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Zeckendorf Representation Analysis on Third Order Fibonacci Sequences that Do Not Satisfy the Uniqueness Property

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Zeckendorf Representation Analysis on Third Order Fibonacci Sequences that Do Not Satisfy the Uniqueness Property

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Abstract

Zeckendorf's Theorem states that every natural number can be expressed uniquely as the sum of distinct non-consecutive terms of the shifted Fibonacci sequence (i.e. 1, 2, 3, 5, ...). This theorem has motivated the study of representation of integers by the sum of nonadjacent terms of N-th order Fibonacci sequences, including the characterization of the uniqueness of Zeckendorf representation based on the initial terms of the sequence. Moreover, when this uniqueness property is satisfied for third order Fibonacci sequences, the ratio of integers less than a given number X that have a Zeckendorf representation has been estimated by Dr. Sungkon Chang in [1]. The following thesis will focus on similar results for third order Fibonacci sequences where the uniqueness clause is not satisfied.

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1 Introduction

1.1 Example and Thesis Motivation

Let us start by constructing the Fibonacci sequence, arguably the most famous of integer sequences. We define the first two terms of the sequence as $F_1 = 1$ and $F_2 = 2$ and let each successive term be the sum of the two previous ones. For example, $F_3 = F_2 + F_1 = 2+1 = 3$, $F_4 = F_3 + F_2 = 5$, $F_5 = F_4 + F_3 = 8$, and so on. In this way, we construct the infinite sequence $\{1,2,3,5,8,13,21,34,55,\ldots\}$. This sequence's interesting properties, useful not only in the realm of Number Theory but also in computer science and biology, have spurred the development of new questions that make scholars delve deeper into its unique characteristics. This thesis will focus on the property of the Fibonacci sequence described in Theorem 2.6, also known as Zeckendorf's Theorem, and its generalizations to other similar sequences.

To prompt the reader to pose this question by himself, let us first consider another well-known sequence, the powers of 2 (i.e. $\{2^0,2^1,2^2,2^3,2^4,2^5,\ldots\} = \{1,2,4,8,16,32,\ldots\}$). Known most commonly for being the "language of modern computers," this sequence, as well as other power sequences, has been used to express numbers as a sum of its terms. Let us take, for example, the number 86, which can be expressed as

$$86 = 64+16+4+2 = 2^6+2^4+2^2+2^1.$$

However, this representation of 86 is not the only one using powers of 2. This number can also be represented as

$$86 = 32+32+16+2+2+1+1 = 2^5+2^5+2^4+2^1+2^1+2^0+2^0.$$

Nevertheless, due to the fact that for every n, we have $2^n + 2^n = 2^{n+1}$, we could repeatedly apply this property to the above representation, or to any other for that matter, and arrive to the first representation of 86 we considered. Additionally, using the property that for every n, $\sum_{k=1}^{n} 2^k = 2^{n+1}-1$ we can prove that this first representation is unique. We call this expression of a number as the sum of *distinct* powers of 2 the *base 2* or *binary* representation of a number. Furthermore, using strong induction, we can prove that every positive integer has a binary representation. Thus, for every number we can find its unique binary representation, a unique expression of the number as the sum of distinct powers of 2.

Going back to the case above, since every power of 2 less than 86 can be or not be in the binary representation of 86, the usual binary representation arises as a way to shorten notation. We express $86 = (1010110)_2$ in an analogous way to the decimal expansion, where each digit represents the value assigned to the corresponding power of 10. That is,

$$8\cdot 10^1 + 6\cdot 10^0 = 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0.$$

Notice, however, that the only "digits" that can be present in this representation are 0 and 1 because of the first property described above. Most importantly, any binary representation we construct has a corresponding number represented in the usual base 10, and every positive integer also has a unique representation in base 2. Consequently, we can always find a simple representation of any number as the sum of terms of a sequence without the need for other coefficients other than 0 or 1.

This perfect correspondence between representations in base 10 and base 2, especially considering the simplicity of the latter, might prompt us to ask if this property is unique to this sequence. In particular, considering the case where we try to represent numbers as the sum of terms of Fibonacci numbers might seem logical, even if the result is not exactly the same as that with powers of 2. However, when we try to express 86, again, as the sum of distinct Fibonacci numbers, we arrive at a similar problem as before. We notice

$$86 = 55 + 21 + 8 + 2 = F_9 + F_7 + F_5 + F_2,$$

but since $F_{n+2} = F_{n+1} + F_n$ for all n,

$$86 = F_9 + F_7 + F_4 + F_3 + F_2 = F_9 + F_6 + F_5 + F_4 + F_3 + F_2 = F_9 + F_7 + 2F_4 = \dots$$

In light of this, we might be inclined to impose one more restriction on this Fibonacci sum representation in search for a parallel result for this sequence. Since consecutive terms add up to yet another term in the sequence, it is only natural to add the condition that not only the terms in the sequence be distinct, but also that no two consecutive terms be present in the representation. This representation will be introduced in section 2 and is known as the Zeckendorf representation of a number.

Furthermore, if a result is achieved for this sequence, what can be said about similar sequences constructed with different initial terms or with recurrence relations that could restrict three or more consecutive terms from being present in the representation? Can all numbers be represented with these conditions? If not, what ratio of numbers, approximately, can we estimate to have such representations? These questions are the main concern of the paper that follows.

1.2 Existing Literaure

As mentioned, mathematicians have always been fascinated with sequences of integers, and one that has captivated many a scholar's attention has been the Fibonacci sequence. In 1939, Édouard Zeckendorf (1901-1983), a Belgian mathematician, discovered an interesting property of this sequence: *all positive integers can be uniquely expressed as the sum of distinct non-consecutive Fibonacci numbers*. However, it was not until 1972 that Zeckendorf published [2] explaining his decades-long ideas claiming this theorem that now holds his name. (see [3]) Independently, Dutch mathematician Gerrit Lekkerkerker published [4] in 1951, producing similar results that later inspired the research team that published [5] in 1960, proving the Fibonacci sequence to be the only sequence of natural numbers to uniquely represent numbers in what we now call Zeckendorf representation.

In recent years, many mathematicians have been diving deeper into the implications of this theorem and its generalizations. A group that stands out is the authors of [6]–a research group from Williams College that has been generalizing this result to other sequences defined by different recurrence relations. They have developed a “Zeroing Algorithm" that might help produce a generalized Zeckendorf representation for sequences generated by homogeneous recurrence relations with positive coefficients, albeit sacrificing the distinct terms condition. In light of the Fibonacci sequence being the only sequence satisfying Zeckendorf's theorem, studied in [1] is the ratio of numbers expected to have Zeckendorf representation for Fibonacci-like sequences where a unique representation is guaranteed. [1]

2 Counting Third Order Zeckendorf Integers

Before introducing the results that lay most of the groundwork for our central question, the reader must first be familiarized with the notation provided in the next subsection. We will start by providing simplified generalized definitions used in the existing literature and will later narrow down the scope of our investigation to third order sequences and representations.

2.1 Preliminary Definitions and Notation

Let $\mathbb{N}$ denote the set of positive integers. A sequence $\{G_k\}_{k=1}^{\infty}$ of numbers is simply denoted by the capital letter G. Given a sequence G and a finite subset A of $\mathbb{N}$, let $\sum_A G_k$ denote $\sum_{k\in A} G_k$.

Definition 2.1. Let $N \geq 2$ be a positive integer. Let a sequence G be defined by the following linear recurrence where $k \in \mathbb{N}$ and a_i are pairwise distinct positive integers for $i \in \{1,2,\ldots,N\}$

$$G_{k+N} = G_{k+N-1} + G_{k+N-2} + \cdots + G_k,\ (G_1, G_2, \ldots, G_N) = (a_1, a_2, \ldots, a_N)$$

Then, G is said to *satisfy the N-th order Fibonacci recurrence*. Furthermore, if $a_i = 2^{i-1}$ for all $i \in \{1,2,\ldots,N\}$, then G is called the *N-th order Fibonacci sequence*.

In particular, if $N = 2$ and $a_1 = 1, a_2 = 2$, this sequence is denoted by F and is called the *Fibonacci sequence*. In addition, in the case where $N = 3,\ a_1 = 1,\ a_2 = 2,$ and $a_3 = 4$, H will denote this sequence, which will be referred to as the *third order Fibonacci sequence*. Notice that in most of the literature, the initial terms of the Fibonacci sequence are usually set to be either $(F_1, F_2) = (0,1)$ or $(1,1)$. However, using this "shifted" sequence facilitates the description of the uniqueness property introduced in Definition 2.5 and its relation to Fibonacci recurrences.

Definition 2.2. Let $\mathscr{F}_N$ be the set of all finite sets $A \subset \mathbb{N}$ such that if $\{k, k+1, \ldots, k+N-2\} \subseteq A$ for $k \in \mathbb{N}$, then $k+N-1 \notin A$. Such set A is then called an *N-th order Zeckendorf index subset* and $\mathscr{F}_N$ the *collection of N-th order Zeckendorf expressions*.

Thus, considering any particular N, Zeckendorf index subsets are sets of positive integers that contain no more than $N-1$ consecutive integers. This construction is then applied to the indices of the terms of any particular sequence G (hence the name) as described in the definition below. Most of the thesis will focus on third order Zeckendorf representations; therefore most of the index subsets we will deal with involve index subsets with no more than two consecutive integers.

Definition 2.3. Let G be a sequence that satisfies the N-th order Fibonacci recurrence. A positive integer m is called an *N-th order Zeckendorf integer* for G if $m = \sum_A G_k$ for some set $A \in \mathscr{F}_N$. Furthermore, the expression $\sum_A G_k$ is called an *N-th order Zeckendorf expansion of m with G*.

As an example, in a third order Zeckendorf representation context,

$$86 = 81 + 4 + 1 = H_8 + H_3 + H_1$$

would be considered a (third order) Zeckendorf expansion of 86 with H, but

$$86 = 44 + 24 + 13 + 4 + 1 = H_7 + H_6 + H_5 + H_3 + H_1$$

would not since it contains 3 consecutive terms of H in the expansion.

Definition 2.4. In the context of N-th order Zeckendorf representations, let

$$R_G := \left\{ n \in \mathbb{N} : \exists A \in \mathscr{F}_N : n = \sum_A G_k \right\}$$

be the set of N-th order Zeckendorf integers for G, and let $R_G(X) = \{m \in R_G \mid m \le X\}$. It is said that G *represents* $\mathbb{N}$ if $R_G = \mathbb{N}$.

As we will discuss later, it will be important to know that the only sequence that satisfies the N-th order Fibonacci recurrence and uniquely represents $\mathbb{N}$ in the context of N-th order representations is the N-th order Fibonacci sequence, result found in Theorem 2.8. For example, if we consider a sequence G that satisfies the third order Fibonacci recurrence with $(G_1, G_2, G_3) = (5, 2, 3)$, the number 86 is a third order Zeckendorf integer for G since it can be represented as

$$86 = 53 + 28 + 5 = G_7 + G_6 + G_1,$$

but there is no third order Zeckendorf expansion of 87 with G. Therefore, for any other sequence G that satisfies the N-th order Fibonacci recurrence, our focus will be on calculating an approximation of $R_G(X)$ for all X.

For the Fibonacci sequence, it is easier to find all Zeckendorf integers since every integer has its unique Zeckendorf representation (see Theorem 2.6), similar to that discussed for the binary representation. Unfortunately, not every sequence has this property. Using the example above,

$$\begin{aligned} 86 &= 53 + 28 + 5 = G_7 + G_6 + G_1 \\ &= 53 + 28 + 3 + 2 = G_7 + G_6 + G_3 + G_2 \end{aligned}$$

Therefore, for this particular sequence G there are some numbers that may have more than one valid Zeckendorf representation. This motivates the following distinction:

Definition 2.5. Let $\{A, B\} \subset \mathscr{F}_N$. It is said that a sequence G *satisfies the unique expansion property under* $\mathscr{F}_N$ if $\sum_A G_k = \sum_B G_k$ implies that $A = B$.

