.

Of course, we want the action to be well defined in the sense that it should not matter which vertex we start with or which paths we take. In order to accomplish this, we see that the following conditions need to hold. Let $c_1, \ldots, c_r$ be the lengths of all of the cycles in G and let $p_1, \ldots, p_s$ be the lengths of all paths connecting two vertices of G which have degree > 2. We require

$$n | \gcd(kc_1, \ldots, kc_s, kp_1, \ldots, kp_s).$$

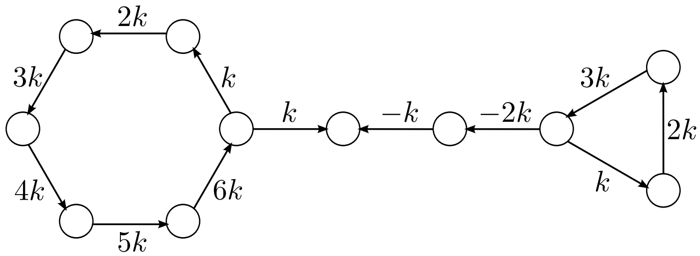

Figure 2.5: Action of k on G'

So, for the example in the figures above, we require that $n|\gcd(3k, 6k)$. Thus, if we assume for this example that n does not divide k, we can simplify the action as seen in figure 2.6. Following the convention used in [5], the grey vertices in figure 2.6 correspond to curves that would be fixed under the corresponding automorphism on the singular fiber. Edges between vertices not representing fixed curves represent isolated fixed points. Those edges are highlighted with a diamond in the middle of the edge.

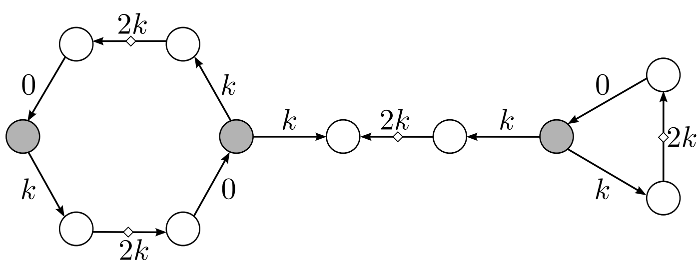

Figure 2.6: Reduced action of k on G'

Regardless of how we choose G', the positions of these highlighted vertices and edges will be the same. So we have a well defined action of k on G (as seen in figure 2.7) that allows us to quickly see how many fixed curves and isolated fixed points there are under a specified automorphism.

Something that this example did not illustrate is what happens with vertices of degree 1. These curves intersect only one other curve in Γ. If the curve ℓ represented by this vertex is not fixed, then by proposition 2.1, there is a second fixed point on

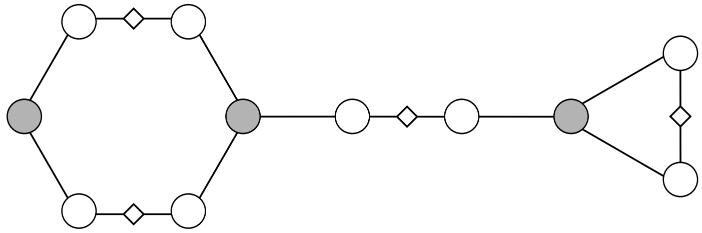

Figure 2.7: Action of k on G

ℓ. For the time being, we will count this as another isolated fixed point, as we do not know what type of curve intersects ℓ here. This is illustrated in the next section, when we look at the action of automorphisms on the I_v^*, II^*, III^* and IV^* fibers.

2.4 Examples: I_v, I_v^*, II^*, III^* and IV^* fibers

As an easy example the process outlined in section 2.3, we examine how automorphisms from $H(n)$ act on some simple singular fibers. Note that the Gram graph of I_v is the affine Dynkin diagram $\widetilde{A}_{v-1}$, which has no vertex of degree > 2. This does not cause a problem for our purposes, since our restrictions and number of fixed points and curves are invariant under rotation of $\widetilde{A}_{v-1}$. We simply choose a vertex to start on, without sacrificing generality. For this section, assume $k = 1$. This is not a dangerous assumption, since if $k > 1$ and $\gcd(n, k) > 1$, then for the purposes of counting, we can just make the transformation $n \mapsto n/\gcd(n, k)$ and get the information attained below. If $k > 1$ and $\gcd(n, k) = 1$, then ζ_n^k is a primitive nth root of unity, and will yield the exact same results as $k = 1$. The only two fibers that place restrictions on n by themselves are the I_v and I_v^* fibers, which both require n to divide v, as summarized in table 2.1. The fixed loci for non-symplectic automorphisms of various orders n are summarized in tables 2.2 and 2.3.

