Summer 2014

A Constructive Proof of the Borel-Weil Theorem for Classical Groups

Kostiantyn Timchenko

Georgia Southern University

Follow this and additional works at: https://digitalcommons.georgiasouthern.edu/etd

Part of the Geometry and Topology Commons

Recommended Citation

https://digitalcommons.georgiasouthern.edu/etd/1144

This thesis (open access) is brought to you for free and open access by the Graduate Studies, Jack N. Averitt College of at Digital Commons@Georgia Southern. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of Digital Commons@Georgia Southern. For more information, please contact digitalcommons@georgiasouthern.edu.
A CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM FOR CLASSICAL GROUPS

by

KOSTIANTYN TIMCHENKO

(Under the Direction of François Ziegler)

ABSTRACT

The Borel-Weil theorem is usually understood as a realization theorem for representations that have already been shown to exist by other means ("Theorem of the Highest Weight"). In this thesis we turn the tables and show that, at least in the case of the classical groups $G = U(n)$, $SO(n)$ and $Sp(2n)$, the Borel-Weil construction can be used to quite explicitly prove existence of an irreducible representation with highest weight $\lambda$, for each dominant integral form $\lambda$ on the Lie algebra of a maximal torus of $G$.

Key Words: Unitary representation, coadjoint orbit, geometric quantization, Kähler manifold

2009 Mathematics Subject Classification: 22E45, 53D50
A CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM FOR CLASSICAL GROUPS

by

KOSTIANTYN TIMCHENKO

B.S. in Mathematics, Odessa National University

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2014
A CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM FOR CLASSICAL GROUPS

by

KOSTIANTYN TIMCHENKO

Major Professor: François Ziegler

Committee: Jimmy Dillies
Yi Lin

Electronic Version Approved:
July 16, 2014
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td><strong>Basic definitions</strong></td>
<td>4</td>
</tr>
<tr>
<td>2.1</td>
<td>4</td>
</tr>
<tr>
<td><strong>Lie groups and Lie algebras</strong></td>
<td>4</td>
</tr>
<tr>
<td>2.2</td>
<td>5</td>
</tr>
<tr>
<td><strong>Exponential map</strong></td>
<td>5</td>
</tr>
<tr>
<td>2.3</td>
<td>6</td>
</tr>
<tr>
<td><strong>Representations</strong></td>
<td>6</td>
</tr>
<tr>
<td>2.4</td>
<td>7</td>
</tr>
<tr>
<td><strong>The adjoint representation</strong></td>
<td>7</td>
</tr>
<tr>
<td>2.5</td>
<td>7</td>
</tr>
<tr>
<td><strong>Matrix Lie groups</strong></td>
<td>7</td>
</tr>
<tr>
<td>2.6</td>
<td>8</td>
</tr>
<tr>
<td><strong>Unitary representations</strong></td>
<td>8</td>
</tr>
<tr>
<td>2.7</td>
<td>8</td>
</tr>
<tr>
<td><strong>Representations of compact groups</strong></td>
<td>8</td>
</tr>
<tr>
<td>2.8</td>
<td>9</td>
</tr>
<tr>
<td><strong>Schur's Lemma</strong></td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td><strong>Highest weight theory for compact groups</strong></td>
<td>11</td>
</tr>
<tr>
<td>3.1</td>
<td>11</td>
</tr>
<tr>
<td><strong>Weights</strong></td>
<td>11</td>
</tr>
<tr>
<td>3.2</td>
<td>12</td>
</tr>
<tr>
<td><strong>Roots</strong></td>
<td>12</td>
</tr>
<tr>
<td>3.3</td>
<td>12</td>
</tr>
<tr>
<td><strong>Weyl group</strong></td>
<td>12</td>
</tr>
<tr>
<td>3.4</td>
<td>13</td>
</tr>
<tr>
<td><strong>Order on t^</strong>*</td>
<td>13</td>
</tr>
<tr>
<td>3.5</td>
<td>13</td>
</tr>
<tr>
<td><strong>Highest weight</strong></td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td><strong>Coadjoint orbits</strong></td>
<td>15</td>
</tr>
<tr>
<td>4.1</td>
<td>15</td>
</tr>
<tr>
<td><strong>The coadjoint action</strong></td>
<td>15</td>
</tr>
</tbody>
</table>
4.2 Hunt’s argument and its consequences ................. 16
4.3 The coadjoint orbits as symplectic manifolds ............. 18
4.4 An example of coadjoint orbits: Grassmannians ............ 19

5 Non-constructive Borel-Weil ................................. 22

5.1 The tautological line bundle over projective space ......... 22
5.2 The orbit of the highest weight vector ..................... 24
5.3 The moment map and coadjoint orbits ....................... 26
5.4 The space \( H^0(L) \) of antiholomorphic sections ........... 27
5.5 Realization in the space \( H^0(\lambda) \) of functions on \( G_C \) ........ 31

6 Constructive Borel-Weil ................................. 33

6.1 \( H^0(\lambda) \) is irreducible or zero ..................... 33
6.2 \( H^0(\lambda) \) is non-zero .................................. 36

6.2.1 Heuristics and \( U(n) \) ............................... 36
6.2.2 \( Sp(2n) \) .................................................. 42
6.2.3 \( SO(2n + 1) \) .............................................. 44
6.2.4 \( SO(2n) \) .................................................. 46

A Bruhat decomposition ................................. 50
CHAPTER 1
INTRODUCTION

The classification of all irreducible representations of a compact connected Lie group $G$ is given by the Theorem of the Highest Weight: *Every irreducible $G$-module $V$ has a unique highest weight $\lambda$, which characterizes $V$ and can be any so-called "dominant integral form".* One assertion of this statement, namely the existence of the highest weight, is relatively easy. Indeed one can prove it in the $SU(2)$ case with raising/lowering operations; then once the $SU(3)$ case is understood it quickly becomes clear how to argue for general compact Lie groups. We refer the reader to B. Hall's book [H03, Thm 7.15] for a nice exposition of this theory, which we briefly review in Chapter 3, after a few preliminaries dealt with in Chapter 2.

The hard part of the Theorem of the Highest Weight, the second part, tells us that for every dominant integral form $\lambda$ there exists an irreducible representation $V_\lambda$ with highest weight $\lambda$. There are four standard approaches to constructing $V_\lambda$:

1. Cartan’s case-by-case construction [C13]
2. Weyl’s theory [W25]
3. Verma modules [C48; H51]
4. Borel-Weil theory [S54]

Historically, Cartan’s approach was the earliest; he constructed irreducible representations for each compact simple Lie group (four classical series and five exceptional), considering them case-by-case. The other methods provide a more unified approach for all compact Lie groups at a time. In this exposition, we are interested in the Borel-Weil theory, but take a case-by-case viewpoint much like Cartan’s.

The Borel-Weil Theorem was presented by Serre at the May 1954 Bourbaki seminar; Borel and Weil themselves never wrote it up for publication. This theorem realizes $V_\lambda$ as the space of (anti-)holomorphic sections of a certain holomorphic line bundle. Namely, let $G$ be a compact connected Lie group and $\lambda$ a dominant integral form. Now $\lambda$ gives us a character on the maximal torus $T$ of $G$ by $\exp(H) \mapsto e^{i(\lambda, H)}$, which extends to a holomorphic character $\chi$ on a Borel subgroup $B$ of $G_\mathbb{C}$. We construct the line bundle over $G_\mathbb{C}/B$ as follows

$$\pi : G_\mathbb{C} \times_B \mathbb{C} \to G_\mathbb{C}/B. \quad (1.1)$$

Antiholomorphic sections of it are the antiholomorphic functions on $G_\mathbb{C}$ such that $F(gb) = \chi(b)F(g)$ for $b \in B$ and $g \in G_\mathbb{C}$. We denote the space of antiholomorphic sections by $H^0(\lambda)$. The group $G$ acts on it by $(gF)(g') = F(g^{-1}g')$ and we have the Borel-Weil Theorem: *The space $H^0(\lambda)$ is non-zero and forms an irreducible representation with highest weight $\lambda$.*
Traditionally this theorem has been understood as a realization theorem for representations that were long known to exist. From this point of view the proof is not hard either: assuming $V_\lambda$ exists, we can map it into $H^0(\lambda)$ by $\varphi \mapsto F,$

$$F(g) = (gv_0, \varphi)$$

where $v_0$ is a highest weight vector and $(\cdot, \cdot)$ is a $G$-invariant inner product on $V_\lambda$. That this map is one-to-one and onto follows from a nice argument of Kobayashi and Kunze [K61; K62], who use the fact that the space $H^0(\lambda)$ has a reproducing kernel (equivalently, point evaluations on it are continuous) to show that it is automatically irreducible (or zero).

Borel and Weil also gave a second realization of $V_\lambda$ as $H^0(L)$, the module of antiholomorphic sections of a line bundle $L \to X$ over a deeper quotient $X = G_\mathbb{C}/P$. Nowadays one recognizes $X$ as the coadjoint orbit of $\lambda$ in $\mathfrak{g}^*$, and this construction as an instance of geometric quantization or the orbit method [K70; S70; K72]. Chapters 4 and 5 are devoted to an exposition (with proofs) of this theory and both realizations of $V_\lambda$.

Now it is natural to ask whether the tables can be turned and the Borel-Weil construction (1.1) used to establish $V_\lambda$’s very existence. In view of the Kobayashi-Kunze result, this amounts to showing that $H^0(\lambda) \neq 0$ without assuming $V_\lambda$ exists. To our knowledge this was first done in [S68, pp. 216–217], by a method also exposed in [H75, 31.4] and modified in [J87, p. 201]. These authors set out to find a nonzero highest weight vector $F \in H^0(\lambda)$, and observe that this $F$ should satisfy

$$F(\pi b) = \overline{\chi(b)}F(1)$$

for all $b \in B$ and $\pi \in N^-$, the unipotent radical of the opposite Borel subgroup. So apart from a constant factor, $F$ is known on the open dense “big cell” $N^- B \subset G_\mathbb{C}$ of the Bruhat decomposition (see Appendix), and matters are reduced to showing that (1.3) extends antiholomorphically from $N^- B$ to all of $G_\mathbb{C}$. This Steinberg does by invoking two rather deep results of Chevalley [H75, 5.3B and 11.2], and Jantzen by first extending (1.3) to codimension 1 Bruhat cells and then applying a version of the Riemann Extension Theorem [G84, p. 132].

The main observation of this Thesis, inspired by Problems of Goodman-Wallach [G98, 12.1.4.5] and Hall [H03, 7.7.6], is that in the case of the classical groups $U(n)$, $SO(n)$ and $Sp(2n)$ one can bypass this tedious and non-constructive procedure, and instead heuristically derive and then prove explicit formulas for $F$. Thus for instance in the case of $G = U(n)$, where $\lambda$ is a diagonal matrix with integer entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we find that the extension of (1.3) to $G_\mathbb{C} = GL(n, \mathbb{C})$—and thus a highest weight vector in $H^0(\lambda)$—is given by the formula (already known to
CHAPTER 1. INTRODUCTION

Cartan [C13, §36]!

\[
F(g) = \frac{\det_1(g)^{\lambda_1 - \lambda_2} \det_2(g)^{\lambda_2 - \lambda_3} \cdots \det_n(g)^{\lambda_n - \lambda_{n+1}}}{(1.4)}
\]

where \( \det_i(g) \) denotes the \( i \)-th principal minor of the matrix \( g \) and we have set \( \lambda_{n+1} = 0 \). Note that since the exponents \( \lambda_i - \lambda_{i+1} \) are integers and non-negative except perhaps the last one, the function (1.4) is indeed well-defined and antiholomorphic on all of \( G_\mathbb{C} \). Our concluding Chapter 6 is devoted to establishing this formula and similar ones for \( \text{SO}(n) \) and \( \text{Sp}(2n) \).

Thereby the non-vanishing of \( H^0(\lambda) \), the existence of \( V_\lambda \) (Theorem of the Highest Weight), and the equality of the two (Borel-Weil Theorem) are simultaneously and constructively proved for each dominant integral form \( \lambda \) on each compact classical group.
In this chapter we give basic definitions and theorems.

2.1 Lie groups and Lie algebras

Definition 2.1. A Lie group $G$ is a differentiable manifold which is also endowed with a group structure such that multiplication and inversion are smooth maps.

A map $\varphi : G \to H$ is a Lie group homomorphism if $\varphi$ is both smooth and a group homomorphism. We call $\varphi$ an isomorphism if, in addition, $\varphi$ is a diffeomorphism.

Definition 2.2. A Lie algebra $g$ over $\mathbb{R}$ is a real vector space $g$ together with a bilinear operator $\left[\cdot, \cdot\right] : g \times g \to g$ (called the bracket) such that for all $X, Y, Z \in g$,

(a) $\left[ X, Y \right] = -\left[ Y, X \right]$ (skew-symmetry)

(b) $\left[ \left[ X, Y \right], Z \right] + \left[ \left[ Y, Z \right], X \right] + \left[ \left[ Z, X \right], Y \right] = 0$ (Jacobi identity).

A morphism of Lie algebras is a linear map $\alpha : g \to h$ which preserves the bracket.

To define the Lie algebra of a Lie group we need the following:

Definition 2.3. Let $g \in G$. Left translation by $g$ is the diffeomorphism $l_g : G \to G$ defined as follows:

$$ l_g(g') = gg' $$

for all $g' \in G$.

Definition 2.4. A vector field $X$ on $G$ is called left invariant, if

$$ dl_g \circ X = X \circ l_g $$

for each $g \in G$.

Proposition 2.5. Let $G$ be a Lie group and $g$ its set of left invariant vector fields.

(a) $g$ is a real vector space isomorphic to the tangent space $T_e(G)$ to $G$ at the identity;

(b) $g$ forms a Lie algebra under the Lie bracket operation on vector fields, namely

$$ \left[ X, Y \right] = X \circ Y - Y \circ X. $$

Definition 2.6. We define the Lie algebra of the Lie group $G$ to be the Lie algebra $g$ of left invariant vector fields on $G$. Alternatively, we could take as the Lie algebra of $G$ the tangent space $T_e(G)$ at the identity.
Example 2.7. The group $GL(n, \mathbb{C})$ of all invertible complex matrices is a Lie group. Its Lie algebra is the vector space $gl(n, \mathbb{C})$ of all $n \times n$ complex matrices. The bracket is given by the commutator

$$[A, B] = AB - BA.$$ 

The Lie groups that are subgroups of $GL(n, \mathbb{C})$ are called matrix Lie groups.

2.2 Exponential map

The existence and uniqueness theorem for first-order differential equations gives us the next result:

**Proposition 2.8.** Let $G$ be a Lie group together with its Lie algebra $g$. Then for each $X \in g$ there exists a unique morphism of Lie groups $\exp_X : \mathbb{R} \to G$ such that $\left. \frac{d}{dt} \exp_X(t) \right|_{t=0} = X$. We call this map the one-parameter subgroup corresponding to $X$.

