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Multiple Solutions With Constant Sign of a Dirichlet Problem for a Class of Elliptic Systems With Variable Exponent Growth

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MULTIPLE SOLUTIONS WITH CONSTANT SIGN OF A DIRICHLET PROBLEM FOR A CLASS OF ELLIPTIC SYSTEMS WITH VARIABLE EXPONENT GROWTH

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Abstract. We investigate the following Dirichlet problem with variable exponents:

$$\begin{cases} -\triangle_{p(x)} u = \lambda\alpha(x)\left|u\right|^{\alpha(x)-2} u\left|v\right|^{\beta(x)} + F_u(x,u,v), \text{ in } \Omega, \\ -\triangle_{q(x)} v = \lambda\beta(x)\left|u\right|^{\alpha(x)}\left|v\right|^{\beta(x)-2} v + F_v(x,u,v), \text{ in } \Omega, \\ u = 0 = v, \text{ on } \partial\Omega. \end{cases}$$

We present here, in the system setting, a new set of growth conditions under which we manage to use a novel method to verify the Cerami compactness condition. By localization argument, decomposition technique and variational methods, we are able to show the existence of multiple solutions with constant sign for the problem without the well-known Ambrosetti–Rabinowitz type growth condition. More precisely, we manage to show that the problem admits four, six and infinitely many solutions respectively.

Key words: $p(x)$-Laplacian, Dirichlet problem, solutions with constant sign, Ambrosetti-Rabinowitz Condition, Cerami condition, Critical point.
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1. Introduction

In this paper, we consider the existence of multiple solutions to the following Dirichlet problem for an elliptic system with variable exponents:

$$(P)\quad \begin{cases} -\triangle_{p(x)} u = \lambda\alpha(x)\left|u\right|^{\alpha(x)-2} u\left|v\right|^{\beta(x)} + F_u(x,u,v), \text{ in } \Omega, \\ -\triangle_{q(x)} v = \lambda\beta(x)\left|u\right|^{\alpha(x)}\left|v\right|^{\beta(x)-2} v + F_v(x,u,v), \text{ in } \Omega, \\ u = 0 = v, \text{ on } \partial\Omega, \end{cases}$$

where $\triangle_{p(x)}u := \operatorname{div}(\left|\nabla u\right|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian which is nonlinear and nonhomogeneous, $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $p(\cdot), q(\cdot), \alpha(\cdot), \beta(\cdot) > 1$ are in the space $C^1(\overline{\Omega})$ which consists of differentiable functions with continuous first order derivatives on $\overline{\Omega}$.

Elliptic equations and systems of elliptic equations with variable exponent growth as in problem (P) arise from applications in electrorheological fluids and image restoration.

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We refer the readers to [1], [5], [24], [39] and the references therein for more details in applications. In particular, see [5] for a model with variable exponent growth and its important applications in image denoising, enhancement, and restoration. Problems with variable exponent growth also brought challenging pure mathematical problems. Compared with the classical Laplacian $\Delta = \Delta_2$ which is linear and homogeneous and the p -Laplacian $\Delta_p \cdot := \mathrm{div}(|\nabla\cdot|^{p-2}\nabla\cdot)$ which is nonlinear but homogeneous for constant p, the $p(x)$-Laplacian is both nonlinear and inhomogeneous. Due to the nonlinear and inhomogeneous nature of the $p(x)$-Laplacian, nonlinear (systems of) elliptic equations involving $p(x)$-Laplacian and nonlinearities with variable growth rates are much more difficult to deal with. Driven by the real-world applications and mathematical challenges, the study of elliptic equations and systems with variable exponent growth has attracted many researchers with different backgrounds, and become a very attracting field. We refer the readers to [3], [9], [10], [12], [15], [20], [23], [25], [35], [36], [37] and the related references to tract the rapid development of the field.

In this paper, our main goal is to obtain some existence results for the problem (P), a Dirichlet problem for elliptic systems with variable exponents, without the famous Ambrosetti-Rabinowitz condition via critical point theory. For this purpose, we propose a new set of growth conditions for the nonlinearities in the current system of elliptic equations setting. Our new set of growth conditions involve only variable growths which naturally match the variable nature of the problem under investigation. Under our growth conditions, we can use a novel method to verify that the corresponding functional to the problem (P) satisfies the Cerami compactness condition which is a weaker compactness condition yet is still sufficient to yield critical points of the functional. See the details of proofs in Section 3. The current study generalizes in particular our former investigations [32] and [37]. However, this generalization from a single elliptic equation to the current system of elliptic equations is by no means trivial. Besides technical complexities, the assumptions in the current study are more involved. In particular, though we still do not need any monotonicity on the nonlinear terms, we do need impose certain monotonicity assumptions on the variable exponents to close our argument in the system setting.