Dealing with sequences that satisfy this uniqueness property facilitates the counting process because using the injective relationship between Zeckendorf expressions and Zeckendorf integers, one can count the number of Zeckendorf expressions and consequently find the number of Zeckendorf integers in return. (see Theorem 2.7) However, for sequences that do not satisfy this property, this process produces double countings of Zeckendorf integers by just considering the count of Zeckendorf expressions. It is the main concern of this paper, then, to determine when these repeated expressions occur in order to find the average number of third order Zeckendorf integers for sequences that do not satisfy the uniqueness property.

2.2 Zeckendorf's Theorem and Some of its Generalizations

This next subsection is dedicated to introducing some important results that prepare us to understand the context of our research question.

Theorem 2.6 ([2], Zeckendorf's Theorem)**.** *Every positive integer m can be uniquely expressed as the sum of distinct, non-consecutive terms of F.*

That is, for every positive integer m, there exists one and only one (second order) Zeckendorf index subset $A \in \mathscr{F}_2$ such that $m = \sum_A F_k$. The proof of this theorem by Zeckendorf himself can be found in his article[2].

We can generalize this result to any sequence that satisfies the N-th order Fibonacci recurrence as follows:

Theorem 2.7 (Generalized Zeckendorf's Theorem)**.** *For every positive integer m and for all $N \in \mathbb{N}$, there exists one and only one N-th order Zeckendorf index subset $A \in \mathscr{F}_N$ such that $m = \sum_A H_k$, where H is the N-th order Fibonacci sequence.*

Furthermore, the following result provides a better understanding of why this property is stated for Fibonacci sequences and not other sequences in general.

Theorem 2.8 ([5], Weak Converse of Zeckendorf's Theorem)**.** *In the context of N-th order Zeckendorf representations, if $R_G = \mathbb{N}$ with unique representation and G is an increasing sequence, then G is the N-th order Fibonacci sequence.*

In other words, the only increasing sequence and uniquely represents represents $\mathbb{N}$ with N-th order Zeckendorf expressions is the N-th order Fibonacci sequence.

Definition 2.9. In the context of $A \in \mathscr{F}_N$, if $A = \varnothing$ then define $\max(A) = 0$ and $\min(A) = \infty$.

Theorem 2.10 ([1], Theorem 6)**.** *A sequence G that satisfies the N-th order Fibonacci recurrence also satisfies the unique expansion property under $\mathscr{F}_N$ if and only if there are disjoint index subsets $A^* \neq \{\}$ and B^* contained in $\{1,2,\dots,N-1\}$ and a sequence $\epsilon'_k = 0,1,2$ such that $j_0 := \max A^* \geq \max B^*$ and*

$$\sum_{k\in A^*} G_k = \sum_{k\in B^*} G_k + \sum_{k=j_0+1}^{N} \epsilon'_k G_k$$

This now gives us a characterization of the sequences that do not satisfy the uniqueness property. Determining with certainty the kinds of sequences that have multiple representations for Zeckendorf integers and describing the shape of those corresponding representations will form a major part of this paper.

2.3 Third Order Fibonacci Sequences that Satisfy the Uniqueness Property

We now shift our attention only to third order Zeckendorf expressions and sequences that satisfy the third order Fibonacci recurrence. In this manner, we will use the notation described above assuming $N = 3$, also omitting the order of Zeckendorf representations or index subsets (e.g. $\mathscr{F} = \mathscr{F}_3$) from now on to simplify readability.

By Theorem 2.10, a Fibonacci recurrence G has the property of unique expansion if and only if none of the following are satisfied:

(1) $G_1 = 2G_2$

(2) $G_1 = G_2 + G_3$

(3) $G_1 = G_2 + 2G_3$

(4) $G_1 = 2G_2 + G_3$

(5) $G_1 = 2G_2 + 2G_3$

(6) $G_1 = 2G_3$

(7) $G_2 = G_1 + G_3$

(8) $G_2 = G_1 + 2G_3$

(9) $G_2 = 2G_3$

(10) $G_1 + G_2 = G_3$

(11) $G_1 + G_2 = 2G_3$

We notice that the sequence $H = \{1, 2, 4, 7, 13, 24, 44, 81, \ldots\}$ as defined in Section 2.1 does not satisfy any of these conditions. Thus, H satisfies the unique expansion property, also clearly observed, since H represents $\mathbb{N}$ uniquely by Theorem 2.7.

Example 2.11. We can extend this notion to other Fibonacci recurrences that are not the Fibonacci sequence. For example, let us consider a truncated version of H, that is, this sequence without its first term $H_1 = 1$. That is

$$\hat{H} = \{2, 4, 7, 13, 24, 44, 81, 149, \ldots\}$$

This sequence has now lost its property of complete representation of $\mathbb{N}$ since numbers like 86, which had a unique representation under H that included H_1 ($86 = H_8 + H_3 + H_1$), now lack a representation under $\hat{H}$. However, all other numbers whose representation with H did not include H_1 now keep their unique representation under $\hat{H}$. Therefore, $\hat{H}$ still maintains the uniqueness property, also observed when comparing the conditions stated at the beginning of this section with the initial terms of $\hat{H}$.

Example 2.12. Another example sequence that satisfies the uniqueness property based on its in-adherence to any of the above conditions is the Fibonacci recurrence G with $(G_1, G_2, G_3) = (2, 5, 6)$. However, a sequence with a slight change in initial terms like $(G_1, G_2, G_3) = (2, 5, 7)$ is such that $G_1 + G_2 = G_3$, creating double representations for numbers such as 21, which can be represented as $G_4 + G_2 + G_1 = 14 + 5 + 2$ or $G_4 + G_3 = 14 + 7$. In these specific cases, Zeckendorf expressions are not in one-to-one correspondence with Zeckendorf integers. Therefore, finding a bijective mapping from Zeckendorf representations to Zeckendorf integers is only possible for sequences that do not satisfy the conditions above and therefore satisfy the unique expansion property.

Now, for those sequences that satisfy the uniqueness representation, we notice that not all represent $\mathbb{N}$. Therefore, a question we might ask ourselves is "what proportion of numbers can we actually represent with this sequence?". For sequences satisfying the uniqueness property, the act of counting how many possible representations it can generate whose sum does not exceed a certain X corresponds exactly to the Zeckendorf integers less than X because of the presence of one and only one representation for each of these integers. This was the method used in [1] and described in detail in Section 3.3.1 to calculate the average number of Zeckendorf integers for sequences that satisfy the uniqueness property. For now, we concentrate on describing the relevant parts of a Fibonacci recurrence sequence G that define when the expansion is not unique to be able to calculate the proportion of representable numbers for any G.

3 Non-unique Expansions for Third Order Sequences

As discussed previously, some Fibonacci recurrence sequences might have more than one representation for each of their Zeckendorf integers. Since the proof of the statements described above relied on the sequence G satisfying the uniqueness property in order to create a bijection between Zeckendorf representations and Zeckendorf integers, a new method has to be established to calculate $|R_G(X)|$ for all other sequences that do not satisfy the unique expansion property. By analyzing the correspondence between initial conditions of the first terms of G and the structure of the repeated representations, we aim to achieve this goal in the following section.

To understand the structure of non-unique expansions, we will discuss further some aspects present in the proof of Theorem 2.10 for third order expansions. We will assume that G is any third order Fibonacci recurrence sequence in this section with pairwise distinct G_1, G_2, G_3.

3.1 Analysis on Body Parts and its Expansions

In this section, we introduce the notions of tails and bodies of a sequence, which will prove useful when describing the conditions when a Zeckendorf integer could have multiple representations. We start by providing some results on the sum of the first terms of a Fibonacci recurrence sequence; for this, recall $\mathscr{F}$ from Definition 2.2 and Section 2.3.

Lemma 3.1. $G_1 < G_4$, and if $A \in \mathscr{F}$ and $m := \max(A) \geq 2$, then $\sum_A G_k < G_{m+2}$.

Proof. Notice $G_1 < G_1 + G_2 + G_3 = G_4$. Now assume that $A \in \mathscr{F}$ and $m := \max(A) \geq 2$. We proceed to prove the lemma by induction on m.
For $m = 2$,

$$\sum_A G_k \leq G_1 + G_2 < G_1 + G_2 + G_3 = G_4 = G_{m+2}.$$

Assume the proposition holds for all $2 \leq m < M$ for some $M > 2$. Let $A \in \mathscr{F}$ be such that $\max(A) = M$ and let $A' = A - \{M\}$. Then, since $\max(A') \leq M - 1$

$$\sum_A G_k = G_M + \sum_{A'} G_k < G_M + G_{M+1} < G_{M-1} + G_M + G_{M+1} = G_{M+2}.$$

Therefore, this result holds for all $m \geq 2$ by induction. □

Corollary 3.2. Let $\{A, B\} \subset \mathscr{F}$ such that A and B are disjoint, $\sum_A G_k = \sum_B G_k$, $m := \max(A) < \max(B)$, and $m \geq 2$. Then, $\max(B) = m + 1$.

Proof. Assume for the sake of contradiction that $\max(B) \geq m + 2$. Then

$$\sum_B G_k \geq G_{m+2} > \sum_A G_k.$$

Thus, $m = \max(A) < \max(B) < m + 2 \Rightarrow \max(B) = m + 1$. □

Definition 3.3. Let $\mathscr{F}_\circ := \{A \in \mathscr{F} : \min(A) \geq 3\}$, and let $\overline{\mathscr{F}} := \{A \in \mathscr{F} : \max(A) \leq 2\}$. Let $G_0 := 0$ and $G_\infty = \infty$.

Lemma 3.4. If $A \in \mathscr{F}_\circ$ and $m := \max(A)$, then $\sum_A G_k < G_{m+1}$.

Proof. Let $A \in \mathscr{F}$. If $m = 0$, with $\min(A) = \infty$ and $A = \emptyset$, then $\sum_A G_k = 0 < G_1$. Consider, then, A to be nonempty and $m \geq \min(A) \geq 3$. We will proceed by induction to prove this proposition.
Assume the base case where $m = 3$. Then, $\sum_A G_k = G_3 < G_1 + G_2 + G_3 = G_4$. Now assume that the proposition is true for all $3 \leq m < M$ for some $M > 3$, and let $A \in \mathscr{F}_\circ$ such that $\max(A) = M$. If $M-1 \in A$, let $A' = A - \{M-1, M\}$. If $A' = \emptyset$ then $\sum_A G_k = G_{M-1} + G_M < G_{M-2} + G_{M-1} + G_M = G_{M+1}$, where $M \geq 4$. If, on the other hand, $A' \neq \emptyset$, then $\sum_A G_k = \sum_{A'} G_k + G_{M-1} + G_M$. Since $A' \in \mathscr{F}_\circ$, by the induction hypothesis with $m' := \max(A') \leq M-3$, $\sum_{A'} G_k < G_{M-2}$. Thus, $\sum_A G_k < G_{M-2} + G_{M-1} + G_M = G_{M+1}$. If $M-1 \notin A$, let $A' = A - \{M\}$. If $A' = \emptyset$ then $\sum_A G_k = G_M < G_{M-2} + G_{M-1} + G_M = G_{M+1}$. If, on the other hand, $A' \neq \emptyset$, then $\sum_A G_k = \sum_{A'} G_k + G_M$. Since $A' \in \mathscr{F}_\circ$, by the induction hypothesis with $m' := \max(A') \leq M-2$, $\sum_{A'} G_k < G_{M-1}$. Thus, $\sum_A G_k < G_{M-1} + G_M < G_{M-2} + G_{M-1} + G_M = G_{M+1}$. Therefore, for all $A \in \mathscr{F}_\circ$, we have $\sum_A G_k < G_{m+1}$. □

Definition 3.5. Let $\mathscr{P}$ be the collection of finite subsets of $\mathbb{N}$. Let $\{A, B\} \subset \mathscr{P}$.
Define $A < B$ if $\max(A - S) < \max(B - S)$ where $S = A \cap B$. We call this order on $\mathscr{P}$ the *lexicographical order* on the collection of finite subsets of $\mathbb{N}$.

