Fall 2013

Income Inequality Measures and Statistical Properties of Weighted Burr-type and Related Distributions

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INCOME INEQUALITY MEASURES AND STATISTICAL PROPERTIES OF WEIGHTED BURR-TYPE AND RELATED DISTRIBUTIONS

by

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(Under the Direction of Broderick O. Oluyede)

ABSTRACT

In this thesis, tail conditional expectation (TCE) in risk analysis, an important measure for right-tail risk, is presented. This value is generally based on the quantile of the loss distribution. Explicit formulas of several tail conditional expectations and inequality measures for Dagum-type models are derived. In addition, a new class of weighted Burr-III (WBIII) distribution is presented. The statistical properties of this distribution including hazard and reverse hazard functions, moments, coefficient of variation, skewness, and kurtosis, inequality measures, entropy are derived. Also, Fisher information and maximum likelihood estimates of the model parameters are obtained.

Key Words: Weighted Burr Distribution; Tail Conditional Expectation

2009 *Mathematics Subject Classification*: 62E15, 60E05

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B.S. in Mathematics

Taif University, Saudi Arabia

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment

of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2013

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Electronic Version Approved:

December 2013

DEDICATION

It is with great honor and admiration that I dedicate this thesis to my wonderful father, Raja Al Buqami. As a Staff Brigadier General in the Saudi Arabian military, my father is a man of high measure and a figure who demands respect. But above all, his has been the role of parent and mentor. Whether his is the voice of ardent encouragement in times of accomplishment or calming reassurance in times of despair, my loving father has always been the foundation of who I am and the beacon of who I want to become. I would also like to recognize the person in my life who's been the embodiment of compassion and care for my whole family, my beautiful mother. Without her direction and grace, none of this would have been possible. And to my dear husband, Mohammed Al Buqami, I also owe many thanks and blessings. Not only is he my partner in life, but he is my colleague as well. Without his steadfast support and devotion, this road would have been all the more difficult to travel, my goals all the more out of reach. This has been quite a run. Thank you, my beloved, for running with me.

ACKNOWLEDGMENTS

It is with a heart full of gratitude that I thank Professor Broderick Oluyede for his outstanding guidance and care throughout this process. Words cannot justly convey how graciously he has invested his time and energy in assisting me, as well as many of my colleagues over the years. The interest he shows in his students is unparalleled, and his enthusiasm and concern have been welcome elements in our conferences. Professor Oluyede has undoubtedly made a mark on my time spent in academia, and I will always remember him for his sincerity and unwavering dedication. Not only has it been a great pleasure and reassurance to have Professor Oluyede as my advisor, but any opportunity to work or study with him in the future would also be an immeasurable honor. Professor Charles Champ and Professor Hani Samawi have both been exceptionally kind and personable with me during my time spent in the Department of Mathematical Sciences. While the academic advisement and hand-on assistance I have been provided are tantamount to my positive experience at Georgia Southern University, the feeling of welcome and comfort Professors Champ and Samawi have fostered within the department has made this time in my life all the more fulfilling. It is also my pleasure to thank all members of faculty and staff in the Mathematical Sciences program at Georgia Southern University, especially Dr.Yan Wu and Dr. Hua Wang, for all their efforts towards the success of all students.

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INTRODUCTION

1.1 Generalized Dagum Distribution

The generalized beta distribution of the second kind (GB2) with four parameters is a flexible distribution for modeling income and wealth distributions. In addition, GB2 includes special cases of other distributions such as inverse Burr with shape parameter p=1, with q=1 for Dagum, and exponential distributions. The probability density function (pdf) of the generalized beta distribution of the second kind (GB2) is given by:

$$f_{GB2}(x; a, b, p, q) = \frac{ax^{ap-1}}{b^{ap}B(p,q)[1+(\frac{x}{b})^a]^{p+q}}, \quad \text{for } x > 0, \tag{1.1}$$

where a,p,q are the shape parameters, b is the scale parameter, $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function, and a, b, p, q are positive real values. See McDonald (1984), McDonald and Xu (1995) for additional details.

Dagum distribution is a special case of GB2 named after Camilo Dagum (1977). The cdf and pdf are given by:

$$G_D(x; \beta, \lambda, \delta) = (1 + \lambda x^{-\delta})^{-\beta}, \tag{1.2}$$

and

$$g_D(x; \beta, \lambda, \delta) = \beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}, \tag{1.3}$$

for $x > 0$, and β, δ, $\lambda > 0$, respectively. Note that λ is a scale parameter, while δ and β are shape parameters. Furthermore, the q-th quantile is

$$x_q = \lambda^{\frac{1}{\delta}}(q^{\frac{-1}{\beta}} - 1)^{\frac{-1}{\delta}}, \tag{1.4}$$

and the r-th moment is given by

$$E[X^r;\beta,\lambda,\delta] = \beta\lambda^{\frac{r}{\delta}} B\left(\frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right), \tag{1.5}$$

where $B(.,.)$ is the beta function and $\delta > r$. See kleiber (2007), Kleiber and Kotz (2003) for additional details.

1.1.1 Mc-Dagum Distribution

The McDonald Dagum distribution (Oluyede and Rajasooriya (2013)) is a generalization of the Dagum distribution and includes the beta Dagum distribution (Domma and Condino (2013)) as a sub model. The cdf of the Mc-Dagum distribution is given by:

$$F_{MD}(x;\xi) = \frac{1}{B(a/c,b)} \int_0^{G_D(x;\theta)} z^{a/c-1}(1-z)^{b-1}dz. \tag{1.6}$$

The corresponding series representation of the cdf is

$$\begin{aligned} F_{MD}(x;\xi) &= \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(a/c+b)}{\Gamma(a/c)\Gamma(b-j)\Gamma(j+1)(a/c+j)} G_D(x;\theta)^{a/c+j} \\ &= \sum_{j=0}^{\infty} p_j G_D(x;\beta(a/c+j),\lambda,\delta), \end{aligned} \tag{1.7}$$

where $G_D(x;\theta)^{a/c+j} = (1+\lambda x^{-\delta})^{-\beta(a/c+j)} = G_D(x;\beta(a/c+j),\lambda,\delta)$, and $p_j = \frac{(-1)^j\Gamma(a/c+b)}{\Gamma(a/c)\Gamma(b-j)\Gamma(j+1)(a/c+j)}$. The pdf is given by:

$$\begin{aligned} f_{MD}(x;\xi) &= \frac{1}{B(a/c,b)}[G_D(x;\theta)]^{(a/c-1)}[1-G_D(x;\theta)]^{(b-1)} g_D(x;\theta) \\ &= \frac{\beta\lambda\delta x^{-\delta-1}}{B(a/c,b)}(1+\lambda x^{-\delta})^{-a\beta/c-1}[1-(1+\lambda x^{-\delta})^{-\beta}]^{b-1}, \end{aligned} \tag{1.8}$$

for $x > 0$, and $\xi = (\beta, \lambda, \delta, a, b, c) > 0$. The corresponding series representation of the pdf is

$$f_{MD}(x;\xi) = \sum_{j=0}^{\infty} p_j g_{_D}(x;\beta(a/c+j),\lambda,\delta). \tag{1.9}$$

However, if b is an integer, then the Mc-Dagum cdf is

$$F_{MD}(x;\xi) = \sum_{j=0}^{b-1} p_j G_D(x;\beta(a/c+j),\lambda,\delta), \tag{1.10}$$

and the pdf is

$$f_{MD}(x;\xi) = \sum_{j=0}^{b-1} p_j g_{_D}(x;\beta(a/c+j),\lambda,\delta). \tag{1.11}$$

The r-th moment is given by:

$$E[X^r;\xi] = \sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{\frac{r}{\delta}} B\left(\frac{r}{\delta} + \beta(a/c+j), 1 - \frac{r}{\delta}\right), \tag{1.12}$$

and the r-th conditional moment is

$$E[X^r|X \leq x] = \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{\frac{r}{\delta}}}{\sum_{j=0}^{\infty} p_j G(x;\beta(a/c+j),\lambda,\delta)} B\left(y^*;\frac{r}{\delta} + \beta(a/c+j), 1 - \frac{r}{\delta}\right), \tag{1.13}$$

for $\delta > r$, $0 < y^* < 1$, $y^* = (1 + \lambda x^{-\delta})^{-1}$, where $B(y^*;\alpha,\beta)$ is incomplete beta function (Gradshteyn and Ryzhik (2000)).

1.1.2 Special Cases

In this section, we present several distributions that can be readily obtained from Mc-Dagum density function:

(1) If $c = 1$, then the random variable X has a beta-Dagum distribution with the pdf:

$$f_{BD}(x; \beta, \lambda, \delta, a, b) = \frac{\beta\lambda\delta x^{-\delta-1}}{B(a,b)}(1 + \lambda x^{-\delta})^{-a\beta-1}[1 - (1 + \lambda x^{-\delta})^{-\beta}]^{b-1},$$

for $x > 0$, and β, λ, δ, a, $b > 0$, where B(.,.) is the beta function.

(2) If $a = b = c = 1$, then X has a Dagum distribution with the pdf:

$$f_D(x; \beta, \lambda, \delta) = \beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}, \quad \text{for } x > 0, \text{ and } \beta,\ \delta,\ \lambda > 0.$$

(3) If $b = c = 1$, then X has a Dagum distribution with parameters $a\beta$, λ, δ and pdf:

$$f_D(x; a\beta, \lambda, \delta) = a\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-a\beta-1}, \quad \text{for } x > 0, \text{ and } \beta,\ \lambda,\ \delta,\ a > 0.$$

(4) If $a = c = 1$, then X has a reduced beta-Dagum distribution with the pdf:

$$f_{BD}(x; \beta, \lambda, \delta, b) = b\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1}[1 - (1 + \lambda x^{-\delta})^{-\beta}]^{b-1},$$

for $x > 0$, and β, λ, δ, $b > 0$.

(5) If $a = c = \lambda = 1$, then X has a beta-BurrIII distribution with the pdf:

$$f_{BB}(x; \beta, \delta, b) = b\beta\delta x^{-\delta-1}(1 + x^{-\delta})^{-\beta-1}[1 - (1 + x^{-\delta})^{-\beta}]^{b-1},$$

for $x > 0$, and β, δ, $b > 0$.

(6) If $c = \beta = 1$, then X has a beta-Fisk distribution with the pdf:

$$f_{BF}(x; \lambda, \delta, a, b) = \frac{\lambda\delta x^{-\delta-1}}{B(a,b)}(1 + \lambda x^{-\delta})^{-a-1}[1 - (1 + \lambda x^{-\delta})^{-1}]^{b-1},$$

for $x > 0$, and λ, δ, $a, b > 0$.

(7) If $b = c = 1$, then X has the Exponentiated-Dagum distribution with the pdf:

$$f_{ED}(x; \alpha\beta, \lambda, \delta) = \alpha\beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\alpha\beta-1}, \quad \text{for } x > 0, \text{ and } \beta,\ \lambda,\ \delta > 0.$$

Note that, when $\alpha = 1$, then we have Dagum distribution.

1.2 Weighted Distribution

Let X be a random variable with pdf $f(x; \theta)$, where $\theta \in \Omega$ is a natural parameter and Ω is the parameter space. Let $w(x; \epsilon)$ be a positive function with parameter ϵ representing the recording (sighting) mechanism. The pdf of the weighted random variable X^w corresponding to $w(x, \epsilon)$ is given by:

$$f^w(x; \theta, \epsilon) = \frac{w(x; \epsilon) f(x; \theta)}{E[w(X; \epsilon)]}, \tag{1.14}$$

where $0 < E[w(X; \epsilon)] < \infty$ is normalizing factor. In addition, the random variable X^w is known as the weighted version of the random variable X with weighted distribution $f^w(x; \theta, \epsilon)$ and weight function w. See Patil and Rao (1978), Rao (1965), Nanda and Jain (1999), Oluyede (1999), and Patil (1991) for additional details.

1.2.1 Some Special Weight Functions

A general class of weight functions (Riabi et al. (2010)) is given by:

$$w(x;\epsilon) = w(x;k,l,m,r) = x^k e^{lx} F^m(x) \overline{F}^r(x), \tag{1.15}$$

where $\overline{F}(x) = 1 - F(x)$.

Remark:

(1) By setting $l = 0$, we get the weights corresponding to the probability weighted moments (PWMs):

$$w(x;k,m,r) = x^k F^m(x) \overline{F}^r(x).$$

(2) If $k = r = m = 0$, we get the weights corresponding to the moment generating functions:

$$w(x;l) = e^{lx}.$$

(3) By putting $l = r = m = 0$, we have the weights for the moments:

$$w(x;k) = x^k.$$

(4) If $k = l = 0$, $m \to m - 1$, and $r \to n - m$, we get the weights for the order statistics as follows:

$$w(x;m,n) = F^{m-1}(x) \overline{F}^{n-m}(x).$$

(5) By setting $k = l = m = 0$, we have proportional hazard weight functions:

$$w(x;r) = \overline{F}^r(x).$$

(6) If $k = l = r = 0$, we get proportional reversed hazard weight functions:

$$w(x; m) = F^m(x).$$

1.3 Some Utility Notions and Basic Results

For $b > 0$, real non-integer, and $|z| < 1$, we have

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\Gamma(j+1)} z^j. \tag{1.16}$$

1.3.1 Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are widely used tools for analyzing and visualizing income inequality. Lorenz and Bonferroni Curves are given by

$$L(F(x)) = \frac{E[X|X \leq x]}{E[X]}, \tag{1.17}$$

and

$$B(F(x)) = \frac{L(F(x))}{F(x)}, \tag{1.18}$$

respectively.

1.3.2 Mean Residual Life Function

The mean residual life function is well-known concept in reliability and survival analysis, denoted by $MRLF(t)$, that is given by:

$$MRLF(t) = E[X - t|X > t] = E[X|X > t] - t. \tag{1.19}$$

1.4 Entropy

The entropy concept plays a vital role in information theory. The entropy of a random variable is a good measure of randomness or uncertainty.

1.4.1 ϵ-Entropy

The applications of ϵ-entropy can be find in many physical systems. Let X be a random variable with the Dagum pdf:

$$g_{_D}(x;\beta,\lambda,\delta)=\beta\lambda\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1},$$

and cdf

$$G_D(x;\beta,\lambda,\delta)=(1+\lambda x^{-\delta})^{-\beta}.$$

The ϵ-entropy is given by:

$$H_\epsilon(g_{_D})=\frac{1}{\epsilon-1}\left[1-\int_0^\infty g_{_D}^\epsilon(x;\beta,\lambda,\delta)dx\right],$$

for $\epsilon>0$ and $\epsilon\neq 1$. Note that for the Dagum pdf:

$$\int_0^\infty g_{_D}^\epsilon(x;\beta,\lambda,\delta)dx=(\beta\lambda\delta)^\epsilon\int_0^\infty x^{-\epsilon(\delta+1)}(1+\lambda x^{-\delta})^{-\epsilon(\beta+1)}dx,$$

we set $t=(1+\lambda x^{-\delta})^{-1}$, so that $dt=(1+\lambda x^{-\delta})^{-2}\lambda\delta x^{-\delta-1}dx$, and

$$\begin{aligned}\int_0^\infty g_{_D}^\epsilon(x;\beta,\lambda,\delta)dx &= \beta^\epsilon\lambda^{\frac{1-\epsilon}{\delta}}\delta^{\epsilon-1}\int_0^1 t^{\frac{1}{\delta}+\epsilon\beta-\frac{\epsilon}{\delta}-1}(1-t)^{\frac{1}{\delta}(\epsilon-1)+\epsilon-1}dt\\ &= \beta^\epsilon\lambda^{\frac{1-\epsilon}{\delta}}\delta^{\epsilon-1}B\left(\frac{1}{\delta}+\epsilon\beta-\frac{\epsilon}{\delta},\frac{1}{\delta}(\epsilon-1)+\epsilon\right),\end{aligned}$$

for $\delta>\epsilon$. Hence, ϵ-entropy for Dagum distribution is given by:

$$H_\epsilon(g_{_D})=\frac{1}{\epsilon-1}\left[1-\beta^\epsilon\lambda^{\frac{1-\epsilon}{\delta}}\delta^{\epsilon-1}B\left(\frac{1}{\delta}+\epsilon\beta-\frac{\epsilon}{\delta},\frac{1}{\delta}(\epsilon-1)+\epsilon\right)\right],$$

for $\delta>\epsilon$, $\epsilon>0$ and $\epsilon\neq 1$.