When we utilize variational argument to obtain existence of weak solutions to elliptic equations, typically we impose the famous Ambrosetti-Rabinowitz growth condition on the nonlinearity to guarantee the boundedness of Palais-Samle sequence. Under the Ambrosetti-Rabinowitz growth condition, one then tries to verify the Palais-Smale condition. However, the Ambrosetti-Rabinowitz type growth condition excludes a number of interesting nonlinearities. In view of this fact, a lot of efforts were made to show the existence of weak solutions in the variational framework without this type of growth condition, especially for the usual p-Laplacian and a single nonlinear elliptic partial differential equation (see, in particular, [14], [17], [18], [19], [21], [29] and the references therein). Our results can be regarded as extensions of the corresponding results for the p-Laplacian problems. There are also some related earlier works which dealt with elliptic variational problems in the variable exponent spaces framework, see [2], [13], [33], [37], [27] and related works. These earlier studies were mainly focused on a single elliptic equation. In the interesting earlier study [33], the author considered the existence of solutions of the following variable exponent differential equations without

Ambrosetti-Rabinowitz condition on bounded domain,

$$\begin{cases} -\triangle_{p(x)} u = f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

However, in some aspects the assumption is even stronger than the Ambrosetti-Rabinowitz condition. In a recent study [2], the authors considered the variable exponent equation in the whole space $\mathbb{R}^N$ under the following assumptions: (1^0) there exists a constant $\theta \geqslant 1$, such that $\theta\mathcal{F}(x,t) \geqslant \mathcal{F}(x,st)$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - p^+F(x,t)$; ($2^0$) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies $\lim\limits_{|t|\to\infty} \frac{F(x,t)}{|t|^{p^+}} = \infty$. In [13], the authors considered the problem (1.1) in a bounded domain under the condition (2^0). In [27], the authors studied a variable exponent differential equation with a potential term in the whole space $\mathbb{R}^N$. The authors proposed conditions under which they could show the existence of infinitely many high energy solutions without the Ambrosetti-Rabinowitz condition. In the above mentioned works, the growths conditions involved either the supremum or the infimum of the variable exponents. In our current study, we are able to provide a number of existence results in the system setting under assumptions that only involve variable exponent growths which match naturally the variable growth of the problem under study.

We point out here that the growth conditions we use here and the method to check the Cerami compactness condition are different from all the above mentioned works. Due to the differences between the p-Laplacian and $p(x)$ -Laplacian mentioned as above, it is usually challenging to judge whether or not results about p-Laplacian can be generalized to $p(x)$-Laplacian. Meanwhile, some new methods and techniques are needed to study elliptic equations involving the non-standard growth, as the commonly known methods and techniques to study elliptic equations involving standard growth may fail. The main reason, as mentioned earlier, is that the principal elliptic operators in the elliptic equations involving the non-standard growth is not homogeneous anymore. To see some new features associated with the $p(x)$ -Laplacian, we first point out that the norms in variable exponent spaces are the so-called Luxemburg norms $|u|_{p(\cdot)}$ (see Section 2) and the integral $\int_\Omega |u(x)|^{p(x)}\,dx$ does not have the usual constant power relation as in the spaces L^p for constants p. Another subtle feature is on the principal Dirichlet eigenvalue. As invetigated in [10], even for a bounded smooth domain $\Omega \subset \mathbb{R}^N$, the principle eigenvalue $\lambda_{p(\cdot)}$ defined by the Rayleigh quotient

$$\lambda_{p(\cdot)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\backslash\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,dx} \tag{1.2}$$

is zero in general, and only under some special conditions $\lambda_{p(\cdot)} > 0$ holds. For example, when $\Omega \subset \mathbb{R}$ ($N = 1$) is an interval, results show that $\lambda_{p(\cdot)} > 0$ if and only if $p(\cdot)$ is monotone. This feature on the $p(x)$-Laplacian Dirichlet principle eigenvalue plays an important role for us in proposing the assumptions on the variable exponents and in our proofs of the main results.

Now we shall first list the assumptions on the nonlinearity F and variable exponents involved in the current system setting. Our assumptions are as follows.

$(H_{\alpha,\beta})$ $\frac{\alpha(x)}{p(x)} + \frac{\beta(x)}{q(x)} < 1, \forall x \in \overline{\Omega}$.