**Definition 2.9.** We define the exponential map

$$\exp : g \to G$$

by setting

$$\exp(X) = \exp_X(1).$$

The exponential map for matrix groups is given by ordinary exponentiation of matrices. For example, the map $\exp : gl(n, \mathbb{C}) \to GL(n, \mathbb{C})$ sends a matrix $A \in gl(n, \mathbb{C})$ to the invertible matrix $e^A$, where

$$e^A = 1 + A + \frac{A^2}{2!} + \cdots + \frac{A^j}{j!} + \cdots.$$ 

Every morphism of Lie groups $\varphi : G \to H$ defines a morphism of Lie algebras $d\varphi : g \to h$. We can construct it as follows. The derivative at the identity of the map

$$\varphi : G \to H$$

is

$$(d\varphi)_e : T_e(G) \to T_e(H)$$

which is essentially the same as

$$(d\varphi)_e : g \to h.$$
We denote this map again by $d\varphi$. It happens that $d\varphi$ preserves the bracket $[H03, \text{Thm 2.21}]$, so it's actually a morphism between the Lie algebras $g$ and $h$.

Using the exponential map we can establish the following: For $\varphi : G \to H$ and corresponding $d\varphi : g \to h$

$$\varphi(e^{tX}) = e^{td\varphi(X)};$$

in particular,

$$\varphi(e^X) = e^{d\varphi(X)} \quad (2.1)$$

and

$$d\varphi(X) = \left. \frac{d}{dt} \varphi(e^{tX}) \right|_{t=0}. \quad (2.2)$$

2.3 Representations

Definition 2.10. A representation of a Lie group $G$ is a complex vector space $V$ together with a Lie group homomorphism $\Pi : G \to GL(V)$. Analogously, we define a representation of a Lie algebra $g$ as a complex vector space $V$ together with a Lie algebra morphism $\pi : g \to gl(V)$.

Often we use a module notation, writing $gv$ for $\Pi(g)v$ and $Zv$ for $\pi(Z)v$. A morphism (or an intertwining operator) between representations $V, W$ is a linear map $\Phi : V \to W$ such that

$$\Phi(gv) = g\Phi(v),$$

for all $g \in G$ and $v \in V$.

Definition 2.11. A subspace $U$ of $V$ is called invariant if for all $u \in U$ and $g \in G$ there holds $gu \in U$.

Definition 2.12. A representation is called irreducible if it is nonzero and has no non-trivial invariant subspaces.

Every finite-dimensional representation $\Pi$ of $G$ gives rise to an infinitesimal representation $\pi = d\Pi$ of $g$:

$$\Pi : G \to GL(V)$$

gives

$$\pi : g \to gl(V).$$

Using the exponential map

$$\exp : g \to G$$
the relations (2.2) and (2.1) for the representation $\Pi : G \to \text{GL}(V)$ become as follows:

$$\pi(X) = \left. \frac{d}{dt} \Pi(e^{tX}) \right|_{t=0}$$  \hspace{1cm} (2.3)

and

$$\Pi(e^X) = e^{\pi(X)}.$$  \hspace{1cm} (2.4)

### 2.4 The adjoint representation

Let $G$ act on itself by conjugation:

$$g \in G \mapsto c_g \quad (c_g : G \to G)$$

where $c_g$ denotes the inner automorphism

$$c_g : g' \mapsto gg'g^{-1}.$$  \hspace{1cm}

Passing to the tangent spaces we obtain a Lie algebra automorphism

$$d(c_g) : g \to \mathfrak{g}.$$  \hspace{1cm}

Denoting by $gXg^{-1}$ the image of $X$ under $d(c_g)$ we get the so-called *adjoint action* of $G$ on $\mathfrak{g}$:

$$g \mapsto \text{Ad}(g), \quad \text{Ad}(g)(X) = gXg^{-1}.$$  \hspace{1cm}

From $\text{Ad}$ we can obtain infinitesimally

$$\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$  \hspace{1cm}

Then

$$\text{ad}(X)(Y) = [X, Y],$$  \hspace{1cm}

where $[X, Y]$ is the Lie bracket in $\mathfrak{g}$.

### 2.5 Matrix Lie groups

Let $G$ be a matrix Lie group, i.e. $G \subset \text{GL}(n, \mathbb{C})$. Then its Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \{ H \in \mathfrak{gl}(n, \mathbb{C}) : e^{th} \in G \text{ for all } t \in \mathbb{R} \}.$$  \hspace{1cm}

The bracket in this case is the commutator of two matrices $X$ and $Y$, or

$$[X, Y] = XY - YX \text{ in } \mathfrak{gl}(n, \mathbb{C}).$$  \hspace{1cm}

And the adjoint action becomes the matrix conjugation

$$\text{Ad}(g)(X) = gXg^{-1}.$$  \hspace{1cm}
2.6 Unitary representations

Definition 2.13. A complex representation $\Pi$ of a group $G$ is called unitary if there is a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ on $V$: 

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w \in V$. Equivalently, $\Pi(G)$ lies in the unitary group $U(V)$.

Proposition 2.14. Every finite-dimensional unitary representation is completely reducible.

Proof. Suppose $W \subset V$ is an invariant subspace. Then $V = W \oplus W^\perp$ and $W^\perp$ is an invariant subspace too. Indeed, for any $w' \in W^\perp$:

$$\langle gw', v \rangle = \langle w', g^{-1}v \rangle = 0, \text{ for all } v \in W,$$

hence $gw' \in W^\perp$. Repeating this procedure, we decompose $V$ into a direct sum of irreducible invariant subspaces (this process stops because $V$ is finite dimensional; we reduce its dimension at each step). \qed

2.7 Representations of compact groups

Proposition 2.15. Let $G$ be a compact Lie group. There is a unique measure on $G$, the so-called left Haar measure $dg$, such that

$$\int_G dg = 1$$

and which is invariant under the left action of $G$ on itself (Definition 2.3).

The existence of Haar measure on compact Lie groups [H63, p. 185] gives us the next result, which is analogous to the case of finite groups:

Proposition 2.16. Any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible.

Proof. This argument is often called Weyl’s unitarian trick. Suppose that $\Pi$ is a finite-dimensional representation of a compact group $G$ acting on a space $V$. We start from any inner product $\langle \cdot, \cdot \rangle$ on $V$ and then “average” it by using the group action:

$$\langle v, w \rangle_G = \int_G \langle gv, gw \rangle dg$$
where $dg$ is Haar measure on $G$. Then $\langle v, v \rangle_G > 0$ as an integral of a positive function, and by the left invariance of Haar measure

$$\langle hv, hw \rangle_G = \int_G \langle ghv, ghw \rangle dg = \langle v, w \rangle_G$$

(2.5)

for $h \in G$. Hence $\Pi$ is a unitary representation with respect to $\langle \cdot, \cdot \rangle_G$ and thus completely reducible by Proposition 2.14.

2.8 Schur’s Lemma

Theorem 2.17 (Schur’s Lemma). Let $V$, $W$ be two complex $G$-modules.

(1) If $V$ and $W$ are irreducible, then any morphism $\Phi : V \to W$ is either zero or isomorphism.

(2) If $V$ is irreducible and $\Phi : V \to V$ is an (auto)morphism, then $\Phi = \lambda 1$ for some $\lambda \in \mathbb{C}$.

(3) Let $V$ and $W$ be irreducible and let $\Phi_1, \Phi_2$ be two nonzero morphisms $V \to W$, then $\Phi_1 = \lambda \Phi_2$ for some $\lambda \in \mathbb{C}$.

Theorem 2.18. Every irreducible representation of a compact group is finite-dimensional.

Proof. (Cf. [H63, p. 344].) Let $V$ be an irreducible $G$-module together with the $G$-invariant Haar measure $dg$. We fix a unit vector $v \in V$ and consider

$$\Phi = \int_G gv\langle gv, \cdot \rangle dg.$$ 

For $h \in G$,

$$h\Phi h^{-1} = \int_G hgv\langle hgv, h^{-1}\cdot \rangle dg$$

$$= \int_G hgv\langle hgv, \cdot \rangle dg = \int_G kv\langle kv, \cdot \rangle dk = \Phi.$$ 

So $h\Phi = \Phi h$ i.e $\Phi : V \to V$ is an intertwining map. By Schur’s lemma (2.17)

$$\Phi = \lambda 1$$

where

$$\lambda = \langle v, \Phi v \rangle = \int_G |\langle v, gv \rangle|^2 dg > 0.$$
Suppose now that \( E \) is any finite-dimensional subspace of \( V \), together with the orthogonal projector \( p : V \to E \). Then \( \Phi = \lambda I \) implies \( p\Phi p = \lambda p^2 = \lambda p \), i.e.

\[
\int_G pgv\langle gv, p\cdot \rangle dg = \lambda p.
\]

Taking the trace of both sides gives us

\[
\int_G \|pgv\|^2 dg = \lambda \dim(E).
\]

At the same time, \( \int_G \|pgv\|^2 dg \leq \int_G \|gv\|^2 dg = \text{Vol}(G) \), so that

\[
\dim(E) \leq \frac{\text{Vol}(G)}{\lambda}
\]

which gives an upper bound on the dimension of any \( E \) and completes the proof. \( \square \)
CHAPTER 3
HIGHEST WEIGHT THEORY FOR COMPACT GROUPS

Definition 3.1. Let $G$ be a compact connected Lie group. A **maximal torus** $T \subset G$ is a maximal connected abelian subgroup of $G$.

Theorem 3.2. In general, in every compact Lie group there are maximal tori and they are conjugate, i.e. if $T$ and $T'$ are two maximal tori then $T' = gTg^{-1}$ for some $g \in G$.

*Proof.* For the existence part, take a nonzero $X \in \mathfrak{g}$, and consider $U = \exp(\mathbb{R}X)$. Clearly $U$ is a connected abelian subgroup of $G$ and hence is a torus in $G$. Now, either $U$ is maximal or $U$ is contained in a larger torus. The existence of a maximal torus now follows from dimension reasons.

We prove the conjugacy part of this theorem in Chapter 4 (Theorem 4.5). 

A maximal torus $T$ has a Lie algebra $\mathfrak{t}$, a so-called **Cartan subalgebra**, together with its dual, denoted by $\mathfrak{t}^*$. We fix a maximal torus $T$ in $G$ throughout. The conjugacy property assures us that all subsequent constructions (root system, Weyl group, etc.) do not depend on the choice of the maximal torus.

3.1 Weights

Definition 3.3. Let $\mathfrak{g}$ be the Lie algebra of a Lie group $G$. The **complexification** $\mathfrak{g}_C$ of $\mathfrak{g}$ is defined as $\mathfrak{g}_C = \mathfrak{g} \oplus i\mathfrak{g}$. Let $\pi : \mathfrak{g} \to \text{gl}(V)$ be a representation of $\mathfrak{g}$. We can extend the domain of $\pi$ to $\mathfrak{g}_C$ by $\mathbb{C}$-linearity, in order to obtain $\pi : \mathfrak{g}_C \to \text{gl}(V)$.

Let $G$ be a compact Lie group and $\Pi : G \to \text{GL}(V)$ a finite-dimensional representation of $G$. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and write $\mathfrak{t}_C$ for its complexification. There exists a $G$-invariant inner product $(\cdot, \cdot)$ on $V$ by Proposition 2.16. Also, $\pi = d\Pi : \mathfrak{g}_C \to \text{gl}(V)$ is skew-Hermitian on $\mathfrak{g}$ and Hermitian on $i\mathfrak{g}$. Hence $\mathfrak{t}_C$ acts on $V$ as a family of commuting normal operators and so $V$ is simultaneously diagonalizable under the action of $\mathfrak{t}_C$. Then we can define:

Definition 3.4. We call a form $\mu \in \mathfrak{t}^*$ a **weight** of $V$ if

$$V^\mu = \{v \in V : Hv = i\langle \mu, H \rangle v \quad \forall H \in \mathfrak{t}_C\}$$

is non-zero. This gives us a decomposition

$$V = \bigoplus_{\mu : \text{weight}} V^\mu,$$

which is called the **weight space decomposition** of $V$ with respect to $\mathfrak{t}_C$. 
3.2 Roots

Consider the adjoint action $\text{Ad}: G \to \text{GL}(g)$. For every $\text{Ad}(g)$ we can extend its domain from $g$ to $g_C$. Then $\text{Ad}: G \to \text{GL}(g_C)$ is a representation of $G$ with differential $\text{ad}$ extended by complex linearity. It has a weight space decomposition

$$g_C = \bigoplus_{\alpha: \text{weight}} g^\alpha$$

Notice the zero weight space is $g^0 = \{ Z \in g_C : [H,Z] = 0, \ H \in t_C \}$. Then $g^0 = t_C$ since $t$ is a maximal abelian subspace of $g$.

**Definition 3.5.** The nonzero weights of $\text{Ad}$ are called the roots. Hence, we have the root space decomposition of $g_C$,

$$g_C = t_C \oplus \bigoplus_{\alpha: \text{root}} g^\alpha$$

where $g^\alpha = \{ Z \in g_C : [H,Z] = \alpha(H)Z, \ H \in t_C \}$. We denote the set of all roots by $R$.

**Theorem 3.6.** Let $\Pi : G \to \text{GL}(V)$ be a representation of $G$, and $t$ a Cartan subalgebra of $g$. For any root $\alpha$ and weight $\mu$

1. $\pi(g^\alpha)V^\mu \subset V^{\mu + \alpha}$

2. in particular, for roots $\alpha$ and $\beta$, $[g^\alpha, g^\beta] \subset g^{\alpha + \beta}$.

**Definition 3.7.** A form $\mu \in t^*$ is called integral if the character $e^{i\langle \mu, \cdot \rangle}$ of $t$ factors through the covering $\exp : t \to T$.

3.3 Weyl group

**Definition 3.8.** The Weyl group of $G$ is $N(T)/T$. Here $T$ is the maximal torus and $N(T) = \{ n \in G : nTn^{-1} \subset T \}$ is its normalizer.

Up to isomorphism, the Weyl group is independent of the choice of maximal torus. Given $w \in N(T)$, $H \in t$, and $\lambda \in t^*$, define an action of $N(T)$ on $t$ and $t^*$ by

$$w(H) = \text{Ad}(w)(H),$$

$$\langle w(\lambda), H \rangle = \langle \lambda, \text{Ad}(w^{-1})(H) \rangle.$$  

As $\text{Ad}(T)$ acts trivially on $t$, the action of $N(T)$ descends to an action of $W = N(T)/T$.

**Theorem 3.9 (Elementary properties of the action of the Weyl group).**
1. \( W \) is a (compact and discrete, hence) finite group.

2. The set of roots \( R \) is \( W \)-invariant.

3. \( \text{Ad}(w)(g^\alpha) = g^{w(\alpha)} \).

**Definition 3.10.** A Weyl chamber \( C \subset \mathfrak{t} \) is a chosen connected component of

\[ \mathfrak{t} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha). \]

**Definition 3.11.** The system of positive roots \( R^+ \) associated to \( C \) is the set

\[ R^+ = \{ \alpha \in R : \langle \alpha, H \rangle > 0 \text{ for all } H \in C \}. \]

In the same way, we define the system of negative roots

\[ R^- = \{ \alpha \in R : \langle \alpha, H \rangle < 0 \text{ for all } H \in C \}. \]

We write

\[ n^\pm = \bigoplus_{\alpha \in R^\pm} g^\alpha, \]

so that we have

\[ \mathfrak{g}_C = n^- \oplus \mathfrak{t}_C \oplus n^+ \]

by the root space decomposition. This is called the triangular decomposition, in analogy with the decomposition of \( \mathfrak{gl}(n, \mathbb{C}) \) into diagonal and strictly upper/lower triangular matrices.