For the remainder of this section, assume $n > 1$. We now demonstrate how the results in table 2.3 were obtained for the fiber III^*. The corresponding Gram graph

Fiber Type	I_v	I_v^*
Gram Graph	$\widetilde{A}_{v-1}$	$\widetilde{D}_{v+4}$
Restriction on n	$v \equiv 0 \pmod{n}$	$v \equiv 0 \pmod{n}$

Table 2.1: Restrictions on n from I_v and I_v^* fibers

Fiber Type	I_v	I_v^*
Gram Graph	$\widetilde{A}_{v-1}$	$\widetilde{D}_{v+4}$
Fixed (lines, points)	$\left(\frac{v}{n}, v - \frac{2v}{n}\right)$	$\left(\frac{v}{n} + 1, 4 + v - \frac{2v}{n}\right)$

Table 2.2: Fixed rational curves and isolated points of I_v and I_v^* fibers

Fiber Type	IV^*		III^*		II^*	
Gram Graph	$\widetilde{E}_6$		$\widetilde{E}_7$		$\widetilde{E}_8$	
Fixed (lines, points)	$n = 2$	(4,0)	$n \leq 3$	(3,3)	$n = 2$	(4,2)
	$n > 2$	(1,6)	$n > 3$	(1,7)	$2 < n \leq 5$	(2,6)
					$n > 5$	(1,8)

Table 2.3: Fixed rational curves and isolated points of IV^*, III^*, and II^* fibers

to III^* is the affine Dynkin diagram $\widetilde{E}_7$ as shown in figure 2.8.

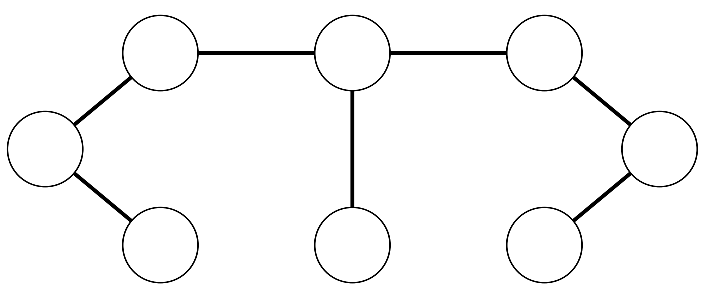

Figure 2.8: Gram graph of the fiber III^*

For the case $n = 2$, we have the action shown in figure 2.9. There are three fixed rational curves and three isolated fixed points. The edges that are shown in dashes represents the second fixed point of the curves with only one intersection point in the fiber as described at the end of section 2.3.

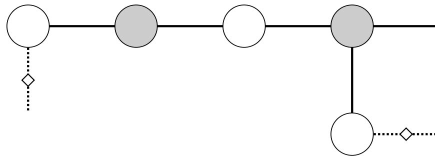

Figure 2.9: Action of $1 \in \mathbb{Z}_2$ on the graph $\widetilde{E}_7$

For the case $n = 3$, we have the action shown in figure 2.10. There are three fixed rational curves and three isolated fixed points.

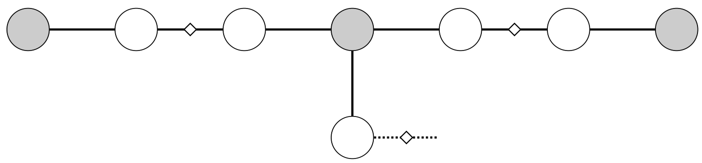

Figure 2.10: Action of $1 \in \mathbb{Z}_3$ on the graph $\widetilde{E}_7$

For the case $n > 3$, we have the action shown in figure 2.11. There is one fixed rational curve and seven isolated fixed points.

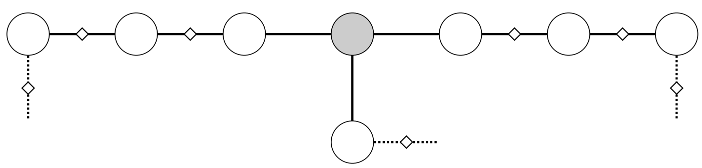

Figure 2.11: Action of $1 \in \mathbb{Z}_n$ $(n > 3)$ on the graph $\widetilde{E}_7$

One can carry out the same process for the remaining Dynkin diagrams to get the rest of the results in tables 2.2 and 2.3.