Note that under the lexicographical order, the set of finite subsets of $\mathbb{N}$ is totally ordered. Let us focus now on the conditions that might cause two Zeckendorf expressions to yield the same numerical value.

Theorem 3.6. *Let $\{A, B\} \subset \mathscr{F}_\circ$. Then $A < B$ if and only if $\sum_A G_k < \sum_B G_k$, and $\sum_A G_k = \sum_B G_k$ implies $A = B$.*

Proof. $(\Rightarrow)$ Suppose that $A < B$. Then $A' < B'$ and $m' = \max(A') < n' = \max(B')$, which implies $\sum_{A'} G_k < G_{m'+1} \leq G_{n'} \leq \sum_{B'} G_k$. Therefore $\sum_A G_k < \sum_B G_k$.
$(\Leftarrow)$ Now assume that $\sum_A G_k < \sum_B G_k$. Then, $\sum_{A'} G_k < \sum_{B'} G_k$. If $n' = \max(B') < m' = \max(A')$, then $\sum_{B'} G_k < G_{n'+1} \leq G_{m'} \leq \sum_{A'} G_k$, which contradicts the supposition. Since $m' \neq n'$, then $m' < n'$, that is $A < B$.
Now, consider the case when $\sum_A G_k = \sum_B G_k$. Since $\sum_A G_k \not> \sum_B G_k$, then $A \not> B$. Similarly, since $\sum_A G_k \not< \sum_B G_k$, then $A \not< B$. Therefore, $A = B$. □

This provides now our first insight into a characterization for when a sequence satisfies the uniqueness property. If for a sequence G there exist distinct $A, B \in \mathscr{F}$ such that $\sum_A G_k = \sum_B G_k$, then A and B cannot be both members of $\mathscr{F}_\circ$. That is, the first two terms in G are relevant to describe the relationship between representation sets of the same Zeckendorf integer. This is the motivation for the following definition, accompanied by three lemmas that describe the behavior of these first terms on sets A and B when considering the case when $\sum_A G_k = \sum_B G_k$,but $A \neq B$ and the restrictions that lead to the full characterization of sequences G that do not satisfy the uniqueness property. Moreover, if a sequence G' is generated by shifting any Fibonacci recurrence sequence by at least two terms, it must satisfy the uniqueness representation property.

Definition 3.7. Let $A \subset \mathbb{N}$. Define $A[k,l] := \{n \in A : k \leq n \leq l\}$ where $k, l \in \mathbb{R} \cup \{\infty\}$.
Define $\mathrm{b}(A) := A[3,\infty]$ and $\mathrm{t}(A) := A[1,2]$. The subset $\mathrm{b}(A)$ is called the *body of A*, and $\mathrm{t}(A)$ is called the *tail of A*.

Lemma 3.8. If $A \in \mathscr{F}$, then $\mathrm{b}(A) \in \mathscr{F}_\circ$ and $\mathrm{t}(A) \in \overline{\mathscr{F}}$.

Proof. The proof of this statement follows from the definitions. □

Note that it is not necessarily true that $A \cup B \in \mathscr{F}$ if $A \in \mathscr{F}_\circ$ and $B \in \overline{\mathscr{F}}$. For example, $A = \{3\} \in \mathscr{F}_\circ$ and $B = \{1,2\} \in \overline{\mathscr{F}}$, but $A \cup B = \{1,2,3\} \notin \mathscr{F}$.

Lemma 3.9. If $\{A,B\} \subset \overline{\mathscr{F}}$, then $\left|\sum_A G_k - \sum_B G_k\right| \in \{0, |G_1 - G_2|, G_1, G_2, G_1 + G_2\}$, and in particular, $\left|\sum_A G_k - \sum_B G_k\right| \le G_1 + G_2$.

Proof. This follows from the fact that $\overline{\mathscr{F}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. □

Lemma 3.10. Let $\{A,B\} \subset \mathscr{F}$ be such that $A < B$ and $\sum_A G_k = \sum_B G_k$. Then, $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k \le G_1 + G_2$. Moreover, if $3 \in A$, then $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k < G_1 + G_2$.

Proof. Since $A < B$, then $\mathrm{b}(A) \le \mathrm{b}(B)$. This is because $\max(A) < \max(B)$ by Definition 3.5.
If $\max(B) \in \mathrm{b}(B)$, then $\mathrm{b}(A) < \mathrm{b}(B)$; otherwise, $\mathrm{b}(A) = \mathrm{b}(B) = \emptyset$. Then, since $\mathrm{b}(A), \mathrm{b}(B) \in \mathscr{F}_\circ$, by Theorem 3.6, $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k \ge 0$. Moreover, if $\sum_A G_k = \sum_B G_k$

$$\begin{aligned}
&\Rightarrow \sum_{\mathrm{b}(A)} G_k + \sum_{\mathrm{t}(A)} G_k = \sum_{\mathrm{b}(B)} G_k + \sum_{\mathrm{t}(B)} G_k \\
&\Rightarrow \sum_{\mathrm{t}(A)} G_k - \sum_{\mathrm{t}(B)} G_k = \sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k \\
&\qquad\qquad\qquad\qquad\qquad \le G_1 + G_2.
\end{aligned}$$

Furthermore, if $3 \in A$, then $\sum_{\mathrm{t}(A)} G_k \ne G_1 + G_2$.

$$\Rightarrow \sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = \sum_{\mathrm{t}(A)} G_k - \sum_{\mathrm{t}(B)} G_k < G_1 + G_2.$$

□

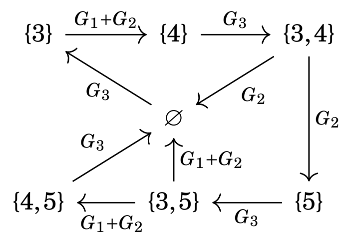
3.2 Double Expansions of Zeckendorf Integers

This section introduces the concept of least upper bounds and provides detailed directed graph diagrams that describe the relationship between the shapes of the first terms of a sequence and the possible structures of double expansions for the same Zeckendorf integer. However, for some of these results, the proof will be omitted and provided in a subsequent paper given that they are out of the scope of this paper.

Definition 3.11. Let $\mathscr{T}$ be an infinite subcollection of $\mathscr{P}$, and let $A \in \mathscr{T}$. The subset $\mathrm{lub}_{\mathscr{T}}(A)$ is called the the least upper bound of A in $\mathscr{T}$ if $B \in \mathscr{T}$ and $A < B$ implies $A < \mathrm{lub}_{\mathscr{T}}(A) \le B$. Given $n \in \mathbb{N}_0$, let $\mathrm{lub}^n_{\mathscr{T}}(A)$ denote the nth iteration of $\mathrm{lub}_{\mathscr{T}}$ on A. Furthermore, if $A \in \mathscr{F}_\circ$, then we denote $\mathrm{lub}_{\mathscr{F}_\circ}(A)$ simply by $\tilde{A}$.

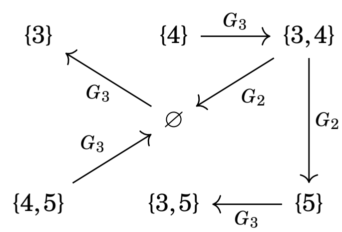
Proposition 3.12. Let $\mathscr{T}$ be an infinite subcollection of $\mathscr{P}$. Then, $\mathrm{lub}_{\mathscr{T}}(A)$ exists for all $A \in \mathscr{T}$.

Proposition 3.13. Let $A \in \mathscr{F}_\circ$. Then, $A[3,5]$ is one of the subsets listed below. The arrows in the diagram below point to $\tilde{A}[3,5]$, and $\sum_{\tilde{A}} G_k = \sum_{\overline{A}} G_k + \sum_A G_k$, where $\overline{A} \subset \{1,2,3\}$ and $\sum_{\overline{A}} G_k$ is the value attached to the arrow:



Proposition 3.14. Let $\{A,B\} \subset \mathscr{F}_\circ$ be such that $A[3,5] = \{3,4\}$ and $B[3,5] = \{3,5\}$. Then, $\tilde{A}[3,5] = \emptyset$ if and only if $\{6,7\} \subset A$, and $\tilde{B}[3,5] = \emptyset$ if and only $\{6\} \subset B$.

Now, consider the situation where $\sum_A G_k = \sum_B G_k$, and take into account the note under Lemma 3.8, particularly when considering $\sum_{\overline{A}} G_k$. By Lemma 3.10 and since $3 \in \{3\}$ and $3 \in \{3,5\}$, for the purposes of calculating $\tilde{A}[3,5] \in \mathscr{F}$, the following diagram, which we will refer to as Λ_G, is more appropriate.



Definition 3.15. We define $\text{Path}(\Lambda_G)$ as the nonrepeated paths in Λ_G. Given $\gamma \in \text{Path}(\Lambda_G)$, let $\text{len}(\gamma)$ denote the number of edges in γ, and let $\text{wt}(\gamma)$ be the sum of the weights along γ.

Definition 3.16. Let $\mathfrak{B} := \{G_2, G_3, G_2 + G_3, G_2 + 2G_3, 2G_3\}$ denote the the set of the sums of the weights of non-repeated paths in Λ, and let $\mathfrak{T} := \{G_1, G_2, G_1 + G_2, G_2 - G_1, G_1 - G_2\}$ denote the set of $\left|\sum_{\text{t}(A)} G_k - \sum_{\text{t}(B)} G_k\right|$ where $\{A,B\} \subset \mathscr{F}$ and $\text{t}(A) \neq \text{t}(B)$.

Theorem 3.17. *Let $\{A,B\} \in \mathscr{F}$ such that $A < B$. Let $A^* = A[3,5]$ and $B^* = B[3,5]$.*
Then, $\sum_A G_k = \sum_B G_k$ if and only if all the following are satisfied:

(1) There is a non-repeated path γ in Λ beginning at A^* and ending at B^* such that $n := \text{len}(\gamma) \geq 1$,

(2) $\text{lub}^n_{\mathscr{F}_\circ}(\text{b}(A)) = \text{b}(B)$,

(3) $\text{wt}(\gamma) \in \mathfrak{B}$ and $\text{wt}(\gamma) = \sum_{\text{b}(B)} G_k - \sum_{\text{b}(A)} G_k$,

(4) $\sum_{\text{t}(A)} G_k - \sum_{\text{t}(B)} G_k \in \mathfrak{T}$ and $\sum_{\text{t}(A)} G_k - \sum_{\text{t}(B)} G_k = \sum_{\text{b}(B)} G_k - \sum_{\text{b}(A)} G_k$.

Using the above theorem, we devise an algorithm that translates paths in G into body and tail shapes corresponding to the conditions described before that create double countings in sequences that do not satisfy the uniqueness property. The following subsection dicusses this in detail.

3.3 Average Number of Zeckendorf Integers

We will use Theorem 3.17 to categorize all paths of Λ and the corresponding body and tail differences possible with those restrictions. The first of these algorithms can be observed in Appendix A, where a path is chosen, which determines the body difference in question. Parting from there, determine the possible existence of tails that do not contradict any of the conditions already imposed on members of $\mathscr{F}$ such as nonzero pairwise distinct initial terms of the sequence G or the absence of three consecutive integers in A.

3.3.1 Counting Formulas

Definition 3.18. Let α denote the only positive real root of $x^3 - x^2 - x - 1 = 0$, which is approximately 1.839.

Recall Definition 2.4. The following theorem describes the cardinality of this set linearly in terms of a positive integer X.