### 3.4 Order on \( \mathfrak{t}^* \)

We can define a partial order \( \leq \) on \( \mathfrak{t}^* \) as follows:

\[ \lambda \leq \mu \iff \mu - \lambda \text{ is non-negative on } \mathfrak{t}. \]

**Definition 3.12.** The dominant chamber \( D \subset \mathfrak{t}^* \) consists of those \( \mu \in \mathfrak{t}^* \) such that \( w(\mu) \leq \mu \) for all \( w \in W \).

### 3.5 Highest weight

Now, as explained in \([B05, \S IX.7]\), \( \tilde{C} \) and \( D \) are fundamental domains for the action of \( W \) on \( \mathfrak{t} \) and \( \mathfrak{t}^* \), in the sense that each orbit intersect \( \tilde{C} \) and \( D \) in a single point, and we have:
Theorem 3.13 (Half the Theorem of the Highest Weight). Every irreducible $G$-module $V$ has a unique $\leq$-maximal weight $\lambda$, which characterizes $V$. The highest weight $\lambda$ is an integral point of $D$, and we have $\dim(V^\lambda) = 1$.

As explained in the Introduction, the harder step (the other half) is to prove the converse of Theorem 3.13: Every dominant integral form is the highest weight of some irreducible representation. In what follows we will show how to construct an irreducible representation with given highest weight by the Borel-Weil method.
CHAPTER 4
COADJOINT ORBITS

4.1 The coadjoint action

We have the natural adjoint action (§2.4) of the Lie group $G$ on its Lie algebra $g$:

$$\text{Ad}(g) : g \to g.$$ 

This gives rise to the coadjoint action of $G$ on the dual space $g^*$ as follows:

$$\text{coAd}(g) : g^* \to g^*$$

given by

$$\langle \text{coAd}(g)(x), Z \rangle = \langle x, \text{Ad}(g^{-1})(Z) \rangle$$

for $x \in g^*$, $g \in G$, $Z \in g$. Infinitesimally,

$$\text{coad}(X) : g^* \to g^*$$

is given by

$$\langle \text{coad}(X)(y), Z \rangle = \langle y, -\text{ad}(X)(Z) \rangle = \langle y, [Z, X] \rangle,$$

where $X, Z \in g$ and $y \in g^*$.

**Theorem 4.1.** Let $G$ be a compact Lie group, $T \subset G$ a maximal torus, $t$ its Lie algebra. Then $t = \{T\text{-fixed points in } g\}$.

**Proof.** "$\subseteq$" part. Let $H \in t$, then $e^{sH} \in T$ for all $s \in \mathbb{R}$. Then

$$\text{Ad}(t)(H) = \frac{d}{ds} (te^{sH}t^{-1})|_{s=0} = \frac{d}{ds} (e^{sH})|_{s=0} = H,$$

since $t$ and $e^{sH}$ are elements in $T$ and thus commute.

"$\supseteq$" part. Take $Z \in g$. Suppose that $\text{Ad}(t)(Z) = Z$ for all $t \in T$. As we know,

$$e^{\text{Ad}(t)(Z)} = te^{sZ}t^{-1}, \quad \forall s \in \mathbb{R}$$

which in our case becomes

$$e^{sZ} = te^{sZ}t^{-1}, \quad \forall t \in T.$$ 

Thus $e^{sZ}$ commutes with all elements in $T$. So $T$ and $\{e^{sZ}\}$ generate a larger torus. But $T$ is maximal, hence $e^{sZ} \in T$, for all $s \in \mathbb{R}$, and therefore $Z \in t$. \qed
Hence we have a natural projection
\[ \Pi = \int_T \Ad(t) \, dt : g \to t, \]
which is onto. Indeed, for all \( t \in T \):
\[ \Ad(t)(\Pi(Z)) = \Ad(t) \int_T \Ad(s)(Z) \, ds = \int_T \Ad(ts)(Z) \, ds = \int_T \Ad(r)(Z) \, dr = \Pi(Z). \]

Therefore we have a natural injection \( \Pi^* : t^* \to g^* \). (We define a transpose map \( \langle \Pi^*(\lambda), Z \rangle := \langle \lambda, \Pi(Z) \rangle \), where \( Z \in g \), \( \Pi(Z) \in t \) and \( \lambda \in t^* \).)

**Lemma 4.2.** \( \Pi^* \) identifies \( t^* \) with the set of all \( T \)-fixed points in \( g^* \).

**Proof.** "\( \subseteq \)" We want to show that \( \Pi^*(t^*) \) consists of \( T \)-fixed points.

Let \( x = \Pi^*(\lambda) \) and \( s \in T \), we want to show \( s(x) = x \). But for \( Z \in g \):
\[
\langle s(x), Z \rangle = \langle x, \Ad(s^{-1}(Z)) \rangle = \langle \lambda, \Pi(\Ad(s^{-1}(Z))) \rangle = \langle \lambda, \int_T \Ad(ts^{-1})(Z) \, dt \rangle
\]
by invariance of \( dt = \langle \lambda, \Pi(Z) \rangle = \langle x, Z \rangle \).

"\( \supseteq \)" We want to show that every \( T \)-fixed point in \( g^* \) belongs to \( \Pi^*(t^*) \).

Let \( x \in g^* \) be \( T \)-fixed, put \( \lambda = x|_t \). Then, for \( Z \in g \) hence for \( \Pi(Z) \in t \) we have:
\[
\langle \Pi^*(\lambda), Z \rangle = \langle \lambda, \Pi(Z) \rangle = \langle x, \Pi(Z) \rangle = \langle x, \int_T \Ad(t)(Z) \, dt \rangle
\]
But \( x \) is \( T \)-fixed, hence \( \langle \Pi^*(\lambda), Z \rangle = \langle x, Z \rangle \). \( \square \)

**Definition 4.3.** Given \( x \in g^* \), the coadjoint orbit \( X = G(x) \) is the image of the map
\[ g \in G \mapsto \coAd(g)(x) \in g^*. \]

Such an orbit has the form \( G/G_x \), where \( G_x \) is stabilizer of \( x \in g^* \) under the coadjoint action.

### 4.2 Hunt's argument and its consequences

**Theorem 4.4.** Every coadjoint orbit \( X \) of \( G \) meets \( t^* \).

**Proof.** (Cf. [H56; B79].) Let \( A \in t \) generate a dense 1-parameter subgroup in \( T \):
\[ \exp(\mathbb{R}A) = \{ e^{sA} : s \in \mathbb{R} \} = T. \]
And let \( x \in X \) be a point where the "hamiltonian" \( H_A(x) := \langle x, A \rangle \) attains its minimum (exists by compactness of the orbit as image
of a compact group). Then \( DH_A(x) = 0, \) \( DH_A(x) : T_x X \to \mathbb{R}. \) Evaluating \( DH_A(x) \) on the tangent vectors\(^1\) \( Z(x) = \frac{d}{dt} e^{tZ(x)} \big|_{t=0} \) we get

\[
0 = DH_A(x)(Z(x)) = \langle Z(x), A \rangle
\]

In our case \( x \in g^* \) and \( \langle Z(x), Z' \rangle = \langle x, [Z', Z] \rangle. \) Then,

\[
\langle Z(x), A \rangle = -\langle x, [Z, A] \rangle = -\langle A(x), Z \rangle.
\]

As this vanishes for all \( Z \in g, \) we conclude that \( A(x) = 0 \) and hence

\( t(x) = x \quad \forall t \in \exp(\mathbb{R}A) = T. \)

Hence, \( x \) is a fixed point and \( x \in \mathfrak{t}^* = \{ \text{T-fixed points in } g^* \}. \)

**Theorem 4.5 (Corollary).** **Any two maximal tori are conjugate.**

**Proof.** Let \( T = \exp(\mathbb{R}A) \) and \( U = \exp(\mathbb{R}B) \) be two maximal tori. We can identify \( g \) with \( g^* \) by a \( G \)-invariant inner product on \( g. \) By the theorem above there is a \( g \in G \) such that \( \text{Ad}(g)(B) \in \mathfrak{t}. \) Thus \( \exp(tB)g^{-1} = \exp(t\text{Ad}(g)(B)) \subset T \quad \forall t \in \mathbb{R}. \) Hence \( gUg^{-1} \subset T \) by continuity. Hence \( gUg^{-1} = T \) by maximality.

**Theorem 4.6 (Corollary).** **The intersection \( G(\lambda) \cap \mathfrak{t}^* = W(\lambda) \) (is an orbit of the Weyl group).**

**Proof.** “\( \supset \)”: Consider the Weyl group \( W = N(T)/T. \) Take \( w = nT \in N(T)/T. \) Then \( w(\lambda) = (nT)(\lambda) = n(\lambda) \in G(\lambda). \)

“\( \subset \)”: Suppose \( g(\lambda) \in \mathfrak{t}^* \), then \( g(\lambda) \) is \( T \)-fixed

\[
t(g(\lambda)) = g(\lambda), \quad \forall t \in T.
\]

Hence \( g^{-1}Tg \) lies in the stabilizer \( G_\lambda \) of \( \lambda. \) Now, \( T \) and \( gTg^{-1} \) are two maximal tori in \( G_\lambda. \) So they are conjugate; let \( h \in G_\lambda \) be such that

\[
T = h^{-1}g^{-1}Tgh.
\]

So \( gh \) belongs to the torus’s normalizer, \( N(T). \) Let \( w \) be the corresponding class in the Weyl group: \( w = ghT. \) The action of \( w \) on \( \lambda \in \mathfrak{t}^* \) gives us

\[
w(\lambda) = gh(\lambda) = g(\lambda),
\]

hence \( g(\lambda) \in W(\lambda) \) which completes the proof.

\(^1\)Given \( x \in g^* \), the vectors \( Z(x) \) with \( Z \in g, \) constitute the tangent space to the coadjoint orbit at the point \( x. \) See Section 4.3 on symplectic manifolds for details.
Theorem 4.7 (Corollary). The (co)adjoint stabilizer $G_B = \{ g \in G : gBg^{-1} = B \}$ of any $B \in g$ (or $g^*$) is connected.

Proof. Let $a \in G_B$. Then $\exp(\mathbb{R}B)$ is a torus contained in the centralizer $Z_G(a) = \{ g \in G : gag^{-1} = a \}$, hence contained in a maximal torus $U$ of $Z_G(a)$. Claim: $a \in U$ is connected.

Proof of $a \in U$: Let $a = \exp(A)$ (such an $A$ exists by surjectiveness of the exponential map [B05, Cor. IX.2.2.1]), then $\exp(\mathbb{R}A)$ is another torus in $Z_G(a)$ ⇒ it is conjugate by Theorem 4.5. $\exists g \in Z_G(a)$ such that $g \exp(\mathbb{R}A)g^{-1} \subset U$. In particular $U$ contains $g \exp(A)g^{-1} = gag^{-1} = a$.

Theorem 4.8 (Corollary). (Co)adjoint orbits of a compact connected Lie group are simply connected.

Proof. Consider the diagram

$$
\begin{array}{ccc}
H & \to & G \\
\text{closed subgroup} & & \text{Lie group} \\
\to & G/H & \\
\end{array}
$$

It gives rise to a long exact sequence of homotopy groups, including the following piece [S51, §17]:

$$
\cdots \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to \cdots
$$

(4.1)

Let $H$ be a stabilizer of the (co)adjoint action. Without loss of generality we can replace $G$ with the universal cover $\tilde{G}$, since both $G$ and $\tilde{G}$ have the same (co)adjoint orbits. Then the above (4.1) becomes

$$
\cdots \to 0 \to \pi_1(G/H) \to 0 \to 0 \to \pi_0(G/H) \to \cdots
$$

and thus forces $\pi_1(G/H)$ to be trivial. □

4.3 The coadjoint orbits as symplectic manifolds

Definition 4.9. A symplectic manifold is a smooth manifold $X$ with a nondegenerate, closed 2-form $\sigma$.

Definition 4.10. A smooth action of $G$ on a manifold $X$ is a group morphism $\rho : G \to \text{Diff}(X)$ of $G$ into the diffeomorphisms of $X$, such that $(g, x) \mapsto \rho(g)(x)$ is a smooth map $G \times X \to X$. 
It is customary to drop $\rho$ from the notation and write $g(x)$ for $\rho(g)(x)$. The orbit of $x$ is

$$G(x) = \{g(x) : g \in G\}.$$ 

Every orbit of a smooth action of $G$ on a manifold $X$ is diffeomorphic to the quotient manifold $G/G_x$, where $G_x = \{g \in G : g(x) = x\}$ is the stabilizer of $x$. If there is just one orbit, the action is called transitive and $X$ is called a homogeneous space of $G$.

Every smooth action $\rho$ of $G$ on a manifold $X$ induces an infinitesimal action of the Lie algebra $\rho^*_*: g \to \text{Vect}(X)$ by $\rho^*_*: Z \mapsto (d\rho)_e(Z)$, or, using the exponential map

$$\rho^*_*(Z)(x) = \left. \frac{d}{dt} e^{tZ}(x) \right|_{t=0}.$$ 

As with $\rho$ one usually drops $\rho^*_*$ from the notation and writes $Z(x)$ instead of $\rho^*_*(Z)(x)$. If the action is transitive, the diffeomorphism $G/G_x \to G(x)$ ensures that the tangent space to an orbit $G(x)$ at $x$ is

$$T_x G(x) = \{Z(x) : Z \in g\} = g/g_x$$

where $g_x$ is the Lie algebra of the stabilizer $G_x$.

Let $X$ be a coadjoint orbit of $G$ acting on $g^*$. Then $X$ admits a $G$-invariant symplectic structure given by the Kirillov-Kostant-Souriau 2-form:

**Theorem 4.11** (The Kirillov-Kostant-Souriau Theorem [K72; K70; S70]).

1. Every coadjoint orbit of a Lie group is a homogeneous symplectic manifold when endowed with the KKS 2-form

$$\sigma(Z(x), Z'(x)) = \langle x, [Z', Z] \rangle.$$ 

2. Conversely, every homogeneous symplectic manifold of a connected Lie group $G$ is, up to a possible covering, a coadjoint orbit of some central extension of $G$.

### 4.4 An example of coadjoint orbits: Grassmannians

In this section we are going to illustrate the notion of coadjoint orbits and exhibit the Kirillov-Kostant-Souriau 2-form on them in the special case of Grassmannians (and projective spaces in particular).

Let $V$ be a finite dimensional Hilbert space, so it comes equipped with a complex inner product. Based on this product we consider the group of all unitary
operators, \( G = \mathbb{U}(V) \), together with its Lie algebra \( g = u(V) \):

\[
G = \mathbb{U}(V) = \{ g \in \text{GL}(V) : \bar{g}g = 1 \}, \\
g = \{ Z \in \text{gl}(V) : \bar{Z} + Z = 0 \}.
\]

Here and in what follows the bar \( - \) denotes adjoint. Thus \( g \) consists of skew-adjoint operators. We identify \( g^* \) with the space of all self-adjoint operators as follows:

\[
ig \rightarrow g^* \\
x \mapsto \frac{1}{i} \text{Tr}(x \cdot)
\]

so that \( g^* \) becomes

\[
g^* = \{ x \in \text{gl}(V) : \bar{x} = x \}.
\]

Now we restrict our attention to the case \( V = \mathbb{C}^n \) in order to be more explicit, however the case of an arbitrary space \( V \) can be treated in the same way. Then \( G \) becomes \( \mathbb{U}(n) \) and the coadjoint orbits are conjugacy classes of self-adjoint matrices:

\[
G(x) = \{ gxg^{-1} \text{ for all } g \in G \}.
\]

Each orbit intersects the dominant chamber

\[
D = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_1 \geq \cdots \geq \lambda_n \right\}
\]

in exactly one point (compare §3.5 and Theorem 4.6). The orbit is integral (i.e. it goes through an integral point), when all \( \lambda_i \) are integers.