1.4.2 Renyi Entropy

Renyi entropy (Renyi, 1961) for Dagum distribution is given by

$$\begin{aligned} H_R(g_{_D}) &= (1-\tau)^{-1} log\left[\int_0^\infty g_D^\tau(x;\lambda,\delta,\beta)dx\right] \\ &= (1-\tau)^{-1} log\left[\beta^\tau \lambda^{\frac{1-\tau}{\delta}} \delta^{\tau-1} B\left(\frac{1}{\delta}+\tau\beta-\frac{\tau}{\delta}, \frac{1}{\delta}(\tau-1)+\tau\right)\right], \end{aligned}$$

for $\tau > 0$, $\tau \neq 1$, $\delta > \tau - 1$.

1.5 Probability Weighted Moments for Dagum Distribution

The probability weighted moments (PWMs) for the Dagum distribution is given by:

$$\begin{aligned} & E[X^k G_D^m(X) \overline{G}_D^r(X)] \\ = & \int_0^\infty x^k (1+\lambda x^{-\delta})^{-\beta m} [1-(1+\lambda x^{-\delta})^{-\beta}]^r \beta\lambda\delta x^{-\delta-1} (1+\lambda x^{-\delta})^{-\beta-1} dx \\ = & \beta\lambda^{\frac{k}{\delta}} \int_0^1 t^{\frac{k}{\delta}+\beta m+\beta-1} (1-t)^{-\frac{k}{\delta}+1-1} (1-t^\beta)^r dt \\ = & \beta\lambda^{\frac{k}{\delta}} \sum_{j=0}^\infty \frac{(-1)^j \Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)} \int_0^1 t^{\beta j+\frac{k}{\delta}+\beta m+\beta-1} (1-t)^{1-\frac{k}{\delta}-1} dt \\ = & \beta\lambda^{\frac{k}{\delta}} \sum_{j=0}^\infty \frac{(-1)^j \Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)} B\left(\beta j+\frac{k}{\delta}+\beta m+\beta, 1-\frac{k}{\delta}\right) \\ = & \beta\lambda^{\frac{k}{\delta}} \sum_{j=0}^\infty p_j B\left(\beta j+\frac{k}{\delta}+\beta m+\beta, 1-\frac{k}{\delta}\right), \end{aligned}$$

where we have set $t = (1+\lambda x^{-\delta})^{-1}$, and used the result $(1-t^\beta)^r = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)} t^{\beta j}$, for $r > 0$ and $|t^\beta| < 1$, and $p_j = \frac{(-1)^j\Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)}$, for $\delta > k$.

Remark:

(1) If $l=0$ in $E[X^k e^{lX} F^m(X)\overline{F}^r(X)]$, we get the PWMs:

$$E[X^k G^m(X)\overline{G}^r(X)] = \beta\lambda^{\frac{k}{\delta}} \sum_{j=0}^{\infty} p_j B\left(\frac{k}{\delta} + \beta m + \beta j + \beta, 1 - \frac{k}{\delta}\right), \quad \text{for } \delta > k.$$

(2) If $m=r=0$, we have the moments of order k:

$$E[X^k] = \beta\lambda^{\frac{k}{\delta}} B\left(\frac{k}{\delta} + \beta, 1 - \frac{k}{\delta}\right), \quad \text{for } k < \delta.$$

(3) By setting $k=m=0$, we have the proportional hazard moments:

$$E[\overline{G}^r(X)] = \beta \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(r+1)}{\Gamma(r+1-j)j!} B(\beta j + \beta, 1).$$

(4) When $r=k=0$, we get the proportional reverse hazards moments:

$$E[G^m(X)] = \beta B(\beta m + \beta, 1) = \frac{1}{m+1}.$$

(5) If $k=0$, $m \to m-1$, $r=n-m$, then we obtain moments corresponding to the order statistics.

1.5.1 Tail Conditional Expectation

The tail conditional expectation of a continuous loss random variable X shares four axioms : subadditivity, monotonicity, positive homogeneity, and translation invariance. The tail conditional expectation (TCE) of a random variable X is

$$TCE_X(x_q) = E[X|X > x_q]. \tag{1.20}$$

This risk measure can be interpreted as the mean of worse possible losses given an average amount of the tail of the distribution. See Landsman and Valdez (2005) for additional details. In addition, this tail is based on the q-th quantile, x_q, of the loss distribution and is defined as $x_q = inf\{x|F(x) \geq q\}$. It is uniquely defined as $x_q = F^{-1}(q)$, for a random variable with monotonic, continuous distribution function and it is usually appropriate to assume that the loss variable X has non-negative support in insurance contexts, and we have assumed that in this section. However, the risk measures that we describe can be applied to random variables with a sample space spanning any part of the real line. To evaluate this tail conditional expectation, we use the following formula

$$TCE_X(x_q) = \frac{1}{\overline{F}(x_q)} \int_{x_q}^{\infty} xf(x)dx, \tag{1.21}$$

where $\overline{F}(x_q) > 0$. In the next result, we obtain $TCE(x_q)$ for Dagum distribution.

1.6 Tail Conditional Expectation and Inequality Measures for Dagum Distribution

In this section, we establish the relationship between TCE and income inequality measures for Dagum-type distributions. We derive the most used point inequality measures such as Lorenz and Bonferroni curves.

Theorem.

Let X be a loss random variable with Dagum distribution and x_q the q-th quantile, where $0 < q < 1$. The TCE_r for Dagum distribution is given by:

$$TCE_r(x_q) = \frac{\beta\lambda^{\frac{r}{\delta}}}{1-(1+\lambda x_q^{-\delta})^{-\beta}}\left[B\left(\frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right) - B\left(y_q; \frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right)\right], \tag{1.22}$$

equivalently,

$$TCE_r(x_q) = \beta\lambda^{\frac{r}{\delta}}\left[B\left(\frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right) - \frac{B\left(y_q; \frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right)}{(1+\lambda x_q^{-\delta})^{-\beta}}\right], \tag{1.23}$$

for $\delta > r$, $0 < y_q < y < 1$, $0 < x_q < x < \infty$, and $y_q = (1+\lambda {x_q}^{-\delta})^{-1}$.

Proof. To prove (1.22), we use the following formula

$$\begin{aligned} TCE_r(x_q) &= \frac{1}{\overline{G}_D(x_q)}\int_{x_q}^{\infty} x^r g_D(x)dx \\ &= \frac{1}{1-(1+\lambda x_q^{-\delta})^{-\beta}}\int_{x_q}^{\infty} \beta\lambda\delta x^{r-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}dx. \end{aligned}$$

Let $y = (1+\lambda x^{-\delta})^{-1}$, $dy = (1+\lambda x^{-\delta})^{-2}\lambda\delta x^{-\delta-1}dx$, $dx = \frac{(\lambda y)^{1+\frac{1}{\delta}}dy}{\lambda\delta(1-y)^{1+\frac{1}{\delta}}y^2}$, and

$$\begin{aligned} TCE_r(x_q) &= \frac{\beta\lambda^{\frac{r}{\delta}}}{1-(1+\lambda x_q^{-\delta})^{-\beta}}\int_{y_q}^{1} y^{\frac{r}{\delta}+\beta-1}(1-y)^{-\frac{r}{\delta}+1-1}dy \\ &= \frac{\beta\lambda^{\frac{r}{\delta}}}{1-(1+\lambda x_q^{-\delta})^{-\beta}}\left[B\left(\frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right) - B\left(y_q; \frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right)\right], \end{aligned}$$

where $y_q = (1+\lambda {x_q}^{-\delta})^{-1}$, for $\delta > r$, $0 < y_q < y < 1$, and $0 < x_q < x < \infty$.

To obtain (1.23), we use the following formula

$$TCE_r(x_q) = [E(X^r) - E(X^r|X \le x_q)].$$

Note that

$$E[X^r] = E[X^r|X > x_q] + E[X^r|X \leq x_q].$$

Now,

$$\begin{aligned} E[X^r|X \leq x_q] &= \frac{1}{G_D(x_q)} \int_0^{x_q} x^r g_D(x)dx \\ &= \frac{1}{(1+\lambda x_q^{-\delta})^{-\beta}} \int_0^{x_q} \beta\lambda\delta x^{r-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}dx, \end{aligned}$$

so that

$$\begin{aligned} E[X^r|X \leq x_q] &= \frac{\beta\lambda^{\frac{r}{\delta}}}{(1+\lambda x_q^{-\delta})^{-\beta}} \int_0^{y_q} y^{\frac{r}{\delta}+\beta-1}(1-y)^{-\frac{r}{\delta}+1-1}dy \\ &= \frac{\beta\lambda^{\frac{r}{\delta}}}{(1+\lambda x_q^{-\delta})^{-\beta}} B\left(y_q; \frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right), \end{aligned}$$

for $0 < y_q < y < 1$, where $y_q = (1+\lambda {x_q}^{-\delta})^{-1}$, and $\delta > r$. Therefore, TCE_r for Dagum distribution is given by:

$$TCE_r(x_q) = \beta\lambda^{\frac{r}{\delta}} \left[B\left(\frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right) - \frac{B\left(y_q; \frac{r}{\delta}+\beta, 1-\frac{r}{\delta}\right)}{(1+\lambda x_q^{-\delta})^{-\beta}} \right],$$

for $\delta > r$, $0 < y_q < y < 1$, and $0 < x_q < x < \infty$. When $r = 1$, we have:

$$TCE(x_q) = \beta\lambda^{\frac{1}{\delta}} \left[B\left(\frac{1}{\delta}+\beta, 1-\frac{1}{\delta}\right) - \frac{B\left(y_q; \frac{1}{\delta}+\beta, 1-\frac{1}{\delta}\right)}{(1+\lambda x_q^{-\delta})^{-\beta}} \right],$$

for $\delta > 1$.

Lorenz curve for Dagum distribution is

$$L(G(x)) = \frac{B\left(y; \frac{1}{\delta}+\beta, 1-\frac{1}{\delta}\right)}{(1+\lambda x^{-\delta})^{-\beta} B\left(\frac{1}{\delta}+\beta, 1-\frac{1}{\delta}\right)},$$

for $0 < y < 1$, $0 < x < \infty$, and $\delta > 1$, so that

$$L(G(x_q)) = \frac{B\left(y_q; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x_q^{-\delta})^{-\beta} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)},$$

for $0 < y_q < y < 1$, $0 < x_q < x < \infty$, and $\delta > 1$.

Tail conditional expectation can be written in terms of Lorenz curve as follows:

$$\begin{aligned} TCE(x_q) &= \mu\left[1 - \frac{L(G(x_q))}{G(x_q)}\right] \\ &= \beta\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right) \\ &* \left[1 - \frac{B\left(y_q; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x_q^{-\delta})^{-2\beta} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}\right], \end{aligned}$$

for $\delta > 1$, $0 < y_q < 1$, and $0 < x_q < \infty$.

Bonferroni curve for Dagum distribution is

$$B(G(x)) = \frac{B\left(y; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x^{-\delta})^{-2\beta} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)},$$

for $0 < y < 1$, $0 < x < \infty$, and $\delta > 1$, so that

$$B(G(x_q)) = \frac{B\left(y_q; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x_q^{-\delta})^{-2\beta} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)},$$

for $\delta > 1$, $0 < y_q < y < 1$, and $0 < x_q < x < \infty$.

Also, tail conditional expectation can be written in terms of Bonferroni curve as follows:

$$
\begin{aligned}
TCE(x_q) &= \mu\left[1 - \frac{G(x_q)B(G(x_q))}{G(x_q)}\right] \\
&= \mu\left[1 - B(G(x_q))\right] \\
&= \beta\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right) \\
&* \quad \left[1 - \frac{B\left(y_q; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x_q^{-\delta})^{-2\beta} B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)}\right],
\end{aligned}
$$

for $\delta > 1$, $0 < y_q < 1$, and $0 < x_q < \infty$.

The mean residual life function for Dagum distribution by:

$$
\begin{aligned}
MRLF(t) &= E[(X - t)|X > t] \\
&= TCE(t) - t \\
&= \frac{\beta\lambda^{\frac{1}{\delta}}}{1 - (1 + \lambda t^{-\delta})^{-\beta}} \\
&* \left[B\left(\frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right) - B\left(y; \frac{1}{\delta} + \beta, 1 - \frac{1}{\delta}\right)\right] - t,
\end{aligned}
$$

for $0 < y < 1$, where $y = (1 + \lambda t^{-\delta})^{-1}$, and $\delta > 1$.

1.7 Outline of Thesis

The remaining of this thesis is organized as follows. Explicit formulas for computing tail conditional expectation (TCE) of Dagum-types models are presented in chapter 2. TCE in term of Inequality measures for Mc-Dagum distribution and its sub-models are given. Chapter 3 contains TCE, mean residual life function(MRLF), and entropy measures for Dagum-Weibull and related distributions. Chapter 4 introduces and presents the statistical properties of a new class of weighted Burr III (WBIII) distribution, including mean, variance, standard deviation, coefficients of variation, skewness, and kurtosis. Inequality and entropy measures are presented. In addition, maximum likelihood estimates (MLE), Fisher information, asymptotic confidence intervals for parameters of the WBIII distribution are obtained. Applications and examples are presented in chapter 5.

CHAPTER 2

TAIL CONDITIONAL EXPECTATION AND INEQUALITY MEASURES FOR GENERALIZED DAGUM-TYPE DISTRIBUTIONS

2.1 Useful Functions

In this section, some useful functions employed in subsequent chapters are presented. The gamma function is given by:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{2.1}$$

The first and the second derivative of the gamma function are given by:

$$\Gamma^{'}(x) = \int_0^\infty t^{x-1} (\log t) e^{-t}\, dt, \quad \text{and} \quad \Gamma^{''}(x) = \int_0^\infty t^{x-1} (\log t)^2 e^{-t} dt, \tag{2.2}$$

respectively. The digamma function is defined by:

$$\Psi(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}. \tag{2.3}$$

The lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t}\, dt, \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t}\, dt, \tag{2.4}$$

respectively.