(H_0) $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 continous and

$$|F_u(x, u, v)u| + |F_v(x, u, v)v| \leqslant C(1 + |u|^{\gamma(x)} + |v|^{\delta(x)}), \forall (x, u, v) \in \Omega \times \mathbb{R},$$

where $\gamma, \delta \in C(\overline{\Omega})$ and $p(x) < \gamma(x) < p^*(x), q(x) < \delta(x) < q^*(x)$ where $p^*(x)$ and $q^*(x)$ are defined as

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geqslant N \end{cases}, \quad q^*(x) = \begin{cases} \frac{Nq(x)}{N-q(x)}, & \text{if } q(x) < N \\ +\infty, & \text{if } q(x) \geqslant N \end{cases}$$

(H_1) There exists constants $M, C_1, C_2 > 0$, $a(\cdot) > p(\cdot)$ and $b(\cdot) > q(\cdot)$ on $\overline{\Omega}$ such that

$$\begin{aligned} & C_1 |u|^{p(x)} [\ln(e + |u|)]^{a(x)-1} + C_1 |v|^{q(x)} [\ln(e + |v|)]^{b(x)-1} \\ \leqslant\; & C_2 \left(\frac{F_u(x, u, v)u}{\ln(e + |u|)} + \frac{F_v(x, u, v)v}{\ln(e + |v|)} \right) \\ \leqslant\; & \frac{1}{p(x)} F_u(x, u, v)u + \frac{1}{q(x)} F_v(x, u, v)v - F(x, u, v), \forall |u| + |v| \geqslant M, \forall x \in \Omega. \end{aligned}$$

(H_2) $F(x, u, v) = o(|u|^{p(x)} + |v|^{q(x)})$ uniformly for $x \in \Omega$ as $u, v \to 0$.

(H_3) F satisfies $F_u(x, 0, v) = 0, F_v(x, u, 0) = 0$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$.

(H_4) $F(x, -u, -v) = F(x, u, v)$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$.

$(H_{p,q})$ There are vectors $l_p, l_q \in \mathbb{R}^N \backslash \{0\}$ such that for any $x \in \Omega$, $\phi_p(t) = p(x + tl_p)$ is monotone for $t \in I_{x,p}(l) = \{t \mid x + tl_p \in \Omega\}$, and $\phi_q(t) = q(x + tl_q)$ is monotone for $t \in I_{x,q}(l) = \{t \mid x + tl_q \in \Omega\}$.

To gain a first understanding, we briefly comment on some of the above assumptions. (H_0) means that the nonlinearity F has a subcritical variable growth rate in the sense of variable exponent Sobolev embedding and in the current system of elliptic equations setting. (H_1) and (H_2) describe the far and near field behaviors of the nonlinearity F. Notice that in the current setting, the far field behavior (H_1) is more involved. We emphasize that $(H_{p,q})$ is crucial for our later arguments, for it guarantees that the Rayleigh quotients for $-\Delta_{p(x)}$ and $-\Delta_{q(x)}$ (see (1.2) for $-\Delta_{p(x)}$) are positive. Finally, the assumptions (H_0)-(H_4) on the nonlinearity F are consistent, which can be seen by the following example:

$$F(x, u, v) = |u|^{p(x)} [\ln(1+|u|)]^{a(x)} + |v|^{q(x)} [\ln(1+|v|)]^{b(x)} + |u|^{\theta_1(x)} |v|^{\theta_2(x)} \ln(1+|u|) \ln(1+|v|),$$

where $1 < \theta_1(x) < p(x), 1 < \theta_2(x) < q(x), \frac{\theta_1(x)}{p(x)} + \frac{\theta_2(x)}{q(x)} = 1, \forall x \in \overline{\Omega}$. In addition, F does not satisfy the Ambrosetti-Rabinowitz condition.

Now we are in a position to state our main results.

Theorem 1.1. *If λ is small enough and the assumptions $(H_{\alpha,\beta})$, (H_0), (H_2)-(H_3) and $(H_{p,q})$ hold, then the problem (P) has at least four nontrivial solutions each with constant sign respectively.*

Theorem 1.2. *If λ is small enough and the assumptions $(H_{\alpha,\beta})$, (H_0)-(H_3) and $(H_{p,q})$ hold, then the problem (P) has at least six nontrivial solutions each with constant sign respectively.*

Theorem 1.3. *If the assumptions $(H_{\alpha,\beta})$, (H_0), (H_1) and (H_4) hold, then there are infinitely many pairs of solutions to the problem (P).*

Remark 1.4. The solutions we obtained in Theorem 1.1 and 1.2 to (P) are of constant sign. Meanwhile, we do not need any monotonicity assumptions on the nonlinearity $F(x, \cdot, \cdot)$.

This rest of the paper is organized as follows. In Section 2, we do some functional-analytic preparations. In Section 3, we give the proofs of our main results.

2. Functional-analytic Preliminary

Throughout this paper, we will use letters $c, c_i, C, C_i,\ i = 1, 2, ...$ to denote generic positive constants which may vary from line to line, and we will specify them whenever it is necessary.