Consider the remarkable orbits

\[
X_k = \{ x \in g^* : x^2 = x, \text{Tr}(x) = k \} \\
= \{ \text{rank } k \text{ self-adjoint projectors } x : V \rightarrow V \} \\
= G \begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix},
\]

where \( n \) is the complex dimension of \( V \). Note that \( X_1 \) is just \( \mathbb{P}(V) \). Indeed, the map \( V \setminus \{0\} \rightarrow X_1 \), defined by

\[
\xi \mapsto \frac{\xi(\bar{\xi} \cdot)}{||\xi||^2},
\]

(where \( (, ) \) is the inner product on \( V \)), induces a bijection between \( \mathbb{P}(V) \) and \( X_1 \). Similarly, \( X_k \) is nothing but the Grassmannian of \( k \)-dimensional subspaces of \( V \).

**Proposition 4.12.** The 2-form on \( X_k \) is given by

\[
\sigma(\delta x, \delta' x) = \frac{1}{i} \text{Tr}(x[\delta x, \delta' x]) = \text{Tr}(\delta' x J \delta x),
\]

where \( \delta x, \delta' x \in T_x X_k \) and \( J \delta x = \frac{1}{i} [x, \delta x] \) is a complex structure on \( X_k \) \( (J^2 = -\text{id}) \).
Proof. Recall that the Kirillov-Kostant-Souriau 2-form (Theorem 4.11) is given by 
\[ \sigma(Z(x), Z'(x)) = \langle x, [Z', Z] \rangle. \] First, given \( \delta x \) we want to find \( Z \) such that \( Z(x) = \delta x \). Consider the pairing 
\[ \langle Z(x), Z' \rangle = \langle x, [Z', Z] \rangle = \frac{1}{i} \text{Tr}(x[Z', Z]) = \frac{1}{i} \text{Tr}(xZZ' - xZZ') \]
\[ = \frac{1}{i} \text{Tr}((Zx - xZ)Z') = \frac{1}{i} \text{Tr}([Z, x]Z') = \langle [Z, x], Z' \rangle. \]
Thus \( Z(x) = [Z, x] \). Our claim is that given \( \delta x \), \( Z \) can be taken equal to \( [\delta x, x] \). Indeed, \( x^2 = x \) implies \( x\delta xx = 0 \), and \( Z(x) \) becomes 
\[ Z(x) = [Z, x] = [[\delta x, x], x] = [\delta xx - x\delta x, x] \]
\[ = \delta xx - x\delta xx - x\delta xx + x\delta x = \delta xx + x\delta x = \delta \text{[}x^2\text{]} = \delta x. \]
Therefore we can compute 
\[ \sigma(\delta x, \delta'x) = \langle x, [Z', Z] \rangle = \langle Z(x), Z' \rangle = \langle \delta x, [\delta'x, x] \rangle = \frac{1}{i} \text{Tr}(\delta x[\delta'x, x]) \]
\[ = \frac{1}{i} \text{Tr}(\delta x\delta'xx - \delta xx\delta'x) = \frac{1}{i} \text{Tr}(x[\delta x, \delta'x]). \]
Also defining a linear operator \( J \) on each tangent space by \( J\delta x = \frac{1}{i}[x, \delta x] \) we get 
\( J^2\delta x = -[x, [x, \delta x]] = -\delta x \) so that \( J \) is an almost complex structure on \( X_k \). That gives another expression for the 2-form \( \sigma \):
\[ \sigma(\delta x, \delta'x) = \frac{1}{i} \text{Tr}(\delta'xx\delta x - \delta'x\delta xx) = \text{Tr}(\delta'x\frac{1}{i}[x, \delta x]) = \text{Tr}(\delta'xJ\delta x). \]
Together \( \sigma \) and \( J \) define a pseudo Riemannian metric \( g(\cdot, \cdot) \) on \( X_k \)
\[ g(\delta x, \delta'x) = \sigma(J\delta x, \delta'x) = -\text{Tr}(\delta x\delta'x) \]
which is negative definite: \( g(\delta x, \delta x) = -\text{Tr}(\delta x^2) < 0 \). Each two structures of \( (\sigma, J, g) \) determine the third. In that way we obtain a Kähler structure on \( X_k \).
CHAPTER 5
NON-CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM

In this chapter we consider the coadjoint orbit \( X \) that goes through the highest weight \( \lambda \) of an irreducible representation \( V \). Then we relate \( V \) to the symplectic \( G \)-manifold \( X \) as follows (cf. [Z96, pp. 47–50]):

First, we embed \( X \) into the projective space \( \mathbb{P}(V) \), which gives us a hermitian line bundle with connection \( L \to X \) with base \( X \). The curvature of the connection of \( L \) equals the orbit’s 2-form. (This is called prequantization.)

Second, the square integrable sections for Liouville measure on \( X \) make a unitary \( G \)-module. We extract the submodule of antiholomorphic sections \( H^0(L) \) from it. (This is called polarization.)

Now in Theorem 5.4 we show that \( H^0(L) \) is isomorphic to \( V \); and in Section 5.5 we show that both are also isomorphic to the representation \( H^0(\lambda) \) discussed in the Introduction.

5.1 The tautological line bundle over projective space

In this section we return our attention to the unitary \( G \)-module \( V \). Adopting the constructions from the previous section, we consider the projective space \( \mathbb{P}(V) \) as the space \( X_1 \) of all rank one self-adjoint projectors \( x \) in \( V \):

\[
\mathbb{P}(V) = \{ x \in u(V)^* : x^2 = x, \text{Tr}(x) = 1 \}.
\]

The tautological line bundle over \( \mathbb{P}(V) \) is

\[
L = \{(x, \xi) \in \mathbb{P}(V) \times V : \xi \in \text{Im}(x)\}
\]

together with the projection \( p : L \to \mathbb{P}(V) \)

\[
p : (x, \xi) \mapsto x.
\]

As we can see the fiber above \( x \) is just the complex line \( \text{Im}(x) = \mathbb{C}\xi \). Note that every \( \xi \neq 0 \) in \( V \) determines a projector \( x = \frac{\xi\bar{\xi}}{||\xi||^2} \), where \( \bar{\xi} = (\xi, \cdot) \). So we can identify \( L^x = L \setminus \{(x, 0) : x \in \mathbb{P}(V)\} \) with \( V^x = V \setminus \{0\} \) using the map \( (x, \xi) \mapsto \xi \).

We continue writing \( p \) for the projection \( L^x \to \mathbb{P}(V) \).

Infinitesimally one can write

\[
T_{(x,\xi)}L^x = T_xV^x = \xi^\perp \oplus \mathbb{C}\xi = \text{Ker}(x) \oplus \text{Im}(x).
\]

We write \( \text{Hor} : T_xV^x \to \text{Ker}(x) \) and \( \text{Vert} : T_xV^x \to \text{Im}(x) \) for the projections associated to this decomposition (i.e. \( \delta \xi = (1-x)\delta \xi + x\delta \xi \), where \( x = \frac{\xi\bar{\xi}}{||\xi||^2} \). This splitting is called a connection on the line bundle \( L \).
By definition $\omega$ is the connection 1-form on $L^\times$ such that

$$\text{Vert}(\delta \xi) = i\xi \omega(\delta \xi).$$

Explicitly,

$$x\delta \xi = i\xi \omega(\delta \xi),$$

hence

$$\omega(\delta \xi) = \frac{[\xi, \delta \xi]}{i||\xi||^2}$$

is a 1-form on $L^\times$.

**Theorem 5.1.** This connection has curvature $\sigma$.

Note: Given a connection $\omega$ on $L \overset{p}{\rightarrow} \mathbb{P}(V)$ its curvature is the 2-form $\omega$ such that

$$d\omega = p^* \omega.$$

**Proof.** We compute

$$[d\omega](\delta \xi, \delta' \xi) = [\delta \omega](\delta' \xi) - [\delta' \omega](\delta \xi)$$

$$= \frac{1}{i} \left\{ \left( \frac{\delta \xi}{||\xi||^2}, \delta' \xi \right) - \left( \frac{\delta' \xi}{||\xi||^2}, \delta \xi \right) \right\}$$

$$= \frac{1}{i} \left\{ \left( \frac{\delta \xi ||\xi||^2 - \xi \left[ (\delta \xi, \xi) + (\xi, \delta \xi) \right]}{||\xi||^4}, \delta' \xi \right) - \left( \frac{\delta' \xi ||\xi||^2 - \xi \left[ (\delta' \xi, \xi) + (\xi, \delta' \xi) \right]}{||\xi||^4}, \delta \xi \right) \right\}$$

$$= \frac{1}{i} \left\{ \frac{(\delta \xi, (1-x)\delta' \xi) - (\delta' \xi, (1-x)\delta \xi)}{||\xi||^2} \right\}$$

Now observe that\(^1 x\xi = \xi, \) gives $\delta x \xi + x \delta \xi = \delta \xi, \) or $\delta x \xi = (1-x)\delta \xi. \) Also $(1-x)^2 = (1-x).$ Hence

$$[d\omega](\delta \xi, \delta' \xi) = \frac{1}{i} \frac{(\delta x \xi, \delta' x \xi) - (\delta' x \xi, \delta x \xi)}{||\xi||^2}.$$

In general, $(u, v) = \text{Tr}(vu)$ gives

$$[d\omega](\delta \xi, \delta' \xi) = \frac{1}{i} \left\{ \text{Tr}(\delta' x \frac{\xi \xi}{||\xi||^2} \delta x) - \text{Tr}(\delta x \frac{\xi \xi}{||\xi||^2} \delta' x) \right\}$$

$$= \frac{1}{i} \text{Tr}(\delta' x x \delta x - \delta x \delta' x)$$

$$= \frac{1}{i} \text{Tr}(\delta' x [x, \delta x])$$

$$= \text{Tr}(\delta' x J \delta x)$$

$$= \sigma(\delta x, \delta' x). \quad \square$$

\(^1 x \xi \) is a product of square matrix and column vector; $\delta x$ and $\delta \xi$ are the derivatives (tangent vectors) in Frechet notation.
Note that every complex submanifold $W$ of $\mathbb{P}(V)$ is symplectic for the restriction of $\sigma$:

(a) The 2-form on the manifold $W$ is $i^* \sigma$, where $i : W \to \mathbb{P}(V)$ is the inclusion map. Now $d[i^* \sigma] = i^* d\sigma = 0$, thus $i^* \sigma$ on $W$ is closed.

(b) This 2-form is non-degenerate. Indeed, $\sigma(J\delta w, \delta w) = g(\delta w, \delta w) < 0$ unless $\delta w = 0$.

5.2 The orbit of the highest weight vector

We saw that $\mathbb{P}(C^n)$ is a coadjoint orbit of $U(n)$. In analogy with it, we consider $\mathbb{P}(V)$ as a coadjoint orbit of $U(V)$.

Now we return to our original notation: $G$ is any compact connected Lie group and $\Pi : G \to U(V)$ is an irreducible representation of it. Let $\lambda$ be its highest weight (so that $V = V_\lambda$) and let $v_0$ be a highest weight vector and $w_0 = \frac{v_0 v_0}{\|v_0\|^2}$ be the corresponding point in $\mathbb{P}(V)$.

**Theorem 5.2.** The orbit $W := G(w_0)$ is a complex (hence symplectic) submanifold of $\mathbb{P}(V)$.

**Proof.** We can replace $G$ and $g$ by their images in $\text{End}(V)$. For $V = V_\lambda$ we have

$$V = \bigoplus_{\mu \text{ : weight}} V^\mu,$$

where

$$V^\mu = \{v \in V : Hv = i(\mu, H)v \quad \forall H \in t\}.$$

Also

$$g_C = \bigoplus_{\alpha \in \mathbb{R} \cup \{0\}} g^\alpha,$$

where

$$g^\alpha = \{Z \in g_C : [H, Z] = i(\alpha, H)Z \quad \forall H \in t\}.$$

We have $g^\alpha V^\mu \subset V^\mu + \alpha$ by Theorem 3.6. Each $\alpha \in \mathbb{R}$ is either $> 0$, or $< 0$ (sign in the chamber $C$).

Claim:

1. if $\alpha > 0$ then $g^\alpha(w_0) = 0$,

2. $g^0(w_0) = 0$,

3. if $\alpha < 0$, then $g^\alpha(w_0) \subset g(w_0)$ where $g$ is the (real!) Lie algebra of $G$. 
CHAPTER 5. NON-CONSTRUCTIVE BOREL-WEIL

From this it follows that \( g_C(w_0) = g(w_0) \) and by equivariance that \( g_C(g(w_0)) = g(g(w_0)) \forall g \in G \). Indeed, suppose \( Z \in g_C \) and \( g \in G \). Then

\[
Z(g(w_0)) = \frac{d}{dt} e^{tZ}(g(w_0))|_{t=0},
\]

here the group action is \( g(w_0) = gw_0g^{-1} \) (composition of operators \( g \in U(V) \)). Now

\[
Z(g(w_0)) = \frac{d}{dt} gg^{-1} e^{tZ}(g(w_0))|_{t=0} = \frac{d}{dt} g(e^{tAd(g^{-1}Z)}|_{t=0}) = Dg(w_0)(Ad(g^{-1})(z(w))) \in g(g(w_0)).
\]

This shows that \( g_C(g(w_0)) \subset g(g(w_0)) \) with trivial reverse inclusion.

Thus we have \( g_C(w) \subset g(w) \) for all \( w \in W = G(w_0) \). It follows that \( G(w_0) \) is open in \( G_C(w_0) \). Also \( G(w_0) \) is connected, hence closed in \( G_C(w_0) \). Since \( G_C(w_0) \) is connected, we conclude that \( W = G(w_0) = G_C(w_0) \). So \( W \) is a complex orbit, hence a complex manifold.

It remains to show our 3 claims:

1) \( \alpha > 0 \Rightarrow g^\alpha(w_0) = 0 \).

The action on \( w_0 \in \mathbb{P}(V) \) is defined by \( Z(w_0) = p_*(Zv_0) \) where \( p_* \) is the derivative of \( p : V \setminus \{0\} \to \mathbb{P}(V) \). But for \( Z \subset g^\alpha, \alpha > 0: \)

\[
Zv_0 \in V^{\lambda+\alpha} = \{0\}
\]

since \( \lambda \) is the \textit{highest} weight. Thus \( Zv_0 = 0 \).

2) \( g^0(w_0) = 0 \)

Indeed, for \( Z \in g^0 = \mathfrak{t}_C \) we have

\[
Z(w_0) = p_*(Zv_0) = p_*(i(\lambda, Z)v_0) = \frac{d}{dt} p(e^{it(\lambda, Z)}|_{t=0}) = \frac{d}{dt} p(v_0)|_{t=0} = 0.
\]

3) If \( \alpha < 0 \), then \( g^\alpha(w_0) \subset g(w_0) \).