2.2 Tail Conditional Expectation and Inequality Measures for Mc-Dagum Distribution and Sub-Models

The TCE_r for Mc-Dagum distribution is

$$\begin{aligned} TCE_r(x_q) &= [E(X^r) - E(X^r|X \leq x_q)] \\ &= \sum_{j=0}^{\infty} p_j \beta(a/c+j) \lambda^{r/\delta} \Big[B(r/\delta + \beta(a/c+j), 1 - r/\delta) \\ &- \frac{B(y_q{}^*; r/\delta + \beta(a/c+j), 1-r/\delta)}{\sum_{j=0}^{\infty} p_j G(x_q; \beta(a/c+j), \lambda, \delta)} \Big], \end{aligned}$$

for $\delta > r$. When $r = 1$, we have:

$$\begin{aligned} TCE(x_q) &= [E(X) - E(X|X \leq x_q)] \\ &= \sum_{j=0}^{\infty} p_j \beta(a/c+j) \lambda^{1/\delta} \Big[B(1/\delta + \beta(a/c+j), 1 - 1/\delta) \\ &- \frac{B(y_q{}^*; 1/\delta + \beta(a/c+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x_q; \beta(a/c+j), \lambda, \delta)} \Big], \end{aligned} \tag{2.5}$$

for $\delta > 1$, $0 < y_q{}^* = (1 + \lambda x_q{}^{-\delta})^{-1} < 1$, where $G(x_q; \beta(a/c+j), \lambda, \delta) = (1 + \lambda x_q{}^{-\delta})^{-\beta(a/c+j)}$, and $p_j = \frac{(-1)^j \Gamma(a/c+b)}{\Gamma(a/c)\Gamma(b-j)\Gamma(j+1)(a/c+j)}$. The model with the parameters ($\lambda = 6.5$, $\beta = 1.5$, $\delta = 5$, $a = 1.005$, $b = 1.04$, $c = 1.9$) in Figure 2.1 corresponds to Mc-Dagum distribution.

We derive income inequality measures for Mc-Dagum distribution. Lorenz

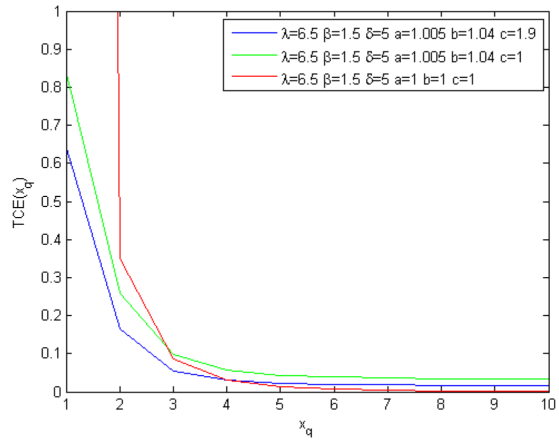

Figure 2.1: TCE for Mc-Dagum and sub-models for different values of parameters.

curve for Mc-Dagum distribution is

$$\begin{aligned} L(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a/c+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x; \beta(a/c+j), \lambda, \delta)} \\ &* \frac{1}{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(1/\delta + \beta(a/c+j), 1-1/\delta)}, \end{aligned} \tag{2.6}$$

for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

Tail conditional expectation for Mc-Dagum distribution can be written in terms of Lorenz curve as follows:

$$
\begin{aligned}
TCE(x_q) &= \mu\left[1-\frac{L(F(x_q))}{F(x_q)}\right] \\
&= \sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta}+\beta(a/c+j),1-\frac{1}{\delta}\right) \\
&* \quad \left[1-\frac{\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(y^*;1/\delta+\beta(a/c+j),1-1/\delta)}{F^2(x_q)\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(1/\delta+\beta(a/c+j),1-1/\delta)}\right],
\end{aligned}
\tag{2.7}
$$

where $F(x_q) = \sum_{j=0}^{\infty} p_j G(x_q;\beta(a/c+j),\lambda,\delta)$, $\overline{F}(x_q) = 1 - F(x_q)$, $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

Bonferroni curve for Mc-Dagum distribution is

$$
\begin{aligned}
B(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(y^*;1/\delta+\beta(a/c+j),1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x;\beta(a/c+j),\lambda,\delta)} \\
&* \quad \frac{1}{F(x)\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(1/\delta+\beta(a/c+j),1-1/\delta)},
\end{aligned}
\tag{2.8}
$$

for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$. Also, tail conditional expectation for Mc-Dagum distribution can be written in terms of Bonferroni curve as follows:

$$
\begin{aligned}
TCE(x_q) &= \mu\Big[1-B(F(x_q))\Big] \\
&= \sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta}+\beta(a/c+j),1-\frac{1}{\delta}\right) \\
&* \quad \left[1-\frac{\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(y^*;1/\delta+\beta(a/c+j),1-1/\delta)}{F^2(x_q)\sum_{j=0}^{\infty} p_j\beta(a/c+j)\lambda^{1/\delta}B(1/\delta+\beta(a/c+j),1-1/\delta)}\right],
\end{aligned}
\tag{2.9}
$$

where $F(x_q) = \sum_{j=0}^{\infty} p_j G(x_q; \beta(a/c+j), \lambda, \delta)$, $\overline{F}(x_q) = 1 - F(x_q)$, $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

The mean residual life function (MRLF) for Mc-Dagum distribution is

$$\begin{aligned} MRLF(t) &= TCE(t) - t \\ &= \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta}}{1 - \sum_{j=0}^{\infty} p_j G(t; \beta(a/c+j), \lambda, \delta)} \Bigg[B(1/\delta + \beta(a/c+j), 1 - 1/\delta) \\ &- B({y_q}^*; 1/\delta + \beta(a/c+j), 1 - 1/\delta) \Bigg] - t, \end{aligned} \tag{2.10}$$

for $0 < {y_q}^* < 1$, and $\delta > 1$.

2.2.1 Sub-Models

Several TCE and inequality measures for the sub-distributions can be derived from TCE and inequality measures of Mc-Dagum distribution as special cases.

Remark:

(1) If $c = 1$ and $b, a > 0$, then $TCE(x_q)$ for beta-Dagum distribution is given by:

$$\begin{aligned} TCE(x_q) &= \sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} \Bigg[B(1/\delta + \beta(a+j), 1 - 1/\delta) \\ &- \frac{B({y_q}^*; 1/\delta + \beta(a+j), 1 - 1/\delta)}{\sum_{j=0}^{\infty} p_j G(x_q; \beta(a+j), \lambda, \delta)} \Bigg], \end{aligned}$$

for $\delta > 1$, $0 < {y_q}^* = (1 + \lambda {x_q}^{-\delta})^{-1} < 1$, where $G(x_q; \beta(a+j), \lambda, \delta) = (1 + \lambda {x_q}^{-\delta})^{-\beta(a+j)}$, and $p_j = \frac{(-1)^j \Gamma(a+b)}{\Gamma(a)\Gamma(b-j)\Gamma(j+1)(a+j)}$.

The model with the parameters ($\lambda = 6.5$, $\beta = 1.5$, $\delta = 5$, $a = 1.005$, $b = 1.04$, $c = 1$) in Figure 2.1 corresponds to beta-Dagum distribution.

Lorenz curve for beta-Dagum distribution is

$$\begin{aligned} L(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x; \beta(a+j), \lambda, \delta)} \\ &* \frac{1}{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(1/\delta + \beta(a+j), 1-1/\delta)}, \end{aligned}$$

for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

TCE can be written in terms of Lorenz curve as follows:

$$\begin{aligned} TCE(x_q) &= \sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta} + \beta(a+j), 1 - \frac{1}{\delta}\right) \\ &* \left[1 - \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a+j), 1-1/\delta)}{F^2(x_q)\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(1/\delta + \beta(a+j), 1-1/\delta)}\right], \end{aligned}$$

where $F(x_q) = \sum_{j=0}^{\infty} p_j G(x_q; \beta(a+j), \lambda, \delta)$ and $\overline{F}(x_q) = 1 - F(x_q)$, for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

Bonferroni curve for beta-Dagum distribution is

$$\begin{aligned} B(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x; \beta(a+j), \lambda, \delta)} \\ &* \frac{1}{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(1/\delta + \beta(a+j), 1-1/\delta) F(x)}, \end{aligned}$$

for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

TCE can be written in terms of Bonferroni curve as follows:

$$\begin{aligned} TCE(x_q) &= \sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta} + \beta(a+j), 1 - \frac{1}{\delta}\right) \\ &* \left[1 - \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a+j), 1-1/\delta)}{F^2(x_q)\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta} B(1/\delta + \beta(a+j), 1-1/\delta)}\right], \end{aligned}$$

where $F(x_q) = \sum_{j=0}^{\infty} p_j G(x_q; \beta(a/c+j), \lambda, \delta)$ and $\overline{F}(x_q) = 1 - F(x_q)$, for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$.

The mean residual life function (MRLF) for beta-Dagum distribution is

$$
\begin{aligned}
MRLF(t) &= TCE(t) - t \\
&= \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{1/\delta}}{1 - \sum_{j=0}^{\infty} p_j G(t; \beta(a+j), \lambda, \delta)} \Bigg[B(1/\delta + \beta(a+j), 1 - 1/\delta) \\
&- B({y_q}^*; 1/\delta + \beta(a+j), 1 - 1/\delta) \Bigg] - t,
\end{aligned}
$$

for $0 < {y_q}^* < 1$, and $\delta > 1$.

(2) If $a = b = c = 1$, then $TCE(x_q)$ and inequality measures for Dagum distribution were mentioned in chapter one. The model with the parameters ($\lambda = 6.5$, $\beta = 1.5$, $\delta = 5$, $a = 1$, $b = 1$, $c = 1$) in Figure 2.1 corresponds to Dagum distribution.

(3) If $b = c = 1$, then $TCE(x_q)$ for Exponentiated-Dagum distribution is

$$
TCE(x_q) = \alpha\beta\lambda^{\frac{1}{\delta}} \left[B\left(\frac{1}{\delta} + \alpha\beta, 1 - \frac{1}{\delta}\right) - \frac{B\left(y_q; \frac{1}{\delta} + \alpha\beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x_q^{-\delta})^{-\alpha\beta}} \right], \quad \text{for } \delta > 1.
$$

Lorenz curve for Exponentiated-Dagum distribution is given by:

$$
L(F(x)) = \frac{B\left(y; \frac{1}{\delta} + \alpha\beta, 1 - \frac{1}{\delta}\right)}{(1 + \lambda x^{-\delta})^{-\alpha\beta} B\left(\frac{1}{\delta} + \alpha\beta, 1 - \frac{1}{\delta}\right)}, \quad \text{for } 0 < y < 1,\ 0 < x < \infty, \text{ and } \delta > 1.
$$

TCE can be written in terms of Lorenz curve as follows:

$$\begin{aligned} TCE(x_q) &= \alpha\beta\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right) \\ &* \left[1-\frac{B\left(y_q; \frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}{(1+\lambda {x_q}^{-\delta})^{-2\alpha\beta} B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}\right], \end{aligned}$$

for $\delta > 1$, $0 < y_q < 1$.

Bonferroni curve for Exponentiated-Dagum distribution is

$$B(F(x)) = \frac{B\left(y; \frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}{(1+\lambda x^{-\delta})^{-2\alpha\beta} B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}, \quad \text{for } \delta > 1,\ 0 < y < 1, \text{ and } 0 < x < \infty.$$

TCE can be written in terms of Bonferroni curve as follows:

$$\begin{aligned} TCE(x_q) &= \alpha\beta\lambda^{\frac{1}{\delta}} B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right) \\ &* \left[1-\frac{B\left(y_q; \frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}{(1+\lambda {x_q}^{-\delta})^{-2\alpha\beta} B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)}\right], \end{aligned}$$

for $\delta > 1$, $0 < y_q < 1$.

The mean residual life function for Exponentiated-Dagum distribution is given by:

$$\begin{aligned} MRLF(t) &= \frac{\alpha\beta\lambda^{\frac{1}{\delta}}}{1-(1+\lambda t^{-\delta})^{-\alpha\beta}} \\ &* \left[B\left(\frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right) - B\left(y; \frac{1}{\delta}+\alpha\beta, 1-\frac{1}{\delta}\right)\right] - t, \end{aligned}$$

for $0 < y < 1$, where $y = (1+\lambda x^{-\delta})^{-1}$, and $\delta > 1$.

2.3 Concluding Remarks

This chapter includes some computations, here are the main results for Mc-Dagum distribution:

- Tail conditional expectation is given by:

$$\begin{aligned} TCE(x_q) &= [E(X) - E(X|X \leq x_q)] \\ &= \sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta}\Bigg[B(1/\delta + \beta(a/c+j), 1-1/\delta) \\ &- \frac{B({y_q}^*; 1/\delta + \beta(a/c+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x_q; \beta(a/c+j), \lambda, \delta)}\Bigg], \quad \text{for } \delta > 1. \end{aligned}$$

- Lorenz and Bonferroni curves for Mc-Dagum distribution are given by:

$$\begin{aligned} L(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a/c+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x; \beta(a/c+j), \lambda, \delta)} \\ &* \frac{1}{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(1/\delta + \beta(a/c+j), 1-1/\delta)}, \end{aligned}$$

  for $\delta > 1$, $0 < y^* < 1$, $0 < x < \infty$, and

$$\begin{aligned} B(F(x)) &= \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(y^*; 1/\delta + \beta(a/c+j), 1-1/\delta)}{\sum_{j=0}^{\infty} p_j G(x; \beta(a/c+j), \lambda, \delta)} \\ &* \frac{1}{F(x)\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta} B(1/\delta + \beta(a/c+j), 1-1/\delta)}, \end{aligned}$$

  for $\delta > 1$, $0 < y^* < 1$, and $0 < x < \infty$, respectively.

  Tail conditional expectation for Mc-Dagum distribution can be written in terms of Lorenz and Bonferroni curves.

- Mean residual life function is given by:

$$
\begin{aligned}
MRLF(t) &= TCE(t) - t \\
&= \frac{\sum_{j=0}^{\infty} p_j \beta(a/c+j)\lambda^{1/\delta}}{1-\sum_{j=0}^{\infty} p_j G(t;\beta(a/c+j),\lambda,\delta)} \Big[B(1/\delta+\beta(a/c+j), 1-1/\delta) \\
&- B({y_q}^*; 1/\delta+\beta(a/c+j), 1-1/\delta) \Big] - t,
\end{aligned}
$$

for $0 < {y_q}^* < 1$, and $\delta > 1$.

Note that we have properties of beta-Dagum distribution when $c = 1$, and Dagum distribution if $a = b = c = 1$ as sub-models.

CHAPTER 3

TCE AND UNCERTAINTY MEASURES FOR DAGUM-WEIBULL DISTRIBUTION

In this chapter, TCE and mean residual life function for the Dagum-Weibull distribution are presented. Entropy measures including Renyi and ϵ-Entropy are also given.

3.1 Dagum-Weibull Distribution

The cdf and pdf of Dagum-Weibull (DW) distribution are given by:

$$F_{DW}(x;\beta,\lambda,\delta,\gamma,c) = (1+\lambda\gamma^{c\delta}x^{-c\delta})^{-\beta}, \tag{3.1}$$

and

$$f_{DW}(x;\beta,\lambda,\delta,\gamma,c) = \beta\lambda\delta\gamma^{c\delta}c\delta x^{-c\delta-1}(1+\lambda\gamma^{c\delta}x^{-c\delta})^{-\beta-1}, \tag{3.2}$$

for $x > 0$, and β, δ, λ, γ, and $c > 0$, respectively, (Oluyede and Kimitei (2013)). With parameters β, $\lambda\gamma^{c\delta}$, and $c\delta$, this can be seen as resulting in $Dagum(\beta,\lambda\gamma^{c\delta},c\delta)$ distribution. In addition, the q-th quantile is

$$x_q = \gamma\lambda^{\frac{1}{c\delta}}(q^{\frac{-1}{\beta}}-1)^{\frac{-1}{c\delta}}. \tag{3.3}$$

Furthermore, the r-th moment is given by:

$$E[X^r;\beta,\lambda,\delta,\gamma,c] = \beta(\gamma^{c\delta}\lambda)^{\frac{r}{c\delta}}B\left(\frac{r}{c\delta}+\beta,1-\frac{r}{c\delta}\right), \tag{3.4}$$

for $c\delta > r$, and β, δ, λ, γ, $c > 0$.