2.5 Flaws of the Model

Of course, this model is missing several important aspects of $\mathrm{Aut}(X)$. We still have to include the symmetries that do not stabilize all of the rational curves. For example, one might notice that $\mathbb{Z}_2$ could also act on the graph $\widetilde{E}_7$ shown in figure 2.12 by reflecting the graph about the middle two vertices. For this action we have one

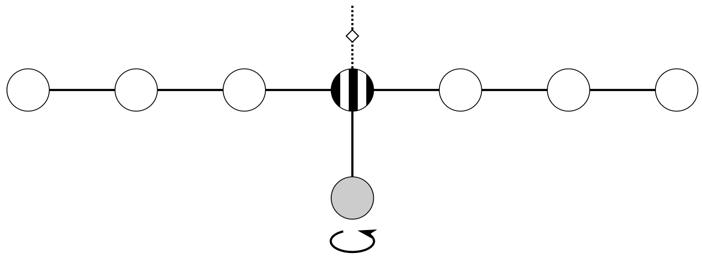

Figure 2.12: Alternate action of $1 \in \mathbb{Z}_2$ on the graph $\widetilde{E}_7$

stable curve (shown with stripes)[2], one isolated fixed point, and one fixed curve. Notice that can we now have vertices of degree > 2 that are not fixed curves. These automorphisms can be combined with our current method in the following way. Apply some symmetry $\tau \notin H(n)$, then use the procedure outlined in section 2.3 on the curves that are stable under τ to deduce what orders are permitted, and describe the fixed locus. We have also not considered the fact that if the fibration $\pi : X \to \mathbb{P}^1$ has a section s, then s is a rational curve and it intersects every fiber exactly once. We also have the possibility of multisections, which intersect every fiber with some multiplicity m. As a consequence, we should be able to decompose each of our graph curves into sections, multisections, and singular fibers with appropriate intersections. If there are more than one section, then the action of an automorphism σ of X must be consistent on all sections, since σ induces an automorphism of the base curve $\mathbb{P}^1$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}^1 & \xrightarrow[\phi]{} & \mathbb{P}^1 \end{array}$$

[2] I am still using the convention from [5]

If σ has finite order, then automorphism ϕ has two fixed points. In particular, this means exactly two fibers are stable under σ.

In the next chapter we work out an example to address these flaws.

CHAPTER 3

EXAMPLE OF FIBRATIONS AND AUTOMORPHISM

3.1 A Picard rank 16 example

S.M. Belcastro has worked out configurations of curves for for 95 types of $K3$ surfaces in [3]. The 62nd surface on the list is a $K3$ surface X with $\mathrm{rank}(\mathrm{Pic}(X)) = 16$ and the configuration of curves shown in figure 3.1, where two different possible fibrations are highlighted.