Suppose \( Z \in g^\alpha \). Claim: \( \tilde{Z} \in g^{-\alpha} \) (adjoint of \( Z \in \text{End}(V) \)).

Indeed, for \( H \in \mathfrak{t} \) we have

\[
[H, \tilde{Z}] = H\tilde{Z} - \tilde{Z}H = -\bar{H}\tilde{Z} + \bar{Z}H = HZ - ZH = i(\bar{\alpha}, H)\bar{Z} = -i(\alpha, H)\tilde{Z}.
\]

In particular \( \tilde{Z}(w_0) = 0 \) hence \( Z(w_0) = (Z - \tilde{Z})(w_0) \subset g(w_0) \). ((\( Z - \tilde{Z} \) is skew-adjoint, hence in \( g \).)\)
5.3 The moment map and coadjoint orbits

In the previous section we showed that the orbit $W$ of the projectivized highest weight vector $w_0 \in \mathbb{P}(V)$ is a homogeneous symplectic manifold. By the Kirillov-Kostant-Souriau Theorem 4.11, the resulting moment map $\Phi : W \to \mathfrak{g}^*$ will make $W = G(w_0)$ (a covering of) a coadjoint orbit of $G$. In this section we verify that this orbit is exactly $G(\lambda)$.

To this end we observe that $G$ preserves $\omega$: indeed

$$(g^*\omega)(\delta \xi) = \frac{(g\xi, g\delta \xi)}{i\|g\xi\|^2} = \frac{(\xi, \delta \xi)}{i\|\xi\|^2} = \omega(\delta \xi).$$

It follows that the Lie derivative $L(Z)\omega$ vanishes for all $Z \in \mathfrak{g}$. By Cartan’s formula for the Lie derivative, $L(Z)\omega = i(Z)d\omega + di(Z)\omega$, it follows that

$$i(Z)d\omega = -d\langle \Phi(\cdot), Z \rangle$$

where $\langle \Phi(\cdot), Z \rangle = i(Z)\omega$, or $\langle \Phi(\xi), Z \rangle = \omega(\xi Z)$. By definition of a moment map [S70], this means that

$$\langle \Phi(\xi), Z \rangle = \omega(\pi(Z)\xi) = \frac{\langle \xi, \pi(Z)\xi \rangle}{i\|\xi\|^2}.$$

defines a moment map $\Phi$ for the action of $G$ on $(L^\times, d\omega)$. We can write it equivalently as

$$\langle \Phi(\xi), Z \rangle = \frac{1}{i} \mathrm{Tr} \left( \frac{\xi \bar{\xi}}{\|\xi\|^2} \pi(Z) \right) = \langle \chi, \pi(Z) \rangle = \langle \pi^*(\chi), Z \rangle$$

where $\pi^* : \mathfrak{u}(V)^* \to \mathfrak{g}^*$ is dual to the representation $\pi : \mathfrak{g} \to \mathfrak{u}(V)$. In other words,

$$\Phi(\xi) = \pi^* \left( \frac{\xi \bar{\xi}}{\|\xi\|^2} \right).$$

This is constant on fibres of $p : L^\times \to \mathbb{P}(V)$, hence descends to a moment map on $\mathbb{P}(V)$, also denoted by $\Phi$ and $\Phi(x) = \pi^*(x)$. This restricts to a moment map on $W = G(\lambda)$, again denoted by $\Phi$:

$$\langle \Phi(w), Z \rangle = \frac{1}{i} \mathrm{Tr}(w\pi(Z)) = \frac{\langle v, Zv \rangle}{i\|v\|^2} \quad (v \in \text{Im}(w)).$$
This moment map is equivariant. Indeed

\[ \langle \Phi(g\xi), Z \rangle = \frac{(g\xi, Zg\xi)}{\|g\xi\|^2} = \frac{(\xi, g^{-1}Zg)}{\|\xi\|^2} = \langle \Phi(\xi), g^{-1}Zg \rangle = \langle g(\Phi(\xi)), Z \rangle. \]

So \( \Phi : W \to g^* \) is a covering map onto a coadjoint orbit, namely \( G(\lambda) \) since we have \( \Phi(w_0) = \lambda \). Indeed:

1) \( \Phi(w_0) \in t^* = (g^*)^T \). To verify it, take \( t \) in \( T \) such that \( t = e^H \), then

\[
\begin{align*}
t(\Phi(w_0)) &= \Phi(t(w_0)) = \pi^* \left( \frac{tv_0tv_0}{\|tv_0\|^2} \right) = \pi^* \left( \frac{e^{i\langle\lambda,H\rangle}v_0e^{i\langle\lambda,H\rangle}v_0}{\|e^{i\langle\lambda,H\rangle}v_0\|^2} \right) \\
&= \pi^* \left( \frac{v_0\delta v_0}{\|v_0\|^2} \right) = \Phi(w_0)
\end{align*}
\]

(\( T \)-invariant).

2) For \( H \in t \)

\[ \langle \Phi(w_0), H \rangle = \frac{\langle v_0, Hv_0 \rangle}{i\|v_0\|^2} = \frac{\langle v_0, i\langle\lambda,H\rangle v_0 \rangle}{i\|v_0\|^2} = \langle \lambda, H \rangle, \]

hence \( \Phi(w_0) = \lambda \).

Finally, this covering is trivial (a diffeomorphism) because it’s a covering of a simply connected orbit (Theorem 4.8).

5.4 The space \( H^0(L) \) of antiholomorphic sections

Again, we now have diffeomorphism \( \Phi : W \to X \) identifying the submanifold \( W = G(w_0) \) of \( \mathbb{P}(V) \) with the coadjoint orbit \( X = G(\lambda) \). Denote its inverse by \( I : X \to W \). We have the following diagram:

\[
\begin{array}{ccc}
L & \downarrow & \\
& \searrow & \\
X & \to & W \subset \mathbb{P}(V)
\end{array}
\]

We can pull-back the bundle \( L = \{(w, \xi) \in W \times V : \xi \in \text{Im}(w)\} \) and its connection \( \omega(\delta \xi) / i\|\delta \xi\|^2 \) which satisfies \( d\omega = p^*\sigma \). Let us denote this pull-back again by \( p : L \to X \). Then \( L^\times \) is a complex submanifold of \( V \setminus \{0\} \).

We are going to identify \( V \) with the space \( H^0(L) \) of antiholomorphic sections of \( L \). A section of \( L \) can be defined...
• either as a map \( s : X \to L \) such that
  \[ p(s(w)) = w \]

• or as a map \( f : L^\times \to \mathbb{C} \) such that
  \[ f(z\xi) = \bar{z}f(\xi). \]

with the relation
  \[ s(w) = \frac{\xi f(\xi)}{\|\xi\|^2}, \quad \xi \in \text{Im}(w) \]

and conversely:
  \[ f(\xi) = (\xi, s(w)) \]

which is antilinear in its first entry \( \xi \).

This is called \textit{antiholomorphic}, if \( f \) is antiholomorphic, i.e. \( Df(\xi) : T_{\xi}L^\times \to \mathbb{C} \) is complex antilinear for each \( \xi \).

We make the space \( H^0(L) \) into a (pre-)Hilbert space by putting

\[
(f, f') = \int_X \frac{\bar{f(\xi)}f'(\xi)}{\|\xi\|^2} dx
\]

(the integrand depends only on \( x, \xi \in \text{Im}(x) \)), where \( dx \) is the volume form on \( X \) coming from \( \sigma \wedge \ldots \wedge \sigma \) (the number of factors is \( \dim(X)/2 \)), normalized so that \( \int_X dx = \dim(V) \) — the so-called \textit{Liouville measure on} \( X \).

\textbf{Lemma 5.3.} \textit{Point evaluations on} \( H^0(L) \) \textit{are continuous. (Point evaluation at} \( \xi \text{, is the linear form} f \mapsto f(\xi) \) \forall \xi \exists K_\xi \text{ s.t.}

\[ |f(\xi)| \leq K_\xi \|f\|. \]

\textit{Proof.} See Remark 6.4. \( \square \)

Note, \( G \) acts on \( H^0(L) \) by \( (gf)(\xi) = f(\bar{g}\xi) \) (naturally and unitarily).

\textbf{Theorem 5.4} \textit{(Borel-Weil, non-constructive proof).} \( H^0(L) \) \textit{is isomorphic to} \( V \).

\textit{Proof.} Define a map

\[ V \to H^0(L) \]

\[ \varphi \mapsto f \]

by

\[ f(\xi) = (\xi, \varphi), \]

where \((\cdot, \cdot)\) is the scalar product on \( V \).
1. This map is well-defined, i.e. $f$ is antiholomorphic in $\xi$:

$$Df(\xi)(\delta\xi) = (\delta\xi, \varphi)$$

is indeed antilinear.

2. This map is isometric (hence injective). Schur's Lemma (2.17) gives us

$$A := \int_X \frac{\xi(\xi, \cdot)}{\|\xi\|^2} \, dx = \int_X dx = \lambda \cdot 1,$$

because

$$gAg^{-1} = \int_X g(x) \, dx = \int_X dx = A.$$  

(dx is invariant on $X$ under the $G$-action). Taking trace

$$\text{Tr}(A) = \int_X \frac{\text{Tr}(\xi(\xi, \cdot))}{\|\xi\|^2} \, dx = \lambda \dim(V).$$

We have chosen $dx$ such that

$$\int_X dx = \dim(V),$$

so $\lambda = 1$. Now with $f(\xi) = (\xi, \varphi)$ we get

$$\|f\|^2 = \int_X \frac{\overline{f}(\xi)f(\xi)}{\|\xi\|^2} \, dx = \int_X \frac{(\varphi, \xi)(\xi, \varphi)}{\|\xi\|^2} \, dx$$

$$= (\varphi, \int_X \frac{\xi(\xi, \cdot)}{\|\xi\|^2} \, dx \varphi) = \|\varphi\|^2.$$

Thus, $\varphi \mapsto f = (\cdot, \varphi)$ preserves norm.

3. The map $\varphi \mapsto f$ is onto $H^0(L)$.

Indeed, by the previous Lemma 5.3 and the Riesz representation theorem: For each $\xi \in L^\times$ there is a vector $e_\xi \in H^0(L)$ such that

$$f(\xi) = (e_\xi, f) \quad \forall f \in H^0(L).$$

Now,

$$(e_\xi, f) = \int_X \frac{\overline{e_\xi}(\eta)f(\eta)}{\|\eta\|^2} \, d\eta = \int_X \frac{(e_\xi, e_\eta)}{\|\eta\|^2} f(\eta) \, d\eta.$$  

(The function $K(\xi, \eta) = (e_\xi, e_\eta)$ is called the reproducing kernel of $H^0(L)$.)
We are going to show that 

\[(e_\xi, e_\eta)_{H^0(L)} = (\xi, \eta)\nu.\]

This will finish the proof since 

\[f(\xi) = \int_X \frac{\langle \xi, \eta \rangle}{\|\eta\|^2} f(\eta) \, dy = (\xi, \varphi),\]

where \[\varphi = \int_X \frac{\eta f(\eta)}{\|\eta\|^2} \, dy.\]

Observe:
1) \(e_{g\xi} = ge_\xi\) \(\forall g \in G,\)
2) \(e_{z\xi} = ze_\xi\) \(\forall z \in S^1.\)

Proof of 1)

\[(e_{g\xi}, f) = f(g\xi) = (\bar{g}f)(\xi) = (e_\xi, \bar{g}f) = (ge_\xi, f) \quad \forall f \in H^0(L).\]

Proof of 2)

\[(e_{z\xi}, f) = f(z\xi) = \bar{z}f(\xi) = \bar{z}(e_\xi, f) = (ze_\xi, f) \quad \forall f \in H^0(L).\]

Claim 5.5. \((e_\xi, e_\xi) = c\|\xi\|^2\) for some \(c > 0.\)

Proof. Consider the function on the orbit \(\Phi(x) = \frac{(e_\xi, e_\xi)}{\|\xi\|^2},\) for \(\xi \in \text{Im}(x).\)

Now \(\Phi(g(x)) = \frac{(e_{g\xi}, e_{g\xi})}{\|g\xi\|^2} = \frac{(ge_\xi, ge_\xi)}{\|g\xi\|^2} = \Phi(x),\) for all \(g \in G.\) Hence, \(\Phi(x)\) is constant. \(\square\)

Consequence of the claim:

\[(e_\xi, e_\eta) = c(\xi, \eta), \quad \forall \xi, \eta.\]

This holds because the LHS \((e_\xi, e_\eta) = e_\eta(\xi) = e_\xi(\eta)\) is antiholomorphic in \(\xi,\)

and holomorphic in \(\eta.\) At the same time the RHS \((\xi, \eta)\) is antiholomorphic in \(\xi,\)

and holomorphic in \(\eta\) too. Hence these functions are equal by analytic continuation \([B67, 5.14.7].\)

Now, \(c \neq 0,\) because if 

\[f(\xi) = \int_X \frac{(e_\xi, e_\eta)}{\|\xi\|^2} f(\eta) \, dy = 0, \quad \forall \xi\]
then $H^0(L) = 0$. Which is not true, because we have already put $V$ injectively there. Finally, $c = 1$:

$$c \| \xi \|^2 = (e_\xi, e_\xi)$$

$$= \int_X \frac{e_\xi(\eta)e_\xi(\eta)}{\| \eta \|^2} \, dy$$

$$= \int_X \frac{(e_\xi, e_\eta)(e_\eta, e_\xi)}{\| \eta \|^2} \, dy$$

$$= c^2 \int_X \frac{(\xi, \eta)(\eta, \xi)}{\| \eta \|^2} \, dy$$

$$= c^2 (\xi, \xi) \quad \text{(by Schur’s Lemma)}$$

$$= c^2 \| \xi \|^2.$$

Together $c = c^2$ and $c \neq 0$ give us $c = 1$. This ends the proof of the Borel-Weil Theorem 5.4. 

5.5 Realization in the space $H^0(\lambda)$ of functions on $G_C$

In what follows it will be convenient to realize our representations more uniformly in spaces of antiholomorphic functions on the group $G_C$ itself. To this end, consider $g_C = n^- \oplus t_C \oplus n^+$, where $n^\pm = \bigoplus_{\alpha \in \mathbb{R}^+} g^\alpha$. We define the Borel subalgebra $b = t_C \oplus n^+$ and the resulting standard Borel subgroup $B = \exp(b)$. As we can see from Claims 1-2 in Theorem 5.2, $B$ acts trivially on $w_0$. Then the action

$$bv_0 = \chi(b)v_0, \quad \chi(b) \in \mathbb{C}^*$$

(5.1)

gives us a character on $B$.