3.2 Submodels

Submodels of the DW distribution are given in this section.

Remark:

(1) If c=1, then we have the Dagum-Exponential (DE) distribution with pdf:

$$f_{DE}(x;\beta,\lambda,\delta,\gamma) \;=\; \beta\lambda\gamma^{\delta}\delta x^{-\delta-1}(1+\lambda\gamma^{\delta}x^{-\delta})^{-\beta-1}, \quad \text{for } x>0,\ \beta,\ \delta,\ \gamma, \text{ and } \lambda>0.$$

(2) If $\lambda=\gamma=c=1$, we have the Burr-III distribution with pdf:

$$f_{BIII}(x,\beta,\delta) \;=\; \beta\delta x^{-\delta-1}(1+x^{-\delta})^{-\beta-1}, \quad \text{for } \delta,\ \beta>0.$$

(3) If $\beta=1$, we have Fisk-Weibull (FW) distribution with pdf:

$$f_{FW}(x;\lambda,\delta,\gamma,c) \;=\; c\delta\lambda\gamma^{c\delta}x^{-c\delta-1}(1+\lambda\gamma^{c\delta}x^{-c\delta})^{-2}, \quad \text{for } c\delta>1, \text{ and } \lambda,\ \delta,\ \gamma,\ c>0.$$

(4) If $\beta=\gamma=c=1$, we have Fisk distribution with pdf:

$$f_{FISK}(x;\lambda,\delta) \;=\; \delta\lambda x^{-\delta-1}(1+\lambda x^{-\delta})^{-2}, \quad \text{for } \delta>0, \text{ and } \lambda>0.$$

(5) If $\gamma=c=1$, we have Dagum distribution with pdf:

$$f_D(x;\beta,\lambda,\delta)=\beta\lambda\delta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}, \quad \text{for } x>0, \text{ and } \beta,\ \delta,\ \lambda>0.$$

(6) If $c=2$, $\gamma=1$, we have the Dagum-Rayleigh distribution with pdf:

$$f_{DAG-RAY}(x;\beta,\lambda,\delta) \;=\; (2\delta)\beta\lambda x^{-2\delta-1}(1+\lambda x^{-2\delta})^{-\beta-1}, \quad \text{for } \beta,\ \lambda,\ \delta>0.$$

3.3 Tail Conditional Expectation for Dagum-Weibull Distribution

The tail conditional expectation (TCE) for Dagum-Weibull Distribution (DW) is presented in this section. Note that

$$E[X^r|X > x_q] = E[X^r] - E[X^r|X \leq x_q],$$

and

$$\begin{aligned} E[X^r|X \leq x_q] &= \frac{1}{F(x_q)} \int_0^{x_q} x^r f_{DW}(x)dx \\ &= \frac{1}{(1+\lambda\gamma^{c\delta}x_q^{-c\delta})^{-\beta}} \int_0^{x_q} \beta\lambda\gamma^{c\delta}c\delta x^{r-c\delta-1}(1+\lambda\gamma^{c\delta}x^{-c\delta})^{-\beta-1}dx. \end{aligned}$$

We set $y = (1+\lambda\gamma^{c\delta}x^{-c\delta})^{-1}$, $0 < y < 1$, so that $dy = (1+\lambda x^{-\delta})^{-2}\lambda\delta x^{-\delta-1}dx$, and $dx = \frac{(\lambda y)^{1+\frac{1}{\delta}}dy}{\lambda\delta(1-y)^{1+\frac{1}{\delta}}y^2}$. Consequently,

$$\begin{aligned} E[X^r|X \leq x_q] &= \frac{\beta(\gamma^{c\delta}\lambda)^{\frac{r}{c\delta}}}{(1+\gamma^{c\delta}\lambda x_q^{-c\delta})^{-\beta}} \int_0^{y_q} y^{\frac{r}{c\delta}+\beta-1}(1-y)^{-\frac{r}{c\delta}+1-1}dy \\ &= \frac{\beta(\gamma^{c\delta}\lambda)^{\frac{r}{c\delta}}}{(1+\gamma^{c\delta}\lambda x_q^{-c\delta})^{-\beta}} B\left(y_q; \frac{r}{c\delta}+\beta, 1-\frac{r}{c\delta}\right), \end{aligned}$$

for $0 < y_q < 1$, $y_q = (1+\gamma^{c\delta}\lambda x_q{}^{-c\delta})^{-1}$, and $c\delta > r$. Therefore, the TCE_r for DW distribution is

$$TCE_r(x_q) = \beta(\gamma^{c\delta}\lambda)^{\frac{r}{c\delta}} \left[B\left(\frac{r}{c\delta}+\beta, 1-\frac{r}{c\delta}\right) - \frac{B\left(y_q; \frac{r}{c\delta}+\beta, 1-\frac{r}{c\delta}\right)}{(1+\gamma^{c\delta}\lambda x_q^{-c\delta})^{-\beta}} \right],$$

for $c\delta > r$, $0 < y_q < 1$, and $0 < x_q < \infty$. When $r = 1$, we have:

$$TCE(x_q) = \beta(\gamma^{c\delta}\lambda)^{\frac{1}{c\delta}} \left[B\left(\frac{1}{c\delta}+\beta, 1-\frac{1}{c\delta}\right) - \frac{B\left(y_q; \frac{1}{c\delta}+\beta, 1-\frac{1}{c\delta}\right)}{(1+\gamma^{c\delta}\lambda x_q^{-c\delta})^{-\beta}} \right],$$

for $c\delta > 1$. Note that tail conditional expectation can be obtained in terms of Lorenz and Bonferroni curves as previously mentioned in chapter 2.

3.4 Mean Residual Life Function for DW Distribution

The mean residual life function $MRLF(t)$ for DW distribution is given by:

$$\begin{aligned} MRLF(t) &= TCE(t) - t \\ &= \frac{\beta(\gamma^{c\delta}\lambda)^{\frac{1}{c\delta}}}{1-(1+\gamma^{c\delta}\lambda t^{-c\delta})^{-\beta}} \\ &* \left[B\left(\frac{1}{c\delta}+\beta, 1-\frac{1}{c\delta}\right) - B\left(y_q; \frac{1}{c\delta}+\beta, 1-\frac{1}{c\delta}\right)\right] - t, \end{aligned} \tag{3.5}$$

for $0 < y_q < 1$, and $c\delta > 1$.

3.5 Entropy

In this section, Renyi (1961) and ϵ-entropies for the DW distribution are presented.

3.5.1 ϵ-Entropy

ϵ-entropy for DW Distribution is given by:

$$H_\epsilon(f_{DW}) = \frac{1}{\epsilon - 1}\left[1 - \int_0^\infty f_{DW}^\epsilon(x;\beta,\lambda,\delta,\gamma,c)dx\right], \quad \epsilon > 0, \text{ and } \epsilon \neq 1.$$

Now,

$$\begin{aligned}\int_0^\infty f^\epsilon_{DW}(x;\beta,\lambda,\delta,\gamma,c)dx &= (\beta\lambda\gamma^{c\delta}c\delta)^\epsilon \\ &* \int_0^\infty x^{-\epsilon(c\delta+1)}(1+\lambda\gamma^{c\delta}x^{-c\delta})^{-\epsilon(\beta+1)}dx.\end{aligned}$$

We set $t=(1+\lambda\gamma^{c\delta}x^{-c\delta})^{-1}$, then $dt=(1+\gamma^{c\delta}\lambda x^{-c\delta})^{-2}\lambda\gamma^{c\delta}c\delta x^{-c\delta-1}dx$, and $dx=\frac{(\lambda\gamma^{c\delta}t)^{1+\frac{1}{c\delta}}dt}{c\delta\lambda\gamma^{c\delta}(1-t)^{1+\frac{1}{c\delta}}t^2}$, so that

$$\begin{aligned}\int_0^\infty f^\epsilon_{DW}(x;\beta,\lambda,\delta,\gamma,c)dx &= \beta^\epsilon(\gamma^{c\delta}\lambda)^{\frac{1-\epsilon}{c\delta}}(c\delta)^{\epsilon-1} \\ &* \int_0^1 t^{\frac{1}{c\delta}+\epsilon\beta-\frac{\epsilon}{c\delta}-1}(1-t)^{\epsilon+\frac{\epsilon}{c\delta}-\frac{1}{c\delta}-1}dt \\ &= \beta^\epsilon(\gamma^{c\delta}\lambda)^{\frac{1-\epsilon}{c\delta}}(c\delta)^{\epsilon-1} \\ &* B\left(\frac{1}{c\delta}+\epsilon\beta-\frac{\epsilon}{c\delta},\epsilon+\frac{\epsilon}{c\delta}-\frac{1}{c\delta}\right),\end{aligned}$$

for $c\delta>\epsilon-1$. Hence, ϵ-entropy for DW Distribution is:

$$\begin{aligned}H_\epsilon(f_{DW}) &= \frac{1}{\epsilon-1}\Bigg[1-\beta^\epsilon(\lambda\gamma^{c\delta})^{\frac{1-\epsilon}{c\delta}}(c\delta)^{\epsilon-1} \\ &* B\left(\frac{1}{c\delta}+\epsilon\beta-\frac{\epsilon}{c\delta},\epsilon+\frac{\epsilon}{c\delta}-\frac{1}{c\delta}\right)\Bigg],\end{aligned}$$

for $\epsilon>0$, $\epsilon\neq1$, and $c\delta>\epsilon-1$.

3.5.2 Renyi Entropy

Renyi entropy (Renyi, 1961) for DW Distribution is given by:

$$\begin{aligned}H_R(f_{DW}) &= (1-\tau)^{-1}log\left[\int_0^\infty f^\tau_{DW}(x;\beta,\lambda,\delta,\gamma,c)dx\right] \\ &= (1-\tau)^{-1}log\left[\beta^\tau(\gamma^{c\delta}\lambda)^{\frac{1-\tau}{c\delta}}(c\delta)^{\tau-1}B\left(\frac{1}{c\delta}+\tau\beta-\frac{\tau}{c\delta},\tau+\frac{\tau}{c\delta}-\frac{1}{c\delta}\right)\right],\end{aligned}$$

for $\tau > 0$, $\tau \neq 1$, and $c\delta > \tau - 1$.

3.6 Concluding Remarks

This chapter includes some computations, here are the main results for DW distribution:

- Tail conditional expectation is given by:

$$TCE(x_q) = \beta(\gamma^{c\delta}\lambda)^{\frac{r}{c\delta}} \left[B\left(\frac{1}{c\delta} + \beta, 1 - \frac{1}{c\delta}\right) - \frac{B\left(y_q; \frac{1}{c\delta} + \beta, 1 - \frac{1}{c\delta}\right)}{(1 + \gamma^{c\delta}\lambda x_q^{-c\delta})^{-\beta}} \right],$$

  for $c\delta > 1$.

- Mean residual life function is given by:

$$\begin{aligned} MRLF(t) &= \frac{\beta(\gamma^{c\delta}\lambda)^{\frac{1}{c\delta}}}{1 - (1 + \gamma^{c\delta}\lambda t^{-c\delta})^{-\beta}} \\ &* \left[B\left(\frac{1}{c\delta} + \beta, 1 - \frac{1}{c\delta}\right) - B\left(y_q; \frac{1}{c\delta} + \beta, 1 - \frac{1}{c\delta}\right) \right] - t, \end{aligned}$$

  for $0 < y_q < 1$, and $c\delta > 1$.

- ϵ-entropy for DW distribution is:

$$\begin{aligned} H_\epsilon(f_{DW}) &= \frac{1}{\epsilon - 1} \Bigg[1 - \beta^\epsilon (\lambda\gamma^{c\delta})^{\frac{1-\epsilon}{c\delta}} (c\delta)^{\epsilon - 1} \\ &* B\left(\frac{1}{c\delta} + \epsilon\beta - \frac{\epsilon}{c\delta}, \epsilon + \frac{\epsilon}{c\delta} - \frac{1}{c\delta} \right) \Bigg], \end{aligned}$$

  for $\epsilon > 0$, $\epsilon \neq 1$, and $c\delta > \epsilon - 1$.

- Renyi entropy is given by:

  $$H_R(f_{DW}) = (1-\tau)^{-1} log\left[\beta^{\tau}(\gamma^{c\delta}\lambda)^{\frac{1-\tau}{c\delta}}(c\delta)^{\tau-1}B\left(\frac{1}{c\delta}+\tau\beta-\frac{\tau}{c\delta}, \tau+\frac{\tau}{c\delta}-\frac{1}{c\delta}\right)\right],$$

  for $\tau > 0$, $\tau \neq 1$, and $c\delta > \tau - 1$.

CHAPTER 4

GENERALIZATIONS VIA WEIGHTING FOR BURR-TYPE III DISTRIBUTION

4.1 Introduction

In 1942, Irving W. Burr constructed the Burr system of distributions. The generalized (log-) logistic-Burr distribution is referred to as Dagum distribution (1983) with an additional scale parameter (λ). Dagum distribution is also known as the generalized (log-) logistic distribution with $\beta = 1$. In this chapter, we present the more important class of weighted BurrIII distribution which is a flexible parametric model. A number of distributions are actually limiting forms of Burr distribution. The Burr-XII distribution is one of the most widely known Burr distribution and has the logistic and Weibull distributions as sub-models. Paranaiba et al. (2013) developed the statistical properties of the Kumaraswamy Burr-XII distribution.

4.2 Burr-Type Distributions

The cdf and reliability function of the BurrIII and BurrXII can be written in closed forms. The cdf and pdf for BurrIII (BIII) distribution are given by:

$$F_{BIII}(x; c, k, s) = (1 + (x/s)^{-c})^{-k}, \tag{4.1}$$

and

$$f_{BIII}(x; c, k, s) = cks^c x^{-c-1}(1 + (x/s)^{-c})^{-k-1}, \tag{4.2}$$

for $x > 0$, and c, k, $s > 0$, respectively. Note that, s is a scale parameter, while c and k are shape parameters. With parameters c, k, and s^c, this can be seen as resulting in $Dagum(c, k, s^c)$ distribution. The l-th moment of BIII distribution is given by:

$$E[X^l] = ks^l B(k + l/c, 1 - l/c),$$

obtained by setting $t = (1 + (x/s)^{-c})^{-1}$, for $c > l$, (Al-Dayian (1999)). The cdf and pdf for BurrXII (BXII) distribution are given by:

$$F_{BXII}(x; c, k, s) = 1 - (1 + (x/s)^c)^{-k}, \tag{4.3}$$

and

$$f_{BXII}(x; c, k, s) = cks^{-c}x^{c-1}(1 + (x/s)^c)^{-k-1}, \tag{4.4}$$

respectively, where $c > 0$ and $k > 0$ are shape parameters and $s > 0$ is a scale parameter. The l-th moment of BXII distribution is given by:

$$E[X^l] = ks^l B(k - l/c, 1 + l/c), \quad \text{for } c > l.$$

4.3 Probability Weighted Moments

Consider the weight function $w(x) = x^l F^m(x)\overline{F}^r(x)$, then the probability weighted moments (PWMs) corresponding to Burr III pdf are given by:

$$
\begin{aligned}
& E[X^l F^m_{BIII}(X)\overline{F}^r_{BIII}(X)] \\
= & \int_0^\infty x^l (1+(x/s)^{-c})^{-km}[1-(1+(x/s)^{-c})^{-k}]^r \\
* & \; cks^c x^{-c-1}(1+(x/s)^{-c})^{-k-1}dx \\
= & \int_0^1 \left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}[l-c-1]} t^{km}(1-t^k)^r cks^c t^{k+1-2}\frac{(s^c t)^{1+\frac{1}{c}}dt}{cs^c(1-t)^{1+\frac{1}{c}}} \\
= & \; ks^l \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)} \int_0^1 t^{kj+\frac{l}{c}+km+k-1}(1-t)^{1-\frac{l}{c}-1}dt \\
= & \; ks^l \sum_{j=0}^{\infty} q_j B\left(kj+\frac{l}{c}+km+k, 1-\frac{l}{c}\right),
\end{aligned}
$$

where $q_j = \frac{(-1)^j\Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)}$, $c > l$, and we have set $t = (1+(x/s)^{-c})^{-1}$, and used the result $(1-t^k)^r = \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(r+1)}{\Gamma(r+1-j)\Gamma(j+1)} t^{kj}$, for $r > 0$ and $|t^k| < 1$.