Theorem 3.19. *Let G be a sequence of positive integers satisfying the third order Fibonacci recurrence and the unique expansion property under $\mathscr{F}$. Then, the number of positive integers $\le X$ that have $\mathscr{F}$-expansions in terms of G is*

$$\left|R_G(X)\right| = \frac{4+6\alpha+7\alpha^2}{G_3+(G_2+G_3)\alpha+G_4\alpha^2}X + O(1).$$

We can observe that this quantity increases linearly in terms of X, and it is therefore meaningful to calculate the average number of Zeckendorf integers with respect to X as X grows larger, at least for sequences that satisfy the uniqueness property.

Definition 3.20. In the context of any third order Fibonacci recurrence G, we define the average number of Zeckendorf integers (with respect to $\mathbb{N}$) to be

$$\mathfrak{R}_G = \lim_{X\to\infty} \frac{\left|R_G(X)\right|}{X}.$$

Using the formula in Theorem 2.12 we observe that this agrees with the result in Theorem 2.7 in the case of H, as $\mathfrak{R}_H = \lim_{X\to\infty} \frac{X+O(1)}{X} = 1$ where we had said that $R_G = \mathbb{N}$. Moreover, for sequences like $\hat{H}$ (as defined in Example 2.11) we can now calculate the average number of Zeckendorf integers using only the first terms of the sequence, which in this case yields

$$\mathfrak{R}_{\hat{H}} = \frac{4+6\alpha+7\alpha^2}{7+11\alpha+13\alpha^2} \approx 0.5437.$$

In light of this example, it will prove to be helpful to calculate the ratio that corresponds to the linear coefficient of X in Theorem 2.14 for more than just the first four terms of a sequence. Thus, we define the following:

Definition 3.21. In the context of any third order Fibonacci recurrence G, for any $n \in \mathbb{N}$ define

$$r_n(G) = \frac{4+6\alpha+7\alpha^2}{G_{n+2}+(G_{n+1}+G_{n+2})\alpha+(G_{n+3})\alpha^2}.$$

We will shorten the notation to r_n if the sequence G is known or implied.

It is important to notice now that for all sequences G that satisfy the unique representation property, $\mathfrak{R}_G = r_1(G)$. In addition, now there is a way of relating these averages whenever the sequences themselves are related. Using the sequence $\hat{H}$ defined in Example 2.11 and H from Definition 2.1, we can see that

$$\mathfrak{R}_{\hat{H}} = r_1(\hat{H}) = r_2(H).$$

However, for other Fibonacci recurrences that do not satisfy the uniqueness property like the one defined by $(G_1, G_2, G_3) = (5, 2, 3)$, incorrectly using Theorem 2.14 would yield an overestimation of the actual average. In the case of this particular sequence, for example, we can observe, as will be shown in Section 4, that $\mathfrak{R}_G = r_2 + r_4 + r_7 + r_8 \approx 0.6261$ while $r_1 \approx 0.8412$. Therefore, in general $\mathfrak{R}_G \neq r_1(G)$, and the necessity arises for us to find the correct value of $\mathfrak{R}_G$ for third order Fibonacci recurrences G that do not satisfy the uniqueness property.

3.3.2 Counting Formulas for Non-Unique Expansions

The following propositions set the basis for the development of further algorithms to determine the value of $\mathfrak{R}_G$ depending on the initial conditions that the sequence G might satisfy. Since we have described in Proposition 3.14 the double representations that can arise and the corresponding shapes of the bodies and difference of tails, our goal is to calculate the proportion of integers that are repeated in these cases by adapting Definition 3.20 to a more general case that takes into account these shapes.

Definition 3.22. In the context of G being a sequence that satisfies the Fibonacci recurrence, let $A \in \mathscr{F}$ and $l \geq \max(A)$ be a positive integer. Define $R_G(X, A, l) := \{\sum_{A'} G_k \leq X : A' \in \mathscr{F}, A'[1, l] = A\}$

Definition 3.23. Let G be a sequence of positive integers satisfying the third order Fibonacci recurrence and let $\mathfrak{F}$ be the set of finite subsets of $\mathbb{N}$. Define $\text{eval}_G : \mathfrak{F} \to \mathbb{N}$ to be the function given by $A \mapsto \sum_A G_k$.

Definition 3.24. Let $\{a_1, a_2, \ldots, a_m, *_n\} \in \mathscr{F}$ denote any ordered representation $A \in \mathscr{F}$ such that $A[1, n-1] = \{a_1, a_2, \ldots, a_m\}$ with $a_m \neq n-1$, and let $\{a_1, a_2, \ldots, a_m, \Delta_n\} \in \mathscr{F}$ denote any representation $A \in \mathscr{F}$ such that $A[1, n-1] = \{a_1, a_2, \ldots, a_m\}$ and $A[n, n+1] \neq \{n, n+1\}$.

Notice that the condition $a_m \neq n-1$ for the $*_n$ case and the condition $A[n, n+1] \neq \{n, n+1\}$ in the Δ_n case are to ensure that $A \in \mathscr{F}$. Without these we could consider a set such as $\{1, 3, 4, 5, 7\}$ as one that could be included in the representations $\{1, 3, 4, *_5\}$ or $\{1, 3, \Delta_4\}$. However, $\{1, 3, 4, 5, 7\} \notin \mathscr{F}$.

Proposition 3.25. Let G be a sequence that satisfies the Fibonacci recurrence and the uniqueness representation property. Let $A \in \mathscr{F}$, and a positive integer $l \geq m = \max(A)$. Then,

$$\lim_{X \to \infty} \frac{|R_G(X, A, l)|}{X} = \begin{cases} r_{l+1} & \text{if } l > m \\ r_{l+2} & \text{if } l = m \text{ and } m-1 \in A \\ r_{l+2} + r_{l+3} & \text{if } l = m \text{ and } m-1 \notin A \end{cases}$$

Proof. If $l > m$ then consider G' to be the sequence such that $G'_k = G_{k+l}$. Notice that G' also satisfies the uniqueness property. Now, since A is fixed and $l \neq m$, every expression in terms G' less than $Y = X - \text{eval}_G(A)$ corresponds to one and only one expression in terms of G less than X such that its first l terms correspond with those of A. That is,

$$\begin{aligned} |R_{G'}(Y)| &= |R_G(X, A, l)| \\ \Rightarrow \lim_{Y \to \infty} \frac{|R_{G'}(Y)|}{Y} &= \lim_{X \to \infty} \frac{|R_G(X, A, l)|}{X} \\ \Rightarrow \quad \mathfrak{R}_{G'} = r_1(G') &= \lim_{X \to \infty} \frac{|R_G(X, A, l)|}{X} \\ \Rightarrow \qquad r_{l+1}(G) &= \lim_{X \to \infty} \frac{|R_G(X, A, l)|}{X} \end{aligned}$$

If $l = m$ and $m-1 \in A$, then $m+1 \notin A$ since $A \in \mathscr{F}$. Then,

$$\lim_{X\to\infty} \frac{|R_G(X,A,m)|}{X} = \lim_{X\to\infty} \frac{|R_G(X,A,m+1)|}{X} = r_{(m+1)+1} = r_{m+2}.$$

For the case when $l = m$ and $m-1 \notin A$ see Corollary 3.27. □

Proposition 3.26. For any $n \in \mathbb{N}$, $r_n = r_{n+1} + r_{n+2} + r_{n+3}$.

Proof. Suppose that G is a sequence that satisfies the uniqueness property. Fix $C = \{a_1, a_2, \ldots, a_m\} \in \mathscr{F}$ with $a_m \neq n-1$. Let $S = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, *_n\}\}$.
Let $P = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, *_{n+1}\}\}$, $Q = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n, *_{n+2}\}\}$,
$R = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n, n+1, *_{n+3}\}\}$.
Since P, Q, R are pairwise disjoint and $S = P \cup Q \cup R$, then

$$\begin{aligned}
& |S| = |P| + |Q| + |R| \\
\Rightarrow\quad & |R_G(X,C,n-1)| = |R_G(X,C,n)| + |R_G(X,C\cup\{n\},n+1)| + |R_G(X,C\cup\{n,n+1\},n+2)| \\
\Rightarrow\quad & \lim_{X\to\infty} \frac{|R_G(X,C,n-1)|}{X} = \lim_{X\to\infty} \frac{|R_G(X,C,n)|}{X} + \lim_{X\to\infty} \frac{|R_G(X,C\cup\{n\},n+1)|}{X} \\
& \qquad + \lim_{X\to\infty} \frac{|R_G(X,C\cup\{n,n+1\},n+2)|}{X} \\
\Rightarrow\quad & r_n = r_{n+1} + r_{n+2} + r_{n+3}.
\end{aligned}$$

□

This property is also satisfied for sequences G that do not satisfy the uniqueness property, although the proof of this will be omitted in this paper.

Corollary 3.27. Fix $C = \{a_1, a_2, \ldots, a_m, n-1\} \in \mathscr{F}$ with $a_m \neq n-2$. Let $S = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n-1, \Delta_n\}\}$. Then, S is equivalent to the union of the following four pairwise disjoint sets:

- $P = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n-1, n, *_{n+2}\}\}$
- $Q = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n-1, n+1, *_{n+3}\}\}$
- $R = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n-1, n+1, n+2, *_{n+4}\}\}$
- $T = \{A \in \mathscr{F} : A = \{a_1, a_2, \ldots, a_m, n-1, *_{n+2}\}\}$,

and thus, $\lim_{X\to\infty} \frac{|R_G(X,C,n-1)|}{X} = 2r_{n+2} + r_{n+3} + r_{n+4} = r_{n+1} + r_{n+2}$.

Example 3.28. Consider the sequence H from Section 2.1 and sets $A = \{2,3\}$ and $B = \{1,3\}$.

- $R_H(X,A,4)$ is the set of all expressions of H of the form $\{2,3,*_5\}$. By proposition 3.25, the limit of the proportion of this set to X as X grows large is $r_5(H) \approx 0.087$. Notice this is the same result as if we were calculating $\mathfrak{R}_G = r_1(G)$, where G is the shifted version of H by 5 terms.
- Similarly, $R_H(X,B,3)$ is the set of all expressions of H of the form $\{1,3,\Delta_4\}$. By proposition 3.25, the limit of the proportion of this set to X as X grows large is $r_5(H) + r_6(H) \approx 0.135$.

The propositions above give way to the procedures now performed in the appendices, which we describe here. The first appendix tries to show the process of finding the possible conditions the body difference and tail difference can satisfy when a path in Λ is fixed. In this way, a correspondence between initial conditions of the sequence G and body and tail shapes of corresponding Zeckendorf representations is established. The second appendix deals with describing the shapes of the corresponding representations more in detail. It also sheds light on the total possible representations with G using a fixed transformation described in Appendix A. Finally, Appendix C uses Proposition 3.26 to simplify the calculations performed with Proposition 3.25 to find the formula error caused by double counting of repeated representations and the actual value of $\mathfrak{R}_G$ for each initial condition case.