The derived action defines an infinitesimal character of $b$:

$$Zv_0 = i\langle \lambda, Z \rangle v_0 = i\langle \lambda, H \rangle v_0$$

(5.2)

where $Z = H + U \in t_C \oplus n^+ \subset g_C$ and we recall that $\lambda \in t^* \subset g^*$ extends to $g_C$ by complex linearity. In addition to our space

$$H^0(L) = \left\{ f : L^\times \to \mathbb{C} : \begin{array}{l} f \text{ is antiholomorphic} \\ f(z\xi) = \overline{zf(\xi)} \quad \forall z \in \mathbb{C}^\times \end{array} \right\},$$

we now consider the space

$$H^0(\lambda) = \left\{ F : G_C \to \mathbb{C} : \begin{array}{l} F \text{ is antiholomorphic} \\ F(gb) = \overline{\chi(b)F(g)} \quad \forall b \in B \end{array} \right\}.$$
Claim 5.6. $H^0(L)$ is isomorphic to $H^0(\lambda)$ with $G$-action\(^4\)

$$(gF)(g') = F(\bar{g}g')$$

by

$$f \mapsto F,$$ where $F(g) = f(gv_0)$.\(^5\)

Proof. Let us denote the above map by $\Phi : H^0(L) \to H^0(\lambda)$, then

$$\Phi : f \mapsto F,$$

$$F(g) = f(gv_0), \text{ for } g \in G_C.$$

First of all, $\Phi$ indeed maps $H^0(L)$ to $H^0(\lambda)$. To check this consider the derivative

$$DF(g)(i\delta g) = [\delta g = Zg] = \frac{d}{dt}F(e^{itZ}g)|_{t=0}$$

$$= \frac{d}{dt}f(e^{itZ}gv_0)|_{t=0} = Df(gv_0)(iZgv_0)$$

$$= -iDf(gv_0)(Zgv_0) \text{ (f is antiholomorphic)}$$

$$= -iDF(g)(Zg) = -iDF(g)(\delta g).$$

Hence, $F$ is antiholomorphic. Also, (5.1) gives us

$$F(gb) = f(gbv_0) = f(g\chi(b)v_0) = \overline{\chi(b)}f(gv_0) = \overline{\chi(b)}F(g).$$

Thus $F$ is in $H^0(\lambda)$.

Second, $\Phi$ is an intertwining map between $H^0(L)$ and $H^0(\lambda)$. Indeed:

(a) Consider $\Phi(gf) \in H^0(\lambda)$. We have $\Phi(gf)(g') = (gf)(g'v_0) = f(\bar{g}g'v_0)$.

(b) At the same time, $g(\Phi f)(g') = (gF)(g') = F(\bar{g}g') = f(\bar{g}g'v_0)$.

Together with the fact that $H^0(\lambda)$ is irreducible as will be proved presently (Theorem 6.2) this completes the proof. \(\square\)

---

\(^4\)We write $g \mapsto \overline{g}$ for the antiautomorphism of $G_C$ whose differential at the identity is $-1$ on $g$ and $1$ on $ig$. Note that for $g \in G$ we have $\overline{g} = g^{-1}$; our formula then extends the action to $G_C$.

\(^5\)Here by $gv_0$ we mean the $G_C$-action on the line bundle $L^\times \subset V^\times$. 
CHAPTER 6
CONSTRUCTIVE PROOF OF THE BOREL-WEIL THEOREM

Our goal now is to prove the following theorem:

**Theorem 6.1 (Borel-Weil construction).** Assume $G$ is one of the classical groups $U(n)$, $SO(n)$, or $Sp(2n)$. Let $\lambda$ be a dominant integral element of $\mathfrak{t}^*$. Then $H^0(\lambda)$ is nonzero and forms an irreducible $G$-module with highest weight $\lambda$.

We have

$$H^0(\lambda) = \left\{ F : G_C \to \mathbb{C} \right. : \begin{array}{ll} F \text{ is antiholomorphic} \\ F(gb) = \chi(b)F(g) \quad \forall b \in B \end{array} \right\} \tag{6.1}$$

together with

$$\chi : B \to \mathbb{C}.$$

Our steps to prove Borel-Weil:

- $H^0(\lambda)$ is irreducible or zero
- $H^0(\lambda)$ is non-zero (in a constructive way).

6.1 $H^0(\lambda)$ is irreducible or zero

We start by making $H^0(\lambda)$ into a unitary $G$-module, as follows. The functions in (6.1) are sections of the bundle $G_C \times_B \mathbb{C} \to G_C/B$ associated to the character $\chi$ of $B$. The Iwasawa decomposition $G_C = GB$ [B04, p. 203] allows us to identify its (complex) base $G_C/B$ with the (compact) manifold $G/(G \cap B) = G/T$, and thus our bundle with the bundle

$$G \times_T \mathbb{C} \to G/T$$

associated to the character $\chi$ restricted to $T$. Sections of it (thought of as functions $F : G \to \mathbb{C}$ such that $F(gt) = \chi(t)F(g)$) then carry a natural inner product given by

$$(F, F') = \int_{G/T} F \overline{F'} \text{vol} \tag{6.2}$$

where $\text{vol}$ is the (unique up to scale) $G$-invariant volume form on $G/T$. This inner product is invariant under the action (5.6)

$$(gF)(g') = F(gg'), \text{ for } F \in H^0(\lambda) \text{ and } g \in G.$$

Using the Kobayashi-Kunze argument [K61; K62] we prove the next

**Theorem 6.2.** $H^0(\lambda)$ is irreducible or zero.
Proof. First we need the following lemma:

**Lemma 6.3.** Point evaluations on $H^0(\lambda)$ are continuous.

**Proof.** The point evaluations are the functionals $A_g : H^0(\lambda) \to \mathbb{C}$ defined by

$$A_g(F) = F(g)$$

where $F \in H^0(\lambda)$ and $g \in G_C$.

It is sufficient to prove that point evaluations functionals are bounded. Take $g$ in $G_C$. We want to prove that

$$|F(g)| \leq C_g \|F\|, \quad \forall F \in H^0(\lambda).$$

Let $(U, \varphi)$ be a holomorphic chart of $G/T = G_C/B$ such that $gT \in U$. Then $\varphi$ maps $U \to \varphi(U) \subset \mathbb{C}^n$ and we have

$$\|F\|^2 = \int_{G/T} |F(uT)|^2 \text{vol}$$

$$(6.3)$$

$$\geq \int_{U} |F(uT)|^2 \text{vol}$$

$$(6.4)$$

$$= \int_{\varphi(U)} (\varphi^{-1})^* (|F(uT)|^2 \text{vol})$$

$$(6.5)$$

Now, $(\varphi^{-1})^* (|F(uT)|^2 \text{vol})$ is the pull-back of the top-degree form $|F(uT)|^2 \text{vol}$, hence

$$(\varphi^{-1})^* (|F(uT)|^2 \text{vol}) = |F \circ \varphi^{-1}(z)|^2 [(\varphi^{-1})^* \text{vol}]$$

$$= |F \circ \varphi^{-1}(z)|^2 p(z) \text{dz},$$

where $p(z)$ is a function on $\varphi(U)$, either positive or negative everywere. Now, (6.5) becomes

$$\|F\|^2 \geq \int_{U} |F(uT)|^2 \text{vol} = \int_{\varphi(U)} |F \circ \varphi^{-1}(z)|^2 p(z) \text{dz}$$

and the last equality forces $p(z)$ to be positive on $\varphi(U)$.

Suppose $z \in \varphi(U)$ is the image of $gT$ under $\varphi : \varphi(gT) = z$. Let $D$ be a ball around $z$ contained in $\varphi(U)$. Denote $F \circ \varphi^{-1}$ by $\tilde{F}$. The application of the Mean Value Theorem for the harmonic function $\tilde{F}$ [A01, p. 6; F00, p. 8] together with the Cauchy-Schwarz inequality gives us

$$|\tilde{F}(z)| \leq \frac{1}{\text{Vol}(D)} \int_D |\tilde{F}(t)| \text{dt} = \frac{1}{\text{Vol}(D)} \int_D \left( \frac{1}{\sqrt{p(t)}} |\tilde{F}(t)| \sqrt{p(t)} \right) \text{dt}$$

$$\leq \frac{1}{\text{Vol}(D)} \left( \int_D \frac{1}{p(t)} \text{dt} \right)^{1/2} \left( \int_D |\tilde{F}(t)|^2 p(t) \text{dt} \right)^{1/2} \leq C_z \|F\|$$

which completes the proof of the lemma. \qed
Remark 6.4. Arguing in the same manner, we obtain the continuity of point evaluations in the space $H^0(L)$ (Lemma 5.3).

Now we shall prove Theorem 6.2. If $H^0(\lambda) = 0$ we are done. Else, we argue much as in the proof of Theorem 5.4, part 3: Consider the $G$-invariant inner product $(\cdot, \cdot)$ on $H^0(\lambda)$, given by (6.2). By the Riesz representation theorem, for every $z \in G_C$ there exists a unique vector $E_z \in H^0(\lambda)$ such that

$$F(z) = (E_z, F), \quad \forall F \in H^0(\lambda).$$

(6.6)

For $w \in G_C$ that gives us

$$E_z(w) = (E_w, E_z).$$

(6.7)

For all $z, w \in G_C$

$$E_z(w) = \bar{E}_w(z).$$

(6.8)

Now, for $g \in G_C$ one has

$$E_{gz} = g E_z.$$  

(6.9)

Indeed, for all $F \in H^0(\lambda)$,

$$(E_{gz}, F) = F(gz) = (\bar{g}F)(z) = (E_z, \bar{g}F) = (gE_z, F).$$

Hence $E_{gz} = g E_z$. Suppose that $H'$ is a nonzero closed subspace of $H^0(\lambda)$ invariant under the $G$-action. Let $E'_z$ be the orthogonal projection of $E_z$ on $H'$. Then $E'_z$ satisfies analogs of (6.6–6.9). Consider the function $\Phi(z) = \frac{E'_z(z)}{E_z(z)}$. Using the Iwasawa decomposition [B04, p. 203] $G_C = GB$ together with the fact that $G$ acts unitarily (but not $G_C$), we get:

For $g \in G_C, g = ub$, where $u \in G$, $b \in B$,

$$\Phi(g) = \frac{E'_g(ub)}{E_g(ub)} = \frac{\chi(b)E'_g(u)}{\chi(b)E_g(u)} = \frac{\bar{\chi}(b)\bar{E'_g}(u)}{\bar{\chi}(b)\bar{E_g}(u)} = \frac{\bar{E}'_u(g)}{\bar{E}_u(g)} = \frac{E'_u(ub)}{E_u(ub)}$$

$$= \frac{\chi(b)E'_u(u)}{\chi(b)E_u(u)} = \frac{E'_u(u)}{E_u(u)} = \frac{(uE'_e, uE'_e)}{(uE_e, uE_e)} = \frac{(E'_e, E'_e)}{(E_e, E_e)} = \Phi(e)$$

so $\Phi(z)$ is constant. That gives us

$$E'_z(z) = cE_z(z), \quad \forall z \in G_C.$$  

(6.10)

For fixed $z$ both $E'_z(w)$ and $cE_z(w)$ are antiholomorphic as functions of $w$ and by (6.8) both are holomorphic functions of $z$, $w$ being fixed. Since they coincide on the diagonal $z = w$, it follows by analytic continuation that they coincide everywhere [B67, 5.14.7]. Therefore (6.10) becomes

$$E'_z(w) = cE_z(w).$$
The condition above easily gives us $H' = H^0(\lambda)$ since any $F$ orthogonal to $H'$ in $H^0(\lambda)$ is orthogonal to $E_z'$ for every $z$ and hence is zero by (6.6). Thus $H^0(\lambda)$ is irreducible.

6.2 $H^0(\lambda)$ is non-zero

In this section we consider a compact Lie group $G$ together with a dominant integral form $\lambda \in t^*$. We want to show that the resulting space $H^0(\lambda)$ does not reduce to the zero function.

6.2.1 Heuristics and $U(n)$

Heuristics 1: Instead of looking for an arbitrary nonzero $F \in H^0(\lambda)$ we will focus on finding a (nonzero) highest weight vector $F \in H^0(\lambda)$. Then $F$ is going to satisfy

$$uF = F \quad \forall u \in N^+ = \exp(\bigoplus_{\alpha > 0} g\alpha).$$

Now,

$$uF = F$$

$$(uF)(g) = F(g)$$

$$F(\bar{u}g) = F(g).$$

Hence we will have

$$F(\bar{u}b) = F(b) = \overline{\chi(b)}F(1)$$

for all $u \in N^+$ and $b \in B$. Picking $F(1) = 1$ defines it entirely in $N^-B$:

$$F(\bar{u}b) = \overline{\chi(b)}. \quad (6.11)$$

But $N^-B$ is known to be open dense in $G_C$ (by the Bruhat decomposition recalled in the Appendix), so our task is to show that (6.11) extends from $N^-B$ to an antiholomorphic function $F$ on all $G_C$.

Heuristics 2: Starting with $G = U(n)$ we consider its complexification $G_C = GL(n, \mathbb{C})$. The Lie algebra $gl(n, \mathbb{C})$ of $G_C = GL(n, \mathbb{C})$ has a root space decomposition

$$gl(n, \mathbb{C}) = n^- \oplus t_C \oplus n^+$$

Here $t_C$ consists of all diagonal matrices. The subalgebras $n^-$ and $n^+$ are of the following form:

$$n^- = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}, \quad n^+ = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$
We denote $t_C \oplus n^+$ by $b$, that gives us

$$b = \begin{cases} 
\begin{pmatrix} 
  d_1 & & \star \\
  & \ddots & \\
 0 & & d_n 
\end{pmatrix} 
\end{cases}.$$ 

Recall that the maximal torus in $U(n)$ consists of all diagonal matrices in $U(n)$. From $n^-$ and $b$ we get $N^-$

$$N^- = \begin{cases} 
\begin{pmatrix} 
  1 & 0 \\
  & \ddots \\
  & & 1 
\end{pmatrix} 
\end{cases}$$

and the Borel subgroup $B$:

$$B = \begin{cases} 
\begin{pmatrix} 
  b_{11} & & \star \\
  & \ddots & \\
 0 & & b_{nn} 
\end{pmatrix} 
\end{cases}.$$ 

Under the identification $(4.2)$, the dominant integral elements $\lambda \in t^*$ take the form

$$\lambda = \begin{pmatrix} 
  \lambda_1 & 0 \\
  & \ddots \\
 0 & & \lambda_n 
\end{pmatrix}$$

with the conditions $\lambda_1 \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. That gives us a character $\chi$ on $T$ in the following way. Write $H$ for the element in $t$,

$$H = \begin{pmatrix} 
  i\theta_1 \\
  & \ddots \\
  & & i\theta_n 
\end{pmatrix}$$

so that $\langle \lambda, H \rangle = \lambda_1 \theta_1 + \cdots + \lambda_n \theta_n$. Then $\chi(\exp(H)) = e^{i\langle \lambda, H \rangle} = (e^{i\theta_1})^{\lambda_1} \cdots (e^{i\theta_n})^{\lambda_n}$ or

$$\chi : \begin{pmatrix} 
  z_1 \\
  & \ddots \\
  & & z_n 
\end{pmatrix} \mapsto (z_1)^{\lambda_1} \cdots (z_n)^{\lambda_n}. \quad (6.12)$$

As soon as we are going to extend the character $\chi$ from $T$ to $B$ and then to a function $F$ on $N^- B$ and eventually on all of $G_C$, we need to use a form other than $(6.12)$ for
CHAPTER 6. CONSTRUCTIVE BOREL-WEIL

38

Putting $l_i = \lambda_i - \lambda_{i+1}$ (with $\lambda_{n+1} := 0$) we have

$$\lambda = \begin{pmatrix} l_1 + \cdots + l_n \\ l_2 + \cdots + l_n \\ \vdots \\ l_n \end{pmatrix}$$

with $l_1, l_2, \ldots, l_{n-1} \in \mathbb{Z}_+$ and $l_n \in \mathbb{Z}$, and (6.12) becomes

$$\chi : \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \mapsto (z_1)^{l_1}(z_2)^{l_2} \cdots (z_n)^{l_n},$$

where $l_1, l_2, \ldots, l_{n-1}$ are non-negative integers and $l_n$ a (possibly negative) integer.