Remark:

(1) If $m = r = 0$, we have the moments of order l:

$$E[X^l] = ks^l B\left(\frac{l}{c}+k, 1-\frac{l}{c}\right), \quad \text{for } c > l.$$

(2) By setting $l = m = 0$, we have the proportional hazard moments:

$$E[\overline{F}^r(X)] = k\sum_{j=0}^{\infty} q_j B\left(kj+k, 1\right).$$

(3) When $r = l = 0$, we get the proportional reverse hazards moments:

$$E[F^m(X)] = kB(km+k, 1) = \frac{1}{m+1}.$$

(4) When $m = 0$, we get :

$$E[X^l \overline{F}^r(X)] = ks^l \sum_{j=0}^{\infty} q_j B\bigg(kj + \frac{l}{c} + k, 1 - \frac{l}{c}\bigg), \quad \text{for } c > l.$$

(5) When $r = 0$, we get :

$$E[X^l F^m(X)] = ks^l B\bigg(\frac{l}{c} + km + k, 1 - \frac{l}{c}\bigg), \quad \text{for } c > l.$$

(6) When $l = 0$, we get :

$$E[F^m(X)\overline{F}^r(X)] = k \sum_{j=0}^{\infty} q_j B\bigg(kj + km + k, 1\bigg).$$

(7) If $l = 0$, $m \to m - 1$, $r = n - m$, then we obtain moments corresponding to the order statistics.

4.4 Weighted Burr-III Distribution with Weight Function $w(x) = x^l F^m(x)$

In this section, the statistical properties of the weighted Burr-III (WBIII) distribution with the weight function $w(x) = x^l F^m(x)$ are presented.

The WBIII pdf with weight function $w(x) = x^l F^m(x)$ is given by:

$$\begin{aligned} g_{WBIII}(x; c, k, s, l, m) &= \frac{x^l(1 + (x/s)^{-c})^{-km} cks^c x^{-c-1}(1 + (x/s)^{-c})^{-k-1}}{ks^l B\Big(\frac{l}{c} + km + k, 1 - \frac{l}{c}\Big)} \\ &= \frac{cs^{c-l} x^{l-c-1}(1 + (x/s)^{-c})^{-km-k-1}}{B\Big(\frac{l}{c} + km + k, 1 - \frac{l}{c}\Big)}, \end{aligned}$$

where $E\big(w(X)\big) = E\big(X^l F^m(X)\big) = ks^l B\Big(\frac{l}{c} + km + k, 1 - \frac{l}{c}\Big)$, for $c > l$, and $c, k, s, l, m > 0$. The graphs of pdf for WBIII distribution are given in Figures 4.1 and 4.2.

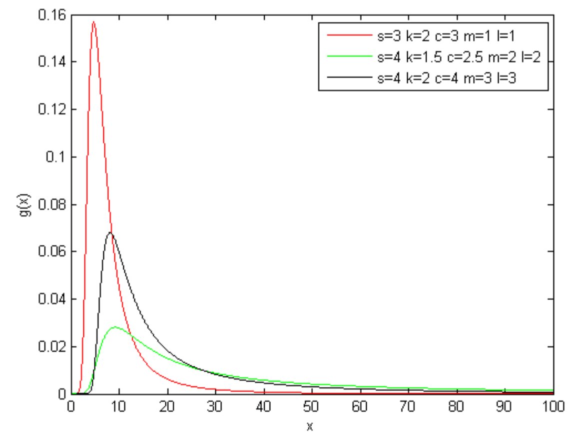

Figure 4.1: WBIII pdf distribution for different values of parameters.

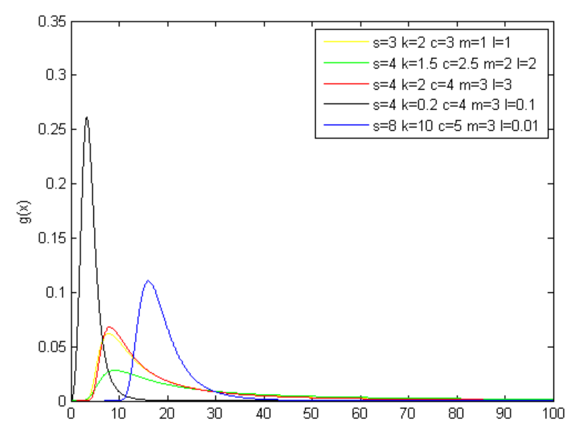

Figure 4.2: WBIII pdf distribution for different values of parameters.

The corresponding WBIII cdf is given by:

$$\begin{aligned} G_{WBIII}(x;c,k,s,l,m) &= \int_0^x g_{WBIII}(y;c,k,s,l,m)dy \\ &= \int_0^x \frac{cs^{c-l}y^{l-c-1}(1+(y/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}dy. \end{aligned}$$

Let $t = (1+(y/s)^{-c})^{-1}$, $0 < t < 1$, then $dy = \frac{(s^c t)^{1+\frac{1}{c}}dt}{cs^c t^2(1-t)^{1+\frac{1}{c}}}$, and $y = \left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}}$. Now,

$$\begin{aligned} G_{WBIII}(x;c,k,s,l,m) &= \int_0^{x^*} \frac{cs^{c-l}\left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}[l-c-1]} t^{km+k-1}(s^c t)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}B\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}dt \\ &= \frac{B\left(x^*;\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}{B\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}, \end{aligned}$$

where $x^* = (1+(x/s)^{-c})^{-1}$, $0 < x < \infty$, $c, k, s, l, m > 0$, and $\frac{B(x^*;a,b)}{B(a,b)}$ is an incomplete beta function ratio, Gradshteyn and Ryzhik (2000). The graph of cdf for WBIII distribution is given in Figure 4.3.

4.4.1 Submodels

Several distributions can be readily obtained from the WBIII distribution.

Remark:

(1) If $l = m = 0$, then we have BurrIII distribution with pdf:

$$g_{BIII}(x;c,k,s) = cks^c x^{-c-1}(1+(x/s)^{-c})^{-k-1}, \quad \text{for } x > 0, \text{ and } c,k,s > 0.$$

(2) If $l = 1$ and $m = 0$, then we have the length-biased BurrIII (LBBIII) distribution with pdf:

$$g_{LBBIII}(x; c, k, s) = \frac{cs^{c-1}x^{-c}(1 + (x/s)^{-c})^{-k-1}}{B\left(\frac{1}{c} + k, 1 - \frac{1}{c}\right)}, \quad \text{for } c > 1,\ x > 0, \text{ and } k, s > 0.$$

(3) If $k = 1$, then we have the weighted Fisk (WF) distribution with pdf:

$$g_{WF}(x; c, l, s, m) = \frac{cs^{c-l}x^{l-c-1}(1 + (x/s)^{-c})^{-m-2}}{B\left(\frac{l}{c} + m + 1, 1 - \frac{l}{c}\right)}, \quad \text{for } c > l,\ x > 0, \text{ and } c, l, s, m > 0.$$

(4) If $k = 1$ and $l = m = 0$, then we have Fisk distribution with pdf:

$$g_{FISK}(x; c, s) = cs^{c}x^{-c-1}(1 + (x/s)^{-c})^{-2}, \quad \text{for } x > 0 \text{ and } c, s > 0.$$

(5) If $k = l = 1$ and $m = 0$, then we have the length-biased Fisk distribution with pdf:

$$g_{LBF}(x; c, s) = \frac{cs^{c-l}x^{-c}(1 + (x/s)^{-c})^{-2}}{B\left(1 + \frac{1}{c}, 1 - \frac{1}{c}\right)}, \quad \text{for } c > 1,\ x > 0 \text{ and } s > 0.$$

The hazard and reverse hazard functions of the WBIII distribution are given by:

$$\begin{aligned} h_{WBIII}(x; c, k, s, l, m) &= \frac{g_{WBIII}(x; c, k, s, l, m)}{\bar{G}_{WBIII}(x; c, k, s, l, m)} \\ &= \frac{cs^{c-l}x^{l-c-1}(1 + (x/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right) - B\left(x^*; \frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}, \end{aligned}$$

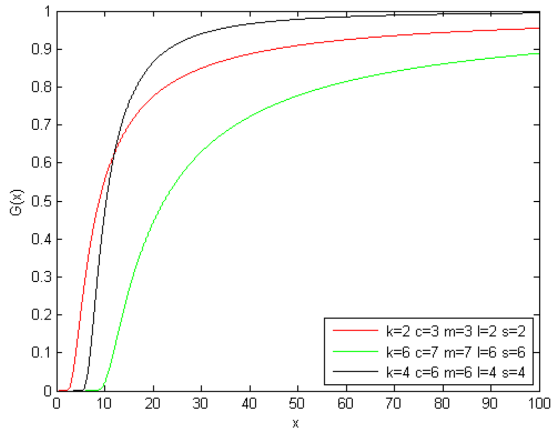

Figure 4.3: WBIII cdf distribution for different values of parameters.

and

$$\begin{aligned}\tau_{WBIII}(x;c,k,s,l,m) &= \frac{g_{WBIII}(x;c,k,s,l,m)}{G_{WBIII}(x;c,k,s,l,m)} \\ &= \frac{cs^{c-l}x^{l-c-1}(1+(x/s)^{-c})^{-km-k-1}}{B\left(x^{*};\frac{l}{c}+km+k,1-\frac{l}{c}\right)},\end{aligned}$$

respectively.

Figure 4.4 shows unimodal shape for the hazard function of WBIII distribution for different values of parameters. The hazard function is decreasing and upside down bathtub shapes in Figure 4.5. Also, Figure 4.6 shows different shapes for hazard function of BIII model as a special case of WBIII distribution for different values of parameters. For example, this plot shows bathtub followed by upside down bathtub shapes and decreasing hazard functions.

In the next result, we study the monotonicity properties of the WBIII

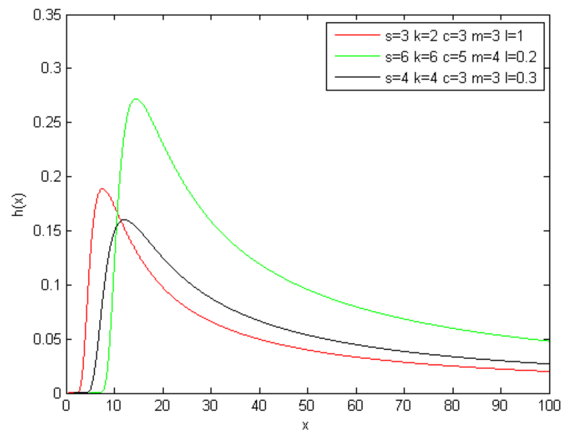

Figure 4.4: Hazard function for different values of parameters.

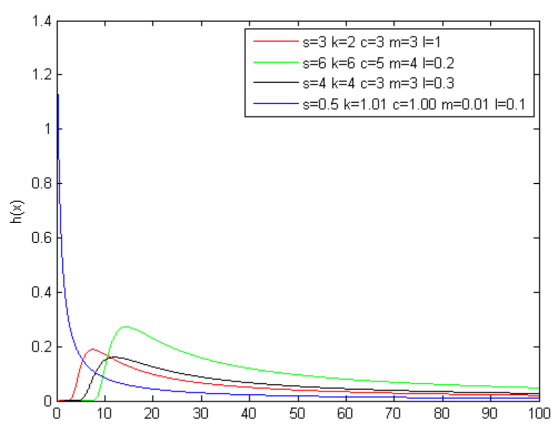

Figure 4.5: Hazard function for different values of parameters.

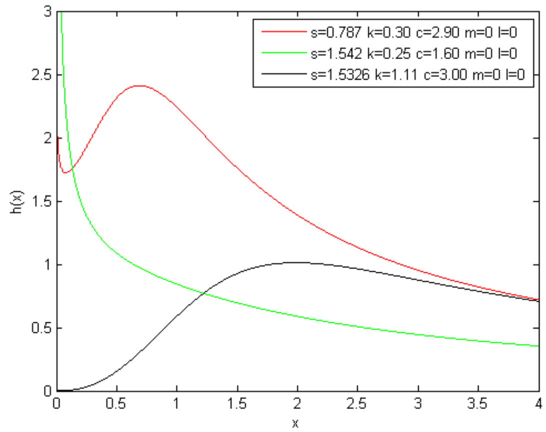

Figure 4.6: Hazard function for different values of parameters.

distribution by taking the logarithm of its pdf as follows:

$$\begin{aligned} ln(g_{WBIII}(x;c,k,s,l,m)) &= ln(c)+(c-l)ln(s)+(l-c-1)ln(x) \\ &- (km+k+1)ln(1+(x/s)^{-c}) \\ &- ln[\Gamma(l/c+km+k)]-ln[\Gamma(1-l/c)] \\ &+ ln[\Gamma(km+k)]. \end{aligned}$$

The derivative of the logarithm is

$$\frac{\partial ln g_{WBIII}(x;c,k,s,l,m)}{\partial x}=\frac{l-c-1+(cmk+ck+l-1)s^c x^{-c}-c-1}{x+s^c x^{-c+1}}.$$

$$\frac{\partial ln g_{WBIII}(x;c,k,s,l,m)}{\partial x}>0 \Leftrightarrow x<\left(\frac{(cmk+ck+l-1)s^c}{c-l+1}\right)^{\frac{1}{c}},$$

$$\frac{\partial ln g_{WBIII}(x;c,k,s,l,m)}{\partial x}=0 \Leftrightarrow x=\left(\frac{(cmk+ck+l-1)s^c}{c-l+1}\right)^{\frac{1}{c}},$$

and

$$\frac{\partial lng_{WBIII}(x;c,k,s,l,m)}{\partial x} < 0 \Leftrightarrow x > \left(\frac{(cmk+ck+l-1)s^c}{c-l+1}\right)^{\frac{1}{c}}.$$

The second derivative of the logarithm is given by:

$$\begin{aligned}
\frac{\partial^2 lng_{WBIII}(x;c,k,s,l,m)}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{l-c-1+(cmk+ck+l-1)s^c x^{-c}-c-1}{x+s^c x^{-c+1}}\right] \\
&= -\left[\frac{(cx^{-c}+cs^c x^{-2c})[s^c(l-1+ckm+ck)]}{(x+s^c x^{-c+1})^2}\right] \\
&- \left[\frac{(x^{-c}+(1-c)s^c x^{-2c})[s^c(l-1+ckm+ck)]}{(x+s^c x^{-c+1})^2}\right].
\end{aligned}$$

The second derivative of the logarithm for WBIII distribution is negative, the mode of WBIII distribution is $x_0 = \left(\frac{(cmk+ck+l-1)s^c}{c-l+1}\right)^{\frac{1}{c}}$.