3.3.3 $\mathfrak{R}_G$ per Initial Conditions of G

Init. Cond.	**Body Diff.**	**Tail Diff.**	$A[3,5]$	$B[3,5]$	**# Double Ct**	$\mathfrak{R}_G$
$G_1=2G_2$	G_2	G_1-G_2	$\{1,3,4\}$	$\{2,5\}$	r_7+r_8	$r_2+r_3+r_5+r_7$
			$\{1,3,4\}$	$\{2\}$	r_9	
$G_1=G_2+G_3$	G_3	G_1-G_2	$\{1,5\}$	$\{2,3,5\}$	r_7+r_8	$r_2+r_4+r_7+r_8$
			$\{1,4,5\}$	$\{2\}$	r_7	
			$\{1\}$	$\{2,3\}$	r_6	
	G_2+G_3	G_1	$\{1,4\}$	$\{5\}$	r_7+r_8	
			$\{1,2,4\}$	$\{2,5\}$	r_7+r_8	
			$\{1,4\}$	$\emptyset$	r_9	
			$\{1,2,4\}$	$\{2\}$	r_9	
			$\{1,3,4\}$	$\{3,5\}$	r_7+r_8	
			$\{1,3,4\}$	$\{3\}$	r_9	
$G_1=G_2+2G_3$	G_2+2G_3	G_1	$\{1,4\}$	$\{3,5\}$	r_7+r_8	$r_2+r_3+r_7+r_8$
			$\{1,2,4\}$	$\{2,3,5\}$	r_7+r_8	
			$\{1,4\}$	$\{3\}$	r_9	
			$\{1,2,4\}$	$\{2,3\}$	r_9	
	$2G_3$	G_1-G_2	$\{1,4,5\}$	$\{2,3\}$	r_7	
$G_1=2G_2+G_3$	G_2+G_3	G_1-G_2	$\{1,4\}$	$\{2,5\}$	r_7+r_8	$r_2+r_3+r_6+r_8+r_{10}$
			$\{1,4\}$	$\{2\}$	r_9	
			$\{1,3,4\}$	$\{2,3,5\}$	r_7+r_8	
			$\{1,3,4\}$	$\{2,3\}$	r_9	
$G_1=2G_2+2G_3$	G_2+2G_3	G_1-G_2	$\{1,4\}$	$\{2,3,5\}$	r_7+r_8	$r_2+r_3+r_5+r_7$
			$\{1,4\}$	$\{2,3\}$	r_9	
$G_1=2G_3$	$2G_3$	G_1	$\{1,4,5\}$	$\{3\}$	r_7	$r_2+r_3+r_6+r_8$
			$\{1,2,4,5\}$	$\{2,3\}$	r_7	
	G_2+2G_3	G_1+G_2	$\{1,2,4\}$	$\{3,5\}$	r_7+r_8	
			$\{1,2,4\}$	$\{3\}$	r_9	
$G_2=G_1+G_3$	G_3	G_2-G_1	$\{2,4\}$	$\{1,3,4\}$	r_6	r_2+r_3
			$\{2,5\}$	$\{1,3,5\}$	r_7+r_8	
			$\{2,4,5\}$	$\{1\}$	r_7	
			$\{2\}$	$\{1,3\}$	r_6	
$G_2=G_1+2G_3$	$2G_3$	G_2-G_1	$\{1,4,5\}$	$\{2,3\}$	r_7	$r_2+r_3+r_5+r_6$
$G_2=2G_3$	$2G_3$	G_2	$\{2,4,5\}$	$\{3\}$	r_7	$r_2+r_3+r_6+r_8$
			$\{1,2,4,5\}$	$\{1,3\}$	r_7	
$G_1+G_2=G_3$	G_3	G_1+G_2	$\{1,2,4\}$	$\{3,4\}$	r_6	r_2+r_3
			$\{1,2,5\}$	$\{3,5\}$	r_7+r_8	
			$\{1,2,4,5\}$	$\emptyset$	r_7	
			$\{1,2\}$	$\{3\}$	r_6	
$G_1+G_2=2G_3$	$2G_3$	G_1+G_2	$\{1,2,4,5\}$	$\{3\}$	r_7	$r_2+r_3+r_5+r_6$

Table A: $\mathfrak{R}_G$ for every Fibonacci recurrence sequence that does not satisfy the uniqueness property by one and only one condition.

Using the previous propositions, the above table summarizes the data recovered from the procedures shown in the three appendices. Most importantly, now for every sequence G that satisfies one and only one of the following 11 initial conditions shown in Table A, we can calculate $\mathfrak{R}_G$ by looking into the last column corresponding to each case and making use of Definition 3.21.

Now, we notice that this only accounts for the cases where there is only one initial value is satisfied. We can observe from the shapes of $A[3,5]$ and $B[3,5]$ in Table A that that there is no overlap where a representation can have multiple duplicates. That is, if there are $A, B \in \mathscr{F}$ such that $A < B$ and $\sum_A G_k = \sum_B G_k$, then there cannot exist a $C \in \mathscr{F}$ such that $B < C$ and $\sum_B G_k = \sum_C G_k$. Therefore, unless there is a sequence G for which two (or more) of the conditions shown in the table above are satisfied simultaneously, then there is no possibility for any sequence to contain more duplicates than the ones shown in Table A. Thus, we focus our attention to those cases where multiple conditions are satisfied by the initial terms of G and how to handle those cases. This is the motivation behind Table B, which contains all the possible groupings of initial conditions that can be met simultaneously without violating any of the guidelines set for a sequence G found in Definition 2.1.

Init. Conds.	**Tot. Double Ct**	$\mathfrak{R}_G$	(G_1, G_2, G_3) **shape**
$G_1 = 2G_2$ $G_1 = G_2 + 2G_3$ $G_2 = 2G_3$	$r_4 + r_6 + r_7 + r_9 + r_{10}$	$r_2 + r_4 + r_6 + r_{11}$	$(4,2,1)$
$G_1 = 2G_2$ $G_1 + G_2 = G_3$	$r_4 + r_6$	$r_2 + r_4 + r_5$	$(2,1,3)$
$G_1 = 2G_2$ $G_1 + G_2 = 2G_3$	$r_6 + r_7$	$r_2 + r_3 + r_5$	$(4,2,3)$
$G_1 = G_2 + G_3$ $G_2 = 2G_3$	$r_3 + r_9 + r_{10}$	$r_2 + r_5 + r_6 + r_8$	$(3,2,1)$
$G_1 = G_2 + G_3$ $G_1 + G_2 = 2G_3$	$r_4 + r_5 + r_7 + r_9$	$r_2 + r_4 + r_8$	$(3,1,2)$
$G_1 = 2G_2 + G_3$ $G_1 = 2G_3$	$r_4 + r_7 + r_9$	$r_2 + r_4 + r_5 + r_8$	$(4,1,2)$
$G_1 = 2G_2 + G_3$ $G_2 = 2G_3$	$r_5 + r_6 + r_9 + r_{10}$	$r_2 + r_3 + r_8$	$(5,2,1)$
$G_1 = 2G_2 + G_3$ $G_1 + G_2 = 2G_3$	$r_5 + r_7 + r_9$	$r_2 + r_3 + r_7 + r_8$	$(5,1,3)$
$G_1 = 2G_2 + 2G_3$ $G_2 = 2G_3$	$r_5 + r_9 + r_{10}$	$r_2 + r_3 + r_6 + r_8$	$(6,2,1)$
$G_1 = 2G_3$ $G_2 = G_1 + G_3$	$r_4 + r_5 + r_9 + r_{10}$	$r_2 + r_4 + r_7 + r_9 + r_{11}$	$(2,3,1)$
$G_1 = 2G_3$ $G_2 = G_1 + 2G_3$	$r_5 + r_7 + r_9 + r_{10}$	$r_2 + r_3 + r_5 + r_7 + r_9 + r_{11}$	$(2,4,1)$
$G_2 = G_1 + G_3$ $G_1 + G_2 = 2G_3$	$r_4 + r_7$	$r_2 + r_4 + r_5 + r_8 + r_9$	$(1,3,2)$

Table B: $\mathfrak{R}_G$ for every Fibonacci recurrence sequence that does not satisfy the uniqueness property by more than one condition.

3.4 Case Studies

- $(G_1, G_2, G_3) = (5, 2, 3)$

  This sequence satisfies the condition $G_1 = G_2 + G_3$, therefore numbers such as $20 = G_5 + G_1 = G_5 + G_3 + G_2$ have two representations. Thus, according to Table A, the value of $\mathfrak{R}_G$ is given by

$$\begin{aligned} r_2 + r_4 + r_7 + r_8 &= \frac{4+6\alpha+7\alpha^2}{G_4+(G_3+G_4)\alpha+G_5\alpha^2} + \frac{4+6\alpha+7\alpha^2}{G_6+(G_5+G_6)\alpha+G_7\alpha^2} \\ &\quad + \frac{4+6\alpha+7\alpha^2}{G_9+(G_8+G_9)\alpha+G_{10}\alpha^2} + \frac{4+6\alpha+7\alpha^2}{G_{10}+(G_9+G_{10})\alpha+G_{11}\alpha^2} \\ &= \frac{4+6\alpha+7\alpha^2}{10+13\alpha+15\alpha^2} + \frac{4+6\alpha+7\alpha^2}{28+43\alpha+53\alpha^2} \\ &\quad + \frac{4+6\alpha+7\alpha^2}{177+273\alpha+326\alpha^2} + \frac{4+6\alpha+7\alpha^2}{326+503\alpha+599\alpha^2} \\ &\approx 0.62607 \end{aligned}$$

  Therefore, whereas $r_1 \approx 0.84119$, $\mathfrak{R}_G \approx 0.62607$.

- $(G_1, G_2, G_3) = (6, 4, 3)$

  This sequence satisfies the condition $G_1 = 2G_3$, therefore numbers such as $23 = G_4 + G_2 + G_1 = G_5 + G_3$ have two representations. Thus, according to Table A, the value of $\mathfrak{R}_G$ is given by

$$\begin{aligned} r_2 + r_3 + r_6 + r_8 &= \frac{4+6\alpha+7\alpha^2}{G_4+(G_3+G_4)\alpha+G_5\alpha^2} + \frac{4+6\alpha+7\alpha^2}{G_5+(G_4+G_5)\alpha+G_6\alpha^2} \\ &\quad + \frac{4+6\alpha+7\alpha^2}{G_8+(G_7+G_8)\alpha+G_9\alpha^2} + \frac{4+6\alpha+7\alpha^2}{G_{10}+(G_9+G_{10})\alpha+G_{11}\alpha^2} \\ &= \frac{4+6\alpha+7\alpha^2}{13+16\alpha+20\alpha^2} + \frac{4+6\alpha+7\alpha^2}{20+33\alpha+36\alpha^2} \\ &\quad + \frac{4+6\alpha+7\alpha^2}{125+194\alpha+230\alpha^2} + \frac{4+6\alpha+7\alpha^2}{424+654\alpha+779\alpha^2} \\ &\approx 0.58271 \end{aligned}$$

  Therefore, whereas $r_1 \approx 0.64685$, $\mathfrak{R}_G \approx 0.58271$.

- $(G_1, G_2, G_3) = (2, 6, 4)$

  Since $(G_1, G_2, G_3) \sim (1, 3, 2)$ (that is, $3G_1 = G_2$ and $2G_1 = G_3$), this sequence satisfies the conditions $G_2 = G_1 + G_3$ and $G_1 + G_2 = 2G_3$. Therefore numbers such as $18 = G_4 + G_2 = G_4 + G_3 + G_1$ and $42 = G_5 + G_4 + G_2 + G_1 = G_6 + G_3$ have two representations. Thus, according to Table B, the value of $\mathfrak{R}_G$ is given by

$$\begin{aligned} r_4 + r_7 &= \frac{4+6\alpha+7\alpha^2}{G_6+(G_5+G_6)\alpha+G_7\alpha^2} + \frac{4+6\alpha+7\alpha^2}{G_9+(G_8+G_9)\alpha+G_{10}\alpha^2} \\ &= \frac{4+6\alpha+7\alpha^2}{38+60\alpha+72\alpha^2} + \frac{4+6\alpha+7\alpha^2}{242+374\alpha+446\alpha^2} \\ &\approx 0.11466 \end{aligned}$$

  Therefore, whereas $r_1 \approx 0.61466$, $\mathfrak{R}_G \approx 0.11466$.

4 Future Work

Some of the results in this paper are missing a proof. Further papers about this topic might delve deeper into the intricacies of every theorem or proposition used in this paper to construct the process of eliminating double counts per initial condition of the sequences. More detailed examples for specific sequences might also be provided in the future.

Also, the process used to arrive at our results was purely mechanical and based on categorization rather than on intrinsic properties of the sequences that hold said initial conditions that cause the double Zeckendorf representations. An in-depth analysis of the properties of the values of r_i, $i \in \mathbb{N}$, might yield interesting results. If a direct link can be found between the initial conditions of the first terms of the sequence and the corresponding values r_i that add up to $\mathfrak{R}_G$, significant improvement can be made to understand the structure of these recurrences and their relation to the ratio of Zeckendorf integers they produce.