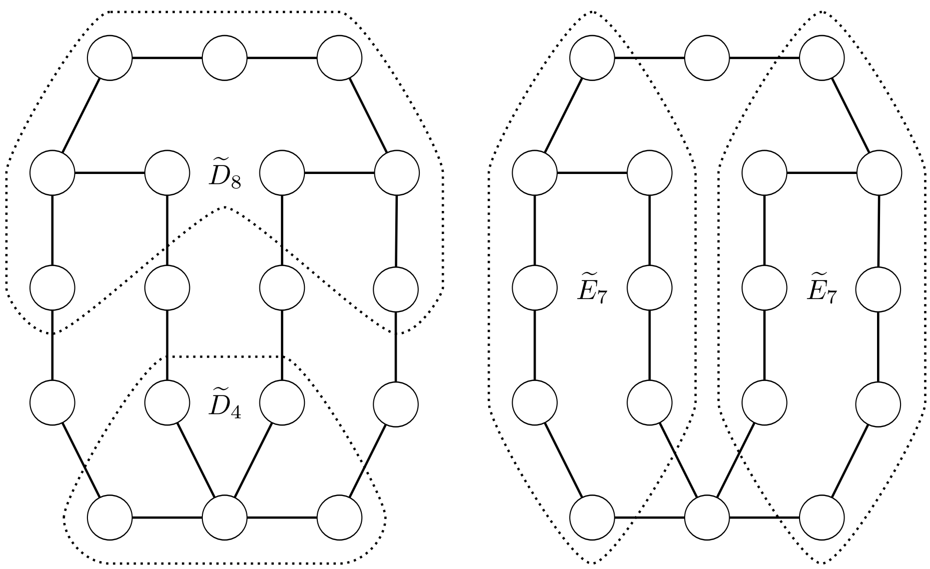

Figure 3.1: Two possible fibrations of X

The fibration on the left has four sections. The fibration on the right has two 2-sections. Let's consider the fibration on the right. We begin by looking at the simplest non-symplectic automorphisms, those that stabilize all of the curves on X. In that case, the bottom 2-section is fixed, and hence, so is the other 2-section. Since the degree 3 vertices in the $\widetilde{E}_7$'s are fixed curves, we must have $n|2k$. The action is as shown in figure 3.2. We will denote the corresponding automorphism by σ_1. There are 8 fixed curves and no isolated fixed points. Now, we consider non-

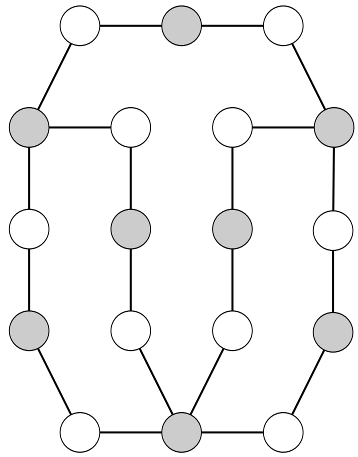

Figure 3.2: $\sigma_1 \in \mathrm{Aut}(X)$

symplectic automorphisms that have some curves that are not stable. Either the $\widetilde{E}_7$'s are permuted, or they are stable. If they are permuted, then we have no fixed curves, the sections are stable, and there are 4 isolate fixed points. The action is shown in figure 3.3. We will call the corresponding automorphism σ_2. Otherwise the $\widetilde{E}_7$'s are

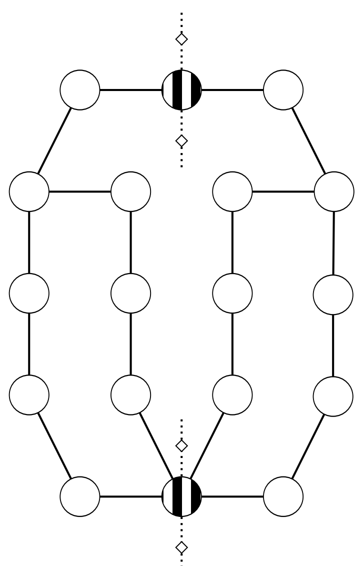

Figure 3.3: $\sigma_2 \in \mathrm{Aut}(X)$

stable. Now each $\widetilde{E}_7$ can either permute its length 3 paths, or be stable. If the curves of one $\widetilde{E}_7$ are stable, then its degree 3 vertex is a fixed curve. We find that the action at the top 2-section is $2k$, while the action at the bottom 2-section is $4k$. Again, we must have $n|2k$, which fixes the degree 3 vertex in the other $\widetilde{E}_7$. Thus, both $\widetilde{E}_7$ fibers permute their length 3 paths. We can have one of the two actions depicted in figure 3.4.

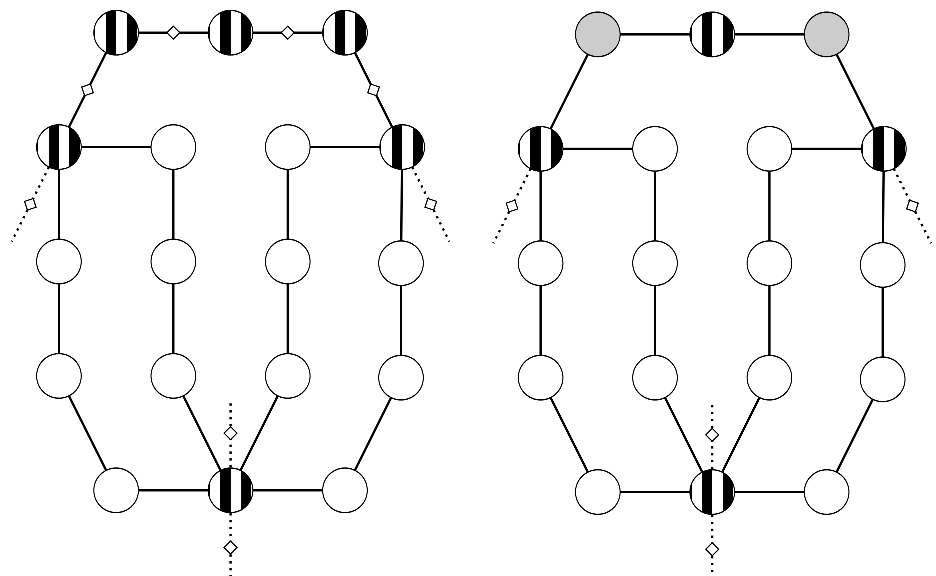

Figure 3.4: σ_3 (left) and σ_4 (right)

Note that $\sigma_3^2 = \sigma_1$, while $\sigma_4^2 = \text{id}$. This characterizes all finite order automorphisms of this fibration. Table 3.1 summarizes these automorphisms.