In order to discuss problem (P), we shall discuss the functional analytic framework. First, we present some results about space $W_0^{1,p(\cdot)}(\Omega)$ which we call variable exponent Sobolev space. These results on the variable exponent spaces will be used later (for details, see [6], [7], [9], [16], [26]). We denote $C(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ with the usual uniform norm, and

$$C_+(\overline{\Omega}) = \left\{ h \,\middle|\, h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for } x \in \overline{\Omega} \right\}.$$

For $h = h(\cdot) \in C(\Omega)$, we denote $h^+ := \max_{x \in \bar{\Omega}} h(x)$ and $h^- := \min_{x \in \bar{\Omega}} h(x)$. For $p = p(\cdot) \in C_+(\overline{\Omega})$, we introduce

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u \text{ is a measurable real-value function, } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

When equipped with the Luxemberg norm

$$|u|_{p(\cdot)} = \inf \left\{ \mu > 0 \,\middle|\, \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leqslant 1 \right\},$$

$(L^{p(\cdot)}(\Omega),\ |\cdot|_{p(\cdot)})$ becomes a Banach space and it is called variable exponent Lebesgue space. For the variable exponent Lebesgue spaces, we have the following Hölder type inequality and simple embedding relation.

Proposition 2.1. *(see [6], [7], [9]). i) The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{(p(\cdot))^0}(\Omega)$, where $(p(\cdot))^0 := \frac{p(\cdot)}{p(\cdot)-1}$ is the conjugate function of $p(\cdot)$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{(p(\cdot))^0}(\Omega)$, we have*

$$\left|\int_\Omega uv dx\right| \leqslant (\frac{1}{p^-} + \frac{1}{q^-}) |u|_{p(\cdot)} |v|_{(p(\cdot))^0} .$$

ii) If $p_1,\ p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leqslant p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.

Denote $Y = \prod_{i=1}^{k} L^{p_i(\cdot)}(\Omega)$ with the norm

$$\| y \|_Y = \sum_{i=1}^{k} |y^i|_{p_i(\cdot)} ,\ \forall y = (y^1, \cdots, y^k) \in Y,$$

where $p_i(x) \in C_+(\overline{\Omega})$, $i = 1, \cdots, m$, then Y is a Banach space. The following proposition can be regarded as a vectorial generalization of the classical proposition on the Nemytsky operator.

Proposition 2.2. *Let $f(x,y) : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ be a Caratheodory function, i.e., f satisfies*

(i) for a.e. $x \in \Omega$, $y \to f(x,y)$ is a continuous function from $\mathbb{R}^k$ to $\mathbb{R}^m$,

(ii) for any $y \in \mathbb{R}^k$, $x \to f(x,y)$ is measurable.

If there exist $\eta(x), p_1(x), \cdots, p_k(x) \in C_+(\overline{\Omega})$, $h(x) \in L^{\eta(\cdot)}(\Omega)$ and positive constant $c > 0$ such that

$$|f(x,y)| \leqslant h(x) + c\sum_{i=1}^{k} |y_i|^{p_i(x)/\eta(x)} \text{ for any } x \in \Omega, y \in \mathbb{R}^k,$$

then the Nemytsky operator from Y to $(L^{\eta(\cdot)}(\Omega))^m$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Proof. Similar to the proof of [4], we omit it here. □

The following two propositions concern the norm-module relations in the variable exponent Lebesgue spaces. Unlike in the usual Lebesgue spaces setting, the norm and module of a function in the variable exponent spaces do not enjoy the usual power equality relation.

Proposition 2.3. *(see [9]). If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx,\ \forall u \in L^{p(\cdot)}(\Omega),$$

then there exists a $\xi \in \overline{\Omega}$ such that $|u|_{p(\cdot)}^{p(\xi)} = \int_\Omega |u|^{p(x)}\, dx$ and *i)* $|u|_{p(\cdot)} < 1(= 1; > 1) \Longleftrightarrow \rho(u) < 1(= 1; > 1)$;

ii) $|u|_{p(\cdot)} > 1 \Longrightarrow |u|_{p(\cdot)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(\cdot)}^{p^+}$; $|u|_{p(\cdot)} < 1 \Longrightarrow |u|_{p(\cdot)}^{p^-} \geqslant \rho(u) \geqslant |u|_{p(\cdot)}^{p^+}$;

iii) $|u|_{p(\cdot)} \to 0 \Longleftrightarrow \rho(u) \to 0$; $|u|_{p(\cdot)} \to \infty \Longleftrightarrow \rho(u) \to \infty$.

Proposition 2.4. *(see [9]). If u, $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \cdots$, then the following statements are equivalent to each other.*

1) $\lim\limits_{k\to\infty} |u_k - u|_{p(\cdot)} = 0$;

2) $\lim\limits_{k\to\infty} \rho(u_k - u) = 0$;

3) $u_k \to u$ *in measure in* Ω *and* $\lim\limits_{k\to\infty} \rho(u_k) = \rho(u)$.