4.5 Moments

The i^{th} moment of WBIII distribution is given by:

$$\begin{aligned}
E(X^i) &= \int_0^\infty \frac{cs^{c-l}x^{i+l-c-1}(1+(x/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}dx \\
&= \int_0^1 \frac{cs^{c-l}\left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}[i+l-c-1]} t^{km+k+1}(s^c t)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}dt \\
&= \frac{s^i B\left(km+k+\frac{i+l}{c},1-\frac{i+l}{c}\right)}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)},
\end{aligned} \tag{4.5}$$

where $t = (1 + (x/s)^{-c})^{-1}$, $0 < t < 1$, $c > i + l$, and $c, k, s, l, m > 0$. The mean and variance of WBIII distribution are given by:

$$\mu_{_{WBIII}} = E_{WBIII}(X) = \frac{sB\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)},$$

and

$$\begin{aligned}
\sigma^2_{WBIII} &= E_{WBIII}(X^2) - (E_{WBIII}(X))^2 \\
&= \frac{s^2B\left(km + k + \frac{2+l}{c}, 1 - \frac{2+l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)} - \frac{s^2B^2\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{B^2\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)},
\end{aligned} \tag{4.6}$$

respectively. The coefficient of variation (CV) is given by:

$$CV = \frac{\sqrt{Var_{_{WBIII}}(X)}}{\mu_{_{WBIII}}}, \tag{4.7}$$

where $\mu_{_{WBIII}} = \frac{sB\left(km+k+\frac{1+l}{c}, 1-\frac{1+l}{c}\right)}{B\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}$, and $Var_{_{WBIII}}(X)$ is given by (4.6). The coefficient of skewness (CS) is given by:

$$CS = E\left[\frac{(X-\mu)^3}{\sigma^3}\right] = \frac{E[X^3] - 3\mu E[X^2] + 2\mu^3}{\sigma^3}, \tag{4.8}$$

where $E(X^i)$ for $i = 1, 2, 3$ is given by (4.5), and $\sigma^3 = (\sqrt{Var_{_{WBIII}}(X)})^3$, respectively. The coefficient of kurtosis (CK) is given by:

$$CK = E\left[\frac{(X-\mu)^4}{\sigma^4}\right] = \frac{E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4}{\sigma^4}, \tag{4.9}$$

where

$$E[X^4] = \frac{s^4 B\left(km + k + \frac{4+l}{c}, 1 - \frac{4+l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}, \tag{4.10}$$

and $\sigma^4 = (\sigma^2_{WBIII})^2 = (Var_{WBIII}(X))^2$. Mean residual life function, denoted by $MRLF(t)$, is given by:

$$\begin{aligned} MRLF(t) &= E[X - t|X > t] \\ &= \frac{sB\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right) - B\left(x^*; km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right) - B\left(x^*; \frac{l}{c} + km + k, 1 - \frac{l}{c}\right)} - t, \end{aligned}$$

where $E_{WBIII}(X) = \frac{sB\left(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}$, $E[X|X \leq t] = \frac{B\left(x^*;km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}{G_{WBIII}(t)B\left(km+k+\frac{l}{c},1-\frac{l}{c}\right)}$, and $G_{WBIII}(t)$ is the cdf of WBIII distribution, $x^* = (1 + (x/s)^{-c})^{-1}$.

4.6 Inequality Measures

Lorenz, Bonferroni and Zenga curves are given in this section. Graphs of these income inequality measures are presented for selected values of the parameters of the WBIII distribution. Lorenz curve for the WBIII distribution is given by

$$\begin{aligned} L(G_{WBIII}(x)) &= \frac{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}{sB\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)} \\ &* \int_0^x \frac{cs^{c-l}t^{l-c-1+1}(1 + (t/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)} dt. \end{aligned}$$

Let $u=(1+(t/s)^{-c})^{-1}$, then $dt=\frac{(s^c u)^{1+\frac{1}{c}}du}{cs^c(1-u)^{1+\frac{1}{c}}u^2}$, and

$$\begin{aligned}
L(G_{WBIII}(x)) &= \frac{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}{sB\left(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)} \\
&* \int_0^{x^*}\frac{cs^{c-l}\left(\frac{1-u}{s^c u}\right)^{\frac{-1}{c}[l-c]}u^{km+k+1}(s^c u)^{1+\frac{1}{c}}}{cs^c(1-u)^{1+\frac{1}{c}}B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}du \\
&= \frac{B\left(x^*;km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}{B\left(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)},
\end{aligned}$$

for $0<u<1$, $x^*=(1+(x/s)^{-c})^{-1}$, and $0<x^*<\infty$. Figure 4.7 shows different shapes of Lorenz curve of WBIII distribution for different values of parameters. These shapes are convex curve that shows inequality measure and the straight line shows equality measures of Lorenz curve between percentage of wealth (x-axis) and percentage of population (y-axis).

Tail conditional expectation can be written in terms of Lorenz curve by:

$$\begin{aligned}
TCE(x) &= \mu\left[1-\frac{L(G(x))}{G(x)}\right] \\
&= \frac{sB\left(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)} \\
&* \left[1-\frac{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)\cdot B\left(x^*;km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}{B\left(x^*;\frac{l}{c}+km+k,1-\frac{l}{c}\right)\cdot B\left(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\right)}\right],
\end{aligned}$$

for $c > 1 + l$, $0 < x^* < \infty$. Figure 4.8 shows TCE in terms of Lorenz curve for different values of parameters.

Bonferroni curve for the WBIII distribution is

$$B(G_{WBIII}(x)) = \frac{B\left(x^*; km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{G_{WBIII}(x) B\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}, \tag{4.11}$$

for $c > l + 1$, where $G_{WBIII}(x)$ is the weighted BIII cdf. As opposed to the Lorenz's singularly convex curve, Bonferroni exhibits both convex and concave properties which show inequality measures between percentage of wealth (x-axis) and percentage of population (y-axis) in Figure 4.9.

TCE can be written in terms of Bonferroni curve by:

$$\begin{aligned} TCE(x) &= \mu\left[1 - B(G(x))\right] \\ &= \frac{sB\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)} \\ &* \left[1 - \frac{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right) \cdot B\left(x^*; km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}{B\left(x^*; \frac{l}{c} + km + k, 1 - \frac{l}{c}\right) \cdot B\left(km + k + \frac{1+l}{c}, 1 - \frac{1+l}{c}\right)}\right], \end{aligned}$$

for $c > 1 + l$, $0 < x^* < \infty$. Figure 4.10 shows TCE in terms of Bonferroni curve for different values of parameters.

Zenga curve for the WBIII distribution is

$$\begin{aligned} A(x) &= 1 - \frac{\int_0^x t g_{WBIII}(t) dt}{\int_x^\infty t g_{WBIII}(t) dt} \cdot \frac{1 - G_{WBIII}(x)}{G_{WBIII}(x)} \\ &= 1 - \frac{L(G_{WBIII}(x))}{1 - L(G_{WBIII}(x))} \cdot \frac{1 - G_{WBIII}(x)}{G_{WBIII}(x)}, \end{aligned}$$

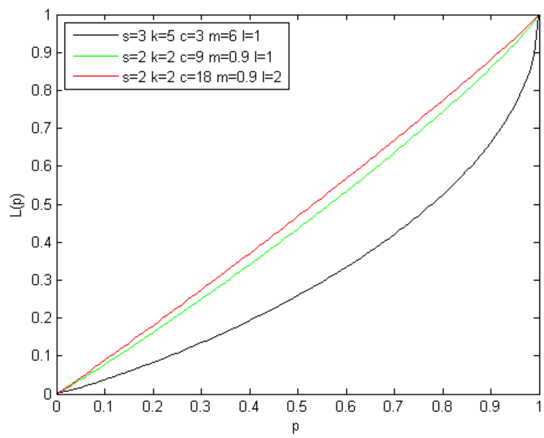

Figure 4.7: Lorenz curve for different values of parameters.

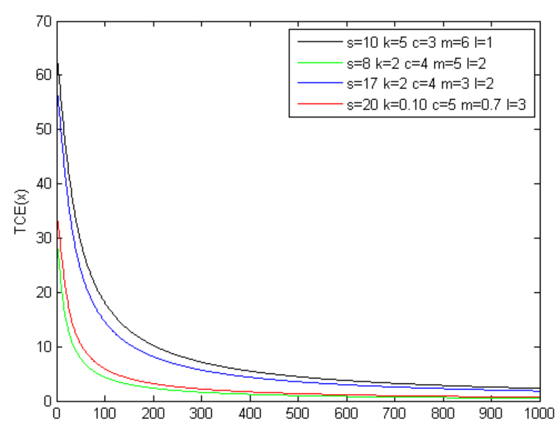

Figure 4.8: TCE in terms of Lorenz curve for different values of parameters.

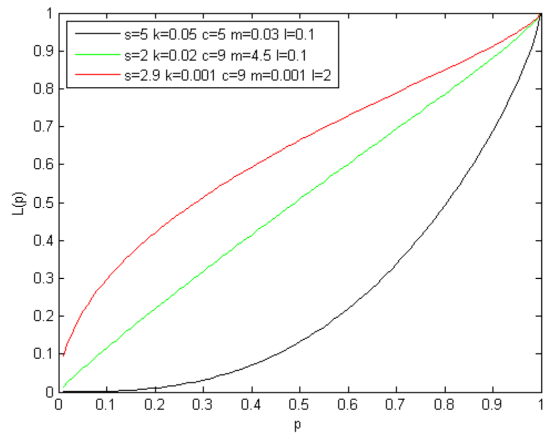

Figure 4.9: Bonferroni curve for different values of parameters.

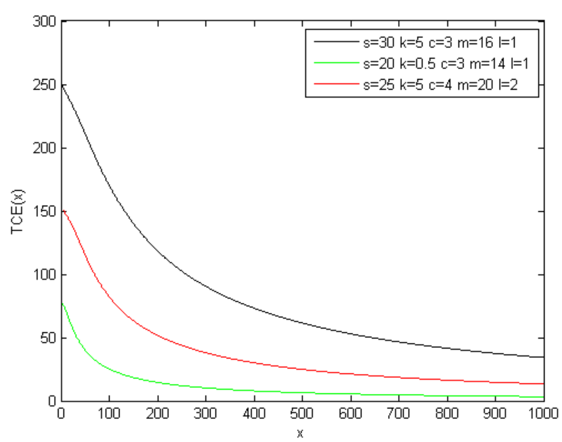

Figure 4.10: TCE in terms of Bonferroni curve for different values of parameters.

where $L(G_{WBIII}(x)) = \frac{B\left(x^*; km+k+\frac{1+l}{c}, 1-\frac{1+l}{c}\right)}{B\left(km+k+\frac{1+l}{c}, 1-\frac{1+l}{c}\right)}$, and $G_{WBIII}(x)$ is the WBIII cdf.

4.7 Entropy

In this section, measure of variation of the uncertainty including Renyi and ϵ-entropies for the WBIII distribution are presented.

4.7.1 ϵ-Entropy

The ϵ-entropy for the WBIII distribution, $\epsilon \neq 1$, $\epsilon > 0$, is given by:

$$\begin{aligned}
H_\epsilon(g_{WBIII}) &= \frac{1}{\epsilon-1}\left[1-\int_0^\infty g^\epsilon_{WBIII}(x;c,k,s,l,m)dx\right] \\
&= \frac{1}{\epsilon-1}\left[1-\int_0^\infty \frac{c^\epsilon s^{\epsilon(c-l)} x^{\epsilon(l-c-1)}(1+(x/s)^{-c})^{-\epsilon(km+k+1)}}{B^\epsilon\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}dx\right] \\
&= \frac{1}{\epsilon-1}\left[1-\int_0^1 \frac{c^\epsilon s^{\epsilon(c-l)}\left(\frac{1-t}{s^c t}\right)^{\frac{-\epsilon}{c}[l-c-1]} t^{\epsilon(km+k+1)}(s^c t)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}B^\epsilon\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}dt\right] \\
&= \frac{1}{\epsilon-1}\left[1-\frac{c^{\epsilon-1}s^{1-\epsilon}B\left(\epsilon km+\epsilon k+\frac{\epsilon(l-1)}{c}+\frac{1}{c}, \epsilon+\frac{\epsilon(1-l)}{c}-\frac{1}{c}\right)}{B^\epsilon\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)}\right].
\end{aligned}$$

4.7.2 Renyi Entropy

Renyi entropy (Renyi (1961)) of a random variable is a measure of variation of the uncertainty and it is a generalization of Shannon entropy (1948). Renyi entropy for WBIII distribution is given by:

$$\begin{aligned} H_R(g_{_{WBIII}}) &= (1-\tau)^{-1} log \int_0^\infty g_{_{WBIII}}^\tau(x;c,k,s,l,m)dx \\ &= (1-\tau)^{-1} log \left[\frac{c^{\tau-1}s^{1-\tau}B\left(\tau km+\tau k+\frac{\tau(l-1)}{c}+\frac{1}{c},\tau+\frac{\tau(1-l)}{c}-\frac{1}{c}\right)}{B^\tau\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)} \right], \end{aligned}$$

for $\tau > 0$, $\tau \neq 1$, $c > l$.

4.8 Fisher Information Matrix

Let X be an observable random variable that has a vector of unknown parameters with the WBIII pdf $g_{_{WBIII}}(x;\Theta)$, where $\Theta = (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)^T = (c,k,s,l,m)^T$. If the density $g_{_{WBIII}}(x;\Theta)$ has second derivative with respect to Θ and under certain regularity conditions, then Fisher information matrix (FIM) is the 5×5 symmetric matrix with elements that are given by:

$$I_{ij}(\Theta) = -E_\Theta\left[\frac{\partial^2 log(g_{_{WBIII}}(x;\Theta))}{\partial\theta_i\partial\theta_j}\right]. \tag{4.12}$$

Recall the pdf for WBIII distribution with weight function $w(x) = x^l F^m(x)$ is given by:

$$g_{_{WBIII}}(x;c,k,s,l,m) = \frac{cs^{c-l}x^{l-c-1}(1+(x/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}, \quad \text{for } c > l.$$

The logarithm of $g_{_{WBIII}}(x;c,k,s,l,m)$ is

$$\begin{aligned} log(g_{_{WBIII}}(x;c,k,s,l,m)) &= log(c)+(c-l)log(s)+(l-c-1)log(x) \\ &- (km+k+1)log(1+(x/s)^{-c}) \\ &- log[\Gamma(l/c+km+k)]-log[\Gamma(1-l/c)] \\ &+ log[\Gamma(km+k)]. \end{aligned}$$

The first, second and mixed partial derivatives for $log(g_{_{WBIII}}(x))$ with respect to Θ are:

$$\begin{aligned} \frac{\partial log(g_{_{WBIII}}(x))}{\partial c} &= \frac{1}{c}+log(s)-log(x)-(km+k+1)\frac{s^c x^{-c} log(\frac{s}{x})}{(1+(x/s)^{-c})} \\ &+ \frac{l}{c^2}\psi(l/c+km+k)-\frac{l}{c^2}\psi(1-l/c), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial c^2} &= \frac{-1}{c^2}-(km+k+1)\frac{s^c x^{-c} log(\frac{s}{x})^2}{(1+(x/s)^{-c})^2}+\frac{2l}{c^3}\psi(l/c+km+k) \\ &- \frac{l^2}{c^4}\psi'(l/c+km+k)+\frac{2l}{c^3}\psi(1-l/c)+\frac{l^2}{c^4}\psi'(1-l/c), \end{aligned}$$

$$\frac{\partial log(g_{_{WBIII}}(x))}{\partial k}=-(m+1)[\psi(l/c+km+k)+\psi(km+k+1)]-(m+1)log(1+(x/s)^{-c}),$$