Finally, extensions of these principles can be made to higher-order Zeckendorf representations. A generalized Zeckendorf count seems to be quite challenging, considering the amount of cases just the third order case produces. However, if a generalizing statement is achieved for third order Fibonacci recurrences that may or may not satisfy the unique expansion property, then these principles can be extrapolated for further orders.

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Appendix A: Body-Tail Difference Comparisons per Path in Λ

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2$	
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2$	✓
G_2	0	G_2	$A \notin \mathscr{F}$	
G_2	G_1	$G_2 - G_1$	$A \notin \mathscr{F}$	
$G_1 + G_2$	0	$G_1 + G_2$	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_1	G_2	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_2	G_1	$A \notin \mathscr{F}$	

Table 1: $\{3,4\} \to \{5\}$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_2$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2$	
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2$	✓
G_2	0	G_2	$A \notin \mathscr{F}$	
G_2	G_1	$G_2 - G_1$	$A \notin \mathscr{F}$	
$G_1 + G_2$	0	$G_1 + G_2$	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_1	G_2	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_2	G_1	$A \notin \mathscr{F}$	

Table 2: $\{3,4\} \to \emptyset$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_2$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_3$	
G_1	G_2	$G_1 - G_2$	$G_1 = G_2 + G_3$	✓
G_2	0	G_2	$G_2 = G_3$	
G_2	G_1	$G_2 - G_1$	$G_2 = G_1 + G_3$	✓
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 + G_2 = G_3$	✓
$G_1 + G_2$	G_1	G_2	$G_2 = G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_3$	

Table 3: $\{5\} \to \{3,5\}$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_3$	
G_1	G_2	$G_1 - G_2$	$A \notin \mathscr{F}$	
G_2	0	G_2	$G_2 = G_3$	
G_2	G_1	$G_2 - G_1$	$G_2 = G_1 + G_3$	✓
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 + G_2 = G_3$	✓
$G_1 + G_2$	G_1	G_2	$G_2 = G_3$	
$G_1 + G_2$	G_2	G_1	$A \notin \mathscr{F}$	

Table 4: $\{4\} \to \{3,4\}$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + G_3$	✓
G_2	0	G_2	$0 = G_3$	
G_2	G_1	$G_2 - G_1$	$0 = G_1 + G_3$	
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 = G_3$	
$G_1 + G_2$	G_1	G_2	$0 = G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_2 + G_3$	✓

Table 5: $\{4\} \to \{5\}$, where $\sum_{b(B)} G_k - \sum_{b(A)} G_k = G_2 + G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + 2G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + 2G_3$	✓
G_2	0	G_2	$0 = 2G_3$	
G_2	G_1	$G_2 - G_1$	$0 = G_1 + 2G_3$	
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 = 2G_3$	✓
$G_1 + G_2$	G_1	G_2	$0 = 2G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_2 + 2G_3$	✓

Table 6: $\{4\} \to \{3,5\}$, where $\sum_{b(B)} G_k - \sum_{b(A)} G_k = G_2 + 2G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + G_3$	✓
G_2	0	G_2	$A \notin \mathscr{F}$	
G_2	G_1	$G_2 - G_1$	$A \notin \mathscr{F}$	
$G_1 + G_2$	0	$G_1 + G_2$	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_1	G_2	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_2	G_1	$A \notin \mathscr{F}$	

Table 7: $\{3,4\} \to \{3,5\}$, where $\sum_{b(B)} G_k - \sum_{b(A)} G_k = G_2 + G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + G_3$	✓
G_2	0	G_2	$A \notin \mathscr{F}$	
G_2	G_1	$G_2 - G_1$	$A \notin \mathscr{F}$	
$G_1 + G_2$	0	$G_1 + G_2$	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_1	G_2	$A \notin \mathscr{F}$	
$G_1 + G_2$	G_2	G_1	$A \notin \mathscr{F}$	

Table 8: $\{3,4\} \to \{3\}$, where $\sum_{b(B)} G_k - \sum_{b(A)} G_k = G_2 + G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + 2G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + 2G_3$	✓
G_2	0	G_2	$0 = 2G_3$	
G_2	G_1	$G_2 - G_1$	$0 = G_1 + 2G_3$	
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 = 2G_3$	✓
$G_1 + G_2$	G_1	G_2	$0 = 2G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_2 + 2G_3$	✓

Table 10: $\{4\} \to \{3\}$, where $\sum_{b(B)} G_k - \sum_{b(A)} G_k = G_2 + 2G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_2 + G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = 2G_2 + G_3$	✓
G_2	0	G_2	$0 = G_3$	
G_2	G_1	$G_2 - G_1$	$0 = G_1 + G_3$	
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 = G_3$	
$G_1 + G_2$	G_1	G_2	$0 = G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_2 + G_3$	✓

Table 9: $\{4\} \to \emptyset$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_2 + G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_3$	
G_1	G_2	$G_1 - G_2$	$G_1 = G_2 + G_3$	✓
G_2	0	G_2	$G_2 = G_3$	
G_2	G_1	$G_2 - G_1$	$G_2 = G_1 + G_3$	✓
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 + G_2 = G_3$	✓
$G_1 + G_2$	G_1	G_2	$G_2 = G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_3$	

Table 11: $\{4,5\} \to \emptyset$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = 2G_3$	✓
G_1	G_2	$G_1 - G_2$	$G_1 = G_2 + 2G_3$	✓
G_2	0	G_2	$G_2 = 2G_3$	✓
G_2	G_1	$G_2 - G_1$	$G_2 = G_1 + 2G_3$	✓
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 + G_2 = 2G_3$	✓
$G_1 + G_2$	G_1	G_2	$G_2 = 2G_3$	✓
$G_1 + G_2$	G_2	G_1	$G_1 = 2G_3$	✓

Table 12: $\{4,5\} \to \{3\}$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = 2G_3$

t(A)	t(B)	Tail Difference	Implied Condition	Validity
G_1	0	G_1	$G_1 = G_3$	
G_1	G_2	$G_1 - G_2$	$G_1 = G_2 + G_3$	✓
G_2	0	G_2	$G_2 = G_3$	
G_2	G_1	$G_2 - G_1$	$G_2 = G_1 + G_3$	✓
$G_1 + G_2$	0	$G_1 + G_2$	$G_1 + G_2 = G_3$	✓
$G_1 + G_2$	G_1	G_2	$G_2 = G_3$	
$G_1 + G_2$	G_2	G_1	$G_1 = G_3$	

Table 13: $\emptyset \to \{3\}$, where $\sum_{\mathrm{b}(B)} G_k - \sum_{\mathrm{b}(A)} G_k = G_3$

Appendix B: Calculations for Double Counts per Initial Condition

Recall Definition 3.23 and Definition 3.24.

(1) $G_1 = 2G_2$

- $\{1,3,4\} \to \{2,5\}$

$$\begin{aligned} &\mathrm{eval}_G\{2,3,4,\Delta_6\} = \mathrm{eval}_G\{5,\Delta_6\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow\ &\mathrm{eval}_G\{1,3,4,\Delta_6\} = \mathrm{eval}_G\{2,5,\Delta_6\} && (G_1-G_2=G_2) \end{aligned}$$

- $\{1,3,4\} \to \{2\}$

$$\begin{aligned} &\mathrm{eval}_G\{2,3,4,6,7,*_9\} = \mathrm{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow\ &\mathrm{eval}_G\{2,3,4,6,7,*_9\} = \mathrm{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow\ &\mathrm{eval}_G\{1,3,4,6,7,*_9\} = \mathrm{eval}_G\{2,8,*_9\} && (G_1-G_2=G_2) \end{aligned}$$

(shape of $B \in \mathscr{F}$ might change depending on G_9, G_{10})

(2) $G_1 = G_2 + G_3$

- $\{1,5\} \to \{2,3,5\}$

$$\mathrm{eval}_G\{1,5,\Delta_6\} = \mathrm{eval}_G\{2,3,5,\Delta_6\} \qquad (G_1=G_2+G_3)$$

- $\{1,4,5\} \to \{2\}$

$$\begin{aligned} &\mathrm{eval}_G\{1,4,5,*_7\} = \mathrm{eval}_G\{2,3,4,5,*_7\} && (G_1=G_2+G_3) \\ \Rightarrow\ &\mathrm{eval}_G\{1,4,5,*_7\} = \mathrm{eval}_G\{2,6,*_7\} && (G_3+G_4+G_5=G_6) \end{aligned}$$

(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

- $\{1\} \to \{2,3\}$

$$\mathrm{eval}_G\{1,*_6\} = \mathrm{eval}_G\{2,3,*_6\} \qquad (G_1=G_2+G_3)$$

- $\{1,4\} \to \{5\}$

$$\begin{aligned} &\mathrm{eval}_G\{2,3,4,\Delta_6\} = \mathrm{eval}_G\{5,\Delta_6\} \\ \Rightarrow\ &\mathrm{eval}_G\{1,4,\Delta_6\} = \mathrm{eval}_G\{5,\Delta_6\} && (G_1-G_2-G_3=0) \end{aligned}$$

- $\{1,2,4\} \to \{2,5\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,\Delta_6\} = \text{eval}_G\{5,\Delta_6\} \\ \Rightarrow\ &\text{eval}_G\{1,4,\Delta_6\} = \text{eval}_G\{2,5,\Delta_6\} && (G_1 - G_3 = G_2) \end{aligned}$$

- $\{1,4\} \to \emptyset$

$$\begin{aligned} &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{2,3,4,6,7,*_9\} && (G_1 = G_2 + G_3) \\ \Rightarrow\ &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} && (G_2 + G_3 + G_4 = G_5) \\ \Rightarrow\ &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} && (G_5 + G_6 + G_7 = G_8) \\ & \text{(shape of } B \in \mathscr{F} \text{ might change depending on } G_9, G_{10}) \end{aligned}$$

- $\{1,2,4\} \to \{2\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} && (G_2 + G_3 + G_4 = G_5) \\ \Rightarrow\ &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} && (G_5 + G_6 + G_7 = G_8) \\ \Rightarrow\ &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{2,8,*_9\} && (G_1 - G_3 = G_2) \\ & \text{(shape of } B \in \mathscr{F} \text{ might change depending on } G_9, G_{10}) \end{aligned}$$

- $\{1,3,4\} \to \{3,5\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,\Delta_6\} = \text{eval}_G\{5,\Delta_6\} && (G_2 + G_3 + G_4 = G_5) \\ \Rightarrow\ &\text{eval}_G\{1,3,4,\Delta_6\} = \text{eval}_G\{3,5,\Delta_6\} && (G_1 - G_2 = G_3) \end{aligned}$$

- $\{1,3,4\} \to \{3\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} && (G_2 + G_3 + G_4 = G_5) \\ \Rightarrow\ &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} && (G_5 + G_6 + G_7 = G_8) \\ \Rightarrow\ &\text{eval}_G\{1,3,4,6,7,*_9\} = \text{eval}_G\{3,8,*_9\} && (G_1 - G_2 = G_3) \\ & \text{(shape of } B \in \mathscr{F} \text{ might change depending on } G_9, G_{10}) \end{aligned}$$

(3) $G_1 = G_2 + 2G_3$

- $\{1,4\} \to \{3,5\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,\Delta_6\} = \text{eval}_G\{5,\Delta_6\} && (G_2 + G_3 + G_4 = G_5) \\ \Rightarrow\ &\text{eval}_G\{1,4,\Delta_6\} = \text{eval}_G\{3,5,\Delta_6\} && (G_1 - G_2 - G_3 = G_3) \end{aligned}$$

- $\{1,2,4\} \to \{2,3,5\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,\Delta_6\} &= \text{eval}_G\{5,\Delta_6\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{1,2,4,\Delta_6\} &= \text{eval}_G\{2,3,5,\Delta_6\} && (G_1-G_3=G_2+G_3) \end{aligned}$$