As in (5.2) this extends to a character on $B$. This defines the function (6.11) (highest weight vector) on the big cell $N^- B \subset GL(n, \mathbb{C})$ as follows:

$$F(\lambda u) = F(\lambda b) = \overline{\chi(b)} = \overline{\chi(d)} \quad (6.13)$$

for $l \in N^-$, $d \in T$, and $u \in N^+$. Explicitly, in $GL(n, \mathbb{C})$, this map is

$$F : \lambda u = \begin{pmatrix} \begin{pmatrix} 1 \\ l_{21} \\ l_{31} \\ \vdots \\ l_{n1} \\ l_{n2} \end{pmatrix} \\ \begin{pmatrix} 1 \\ d_{12} \\ d_{13} \\ \vdots \\ d_{1n} \end{pmatrix} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} \begin{pmatrix} 1 \\ u_{12} \\ u_{13} \\ \vdots \\ u_{1n} \end{pmatrix} \begin{pmatrix} 1 \\ u_{23} \\ u_{24} \\ \vdots \\ u_{2n} \end{pmatrix} \begin{pmatrix} 1 \\ u_{34} \\ u_{35} \\ \vdots \\ u_{3n} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ u_{n-1n} \end{pmatrix}$$

$$\mapsto (d_1)^{l_1}(d_1 d_2)^{l_2} \cdots (d_1 d_2 \cdots d_n)^{l_n}.$$

We have defined the function $F$ on the “big cell” $N^- B \subset G_C$ and our task is to show that it extends to an antiholomorphic function on all of $G_C$. We know by the Bruhat decomposition (reviewed in the Appendix) that Weyl group translates $wN^- B$ of the big cell make a covering of $G_C$ by open dense sets. So it is enough to show that (6.13) extends to each translate.

Heuristics 3: Consider the case $G_C = GL(4, \mathbb{C})$. As we know on the big cell $N^- B$ the function $F$ sends

$$\lambda u = \begin{pmatrix} 1 \\ a \\ b \\ d \\ a' \\ b' \\ c' \\ d' \\ e' \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 \\ d' \\ e' \\ f' \end{pmatrix}$$
to

\[(x)^l_1(xy)^l_2(xyz)^l_3(xyzt)^l_4.\]

We consider different translates \(\hat{w}ldu\) of the element \(ldu\), assuming they belong to the big cell, which is open dense in \(G_{Cl}\). Hereafter \(R.E.F.\) stands for row echelon form: \(R.E.F.(ldu) = du\). Note that under our assumptions, \(\hat{w}\) is in the big cell too, so we can decompose it as \(\hat{w}l = l_1d_1u_1\). Now \(F(\hat{w}ldu) = F(l_1d_1u_1du) = F(d_1u_1du) = F(R.E.F.(\hat{w}l)du)\) for \(\hat{w}\) from the big cell \(N^{-B}\).

- Case \(w = (12)\) : then

\[
F(\hat{w}ldu) = F \left( \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 & b & c \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right) = F \left( \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right)
\]

\[
= F \left( \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right) = F(\hat{w}) = (ax)^l_1(xy)^l_2(xyz)^l_3(xyzt)^l_4
\]

- Case \(w = (123) = (12)(23)\) :}

\[
F(\hat{w}ldu) = F \left( \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 & b & c \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right) = F \left( \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right)
\]

\[
= F \left( \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 1 & d' & e' & f' \end{pmatrix} \right) = F(\hat{w}) = (ax)^l_1(xy)^l_2(xyz)^l_3(xyzt)^l_4
\]
\[ F \begin{pmatrix} a & 1 \\ c - a^{-1}b & 0 \\ (c - a^{-1}b)^{-1}a^{-1} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & d' & e' \\ 1 & f' \end{pmatrix} \]

\[ = F \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast \end{pmatrix} \begin{pmatrix} ax \\ (c - a^{-1}b)y \\ (c - a^{-1}b)^{-1}a^{-1}z \end{pmatrix} \begin{pmatrix} \ast \\ \ast \end{pmatrix} \]

\[ = (ax)^{l_1}((ac - b)xy)^{l_2}((acf - ae - bf + d)xyz)^{l_3}(xyzt)^{l_4} \]

- Case \( w = (1234) = (12)(23)(34) \):

\[ F(\bar{w}ldu) = F \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \ast & \ast & \ast \\ a & 1 & \ast & \ast \\ b & c & 1 & \ast \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \begin{pmatrix} 1 & a' & b' & c' \\ 0 & 1 & d' & e' \\ 0 & 1 & f' \end{pmatrix} \]

\[ = F \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{pmatrix} \begin{pmatrix} ax \\ (c - a^{-1}b)y \\ (f - (c - a^{-1}b)^{-1}(e - a^{-1}d))z \end{pmatrix} \begin{pmatrix} \ast \\ \ast \\ \ast \end{pmatrix} \begin{pmatrix} \ast \\ \ast \\ \ast \end{pmatrix} \]

\[ = (ax)^{l_1}((ac - b)xy)^{l_2}((acf - ae - bf + d)xyz)^{l_3}(xyzt)^{l_4} \]

\[ = a^{l_1} \begin{vmatrix} a & 1 \\ b & c \end{vmatrix}^{l_2} \begin{vmatrix} a & 0 \\ b & c \end{vmatrix}^{l_3} (x)^{l_1}((xy)^{l_2}((xyz)^{l_3}(xyzt)^{l_4}. \]

We can see that in each case we have

\[ F(\bar{w}g) = F(\bar{w}ldu) = \frac{\det_1(\bar{w}l)^{l_1} \det_2(\bar{w}l)^{l_2} \det_3(\bar{w}l)^{l_3} x^{l_1}(xy)^{l_2}(xyz)^{l_3}(xyzt)^{l_4}}{l_4}, \]
where $\det_i(m)$ denotes the $i$-th principal minor of a matrix $m$. Observe that $\det_i(\omega ldu) = \det_i(\omega l)\det_i(d)$. Indeed, writing matrices in block form, where the first block has size $i \times i$, we have

$$\omega ldu = \begin{pmatrix} (\omega l)_1 & (\omega l)_2 \\ (\omega l)_3 & (\omega l)_4 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} u_1 & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} (\omega l)_1 & d_1 u_1 & * \\ (\omega l)_3 & 0 & * \end{pmatrix} \begin{pmatrix} (\omega l)_1 & d_1 u_1 & * \\ 0 & 0 & * \end{pmatrix}.$$

This confirms $\det_i(\omega ldu) = \det_i(\omega l)\det_i(d)$.

After this consideration, $F(\omega g)$ above becomes $F(\omega g) = \det_1(\omega g)^{l_1} \ldots \det_n(\omega g)^{l_n}$ or in other words, renaming $\omega g$ as $g$,

$$F(g) = \det_1(g)^{l_1} \det_2(g)^{l_2} \ldots \det_n(g)^{l_n}. \quad (6.14)$$

Now, taking $F : G_C \to \mathbb{C}$ of the form (6.14) we get an antiholomorphic function that clearly extends $F$ from the big cell. This concludes our heuristics and we state:

**Theorem 6.5.** $H^0(\lambda)$ is non-zero for the $U(n)$ case.

**Proof.** We check that the function (6.14) is really in $H^0(\lambda)$. Indeed, by the same block multiplication argument as above we have

$$\det_i(gb) = \det_i(gd'u') = \det_i(gd') = \det_i(g)\det_i(d') = \det_i(g)\det_i(b).$$

Hence

$$F(gb) = \prod_{i=1}^n \det_i(gb)^{l_i} = \prod_{i=1}^n (\det_i(g)\det_i(b))^{l_i} = \chi(b) \prod_{i=1}^n \det_i(g)^{l_i} = \chi(b)F(g).$$

We have constructed a nonzero vector $F$ in the $G$-module $H^0(\lambda)$. \qed

**Remark 6.6.** The function $F$ in (6.14) is indeed antiholomorphic, as the conjugate of a polynomial map. Moreover, if $\lambda \in \mathfrak{t}^*$ is not dominant, then some of the $l_i$ in (6.14) would be negative, forcing $F$ (or even the character (6.12)) to not be (anti)holomorphic. Hence, if $\lambda$ is not dominant, then the space $H^0(\lambda)$ is zero. Also note that Theorems 6.2 and 2.18 imply that $H^0(\lambda)$ is finite-dimensional.
6.2.2 \( \text{Sp}(2n) \)

Let \( J \) denote the \( n \times n \) matrix
\[
\begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
1 & 0
\end{pmatrix}
\]
. Let \( M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \). Define
\[
\text{Sp}(2n, \mathbb{C}) = \{ g \in \text{SL}(2n, \mathbb{C}) : t^gMg = M \}.
\]
In other words, \( \text{Sp}(2n, \mathbb{C}) \) is the set of fixed points of the involution \( \sigma \) of \( \text{SL}(2n, \mathbb{C}) \):
\[
\text{Sp}(2n, \mathbb{C}) = \{ g \in \text{SL}(2n, \mathbb{C}) : \sigma(g) = g \},
\]
given by \( \sigma(g) = M(t^g g^{-1})M^{-1} \).

In this setting, the maximal torus \( T \) of \( \text{Sp}(2n) = \text{Sp}(2n, \mathbb{C}) \cap \text{U}(2n) \) consists of the following matrices
\[
\begin{pmatrix}
e^{i\theta_1} & & \\
& \ddots & \\
& & e^{i\theta_n}
\end{pmatrix}
\]
\[
\begin{pmatrix}
e^{-i\theta_n} & & \\
& \ddots & \\
& & e^{-i\theta_1}
\end{pmatrix}, \quad \theta_i \in \mathbb{R}
\]

Together with the Cartan subalgebra
\[
t = \left\{ \begin{pmatrix} i\theta_1 & & \\
& \ddots & \\
& & i\theta_n \\
& & -i\theta_n \\
& & \ddots \\
& & -i\theta_1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}.
\]

The groups \( N^- \) and \( B \) in \( \text{SL}(2n, \mathbb{C}) \) are the unipotent lower triangular and the upper triangular matrices of determinant 1. Taking their \( \sigma \)-fixed elements we obtain \( N^- \) and \( B \) for \( \text{Sp}(2n, \mathbb{C}) \):
\[
N^- = \left\{ \bar{u} = \begin{pmatrix} 1 & & \\
& \ddots & \\
& & 0 \\
& & \ast \\
& & & 1 \end{pmatrix} : \bar{u} \in \text{Sp}(2n, \mathbb{C}) \right\},
\]
Now, the dominant integral elements $\lambda \in t^*$ under the identification
\[
\text{id} \to g^* \\
\chi \mapsto \frac{1}{2i} \text{Tr}(\chi \cdot),
\]
are as follows
\[
\lambda = \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n \\
-\lambda_n \\
\vdots \\
-\lambda_1
\end{pmatrix}
\]
where $\lambda_1 = l_1 + \cdots + l_n$, $\lambda_2 = l_2 + \cdots + l_n$, \ldots , $\lambda_n = l_n$ for non-negative integers $l_i$.
That gives us a character on $T$ as follows. For $H \in t$,
\[
H = \begin{pmatrix}
i\theta_1 \\
\vdots \\
i\theta_n \\
-i\theta_n \\
\vdots \\
-i\theta_1
\end{pmatrix}
\]
we have $\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i$. Now, $\chi(\exp(H)) = e^{i(\lambda, H)} = (e^{i\theta_1})^{\lambda_1} \cdots (e^{i\theta_n})^{\lambda_n}$ and on $T$:
\[
\chi : \begin{pmatrix}
e^{i\theta_1} \\
\vdots \\
e^{i\theta_n} \\
e^{-i\theta_n} \\
\vdots \\
e^{-i\theta_1}
\end{pmatrix} \mapsto (e^{i\theta_1})^{l_1}(e^{i\theta_1}e^{i\theta_2})^{l_2} \cdots (e^{i\theta_1}e^{i\theta_2} \cdots e^{i\theta_n})^{l_n}
As in (5.2) this extends to a character of $B$:

$$
\chi: \begin{pmatrix}
    b_1 \\ \\
    \vdots \\ \\
    b_n \\ \\
    b_n^{-1} \\ \\
    0 \\ \\
    \vdots \\ \\
    b_1^{-1}
\end{pmatrix} \mapsto (b_1)^{l_1}(b_1b_2)^{l_2} \cdots (b_1b_2\cdots b_n)^{l_n}.
$$

This defines the function (6.11) (highest weight vector) on the big cell $N^{-B} \subset \text{Sp}(2n, \mathbb{C})$ as follows:

$$
F(l du) = F(l b) = \chi(b) = \chi(d)
$$

for $l \in N^{-}$, $d \in T_C$, and $u \in N^{+}$.

Arguing in the same way as for the $\text{GL}(n, \mathbb{C})$ case we obtain the antiholomorphic function $F \in H^0(\lambda)$ of the following form

$$
F(g) = \overline{\det_1(g)^{l_1}\det_2(g)^{l_2}\cdots\det_n(g)^{l_n}},
$$

and we prove

**Theorem 6.7.** $H^0(\lambda)$ is non-zero for the $\text{Sp}(2n)$ case.

### 6.2.3 $\text{SO}(2n + 1)$

Let $N$ be the $(2n + 1) \times (2n + 1)$ matrix

$$
\begin{pmatrix}
    1 \\ \\
    \vdots \\ \\
    1
\end{pmatrix}
$$

Replacing $M$ by $N$ in the definition (6.15) of $\text{Sp}(2n, \mathbb{C})$ we obtain the definition of

$$
\text{SO}(2n + 1, \mathbb{C}) = \{ g \in \text{SL}(2n + 1, \mathbb{C}) : \sigma(g) = g \},
$$

where $\sigma(g) = N^t g^{-1} N^{-1}$.

In the case of $\text{SO}(2n + 1) = \text{SO}(2n + 1, \mathbb{C}) \cap U(2n + 1)$, the maximal torus $T$ consists of the following matrices

$$
\begin{pmatrix}
    e^{i\theta_1} \\ \\
    \vdots \\ \\
    e^{i\theta_n} \\
    1 \\ \\
    e^{-i\theta_n} \\ \\
    \vdots \\ \\
    e^{-i\theta_1}
\end{pmatrix}
$$
The corresponding Cartan subalgebra \( \mathfrak{t} \) is

\[
\begin{pmatrix}
   i\theta_1 \\
   \vdots \\
   i\theta_n \\
   0 \\
   -i\theta_n \\
   \vdots \\
   -i\theta_1
\end{pmatrix},
\]

where \( \theta_i \in \mathbb{R} \). The dominant integral elements \( \lambda \in \mathfrak{t}^* \) under the identification (6.16) take the form

\[
\begin{pmatrix}
   \lambda_1 \\
   \vdots \\
   \lambda_n \\
   0 \\
   -\lambda_n \\
   \vdots \\
   -\lambda_1
\end{pmatrix},
\]

where \( \lambda_1 = l_1 + \cdots + l_n, \lambda_2 = l_2 + \cdots + l_n, \ldots, \lambda_n = l_n \) for non-negative integers \( l_1, \ldots, l_n \).