The spaces $W^{1,p(\cdot)}(\Omega)$ and $W^{1,q(\cdot)}(\Omega)$ are defined by

$$\begin{aligned} W^{1,p(\cdot)}(\Omega) &= \left\{u \in L^{p(\cdot)}(\Omega) \,\middle|\, \nabla u \in (L^{p(\cdot)}(\Omega))^N\right\}, \\ W^{1,q(\cdot)}(\Omega) &= \left\{v \in L^{q(\cdot)}(\Omega) \,\middle|\, \nabla v \in (L^{q(\cdot)}(\Omega))^N\right\}, \end{aligned}$$

and be endowed with the following norm

$$\begin{aligned} \|u\|_{p(\cdot)} &= \inf\left\{\mu > 0 \,\middle|\, \int_\Omega \left|\frac{\nabla u}{\mu}\right|^{p(x)} dx + \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}, \\ \|u\|_{q(\cdot)} &= \inf\left\{\mu > 0 \,\middle|\, \int_\Omega \left|\frac{\nabla u}{\mu}\right|^{q(x)} dx + \int_\Omega \left|\frac{u(x)}{\mu}\right|^{q(x)} dx \leqslant 1\right\}. \end{aligned}$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have in particular the following Sobolev embedding relation and Poincaré type inequality.

Proposition 2.5. *(see [6], [9]). i) $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable reflexive Banach spaces;*

ii) If $\eta \in C_+(\overline{\Omega})$ and $\eta(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{\eta(\cdot)}(\Omega)$ is compact and continuous;

iii) There is a constant $C > 0$, such that

$$|u|_{p(\cdot)} \leqslant C\,|\nabla u|_{p(\cdot)}, \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

We know from iii) of Proposition 2.5 that $|\nabla u|_{p(\cdot)}$ and $\|u\|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. From now on we will use $|\nabla u|_{p(\cdot)}$ to replace $\|u\|_{p(\cdot)}$ as the norm on $W_0^{1,p(\cdot)}(\Omega)$, and use $|\nabla v|_{q(\cdot)}$ to replace $\|v\|_{q(\cdot)}$ as the norm on $W_0^{1,q(\cdot)}(\Omega)$.

Under the assumption ($\mathrm{H}_{p,q}$), $\lambda_{p(\cdot)}$ defined in (1.2) is positive, i.e., we have the following proposition.

Proposition 2.6. *(see [10]). If the assumption ($H_{p,q}$) is satisfied, then $\lambda_{p(\cdot)}$ defined in (1.2) is positive.*

Denote $X = W_0^{1,p(\cdot)}(\Omega) \times W_0^{1,q(\cdot)}(\Omega)$. The norm $\|\cdot\|$ on X is defined by

$$\|(u,v)\| = \max\{\|u\|_{p(\cdot)}, \|v\|_{q(\cdot)}\}.$$

For any (u,v) and (ϕ,ψ) in X, let

$$\begin{aligned}
\Phi_1(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \\
\Phi_2(v) &= \int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)} dx, \\
\Phi(u,v) &= \Phi_1(u) + \Phi_2(v), \\
\Psi(u,v) &= \int_\Omega \lambda |u|^{\alpha(x)} |v|^{\beta(x)} + F(x,u,v) dx.
\end{aligned}$$

From Proposition 2.2, Proposition 2.5 and condition (H_0), it is easy to see that $\Phi_1, \Phi_2, \Phi, \Psi \in C^1(X, \mathbb{R})$ and then

$$\begin{aligned}
\Phi'(u,v)(\phi,\psi) &= D_1\Phi(u,v)(\phi) + D_2\Phi(u,v)(\psi), \\
\Psi'(u,v)(\phi,\psi) &= D_1\Psi(u,v)(\phi) + D_2\Psi(u,v)(\psi),
\end{aligned}$$

where

$$\begin{aligned}
D_1\Phi(u,v)(\phi) &= \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx = \Phi_1'(u)(\phi), \\
D_2\Phi(u,v)(\psi) &= \int_\Omega |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx = \Phi_2'(v)(\psi), \\
D_1\Psi(u,v)(\phi) &= \int_\Omega \lambda\alpha(x) |u|^{\alpha(x)-2} u |v|^{\beta(x)} + \frac{\partial}{\partial u} F(x,u,v)\phi dx, \\
D_2\Psi(u,v)(\psi) &= \int_\Omega \lambda\beta(x) |u|^{\alpha(x)} |v|^{\beta(x)-2} v + \frac{\partial}{\partial v} F(x,u,v)\psi dx.
\end{aligned}$$

The integral functional associated with the problem (P) is

$$\varphi(u,v) = \Phi(u,v) - \Psi(u,v).$$

Without loss of generality, we may assume that $F(x,0,0) = 0, \forall x \in \overline{\Omega}$. Obviously, we have

$$F(x,u,v) = \int_0^1 [u\partial_2 F(x,tu,tv) + v\partial_3 F(x,tu,tv)]dt, \forall x \in \overline{\Omega},$$

where ∂_j denotes the partial derivative of F with respect to its j-th variable, then the condition (H_0) holds

$$|F(x,u,v)| \leqslant c(|u|^{\gamma(x)} + |v|^{\delta(x)} + 1), \forall x \in \overline{\Omega}. \tag{2.1}$$