	σ_1	σ_2	σ_3	σ_4
Order	2	2	4	2
Fixed (lines, points)	(8,0)	(0,4)	(0,8)	(2,4)

Table 3.1: Summary of finite automorphisms

3.2 A Picard rank 10 example

In [16], S. Taki shows that there is a Picard rank 10 $K3$ surface which admits a non-symplectic automorphism σ of order 3. The Picard lattice of this surface is $U \oplus E_6 \oplus A_2$. He also gives the fixed locus of the automorphism, which is a genus 2 curve, 2 rational curves, and 4 isolated point. We can find the Gram graph of rational curves in this $K3$ surface in [3], as it is the 22nd surface on her list. The graph is shown in figure 3.5.

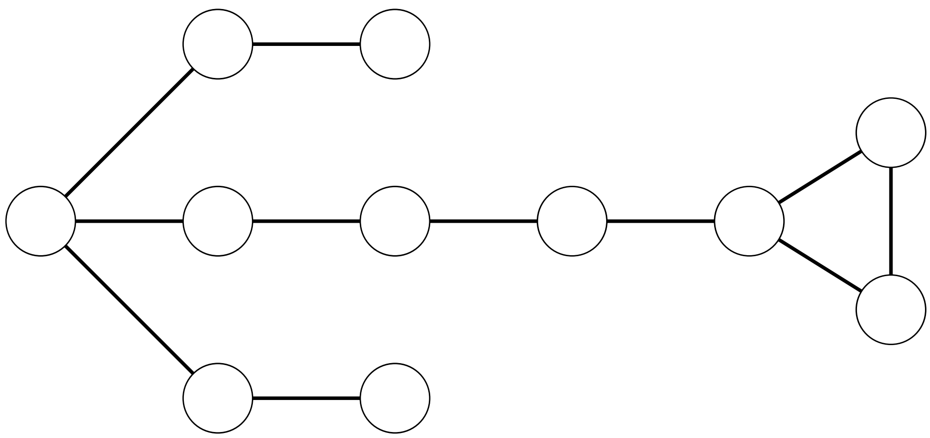

Figure 3.5: The Gram graph of X

We will use this diagram to describe σ. The fibration we use is a type IV^* fiber on the left and a type IV fiber on the right connected by a single section in the middle of the graph. Since σ has order 3, then the type IV^* fiber is stable, so its degree 3 vertex represents a fixed curve. From here we see that the section is also fixed. Note that the $\widetilde{A}_2$ in our graph could have also been a type I_3 fiber. We can now see that this is impossible since that would force the curve from the type I_3 fiber that intersects the section to also be fixed. The action is as shown in figure 3.2.

It would appear that we have 4 too many isolated fixed points (the three edges in the $\widetilde{A}_2$ graph represent the same intersection point), but this is where the fixed genus 2 curve C that Taki found to be in the fixed locus of σ fits perfectly. Each fiber

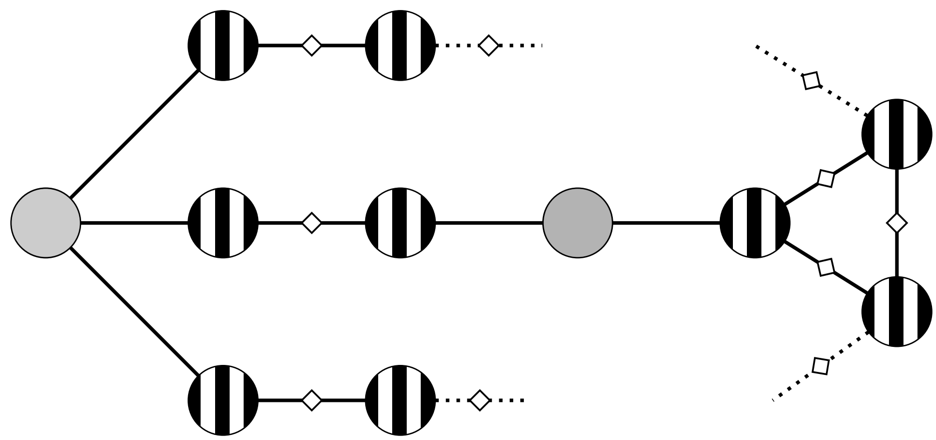

Figure 3.6: The action of σ

intersects C at two points (it is a 2-section) as shown in figure 3.7.