- $\{1,4\} \to \{3\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow \text{eval}_G\{1,4,6,7,*_9\} &= \text{eval}_G\{3,8,*_9\} && (G_1-G_2-G_3=G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_9, G_{10}) \end{aligned}$$

- $\{1,2,4\} \to \{2,3\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow \text{eval}_G\{1,2,4,6,7,*_9\} &= \text{eval}_G\{2,3,8,*_9\} && (G_1-G_3=G_2+G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_9, G_{10}) \end{aligned}$$

- $\{1,4,5\} \to \{2,3\}$

$$\begin{aligned} \text{eval}_G\{3,4,5,*_7\} &= \text{eval}_G\{6,*_7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow \text{eval}_G\{1,4,5,*_7\} &= \text{eval}_G\{2,3,6,*_7\} && (G_1-G_3=G_2+G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_7, G_8) \end{aligned}$$

(4) $G_1 = 2G_2 + G_3$

- $\{1,4\} \to \{2,5\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,\Delta_6\} &= \text{eval}_G\{5,\Delta_6\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{1,4,\Delta_6\} &= \text{eval}_G\{2,5,\Delta_6\} && (G_1-G_2-G_3=G_2) \end{aligned}$$

- $\{1,4\} \to \{2\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{2,3,4,6,7,*_9\} &= \text{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow \text{eval}_G\{1,4,6,7,*_9\} &= \text{eval}_G\{2,8,*_9\} && (G_1-G_2-G_3=G_2) \end{aligned}$$

- $\{1,3,4\} \to \{2,3,5\}$

$$\begin{aligned} \text{eval}_G\{2,3,4,\Delta_6\} &= \text{eval}_G\{5,\Delta_6\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow \text{eval}_G\{1,3,4,\Delta_6\} &= \text{eval}_G\{2,3,5,\Delta_6\} && (G_1-G_2=G_2+G_3) \end{aligned}$$

- $\{1,3,4\} \to \{2,3\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow\ &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow\ &\text{eval}_G\{1,3,4,6,7,*_9\} = \text{eval}_G\{2,3,8,*_9\} && (G_1-G_2=G_2+G_3) \end{aligned}$$

(5) $G_1 = 2G_2 + 2G_3$

- $\{1,4\} \to \{2,3,5\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,\Delta_6\} = \text{eval}_G\{5,\Delta_6\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow\ &\text{eval}_G\{1,4,\Delta_6\} = \text{eval}_G\{2,3,5,\Delta_6\} && (G_1-G_2-G_3=G_2+G_3) \end{aligned}$$

- $\{1,4\} \to \{2\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} && (G_2+G_3+G_4=G_5) \\ \Rightarrow\ &\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} && (G_5+G_6+G_7=G_8) \\ \Rightarrow\ &\text{eval}_G\{1,4,6,7,*_9\} = \text{eval}_G\{2,3,8,*_9\} && (G_1-G_2-G_3=G_2+G_3) \end{aligned}$$

(6) $G_1 = 2G_3$

- $\{1,4,5\} \to \{3\}$

$$\begin{aligned} &\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*_7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow\ &\text{eval}_G\{1,4,5,*_7\} = \text{eval}_G\{3,6,*_7\} && (G_1-G_3=G_3) \end{aligned}$$

(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

- $\{1,2,4,5\} \to \{2,3\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,5,*_7\} = \text{eval}_G\{2,6,*_7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow\ &\text{eval}_G\{1,2,4,5,*_7\} = \text{eval}_G\{2,3,6,*_7\} && (G_1-G_3=G_3) \end{aligned}$$

(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

- $\{1,2,4\} \to \{3,5\}$

$$\begin{aligned} &\text{eval}_G\{2,3,4,\Delta_6\} = \text{eval}_G\{5,\Delta_6\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow\ &\text{eval}_G\{1,2,4,\Delta_6\} = \text{eval}_G\{3,5,\Delta_6\} && (G_1-G_3=G_3) \end{aligned}$$

- $\{1,2,4\} \to \{5\}$

$$\text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{5,6,7,*_9\} \qquad (G_2+G_3+G_4=G_5)$$
$$\Rightarrow \text{eval}_G\{2,3,4,6,7,*_9\} = \text{eval}_G\{8,*_9\} \qquad (G_5+G_6+G_7=G_5)$$
$$\Rightarrow \text{eval}_G\{1,2,4,6,7,*_9\} = \text{eval}_G\{3,8,*_9\} \qquad (G_1-G_3=G_3)$$
(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

(7) $G_2 = G_1 + G_3$

- $\{2,4\} \to \{1,3,4\}$

$$\text{eval}_G\{2,4,*_6\} = \text{eval}_G\{1,3,4,*_6\} \qquad (G_2=G_1+G_3)$$

- $\{2,5\} \to \{1,3,5\}$

$$\text{eval}_G\{2,5,\Delta_6\} = \text{eval}_G\{1,3,5,\Delta_6\} \qquad (G_2=G_1+G_3)$$

- $\{2,4,5\} \to \{1\}$

$$\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*7\} \qquad (G_3+G_4+G_5=G_6)$$
$$\Rightarrow \text{eval}_G\{2,4,5,*_7\} = \text{eval}_G\{1,6,*_7\} \qquad (G_2-G_3=G_1)$$
(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

- $\{2\} \to \{1,3\}$

$$\text{eval}_G\{2,*_6\} = \text{eval}_G\{1,3,*_6\} \qquad (G_2=G_1+G_3)$$

(8) $G_2 = G_1 + 2G_3$

- $\{2,4,5\} \to \{1,3\}$

$$\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*7\} \qquad (G_3+G_4+G_5=G_6)$$
$$\Rightarrow \text{eval}_G\{2,4,5,*_7\} = \text{eval}_G\{1,3,6,*_7\} \qquad (G_2-G_3=G_1+G_3)$$
(shape of $B \in \mathscr{F}$ might change depending on G_7, G_8)

(9) $G_2 = 2G_3$

- $\{2,4,5\} \to \{3\}$

$$\begin{aligned} &\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow &\text{eval}_G\{2,4,5,*_7\} = \text{eval}_G\{3,6,*_7\} && (G_2-G_3=G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_7, G_8) \end{aligned}$$

- $\{1,2,4,5\} \to \{1,3\}$

$$\begin{aligned} &\text{eval}_G\{1,3,4,5,*_7\} = \text{eval}_G\{1,6,*7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow &\text{eval}_G\{1,2,4,5,*_7\} = \text{eval}_G\{1,3,6,*_7\} && (G_2-G_3=G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_7, G_8) \end{aligned}$$

(10) $G_1 + G_2 = G_3$

- $\{1,2,4\} \to \{3\}$

$$\text{eval}_G\{1,2,4,*_6\} = \text{eval}_G\{3,4,*_6\} \qquad (G_1+G_2=G_3)$$

- $\{1,2,5\} \to \{3,5\}$

$$\text{eval}_G\{1,2,5,\Delta_6\} = \text{eval}_G\{3,5,\Delta_6\} \qquad (G_1+G_2=G_3)$$

- $\{1,2,4,5\} \to \emptyset$

$$\begin{aligned} &\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow &\text{eval}_G\{1,2,4,5,*_7\} = \text{eval}_G\{6,*_7\} && (G_1+G_2-G_3=0) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_7, G_8) \end{aligned}$$

- $\{1,2\} \to \{3\}$

$$\text{eval}_G\{1,2,*_6\} = \text{eval}_G\{3,*a_6\} \qquad (G_1+G_2=G_3)$$

(11) $G_1 + G_2 = 2G_3$

- $\{1,2,4,5\} \to \{3\}$

$$\begin{aligned} &\text{eval}_G\{3,4,5,*_7\} = \text{eval}_G\{6,*7\} && (G_3+G_4+G_5=G_6) \\ \Rightarrow &\text{eval}_G\{1,2,4,5,*_7\} = \text{eval}_G\{3,6,*_7\} && (G_1+G_2-G_3=G_3) \\ & && (\text{shape of } B \in \mathscr{F} \text{ might change depending on } G_7, G_8) \end{aligned}$$

Appendix C: Counts of Zeckendorf Integers per Initial Condition

(1) $G_1 = 2G_2$

\# Double Counts:

$$(r_7 + r_8) + r_9 = r_6$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_6 &= r_2 + r_3 + r_4 - r_6 \\ &= r_2 + r_3 + r_5 + r_7 \end{aligned}$$

(2) $G_1 = G_2 + G_3$

\# Double Counts:

$$\begin{aligned} r_6 + 4(r_7 + r_8) + r_7 + 3r_9 &= 4r_6 + 2r_7 + r_8 \\ &= r_5 + 3r_6 + r_7 \\ &= r_4 + 2r_6 \\ &= r_4 + r_6 + r_7 + r_8 + r_9 \\ &= r_4 + r_5 + r_9 \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_5 - r_9 &= r_2 + r_3 - r_5 - r_9 \\ &= r_2 + r_4 + r_6 - r_9 \\ &= r_2 + r_4 + r_7 + r_8 \end{aligned}$$

(3) $G_1 = G_2 + 2G_3$

\# Double Counts:

$$\begin{aligned} 2(r_7 + r_8) + r_7 + 2r_9 &= 2r_6 + r_7 \\ &= r_6 + 2r_7 + r_8 + r_9 \\ &= r_5 + r_7 + r_9 \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_7 - r_9 &= r_2 + r_3 + r_4 - r_5 - r_7 - r_9 \\ &= r_2 + r_3 + r_6 - r_9 \\ &= r_2 + r_3 + r_7 + r_8 \end{aligned}$$

(4) $G_1 = 2G_2 + G_3$

Double Counts:

$$\begin{aligned} 2(r_7 + r_8) + 2r_9 &= 2r_6 \\ &= r_6 + r_7 + r_8 + r_9 \\ &= r_5 + r_9 \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_9 &= r_2 + r_3 + r_4 - r_5 - r_9 \\ &= r_2 + r_3 + r_6 + r_7 - r_9 \\ &= r_2 + r_3 + r_6 + r_8 + r_{10} \end{aligned}$$

(5) $G_1 = 2G_2 + 2G_3$

Double Counts:

$$(r_7 + r_8) + r_9 = r_6$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_6 &= r_2 + r_3 + r_4 - r_6 \\ &= r_2 + r_3 + r_5 + r_7 \end{aligned}$$

(6) $G_1 = 2G_3$

Double Counts:

$$\begin{aligned} (r_7 + r_8) + 2r_7 + r_9 &= r_6 + 2r_7 \\ &= r_6 + r_7 + r_8 + r_9 + r_{10} \\ &= r_5 + r_9 + r_{10} \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_9 - r_{10} &= r_2 + r_3 + r_4 - r_5 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_7 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_8 \end{aligned}$$

(7) $G_2 = G_1 + G_3$
Double Counts:

$$\begin{aligned}(r_7 + r_8) + 2r_6 + r_7 &= r_5 + r_6 + r_7 \\ &= r_4\end{aligned}$$

$\Re_G$:

$$r_1 - r_4 = r_2 + r_3$$

(8) $G_2 = G_1 + 2G_3$
Double Counts:

$$r_7$$

$\Re_G$:

$$\begin{aligned}r_1 - r_7 &= r_2 + r_3 + r_4 - r_7 \\ &= r_2 + r_3 + r_5 + r_6\end{aligned}$$

(9) $G_2 = 2G_3$
Double Counts:

$$\begin{aligned}2r_7 &= r_7 + r_8 + r_9 + r_{10} \\ &= r_6 + r_{10}\end{aligned}$$

$\Re_G$:

$$\begin{aligned}r_1 - r_5 - r_9 - r_{10} &= r_2 + r_3 + r_4 - r_5 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_7 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_8\end{aligned}$$

(10) $G_1 + G_2 = G_3$
Double Counts:

$$\begin{aligned}(r_7 + r_8) + 2r_6 + r_7 &= r_5 + r_6 + r_7 \\ &= r_4\end{aligned}$$