From Proposition 2.2, Proposition 2.5 and condition (H_0), it is easy to see that $\varphi \in C^1(X,\mathbb{R})$ and satisfies

$$\varphi'(u,v)(\phi,\psi) = D_1\varphi(u,v)(\phi) + D_2\varphi(u,v)(\psi),$$

where

$$\begin{aligned} D_1\varphi(u,v)(\phi) &= D_1\Phi(u,v)(\phi) - D_1\Psi(u,v)(\phi), \\ D_2\varphi(u,v)(\psi) &= D_2\Phi(u,v)(\psi) - D_2\Psi(u,v)(\psi). \end{aligned}$$

We say $(u,v) \in X$ is a critical point of φ if

$$\varphi'(u,v)(\phi,\psi) = 0, \forall(\phi,\psi) \in X.$$

The dual space of X will be denoted as X^*, then for any $H \in X^*$, there exists $f \in (W_0^{1,p(\cdot)}(\Omega))^*$, $g \in (W_0^{1,q(\cdot)}(\Omega))^*$ such that $H(u,v) = f(u) + g(v)$. We denote $\|\cdot\|_*$, $\|\cdot\|_{*,p(\cdot)}$ and $\|\cdot\|_{*,q(\cdot)}$ the norms of X^*, $(W_0^{1,p(\cdot)}(\Omega))^*$ and $(W_0^{1,q(\cdot)}(\Omega))^*$, respectively. Holds $X^* = (W_0^{1,p(\cdot)}(\Omega))^* \times (W_0^{1,q(\cdot)}(\Omega))^*$, and

$$\|H\|_* = \|f\|_{*,p(\cdot)} + \|g\|_{*,q(\cdot)}.$$

Therefore

$$\|\varphi'(u,v)\|_* = \|D_1\varphi(u,v)\|_{*,p(\cdot)} + \|D_2\varphi(u,v)\|_{*,q(\cdot)}.$$

It's easy to see that Φ is a convex functional, and we have the following proposition.

Proposition 2.7. *(see* [9], [15]*). i)* $\Phi' : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*

ii) Φ' *is a mapping of type* (S_+)*, i.e., if* $(u_n,v_n) \rightharpoonup (u,v)$ *in* X *and* $\overline{\lim}_{n\to+\infty} (\Phi'(u_n,v_n) - \Phi'(u,v),(u_n-u,v_n-v)) \leqslant 0$*, then* $(u_n,v_n) \to (u,v)$ *in* X*;*

iii) $\Phi' : X \to X^*$ *is a homeomorphism.*

Remark 2.8. A proof of a simple version of the above proposition can be found in the references [9], [15]. In the system setting here, the idea of proof is essentially the same. For readers' convenience and for completeness, we present it here.

Proof. i) It follows from Proposition 2.2 that Φ' is continuous and bounded. For any $\xi,\eta \in \mathbb{R}^N$, we have the following inequalities (see [9]) from which we can get the strict monotonicity of Φ':

$$\left[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta)\right]\cdot(|\xi|+|\eta|)^{2-p} \geqslant (p-1)\,|\xi-\eta|^2, 1<p<2, \tag{2.2}$$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \geqslant (\frac{1}{2})^p\,|\xi-\eta|^p, p \geqslant 2. \tag{2.3}$$

ii) From i), if $u_n \rightharpoonup u$ and $\overline{\lim}_{n\to+\infty}(\Phi'(u_n,v_n) - \Phi'(u,v),(u_n-u,v_n-v)) \leqslant 0$, then

$$\lim_{n\to+\infty}(\Phi'(u_n,v_n) - \Phi'(u,v),(u_n-u,v_n-v)) = 0. \tag{2.4}$$

We claim that $\nabla u_n(x) \to \nabla u(x)$ in measure.

Denote

$$\begin{aligned}
U &= \{x \in \Omega \mid p(x) \geqslant 2\}, V = \{x \in \Omega \mid 1 < p(x) < 2\},\\
V_n^- &= \{x \in V \mid |\nabla u_n| + |\nabla u| < 1\}, V_n^+ = \{x \in V \mid |\nabla u_n| + |\nabla u| \geqslant 1\},\\
\Phi_n &= (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_n - \nabla u), \Psi_n = (|\nabla u_n| + |\nabla u|).
\end{aligned}$$

From (2.3), we have

$$\int_U |\nabla u_n - \nabla u|^{p(x)}\, dx \leqslant 2^{p^+} \int_U \Phi_n dx \leqslant 2^{p^+} \int_\Omega \Phi_n dx \to 0. \tag{2.5}$$