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial k^2}=(m+1)^2[\psi'(l/c+km+k)-\psi'(km+k+1)],$$

$$\frac{\partial log(g_{_{WBIII}}(x))}{\partial s} = \frac{(c-l)}{s} - (km+k+1)\frac{cs^{c-1}x^{-c}}{(1+(x/s)^{-c})},$$

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial s^2} = \frac{(c-l)}{s^2} - (km+k+1)\left[\frac{c(c-1)s^{c-2}x^{-c}}{(1+(x/s)^{-c})} - \frac{c^2 s^{2(c-1)}x^{-2c}}{(1+(x/s)^{-c})^2}\right],$$

$$\frac{\partial log(g_{_{WBIII}}(x))}{\partial l} = -log(s) + logx - \frac{1}{c}[\psi(l/c+km+k) + \psi(1-l/c)],$$

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial l^2} = -\frac{1}{c^2}[\psi'(l/c+km+k) + \psi'(1-l/c)],$$

$$\frac{\partial log(g_{_{WBIII}}(x))}{\partial m} = -klog(1+(x/s)^{-c}) - k\psi(l/c+km+k) + k\psi(km+k+1),$$

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial m^2} = -k^2\psi'(l/c+km+k) + k^2\psi'(km+k+1).$$

Note that $\partial\theta_i\partial\theta_j = \partial\theta_j\partial\theta_i$ for mixed partial derivatives of $log(g_{_{WBIII}}(x))$ with respect to Θ, so that

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial c\partial k} = \frac{l(m+1)}{c^2}\psi(l/c+km+k) - (m+1)\frac{s^c x^{-c} log(\frac{s}{x})}{(1+(x/s)^{-c})},$$

$$\begin{aligned}\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial s\partial c} &= \frac{1}{s} - (km+k+1)\left[\frac{cs^{c-1}x^{-c}log(s)}{(1+(x/s)^{-c})^2} + \frac{cs^{c-1}x^{-c}log(x)}{(1+(x/s)^{-c})^2}\right. \\ &- \left.\frac{s^{2c-1}x^{-2c}}{(1+(x/s)^{-c})^2} - \frac{s^{c-1}x^{-c}}{(1+(x/s)^{-c})^2}\right],\end{aligned}$$

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial s\partial k} = (m+1)\frac{cs^{c-1}x^{-c}}{(1+(x/s)^{-c})},$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial l\partial c} = \frac{1}{c^3}\psi'(l/c+km+k) - \frac{l}{c^3}\psi'(1-l/c),$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial l\partial k} = -\frac{(m+1)}{c}\psi'(l/c+km+k),$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial l\partial s} = -\frac{1}{s},$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial m\partial c} = -k\frac{s^c x^{-c} log(\frac{s}{x})}{(1+(x/s)^{-c})} + \frac{lk}{c^2}\psi'(l/c+km+k),$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial m\partial k} = -k(m+1)[\psi'(l/c+km+k) - \psi'(km+k+1)] - log(1+(x/s)^{-c}),$$
$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial m\partial s} = -k\frac{cs^{c-1}x^{-c}}{(1+(x/s)^{-c})},$$

and

$$\frac{\partial^2 log(g_{_{WBIII}}(x))}{\partial m\partial l} = -\frac{k}{c}\psi'(l/c+km+k).$$

The entries of FIM for the WBIII distribution are computed with the assistance of the following expectations, denoted by E_1, E_2, E_3, E_4, E_5, E_6, E_7 and E_8.

$$\begin{aligned} E_1 &= E\left(\frac{X^{-c}}{1+(X/s)^{-c}}\right) \\ &= \frac{1}{B(\frac{l}{c}+km+k, 1-\frac{l}{c})}\int_0^\infty cs^{c-l}x^{l-2c-1}(1+(x/s)^{-c})^{-km-k-2}dx. \end{aligned}$$

Let $t=(1+(x/s)^{-c})^{-1}$, $0<t<1$, then $dx = \frac{(s^c t)^{1+\frac{1}{c}}dt}{cs^c t^2(1-t)^{1+\frac{1}{c}}}$, and $x = \left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}}$.

Now,

$$
\begin{aligned}
E_1 &= \frac{1}{B(\frac{l}{c}+km+k,1-\frac{l}{c})}\int_0^1 \frac{cs^{c-l}\left(\frac{1-t}{s^ct}\right)^{\frac{-1}{c}[l-c-1]}t^{km+k-1}(s^ct)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}}dt\\
&= \frac{s^{-2c}B\left(\frac{l}{c}+km+k,2-\frac{l}{c}\right)}{B\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}.
\end{aligned}
$$

Similarly,

$$
E_2 = E\left(\frac{X^{-c}}{(1+(X/s)^{-c})^2}\right) = \frac{s^{-2c}B\left(\frac{l}{c}+km+k,2-\frac{l}{c}\right)}{B\left(\frac{l}{c}+km+k+1,1-\frac{l}{c}\right)},
$$

and

$$
E_3 = E\left(\frac{X^{-2c}}{(1+(X/s)^{-c})^2}\right) = \frac{s^{-3c}B\left(\frac{l}{c}+km+k,2-\frac{l}{c}\right)}{B\left(\frac{l}{c}+km+k+1,3-\frac{l}{c}\right)}.
$$

Also,

$$
\begin{aligned}
E_4 &= E\left(\frac{X^{-c}log(X)}{1+(X/s)^{-c}}\right)\\
&= \frac{1}{B(\frac{l}{c}+km+k,1-\frac{l}{c})}\int_0^\infty cs^{c-l}x^{l-2c-1}log(x)(1+(x/s)^{-c})^{-km-k-2}dx.
\end{aligned}
$$

Let $t = (1+(x/s)^{-c})^{-1}$, $0 < t < 1$, then $x = \left(\frac{1-t}{s^ct}\right)^{\frac{-1}{c}}$, and $log(x) =$

$-\frac{1}{c}[log(1-t) - log(s^c) - log(t)]$. Now,

$$
\begin{aligned}
E_4 &= \frac{1}{B(\frac{l}{c}+km+k, 1-\frac{l}{c})} \int_0^1 \frac{cs^{c-l}\left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}[l-c-1]} t^{km+k-1}(s^c t)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}} \\
&* \ [log(t) + log(s^c) - log(1-t)]dt \\
&= \frac{s^{-2c}}{cB\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)} \Bigg[\int_0^1 (1-t)^{1-\frac{l}{c}} t^{\frac{l}{c}+km+k-1} log(t)dt \\
&+ \ log(s^c) \int_0^1 (1-t)^{1-\frac{l}{c}} t^{\frac{l}{c}+km+k-1} dt - \int_0^1 (1-t)^{1-\frac{l}{c}} t^{\frac{l}{c}+km+k-1} log(1-t)dt \Bigg] \\
&= \frac{s^{-2c}B\left(l/c+km+k, 2-l/c\right)}{cB\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)} \Bigg[\psi(l/c+km+k) + log(s^c) - \psi(2-l/c)\Bigg].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E_5 &= E\left(\frac{X^{-2c}log(X)}{(1+(X/s)^{-c})^2}\right) \\
&= \frac{s^{-3c}B\left(\frac{l}{c}+km+k, 3-\frac{l}{c}\right)}{cB\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)} \Bigg[\psi(l/c+km+k) + log(s^c) - \psi(3-l/c)\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
E_6 &= E\left(\frac{X^{-c}log(X)}{(1+(X/s)^{-c})^2}\right) \\
&= \frac{s^{-2c}B\left(\frac{l}{c}+km+k+1, 2-\frac{l}{c}\right)}{cB\left(\frac{l}{c}+km+k, 1-\frac{l}{c}\right)} \\
&* \ \Bigg[\psi(l/c+km+k+1) + log(s^c) - \psi(3-l/c)\Bigg].
\end{aligned}
$$

In addition,

$$
\begin{aligned}
E_7 &= E\Bigg(log(X)\Bigg) \\
&= \frac{1}{B(\frac{l}{c}+km+k,1-\frac{l}{c})}\int_0^\infty cs^{c-l}x^{l-c-1}log(x)(1+(x/s)^{-c})^{-km-k-1}dx.
\end{aligned}
$$

Let $t=(1+(x/s)^{-c})^{-1}$, $0<t<1$, then $x=\left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}}$,

and $log(x)=-\frac{1}{c}[log(1-t)-log(s^c)-log(t)]$. Now,

$$
\begin{aligned}
E_7 &= \frac{1}{B(\frac{l}{c}+km+k,1-\frac{l}{c})}\int_0^1 \frac{cs^{c-l}\left(\frac{1-t}{s^c t}\right)^{\frac{-1}{c}[l-c-1]} t^{km+k-1}(s^c t)^{1+\frac{1}{c}}}{cs^c(1-t)^{1+\frac{1}{c}}} \\
&* \quad [log(t)+log(s^c)-log(1-t)]dt \\
&= \frac{1}{cB\left(\frac{l}{c}+km+k,1-\frac{l}{c}\right)}\Bigg[\int_0^1(1-t)^{1-\frac{l}{c}}t^{\frac{l}{c}+km+k-1}log(t)dt \\
&+ \quad log(s^c)\int_0^1(1-t)^{1-\frac{l}{c}}t^{\frac{l}{c}+km+k-1}dt-\int_0^1(1-t)^{1-\frac{l}{c}}t^{\frac{l}{c}+km+k-1}log(1-t)dt\Bigg] \\
&= \frac{1}{c}\Bigg[\psi(l/c+km+k)+log(s^c)-\psi(2-l/c)\Bigg].
\end{aligned}
$$

Similarly,

$$
E_8 = E\Bigg(log(1+(X/s)^{-c})\Bigg) = \Bigg[\psi'(km+k+1)-\psi(l/c+km+k)\Bigg].
$$

FIM for WBIII distribution, using E_1 to E_8 is given by:

$$
I(c,k,s,l,m) = \begin{bmatrix}
I_{cc} & I_{kc} & I_{sc} & I_{lc} & I_{mc} \\
I_{ck} & I_{kk} & I_{sk} & I_{lk} & I_{mk} \\
I_{cs} & I_{ks} & I_{ss} & I_{ls} & I_{ms} \\
I_{cl} & I_{kl} & I_{sl} & I_{ll} & I_{ml} \\
I_{cm} & I_{km} & I_{sm} & I_{lm} & I_{mm}
\end{bmatrix},
$$

where

$$\begin{aligned} I_{cc} &= \frac{1}{c^2} + 2(km+k+1)s^c log(s)E_2 - 2(km+k+1)s^c E_6 + \frac{2l}{c^3}\psi(l/c+km+k) \\ &+ \frac{l^2}{c^4}\psi'(l/c+km+k) - \frac{2l}{c^3}\psi(1-l/c) - \frac{l^2}{c^4}\psi'(1-l/c), \end{aligned}$$

$$I_{kk} = -(m+1)^2[\psi'(l/c+km+k) - \psi'(km+k+1)],$$

$$I_{ss} = -\frac{(c-l)}{s^2} - (km+k+1)\left[c(c-1)s^{c-2}E_2 - c^2 s^{2(c-1)}E_5\right],$$

$$I_{ll} = \frac{1}{c^2}[\psi'(l/c+km+k) + \psi'(1-l/c)],$$

$$I_{mm} = k^2\psi'(l/c+km+k) - k^2\psi'(km+k+1),$$

$$I_{ck} = -\frac{l(m+1)}{c^2}\psi(l/c+km+k) + (m+1)s^c log(s)E_1 - s^c E_4,$$

$$I_{cs} = -\frac{1}{s} + (km+k+1)\left[cs^{c-1}log(s)E_2 + cs^{c-1}E_6 - s^{2c-1}E_3 - s^{c-1}E_2\right],$$

$$I_{cl} = -\frac{1}{c^3}\psi'(l/c+km+k) + \frac{l}{c^3}\psi'(1-l/c),$$

$$I_{cm} = ks^c log(s)E_1 - s^c E_4 - \frac{lk}{c^2}\psi'(l/c+km+k),$$

$$I_{ks} = -(m+1)cs^{c-1}E_1, \quad I_{kl} = \frac{(m+1)}{c}\psi'(l/c+km+k),$$

$$I_{km} = k(m+1)[\psi'(l/c+km+k)+\psi'(km+k+1)]+E_8,$$

$$I_{sl} = \frac{1}{s}, \quad I_{sm} = kcs^{c-1}E_1,$$

and

$$I_{lm} = \frac{k}{c}\psi'(l/c+km+k).$$

4.9 Maximum Likelihood Estimation

In this section, we obtain the maximum likelihood estimates (MLE) of the parameters of the WBIII distribution. Let $x_1, x_2, ..., x_n$ be a random sample from a WBIII distribution, then the likelihood function of WBIII is given by:

$$\begin{aligned} L(x_1, x_2, ..., x_n; c, k, s, l, m) &= \prod_{i=1}^{n}(g_{_{WBIII}}(x; c, k, s, l, m)) \\ &= \prod_{i=1}^{n}\left[\frac{cs^{c-l}{x_i}^{l-c-1}(1+(x_i/s)^{-c})^{-km-k-1}}{B\Big(\frac{l}{c}+km+k, 1-\frac{l}{c}\Big)}\right]. \end{aligned}$$

The log likelihood function

$$log(L(x_1, x_2, ..., x_n; c, k, s, l, m)) = L^*(x_1, x_2, ..., x_n; c, k, s, l, m)$$

is given by:

$$\begin{aligned} L^*(x_1, x_2, ..., x_n; c, k, s, l, m) &= nln(c) + n(c-l)ln(s) + (l-c-1)\sum_{i=0}^{n} ln(x_i) \\ &- (km+k+1)\sum_{i=0}^{n} ln(1+(x_i/s)^{-c}) \\ &- nln[\Gamma(l/c+km+k)] - nln[\Gamma(1-l/c)] \\ &+ nln[\Gamma(km+k)]. \end{aligned}$$

The partial derivatives of $L^*(x_1, x_2, ..., x_n; c, k, s, l, m)$ with respect to c, k, s, l and m are

$$\frac{\partial L^*}{\partial k} = n(m+1)[\psi(l/c+km+k)+\psi(km+k+1)]-(m+1)\sum_{i=0}^{n} ln(1+(x_i/s)^{-c}),$$

$$\frac{\partial L^*}{\partial s} = \frac{n(c-l)}{s} - (km+k+1)\frac{c{x_i}^{-c}s^{c-1}}{(1+(x_i/s)^{-c})},$$

$$\begin{aligned} \frac{\partial L^*}{\partial c} &= \frac{n}{c} + nln(s) - \sum_{i=0}^{n} ln(x_i) - (km+k+1)\sum_{i=0}^{n}\frac{{x_i}^{-c}s^c ln(\frac{s}{x_i})}{(1+(x_i/s)^{-c})} \\ &+ \frac{nl}{c^2}\psi(l/c+km+k) - \frac{\frac{nl}{c^2}}{\Gamma(1-l/c)}, \end{aligned}$$

$$\frac{\partial L^*}{\partial l} = -nln(s) + \sum_{i=0}^{n} lnx_i - \frac{n}{c}[\psi(l/c+km+k)+\psi(1-l/c)],$$

and

$$\frac{\partial L^*}{\partial m} = -k\sum_{i=0}^{n} ln(1+(x_i/s)^{-c}) - nk\psi(l/c+km+k) + nk\psi(km+k+1).$$

Equating $\frac{\partial L^*}{\partial k}$, $\frac{\partial L^*}{\partial s}$, $\frac{\partial L^*}{\partial c}$, $\frac{\partial L^*}{\partial l}$, $\frac{\partial L^*}{\partial m}$ all to zero, and solving them leads to the MLE of the parameters c,k, s, l *and* m, say $\widehat{c_n}$, $\widehat{k_n}$,$\widehat{s_n}$,$\widehat{l_n}$, and $\widehat{m_n}$. There is no closed form solution to these equations, so in this case a numerical technique must be applied to obtain the solution.