Write \( H \) for element in \( \mathfrak{t} \) of the form (6.20) so that

\[
\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i = (l_1 + \cdots + l_n)\theta_1 + (l_2 + \cdots + l_n)\theta_2 + \cdots + l_n\theta_n.
\]

Then \( \chi(\exp(H)) = e^{i \langle \lambda, H \rangle} \) or

\[
\chi: \begin{pmatrix}
   z_1 \\
   \vdots \\
   z_n \\
   1 \\
   z_n^{-1} \\
   \vdots \\
   z_1^{-1}
\end{pmatrix} \mapsto (z_1)^{l_1} (z_1 z_2)^{l_2} \cdots (z_1 z_2 \cdots z_{n-1})^{l_{n-1}} (z_1 z_2 \cdots z_n)^{l_n}.
\]

This defines the function (6.11) (highest weight vector) on the big cell \( N^- B \subset SO(2n + 1, \mathbb{C}) \) as follows:

\[
F(1du) = F(1b) = \overline{\chi(b)} = \overline{\chi(d)}
\] (6.21)
for \( l \in \mathbb{N}^-, d \in T_C, \) and \( u \in \mathbb{N}^+. \)

We argue in the same way as for the \( \text{GL}(n,C) \) case to obtain the antiholomorphic function \( F \in H^0(\lambda) \) in the following form

\[
F(g) = \det_1(g)^{l_1} \det_2(g)^{l_2} \cdots \det_n(g)^{l_n},
\]

and we prove

**Theorem 6.8.** \( H^0(\lambda) \) is non-zero for the \( \text{SO}(2n + 1) \) case.

### 6.2.4 \( \text{SO}(2n) \)

Let \( N \) be the \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{pmatrix}
\]

Replacing \( M \) by \( N \) in the definition (6.15) of \( \text{Sp}(2n,C) \) we obtain the definition of

\[
\text{SO}(2n,C) = \{ g \in \text{SL}(2n,C) : \sigma(g) = g \},
\]

where \( \sigma(g) = N(i^{-1}g^{-1})N^{-1} \).

The maximal torus \( T \) of \( \text{SO}(2n) = \text{SO}(2n,C) \cap U(2n) \) consists of the following matrices:

\[
\begin{pmatrix}
e^{i\theta_1} & & & \\
& \ddots & & \\
& & e^{i\theta_n} & \\
& & & e^{-i\theta_n}
\end{pmatrix}.
\]

The corresponding Cartan subalgebra \( t \) is

\[
\begin{pmatrix}
i\theta_1 & & & \\
& \ddots & & \\
& & i\theta_n & \\
& & & -i\theta_n
\end{pmatrix},
\]

\[
\begin{pmatrix}
& & & -i\theta_1
\end{pmatrix},
\]

(6.24)
where $\theta_i \in \mathbb{R}$. The dominant integral elements $\lambda \in \mathfrak{t}^*$ under the identification (6.16) take the form

$$
\begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_n \\
-\lambda_n \\
\vdots \\
-\lambda_1
\end{pmatrix},
$$

where $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$. Defining $l_i = \lambda_i - \lambda_{i+1}$ for $i < n$ and $l_n = \lambda_{n-1} + \lambda_n$ we have

$$
\begin{align*}
\lambda_1 &= l_1 + \cdots + l_{n-1} + \frac{l_n-l_{n-1}}{2} \\
\lambda_2 &= l_2 + \cdots + l_{n-1} + \frac{l_n-l_{n-1}}{2} \\
&\vdots \\
\lambda_{n-1} &= l_{n-1} + \frac{l_n-l_{n-1}}{2} \\
\lambda_n &= \frac{l_n-l_{n-1}}{2}
\end{align*}
$$

where $l_1, \ldots, l_n$ are non-negative integers such that $l_n - l_{n-1} \in 2\mathbb{Z}$. Writing $H$ for the element in $\mathfrak{t}$ of the form (6.24), we have $\langle \lambda, H \rangle = \sum_i \lambda_i \theta_i$ and the character is given on $\mathfrak{t}$ by $\chi(\exp(H)) = e^{i\langle \lambda, H \rangle}$ for $H \in \mathfrak{t}$. Take the element of $\mathfrak{t}$ of the form

$$
t = \begin{pmatrix}
z_1 \\
\vdots \\
z_n \\
\dfrac{1}{z_n} \\
\dfrac{1}{z_n} \\
\vdots \\
\dfrac{1}{z_1}
\end{pmatrix}.
$$

Now we want to construct an antiholomorphic function on $G_{\mathbb{C}}$ out of the character $\chi$. To succeed in that we write $\chi$ in two different ways depending on the sign of $\lambda_n = \frac{l_n-l_{n-1}}{2}$. Namely,

$$
\chi(t) = \begin{cases}
(z_1)^{l_1}(z_1z_2)^{l_2} \cdots (z_1z_2 \cdots z_{n-1})^{l_{n-1}}(z_1z_2 \cdots z_n)^{\frac{l_n-l_{n-1}}{2}} & \text{if } \lambda_n = \frac{l_n-l_{n-1}}{2} \geq 0, \\
(z_1)^{l_1}(z_1z_2)^{l_2} \cdots (z_1z_2 \cdots z_{n-1})^{l_{n-1}}(z_1z_2 \cdots z_n^{-1})^{\frac{l_n-l_{n-1}}{2}} & \text{if } \lambda_n = \frac{l_n-l_{n-1}}{2} < 0.
\end{cases}
$$

This defines the function (6.11) (highest weight vector) on the big cell $N^-B \subset \text{SO}(2n, \mathbb{C})$ as follows:

$$
F(1du) = F(1b) = \overline{\chi(b)} = \overline{\chi(d)} \quad (6.25)
$$
for $l \in \mathbb{N}^-$, $d \in T_C$, and $u \in \mathbb{N}^+$. 

Arguing in the same way as for the $\text{GL}(n, \mathbb{C})$ case we obtain the antiholomorphic function $F \in H^0(\lambda)$ in the following form

$$F(g) = \begin{cases} 
\det_1(g)^l_1 \det_2(g)^l_2 \cdots \det_{n-1}(g)^{l_{n-1}} \det_n(g)^{l_{n-1}-n} & \text{if } \lambda_n = \frac{l_{n-1}}{2} \geq 0, \\
\det_1(g)^l_1 \det_2(g)^l_2 \cdots \det_{n-1}(g)^{l_{n-1}} \det_n(g_0 g_0)^{l_{n-1}-n} & \text{if } \lambda_n = \frac{l_{n-1}}{2} < 0,
\end{cases}$$

where $g_0$ interchanges $e_n$ with $e_{n+1}$:

$$g_0 = \begin{pmatrix}
1 \\
& \ddots \\
& & 0 & 1 \\
& & & 1 & 0 \\
& & & & \ddots \\
& & & & & 1
\end{pmatrix}^{n-n+1}.$$

**Lemma 6.9.** The function $F(g)$, as defined above, indeed extends the function (6.25) from the big cell.

**Proof.** When $\lambda_n \geq 0$, direct computation and the observation that $\det_i(ldu) = d_{i1} \cdots d_{i1}$ complete the proof. When $\lambda_n < 0$, the only difference is that there remains to show that $\det_n(g_0 g_0)$ extends $F(ldu) = d_{11} d_{22} \cdots d_{nn}$. To this end, note that the lower (resp. upper) triangular matrices $l \in \mathbb{N}^-$ and $u \in \mathbb{N}^+$ in $\text{SO}(2n, \mathbb{C})$ have zeros in the $(n+1, n)$, resp. $(n, n+1)$ entry (compare [B05, p. 212]). Indeed, take $l \in \mathbb{N}^-$ of the form

$$l = \begin{pmatrix}
A & 0 \\
C & D
\end{pmatrix}, \text{ so } \quad l^{-1} = \begin{pmatrix}
A^{-1} & 0 \\
-D^{-1} C A^{-1} & D^{-1}
\end{pmatrix}.$$ 

We know that $\sigma(l) = l$ and this gives us $C = -DJ^t C A$. Now let $e_1, e_n$ be vectors from the standard basis in $\mathbb{C}^n$ and compute

$$C_{1n} = {}^t e_1 C e_n$$

$$= -{}^t e_1 DJ^t C A e_n$$

$$= -{}^t e_1 J^t C e_n \quad \text{(because } {}^t e_1 D = {}^t e_1 \text{ and } A e_n = e_n \text{ by triangularity})$$

$$= -{}^t e_n {}^t C e_1$$

$$= -(^t C)_{n1}$$

$$= -C_{1n}.$$
Hence \( l_{n+1,n} = C_{1n} = 0 \). Likewise (taking the transpose) \( u_{n,n+1} = 0 \) for \( u \in \mathbb{N}^+ \) in \( SO(2n, \mathbb{C}) \). This observation confirms that \( g_0 l g_0 \) and \( g_0 u g_0 \) are again unipotent lower (resp. upper) triangular matrices in \( SO(2n, \mathbb{C}) \). Now take \( g = l d u \) in the big cell \( N^- B \) of \( SO(2n, \mathbb{C}) \) and compute

\[
\det_n(g_0 g g_0) = \det_n(g_0 l g_0 d g_0 g_0 u g_0) \\
= \det_n(g_0 l g_0) \det_n(g_0 d g_0) \det_n(g_0 u g_0) \\
= \det_n(g_0 d g_0) \\
= d_{11} d_{22} \cdots d_{nn}^{-1}.
\]

Thus we can state our final result, which completes the proof of Theorem 6.1:

**Theorem 6.10.** \( H^0(\lambda) \) is non-zero for the \( SO(2n) \) case.

**Remark 6.11.** In the case \( \lambda_n < 0 \) we could also write

\[
\chi(t) = (z_1)_{l_1} (z_1 z_2)_{l_2} \cdots (z_1 z_2 \cdots z_{n-1})_{l_{n-1}} (z_1^{-1} z_2^{-1} \cdots z_n^{-1})_{l_{n-1}^{-1} l_n}.
\]

That gives us the function

\[
F(g) = \det_1(g)_{l_1} \det_2(g)_{l_2} \cdots \det_{n-1}(g)_{l_{n-1}} \det_n(g^{-1})_{l_{n-1}^{-1} l_n}
\]

on \( SO(2n, \mathbb{C}) \) which is antiholomorphic for \( \frac{l_{n-1} - l_{n-1}}{2} < 0 \) and coincides with \( (6.25) \) on the big cell.
Appendix A

BRUHAT DECOMPOSITION

In this appendix we review the Bruhat decomposition of the classical complex groups, namely, \( GL(n, \mathbb{C}) \), \( Sp(2n, \mathbb{C}) \), \( SO(2n, \mathbb{C}) \) and \( SO(2n + 1, \mathbb{C}) \), following the exposition in [S74, pp. 72–75].

Let \( J \) denote the \( n \times n \) matrix
\[
\begin{pmatrix}
0 & 1 \\
1 & \ddots \\
& & 1 \\
& & & 0
\end{pmatrix}
\]
Let \( M = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \). Define \( Sp(2n, \mathbb{C}) \) as the set of fixed points of the involution \( \sigma \) of \( SL(2n, \mathbb{C}) \):
\[
Sp(2n, \mathbb{C}) = \{ g \in SL(2n, \mathbb{C}) : \sigma(g) = g \},
given by \( \sigma(g) = M^t g^{-1} M^{-1} \).

Recall that for a compact Lie group \( G \) we use \( T, W, \) and \( B \) to denote a maximal torus of \( G \), the resulting Weyl group, and a standard Borel subgroup of \( G_C \).

**Theorem A.1** (Bruhat Lemma). Let \( G_C = Sp(2n, \mathbb{C}) \). Then,

(a) \( G_C \) can be written as \( G_C = \bigsqcup_{w \in W} BwB \),

(b) \( BwB = Bw'B \iff w = w' \).

**Proof.** Recall that the Weyl group is \( W = N(T)/T \) and denote a representative of the element \( w \in W \) in \( N(T) \) by \( \hat{w} \). In \( GL(2n, \mathbb{C}) \) the Bruhat lemma follows from the Gaussian elimination process and can be stated as follows [H92, pp. 107–109]:
\[
GL(2n, \mathbb{C}) = \bigsqcup_{w \in W} U\hat{w}B,
\]
and each element of \( U\hat{w}B \) is uniquely expressible in the form \( uvb \) with \( b \in B \) and \( u \in U_{\hat{w}} = U \cap \hat{w}U\hat{w}^{-1} \). (Here \( U \) and \( \hat{U} \) are respectively the unipotent upper and lower triangular groups.)

Note that \( \sigma \) keeps \( B, T \) and \( N(T) \) (of \( GL(2n, \mathbb{C}) \)) invariant:

(a) Let \( x \in B \). Then \( \sigma(x) = M(\hat{x}^{-1})M^{-1} \). Now \( x^{-1} \in B \) again, so let \( x^{-1} =
\begin{pmatrix}
X & Y \\
0 & Z
\end{pmatrix}. \) Hence \( \sigma(x) = \begin{pmatrix} J^tZJ & -J^tYJ \\ 0 & J^tXJ \end{pmatrix} \) is in \( B \) too.

(b) Similarly, \( \sigma(T) \subset T \).

(c) Let \( n \in N(T) \). Then \( nTn^{-1} \subset T \) gives us \( \sigma(n)T\sigma(n)^{-1} \subset T \) and thus \( \sigma(n) \in N(T) \).
The automorphism $\sigma$ acts on $U_w$ as follows. If $u \in U_w = U \cap \tilde{w}U\tilde{w}^{-1}$ then

$$\sigma(u) \in \sigma(U) \cap \sigma(\tilde{w})\sigma(\tilde{U})\sigma(\tilde{w})^{-1}.$$ 

Now, $\sigma(\tilde{w}) \in N(T)$, $\sigma(\tilde{U}) \subset \tilde{U}$ and

$$\sigma(u) \in U_{w'}.$$ 

Let $g$ be an element in $Sp(2n, \mathbb{C})$, then

$$g = uw'b \quad \text{uniquely in } GL(2n, \mathbb{C})$$

and

$$g = \sigma(g) = \sigma(u)\sigma(\tilde{w})\sigma(b).$$

Hence

$$g = \sigma(g) = u_{w'}n_{w'}b'$$

(with $u_{w'} \in U_{w'}$, $n_{w'} \in N(T)$ corresponding to $w' \in W$, $b' \in B$).

Using uniqueness of the decomposition, we get $w = w'$, $\sigma(\tilde{w}) \in T\tilde{w}$, $\sigma(u) = u$ and $\sigma(b) = b (\mod T)$. Thus from the Bruhat decomposition in $GL(2n, \mathbb{C})$ we obtain the (unique) decomposition in $Sp(2n, \mathbb{C})$. \qed

Remark A.2. Replacing $-J$ by $J$ in $M$ we obtain the Bruhat Lemma and its proof for $SO(2n, \mathbb{C})$ and $SO(2n + 1, \mathbb{C})$. 

Bibliography