In view of (2.2), we have

$$\int_{V_n^-} |\nabla u_n - \nabla u|^2\, dx \leqslant \int_{V_n^-} \frac{1}{p(x)-1} \Phi_n dx \leqslant \frac{1}{p^- - 1} \int_\Omega \Phi_n dx \to 0. \tag{2.6}$$

Without loss of generality, we may assume that $0 < \int_{V_n^+} \Phi_n dx < 1$. Then we have

$$\begin{aligned}
\int_{V_n^+} |\nabla u_n - \nabla u|^{p(x)}\, dx &\leqslant C \int_{V_n^+} \Phi_n^{p(x)/2} \Psi_n^{(2-p(x))p(x)/2} dx\\
&= C\Big(\int_{V_n^+} \Phi_n dx\Big)^{1/2} \int_{V_n^+} \Big(\int_{V_n^+} \Phi_n dx\Big)^{-1/2} \Phi_n^{p(x)/2} \Psi_n^{(2-p(x))p(x)/2} dx\\
&\leqslant C\Big(\int_{V_n^+} \Phi_n dx\Big)^{1/2} \int_{V_n^+} \Big[\Big(\int_{V_n^+} \Phi_n dx\Big)^{-1/p(x)} \Phi_n + \Psi_n^{p(x)}\Big] dx\\
&\leqslant C\Big(\int_{V_n^+} \Phi_n dx\Big)^{1/2} \Big[1 + \int_{V_n^+} \Psi_n^{p(x)} dx\Big]\\
&= C\Big(\int_{V_n^+} \Phi_n dx\Big)^{1/2} \Big[1 + \int_{V_n^+} \big(|\nabla u_n| + |\nabla u|\big)^{p(x)} dx\Big].
\end{aligned}$$

From (2.4) and the bounded property of $\{u_n\}$ in X, we have

$$\int_{V_n^+} |\nabla u_n - \nabla u|^{p(x)}\, dx \leqslant \Big(\int_{V_n^+} \Phi_n dx\Big)^{1/2} \Big[1 + \int_{V_n^+} \big(|\nabla u_n| + |\nabla u|\big)^{p(x)} dx\Big] \to 0. \tag{2.7}$$

Thus $\{\nabla u_n\}$ converges in measure to ∇u in Ω, so we have by Egorov's Theorem that $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \Omega$ up to a subsequence. Consequently, we obtain, by Fatou's Lemma, that

$$\varliminf_{n\to+\infty} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx \geqslant \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx. \tag{2.8}$$

Similarly, we have

$$\varliminf_{n\to+\infty} \int_\Omega \frac{1}{q(x)} |\nabla v_n|^{q(x)}\, dx \geqslant \int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)}\, dx. \tag{2.9}$$

From $(u_n, v_n) \rightharpoonup (u, v)$ in X, we have

$$\lim_{n\to+\infty}(\Phi'(u_n,v_n),(u_n-u,v_n-v)) = \lim_{n\to+\infty}(\Phi'(u_n,v_n)-\Phi'(u,v),(u_n-u,v_n-v)) = 0. \tag{2.10}$$

We also have

$$\begin{aligned}
&(\Phi'(u_n,v_n),(u_n-u,v_n-v))\\
= &\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega |\nabla u_n|^{p(x)-2}\,\nabla u_n \nabla u dx\\
&+\int_\Omega |\nabla v_n|^{q(x)}\,dx - \int_\Omega |\nabla v_n|^{q(x)-2}\,\nabla v_n \nabla v dx\\
\geqslant &\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega |\nabla u_n|^{p(x)-1}\,|\nabla u|\,dx\\
&+\int_\Omega |\nabla v_n|^{q(x)}\,dx - \int_\Omega |\nabla v_n|^{q(x)-1}\,|\nabla v|\,dx\\
\geqslant &\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega \Big(\frac{p(x)-1}{p(x)}\,|\nabla u_n|^{p(x)} + \frac{1}{p(x)}\,|\nabla u|^{p(x)}\Big)dx\\
&+\int_\Omega |\nabla v_n|^{q(x)}\,dx - \int_\Omega \Big(\frac{q(x)-1}{q(x)}\,|\nabla v_n|^{q(x)} + \frac{1}{q(x)}\,|\nabla v|^{q(x)}\Big)dx
\end{aligned}$$

$$\begin{aligned}
\geqslant &\int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}\,|\nabla v_n|^{q(x)}\,dx\\
&-\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx.
\end{aligned} \tag{2.11}$$

According to (2.8)-(2.11), we obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\,dx = \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx. \tag{2.12}$$