The approximate $100(1-\alpha)\%$ two-sided confidence intervals for c, k, s, l and m are given by: $\widehat{c} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{cc}^{-1}(\widehat{\Theta})}$, $\widehat{k} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{kk}^{-1}(\widehat{\Theta})}$, $\widehat{s} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{ss}^{-1}(\widehat{\Theta})}$, $\widehat{l} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{ll}^{-1}(\widehat{\Theta})}$ and $\widehat{m} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{mm}^{-1}(\widehat{\Theta})}$, respectively, where $Z_{\frac{\alpha}{2}}$ is the $(\frac{\alpha}{2})^{th}$ percentile of a standard normal distribution. Note that we can obtain the FIM of the WBIII distribution with other weight functions. For example, by setting $k = 0$, we get the FIM for WBIII distribution with weight function $w(x) = x^l$.

We can use the likelihood ratio (LR) test to compare the fit of the WBIII distribution with its sub-models. Specifically, to test $l = m = 0$, the LR statistic is $w = 2[ln(\widehat{c}, \widehat{k}, \widehat{s}, \widehat{l}, \widehat{m}) - ln(\tilde{c}, \tilde{k}, \tilde{s}, 0, 0)]$, where $\widehat{c}$, $\widehat{k}$, $\widehat{s}$, $\widehat{l}$ and $\widehat{m}$ are the unrestricted MLEs and $\tilde{c}$, $\tilde{k}$ and $\tilde{s}$ are the restricted estimates. The LR statistic is asymptotically distributed under the null model as ${\chi_d}^2$ and rejects the null hypothesis if $w > {\chi_d}^2$, where ${\chi_d}^2$ denotes the upper $100d\%$ of the χ^2 with 2 degrees of freedom.

4.10 Concluding Remarks

This chapter includes some computations, here are the main results for WBIII distribution:

- Hazard and reverse hazard functions are given by:

$$\begin{aligned} h_{WBIII}(x;c,k,s,l,m) &= \frac{g_{WBIII}(x;c,k,s,l,m)}{\bar{G}_{WBIII}(x;c,k,s,l,m)} \\ &= \frac{cs^{c-l}x^{l-c-1}(1+(x/s)^{-c})^{-km-k-1}}{B\Bigg(\frac{l}{c}+km+k,1-\frac{l}{c}\Bigg)-B\Bigg(x^*;\frac{l}{c}+km+k,1-\frac{l}{c}\Bigg)}, \end{aligned}$$

and

$$\begin{aligned} \tau_{WBIII}(x;c,k,s,l,m) &= \frac{g_{WBIII}(x;c,k,s,l,m)}{G_{WBIII}(x;c,k,s,l,m)} \\ &= \frac{cs^{c-l}x^{l-c-1}(1+(x/s)^{-c})^{-km-k-1}}{B\Bigg(x^*;\frac{l}{c}+km+k,1-\frac{l}{c}\Bigg)}, \end{aligned}$$

for $c > l$, respectively.

- The i^{th} moment is given by:

$$E(X^i) = \frac{s^i B\Bigg(km+k+\frac{i+l}{c},1-\frac{i+l}{c}\Bigg)}{B\Bigg(\frac{l}{c}+km+k,1-\frac{l}{c}\Bigg)}, \quad \text{for } c > l.$$

- Lorenz and Bonferroni curves are given by:

$$L(G_{WBIII}(x)) = \frac{B\Bigg(x^*;km+k+\frac{1+l}{c},1-\frac{1+l}{c}\Bigg)}{B\Bigg(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\Bigg)},$$

and

$$B(G_{WBIII}(x)) = \frac{B\Bigg(x^*;km+k+\frac{1+l}{c},1-\frac{1+l}{c}\Bigg)}{G_{WBIII}(x)B\Bigg(km+k+\frac{1+l}{c},1-\frac{1+l}{c}\Bigg)},$$

for $c > l+1$, respectively.

- Entropy

  ϵ-entropy is given by:

$$H_\epsilon(g_{WBIII}) = \frac{1}{\epsilon-1}\left[1 - \frac{c^{\epsilon-1}s^{1-\epsilon}B\left(\epsilon km + \epsilon k + \frac{\epsilon(l-1)}{c} + \frac{1}{c}, \epsilon + \frac{\epsilon(1-l)}{c} - \frac{1}{c}\right)}{B^\epsilon\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}\right],$$

  $\epsilon \neq 1$, $\epsilon > 0$, $c > l$. Renyi entropy for WBIII distribution is given by:

$$H_R(g_{WBIII}) = (1-\tau)^{-1}log\left[\frac{c^{\tau-1}s^{1-\tau}B\left(\tau km + \tau k + \frac{\tau(l-1)}{c} + \frac{1}{c}, \tau + \frac{\tau(1-l)}{c} - \frac{1}{c}\right)}{B^\tau\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}\right],$$

  for $\tau > 0$, $\tau \neq 1$, $c > l$.

- Maximum Likelihood Estimation

$$\frac{\partial L^*}{\partial k} = n(m+1)[\psi(l/c+km+k)+\psi(km+k+1)]-(m+1)\sum_{i=0}^{n} ln(1+(x_i/s)^{-c}),$$

$$\frac{\partial L^*}{\partial s} = \frac{n(c-l)}{s} - (km+k+1)\frac{cx_i{}^{-c}s^{c-1}}{(1+(x_i/s)^{-c})},$$

$$\begin{aligned}\frac{\partial L^*}{\partial c} &= \frac{n}{c} + nln(s) - \sum_{i=0}^{n} ln(x_i) - (km+k+1)\sum_{i=0}^{n}\frac{x_i{}^{-c}s^c ln(\frac{s}{x_i})}{(1+(x_i/s)^{-c})} \\ &+ \frac{nl}{c^2}\psi(l/c+km+k) - \frac{\frac{nl}{c^2}}{\Gamma(1-l/c)},\end{aligned}$$

$$\frac{\partial L^*}{\partial l} = -nln(s) + \sum_{i=0}^{n} lnx_i - \frac{n}{c}[\psi(l/c+km+k) + \psi(1-l/c)],$$

  and

$$\frac{\partial L^*}{\partial m} = -k\sum_{i=0}^{n} ln(1+(x_i/s)^{-c}) - nk\psi(l/c+km+k) + nk\psi(km+k+1).$$

  There is no closed form solution to these equations, so in this case a numerical technique must be applied to obtain the solution.

CHAPTER 5

APPLICATIONS OF WEIGHTED BURR-TYPE III AND RELATED DISTRIBUTIONS

5.1 Introduction

In this chapter, we present applications and examples involving the class of weighted BurrIII distribution, which are flexible parametric models with applications in reliability, actuarial science, economics, finance and telecommunications.

5.2 Applications

In this section, applications based on real data, as well as comparisons of the WBIII distribution with its sub-models are given. The MLE of the model parameters are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike information criterion ($AIC = 2p - 2ln(L)$) and Bayesian information criterion ($BIC = pln(n) - 2ln(L)$), where $L = L(\widehat{\Theta})$ is the value of likelihood function evaluated at the parameter estimates, p is the number of estimated parameters and n is the number of observations are tabulated.

Recall the cdf and pdf for BurrIII distribution (BIII) are given by:

$$F_{BIII}(x) = (1 + (x/s)^{-c})^{-k},$$

and

$$f_{BIII}(x) = cks^c x^{-c-1}(1 + (x/s)^{-c})^{-k-1},$$

for $x > 0$, and c, k, $s > 0$, respectively. The weighted Burr-III (WBIII) pdf and cdf with weight function $w(x) = x^l F^m(x)$ are given by:

$$g_{WBIII}(x; c, k, s, l, m) = \frac{cs^{c-l} x^{l-c-1}(1 + (x/s)^{-c})^{-km-k-1}}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)},$$

$$G_{WBIII}(x; c, k, s, l, m) = \frac{B\left(x^*; \frac{l}{c} + km + k, 1 - \frac{l}{c}\right)}{B\left(\frac{l}{c} + km + k, 1 - \frac{l}{c}\right)},$$

respectively, where $x^* = (1 + (x/s)^{-c})^{-1}$, and $\frac{B(x^*;a,b)}{B(a,b)}$ is an incomplete beta function ratio.

The first data set is given in Table 5.3 with $n = 51$ observations is on the strengths of 1.5 cm glass fibers. This data set was given in Smith and Naylor (1987) and Cordeiro and Lemonte (2011). The data was obtained from the National Physical Laboratory in England. The LR test statistic of the hypothesis $H_0 : WBIII(c, k, 1, 0.5, 0)$ vs $H_a : WBIII(c, k, s, 0.5, 0)$ is $w = 36.3$, $(p - value = 1.692 \times 10^{-9})$. We reject the null hypothesis in favor of $WBIII(c, k, s, 0.5, 0)$ distribution, see fitted densities in Figure 5.1. The second data set is uncensored data on breaking stress of carbon fiber (GPa) with $n = 66$, (Nichols and Padgett (2006)) is given in the Table 5.4. The LR test statistic of the hypothesis $H_0 : WBIII(c, k, 1, 0.5, 0)$ vs $H_a : WBIII(c, k, s, 0.5, 0)$ for this data set is $w = 54.5$, $(p - value =$

1.55×10^{-13}). We reject the null hypothesis in favor of $WBIII(c,k,s,0.5,0)$. The LR test statistic of the hypothesis $H_0 : WBIII(c,1,s,0,0)$ vs $H_a : WBIII(c,k,s,0,0)$ is $w = 13.4$, ($p-value = 2.516\times10^{-4} < 0.001$). We reject the null hypothesis in favor of $WBIII(c,k,s,0,0)$ distribution. Based on LR statistic in Tables 5.1 and 5.2, there is no different between the first five models, while the last model with s=1 is a poor fit. The values of the statistics (AIC and BIC) are smaller for the $WBIII(c,k,s,0,0)$ distribution compared to $WBIII(c,1,s,0,0)$ and $WBIII(c,k,1,0.5,0)$ distributions. The model $WBIII(c,1,s,0,0)$ is a better fit compared to the model $WBIII(c,k,1,0.5,0)$ based on the values of the statistic in Table 5.2, (see fitted densities in Figure 5.2).

	Parameters					Statistic		
Models	c	k	s	l	m	-2lnL	AIC	BIC
$WBIII(c,k,s,0,0)$	18.5157	0.2479	1.7013	0	0	20.6	26.6	32.4
	(4.3019)	(0.07370)	(0.04599)					
$WBIII(c,k,s,1,1)$	19.1232	0.09383	1.6955	1	1	20.4	26.4	32.2
	(4.3590)	(0.03058)	(0.04591)					
$WBIII(c,k,s,1,0)$	19.1232	0.1877	1.6955	1	0	20.4	26.4	32.2
	(4.3125)	(0.06034)	(0.04572)					
$WBIII(c,k,s,0,1)$	18.5157	0.1240	1.7013	0	1	20.6	26.6	32.4
	(4.3048)	(0.03689)	(0.04600)					
$WBIII(c,k,s,0.5,0)$	18.8165	0.2173	1.6984	0.5	0	20.5	26.5	32.3
	(4.2892)	(0.06678)	(0.04591)					
$WBIII(c,k,1,0.5,0)$	4.4626	2.6358	1	0.5	0	56.8	60.8	64.7
	(0.3946)	(0.3860)						

Table 5.1: MLEs of Weighted BIII and related distributions for glass fibers data.

	Parameters					Statistic		
Models	c	k	s	l	m	-2lnL	AIC	BIC
$WBIII(c,k,s,0,0)$	10.1895	0.2820	3.5112	0	0	169.9	175.9	182.4
	(2.2944)	(0.08906)	(0.1865)					
$WBIII(c,k,s,1,1)$	10.7878	0.08646	3.4753	1	1	169.8	175.8	182.3
	(2.3171)	(0.03223)	(0.1888)					
$WBIII(c,k,s,1,0)$	10.7878	0.1729	3.4753	1	0	169.8	175.8	182.3
	(2.3789)	(0.06585)	(0.1914)					
$WBIII(c,k,s,0,1)$	10.1895	0.1410	3.5112	0	1	169.9	175.9	182.4
	(2.3004)	(0.04469)	(0.1869)					
$WBIII(c,k,s,0.5,0)$	10.4824	0.2261	3.4927	0.5	0	169.8	175.8	182.4
	(2.1479)	(0.07142)	(0.1817)					
$WBIII(c,1,s,0,0)$	4.8958	1	2.7109	0	0	183.3	187.3	191.7
	(0.5137)		(0.1161)					
$WBIII(c,k,1,0.5,0)$	2.4884	5.0428	1	0.5	0	224.3	228.3	232.7
	(0.1521)	(0.4069)						

Table 5.2: MLEs of Weighted BIII and related distributions for carbon fiber data (GPa).

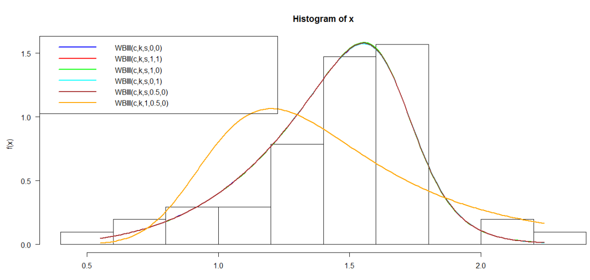

Figure 5.1: Estimated densities of the models for glass fibers data.

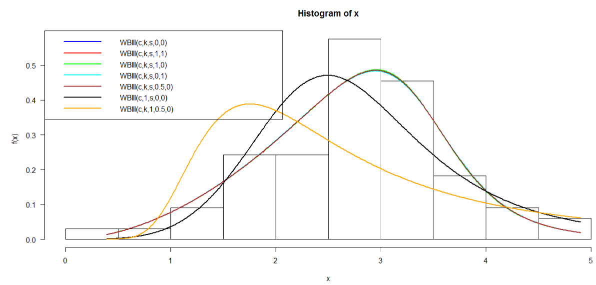

Figure 5.2: Estimated densities of the models for carbon fiber data (GPa).

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13
1.24	1.25	1.27	1.28	1.29	1.30	1.36	1.39	1.42
1.48	1.48	1.49	1.49	1.50	1.50	1.51	1.52	1.53
1.54	1.55	1.55	1.58	1.59	1.60	1.61	1.61	1.61
1.61	1.62	1.62	1.63	1.64	1.66	1.66	1.66	1.67
1.68	1.68	1.69	2.00	2.01	2.24			

Table 5.3: Glass fibers data

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47
3.11	3.56	4.42	2.41	3.19	3.22	1.69	3.28	3.09
1.87	3.15	4.90	1.57	2.67	2.93	3.22	3.39	2.81
4.20	3.33	2.55	3.31	3.31	2.85	1.25	4.38	1.84
0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.89
2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39
2.96	2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08
2.56	1.80	2.53						

Table 5.4: Data set of breaking stress of carbon fiber data (GPa)

5.3 Concluding Remarks

Applications and numerical examples on the estimation and fit of the WBIII distribution to real data are used to illustrate the usefulness of the developed model.

5.4 Future Research

In the future, we hope to study further generalizations of Burr-Type distributions and obtain parameter estimates from the Bayesian perspective. McDonald generalizations of the weighted BurrIII and BurrXII distributions will be considered, including income inequality and entropy measures for these models.

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