$\Re_G$:

$$r_1 - r_4 = r_2 + r_3$$

(11) $G_1 + G_2 = 2G_3$

Double Counts:

$$r_7$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_7 &= r_2 + r_3 + r_4 - r_7 \\ &= r_2 + r_3 + r_5 + r_6 \end{aligned}$$

(12) $G_1 = 2G_2,\ G_1 = G_2 + 2G_3,\ G_2 = 2G_3$

Double Counts:

$$r_6 + r_5 + r_7 + r_9 + r_6 + r_{10} = r_4 + r_6 + r_7 + r_9 + r_{10}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_6 - r_7 - r_9 - r_{10} &= r_2 + r_3 - r_6 - r_7 - r_9 - r_{10} \\ &= r_2 + r_4 + r_5 - r_7 - r_9 - r_{10} \\ &= r_2 + r_4 + r_6 + r_8 - r_9 - r_{10} \\ &= r_2 + r_4 + r_6 + r_{11} \end{aligned}$$

(13) $G_1 = 2G_2,\ G_1 + G_2 = G_3$

Double Counts:

$$r_6 + r_4 = r_4 + r_6$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_6 &= r_2 + r_3 - r_6 \\ &= r_2 + r_4 + r_5 \end{aligned}$$

(14) $G_1 = 2G_2,\ G_1 + G_2 = 2G_3$

Double Counts:

$$r_6 + r_7$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_6 - r_7 &= r_2 + r_3 + r_4 - r_6 - r_7 \\ &= r_2 + r_3 + r_5 \end{aligned}$$

(15) $G_1 = G_2 + G_3,\ G_2 = 2G_3$
Double Counts:

$$r_4 + r_5 + r_9 + r_6 + r_{10} = r_3 + r_9 + r_{10}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_3 - r_9 - r_{10} &= r_2 + r_4 - r_9 - r_{10} \\ &= r_2 + r_5 + r_6 + r_7 - r_9 - r_{10} \\ &= r_2 + r_5 + r_6 + r_8 \end{aligned}$$

(16) $G_1 = G_2 + G_3,\ G_1 + G_2 = 2G_3$
Double Counts:

$$r_4 + r_5 + r_9 + r_7 = r_4 + r_5 + r_7 + r_9$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_5 - r_7 - r_9 &= r_2 + r_3 - r_5 - r_7 - r_9 \\ &= r_2 + r_4 + r_6 - r_7 - r_9 \\ &= r_2 + r_4 + r_8 \end{aligned}$$

(17) $G_1 = G_2 + 2G_3,\ G_1 = 2G_3$
Double Counts:

$$\begin{aligned} r_5 + r_9 + r_5 + r_9 + r_{10} &= r_5 + r_6 + r_7 + r_8 + r_9 + r_9 + r_{10} \\ &= r_4 + r_7 + r_9 \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_7 - r_9 &= r_2 + r_3 - r_7 - r_9 \\ &= r_2 + r_4 + r_5 + r_6 - r_7 - r_9 \\ &= r_2 + r_4 + r_5 + r_8 \end{aligned}$$

(18) $G_1 = G_2 + 2G_3,\ G_2 = 2G_3$

Double Counts:

$$r_5 + r_9 + r_6 + r_{10} = r_5 + r_6 + r_9 + r_{10}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_6 - r_9 - r_{10} &= r_2 + r_3 + r_4 - r_5 - r_6 - r_9 - r_{10} \\ &= r_2 + r_3 + r_7 - r_9 - r_{10} \\ &= r_2 + r_3 + r_8 \end{aligned}$$

(19) $G_1 = 2G_2 + 2G_3,\ G_2 = 2G_3$

Double Counts:

$$\begin{aligned} r_6 + r_6 + r_{10} &= r_6 + r_7 + r_8 + r_9 + r_{10} \\ &= r_5 + r_9 + r_{10} \end{aligned}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_9 - r_{10} &= r_2 + r_3 + r_4 - r_5 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_7 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 + r_8 \end{aligned}$$

(20) $G_1 = 2G_3,\ G_2 = G_1 + G_3$

Double Counts:

$$r_5 + r_9 + r_{10} + r_4 = r_4 + r_5 + r_9 + r_{10}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_5 - r_9 - r_{10} &= r_2 + r_3 - r_5 - r_9 - r_{10} \\ &= r_2 + r_4 + r_6 - r_9 - r_{10} \\ &= r_2 + r_4 + r_7 + r_8 - r_{10} \\ &= r_2 + r_4 + r_7 + r_9 + r_{11} \end{aligned}$$

(21) $G_1 = 2G_3,\ G_2 = G_1 + 2G_3$

\# Double Counts:

$$r_5 + r_9 + r_{10} + r_7 = r_5 + r_7 + r_9 + r_{10}$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_5 - r_7 - r_9 - r_{10} &= r_2 + r_3 + r_4 - r_5 - r_7 - r_9 - r_{10} \\ &= r_2 + r_3 + r_6 - r_9 - r_{10} \\ &= r_2 + r_3 + r_7 + r_8 - r_{10} \\ &= r_2 + r_3 + r_7 + r_9 + r_{11} \end{aligned}$$

(22) $G_2 = G_1 + G_3,\ G_1 + G_2 = 2G_3$

\# Double Counts:

$$r_4 + r_7$$

$\mathfrak{R}_G$:

$$\begin{aligned} r_1 - r_4 - r_7 &= r_2 + r_3 - r_7 \\ &= r_2 + r_4 + r_5 + r_6 - r_7 \\ &= r_2 + r_4 + r_5 + r_8 + r_9 \end{aligned}$$

Appendix D: Evaluation of Plausibility of Multiple Initial Conditions on (G_1, G_2, G_3)

Condition 1	Condition 2	Implied Condition	Validity
$G_1 = 2G_2$	$G_1 = G_2 + G_3$	$G_2 = G_3$	
$G_1 = 2G_2$	$G_1 = G_2 + 2G_3$	$(G_1, G_2, G_3) \sim (4,2,1)$	✓
$G_1 = 2G_2$	$G_1 = 2G_2 + G_3$	$G_3 = 0$	
$G_1 = 2G_2$	$G_1 = 2G_2 + 2G_3$	$G_3 = 0$	
$G_1 = 2G_2$	$G_1 = 2G_3$	$G_2 = G_3$	
$G_1 = 2G_2$	$G_2 = G_1 + G_3$	$G_1 + 2G_3 = 0$	
$G_1 = 2G_2$	$G_2 = G_1 + 2G_3$	$G_1 + 3G_3 = 0$	
$G_1 = 2G_2$	$G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (4,2,1)$	✓
$G_1 = 2G_2$	$G_1 + G_2 = G_3$	$(G_1, G_2, G_3) \sim (2,1,3)$	✓
$G_1 = 2G_2$	$G_1 + G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (4,2,3)$	✓
$G_1 = G_2 + G_3$	$G_1 = G_2 + 2G_3$	$G_3 = 0$	
$G_1 = G_2 + G_3$	$G_1 = 2G_2 + G_3$	$G_2 = 0$	
$G_1 = G_2 + G_3$	$G_1 = 2G_2 + 2G_3$	$G_2 + G_3 = 0$	
$G_1 = G_2 + G_3$	$G_1 = 2G_3$	$G_2 = G_3$	
$G_1 = G_2 + G_3$	$G_2 = G_1 + G_3$	$G_3 = 0$	
$G_1 = G_2 + G_3$	$G_2 = G_1 + 2G_3$	$G_3 = 0$	
$G_1 = G_2 + G_3$	$G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (3,2,1)$	✓
$G_1 = G_2 + G_3$	$G_1 + G_2 = G_3$	$G_2 = 0$	
$G_1 = G_2 + G_3$	$G_1 + G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (3,1,2)$	✓
$G_1 = G_2 + 2G_3$	$G_1 = 2G_2 + G_3$	$G_2 = G_3$	
$G_1 = G_2 + 2G_3$	$G_1 = 2G_2 + 2G_3$	$G_2 = 0$	
$G_1 = G_2 + 2G_3$	$G_1 = 2G_3$	$G_2 = 0$	
$G_1 = G_2 + 2G_3$	$G_2 = G_1 + G_3$	$G_3 = 0$	
$G_1 = G_2 + 2G_3$	$G_2 = G_1 + 2G_3$	$G_3 = 0$	
$G_1 = G_2 + 2G_3$	$G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (4,2,1)$	✓
$G_1 = G_2 + 2G_3$	$G_1 + G_2 = G_3$	$2G_2 + G_3 = 0$	
$G_1 = G_2 + 2G_3$	$G_1 + G_2 = 2G_3$	$G_2 = 0$	
$G_1 = 2G_2 + G_3$	$G_1 = 2G_2 + 2G_3$	$G_3 = 0$	
$G_1 = 2G_2 + G_3$	$G_1 = 2G_3$	$(G_1, G_2, G_3) \sim (4,1,2)$	✓
$G_1 = 2G_2 + G_3$	$G_2 = G_1 + G_3$	$G_1 + 3G_3 = 0$	
$G_1 = 2G_2 + G_3$	$G_2 = G_1 + 2G_3$	$G_1 + 5G_3 = 0$	
$G_1 = 2G_2 + G_3$	$G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (5,2,1)$	✓
$G_1 = 2G_2 + G_3$	$G_1 + G_2 = G_3$	$G_2 = 0$	
$G_1 = 2G_2 + G_3$	$G_1 + G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (5,1,3)$	✓
$G_1 = 2G_2 + 2G_3$	$G_1 = 2G_3$	$G_2 = 0$	
$G_1 = 2G_2 + 2G_3$	$G_2 = G_1 + G_3$	$G_1 + 4G_3 = 0$	
$G_1 = 2G_2 + 2G_3$	$G_2 = G_1 + 2G_3$	$G_1 + 6G_3 = 0$	
$G_1 = 2G_2 + 2G_3$	$G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (6,2,1)$	✓
$G_1 = 2G_2 + 2G_3$	$G_1 + G_2 = G_3$	$3G_2 + G_3 = 0$	
$G_1 = 2G_2 + 2G_3$	$G_1 + G_2 = 2G_3$	$G_2 = 0$	
$G_1 = 2G_3$	$G_2 = G_1 + G_3$	$(G_1, G_2, G_3) \sim (2,3,1)$	✓
$G_1 = 2G_3$	$G_2 = G_1 + 2G_3$	$(G_1, G_2, G_3) \sim (2,4,1)$	✓
$G_1 = 2G_3$	$G_2 = 2G_3$	$G_1 = G_2$	
$G_1 = 2G_3$	$G_1 + G_2 = G_3$	$G_2 + G_3 = 0$	
$G_1 = 2G_3$	$G_1 + G_2 = 2G_3$	$G_2 = 0$	

Condition 1	Condition 2	Implied Condition	Validity
$G_2 = G_1 + G_3$	$G_2 = G_1 + 2G_3$	$G_3 = 0$	
$G_2 = G_1 + G_3$	$G_2 = 2G_3$	$G_1 = G_3$	
$G_2 = G_1 + G_3$	$G_1 + G_2 = G_3$	$G_1 = 0$	
$G_2 = G_1 + G_3$	$G_1 + G_2 = 2G_3$	$(G_1, G_2, G_3) \sim (1, 3, 2)$	✓
$G_2 = G_1 + 2G_3$	$G_2 = 2G_3$	$G_1 = 0$	
$G_2 = G_1 + 2G_3$	$G_1 + G_2 = G_3$	$2G_1 + G_3 = 0$	
$G_2 = G_1 + 2G_3$	$G_1 + G_2 = 2G_3$	$G_1 = 0$	
$G_2 = 2G_3$	$G_1 + G_2 = G_3$	$G_1 + G_3 = 0$	
$G_2 = 2G_3$	$G_1 + G_2 = 2G_3$	$G_1 = 0$	
$G_1 + G_2 = G_3$	$G_1 + G_2 = 2G_3$	$G_3 = 0$	