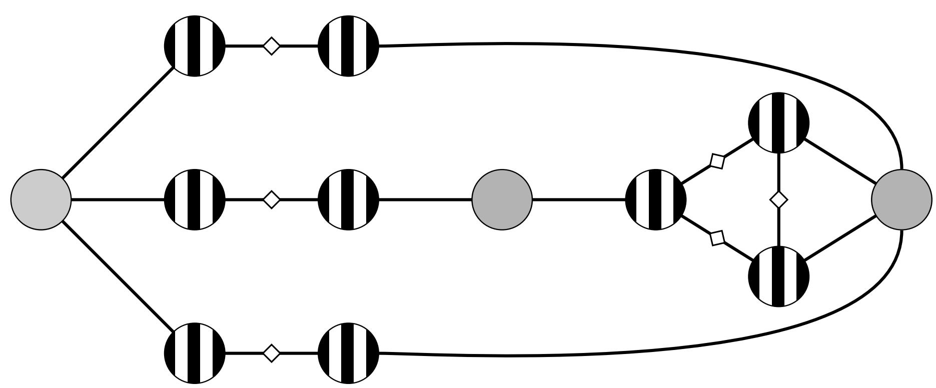

Figure 3.7: The genus 2 curve had been added.

Now, we proceed to complete the fibration so that we can write an explicit Weierstrass equation and write out σ in terms of these coordinates. Since C is a 2-section with genus 2, then it is a double cover of $\mathbb{P}^1$ branched at 6 points. Neither of our exhibited fibers intersect C at these branch points, as they intersect C at two distinct points each. The remaining singular fibers that do not show up in the graph curve found in [3] must be type II fibers whose cuspidal points intersect C at its branch points. We can compute the Euler characteristic of the fibration:

$$\chi(X) = \chi(IV^*) + \chi(IV) + 6 \cdot \chi(II) = 8 + 4 + 6 \cdot 2 = 24,$$

as required. Using table (IV.3.1) in [12], we can construct a Weierstrass model using the number of each singular fiber in our fibration. The model is

$$y^2 = x^3 + t^4(t^6 - 1).$$

Since σ acts trivially on the base $\mathbb{P}^1$, we have the following action.

$$\sigma : (x, y, t) \mapsto (\zeta_3 x, y, t),$$

where ζ_3 is a primitive third root of unity. We can quickly verify that the action is primitive by checking that the volume form

$$\frac{dx \wedge dt}{y} \mapsto \zeta_3 \frac{dx \wedge dt}{y}.$$

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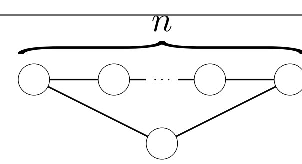
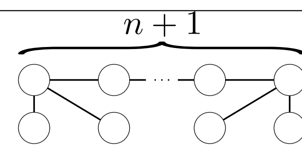
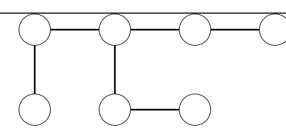
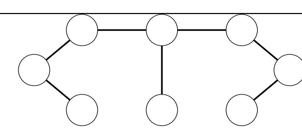
Appendix A

APPENDIX

A.1 Kodaira Singular Fibers

Fiber Type	Description	Graph
I_0	Smooth elliptic curve	
I_1	Nodal rational curve	
I_{n+1}	$n+1$ smooth rational curves forming the graph $\widetilde{A}_n$	
I_n^*	$n+5$ smooth rational curves forming the graph $\widetilde{D}_{n+4}$	
${}_mI_n$	The fiber I_n with multiplicity m	
II	cuspidal rational curve	
III	Two smooth rational curves intersecting with multiplicity two	
IV	Three rational curves intersecting at one point	
IV^*	7 smooth rational curves forming the graph $\widetilde{E}_6$	
III^*	8 smooth rational curves forming the graph $\widetilde{E}_7$	
II^*	9 smooth rational curves forming the graph $\widetilde{E}_8$