It follows from (2.12) that the integrals of the functions family $\left\{\frac{1}{p(x)}\,|\nabla u_n|^{p(x)}\right\}$ possess absolute equicontinuity on Ω (see [22], Ch.6, Section 3, Corollary 1, Theorem 4-5). Since

$$\frac{1}{p(x)}\,|\nabla u_n(x) - \nabla u(x)|^{p(x)} \leqslant C(\frac{1}{p(x)}\,|\nabla u_n(x)|^{p(x)} + \frac{1}{p(x)}\,|\nabla u(x)|^{p(x)}), \tag{2.13}$$

the integrals of the family $\left\{\frac{1}{p(x)}\,|\nabla u_n(x) - \nabla u(x)|^{p(x)}\right\}$ is also absolutely equicontinuous on Ω (see [22], Ch.6, Section3, Theorem 2) and therefore

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\,|\nabla u_n(x) - \nabla u(x)|^{p(x)}\,dx = 0. \tag{2.14}$$

By (2.14), we conclude that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n(x) - \nabla u(x)|^{p(x)}\,dx = 0. \tag{2.15}$$

From Proposition 2.4 and (2.15), we have $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. Similarly, we have $v_n \to v$ in $W_0^{1,q(\cdot)}(\Omega)$. Therefore, $(u_n, v_n) \to (u, v)$ in X. Thus, Φ' is of type (S_+).

iii) By the strict monotonicity, Φ' is an injection. Since

$$\lim_{\|(u,v)\|\to+\infty} \frac{(\Phi'(u,v),(u,v))}{\|(u,v)\|} = \lim_{\|(u,v)\|\to+\infty} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |\nabla v|^{q(x)}\,dx}{\|(u,v)\|} = +\infty,$$

Φ' is coercive, thus Φ' is a surjection in view of Minty-Browder's Theorem (see [34], Th.26A). Hence Φ' has an inverse mapping $(\Phi')^{-1} : X^* \to X$. Therefore, the continuity of $(\Phi')^{-1}$ is sufficient to ensure Φ' to be a homeomorphism.

If $f_n,\ f \in X^*$, $f_n \to f$ in X^*, let $(u_n, v_n) = (\Phi')^{-1}(f_n)$, $(u, v) = (\Phi')^{-1}(f)$, then $\Phi'(u_n, v_n) = f_n$, $\Phi'(u, v) = f$. So $\{(u_n, v_n)\}$ is bounded in X. Without loss of generality, we can assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in X. Since $f_n \to f$ in X^*, we have

$$\lim_{n\to+\infty} (\Phi'((u_n, v_n)) - \Phi'(u_0, v_0), (u_n - u_0, v_n - v_0) = \lim_{n\to+\infty} (f_n, (u_n - u_0, v_n - v_0)) = 0. \tag{2.16}$$

Since Φ' is of type (S_+), $(u_n, v_n) \to (u_0, v_0)$, we conclude that $(u_n, v_n) \to (u, v)$ in X, so $(\Phi')^{-1}$ is continuous. The proof of Proposition 2.7 is complete. □

Denote $B(x_0, \varepsilon, \delta, \theta) = \{x \in \mathbb{R}^N \mid \delta \leqslant |x - x_0| \leqslant \varepsilon, \frac{x-x_0}{|x-x_0|} \cdot \frac{\nabla p(x_0)}{|\nabla p(x_0)|} \geqslant \cos\theta\}$, where $\theta \in (0, \frac{\pi}{2})$. Then we have

Lemma 2.9. *If $p \in C^1(\overline{\Omega})$, $x_0 \in \Omega$ satisfy $\nabla p(x_0) \neq 0$, then there exist a positive ε small enough such that*

$$(x - x_0) \cdot \nabla p(x) > 0, \forall x \in B(x_0, \varepsilon, \delta, \theta), \tag{2.17}$$

and

$$\max\{p(x) \mid x \in \overline{B(x_0, \varepsilon)}\} = \max\{p(x) \mid x \in B(x_0, \varepsilon, \varepsilon, \theta)\}. \tag{2.18}$$

Proof. Since $p \in C^1(\overline{\Omega})$, for any $x \in B(x_0, \varepsilon, \delta, \theta)$, when ε is small enough, it is easy to see that

$$\begin{aligned} \nabla p(x) \cdot (x - x_0) &= (\nabla p(x_0) + o(1)) \cdot (x - x_0) \\ &= \nabla p(x_0) \cdot (x - x_0) + o(|x - x_0|) \\ &\geqslant |\nabla p(x_0)|\, |x - x_0| \cos\theta + o(|x - x_0|) > 0, \end{aligned}$$

where $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ uniformly as $|x - x_0| \to 0$.

When ε is small enough, (2.17) is valid. Since $p \in C^1(\overline{\Omega})$, there exist a small enough positive ε such that

$$p(x) - p(x_0) = \nabla p(y) \cdot (x - x_0) = (\nabla p(x_0) + o(1)) \cdot (x - x_0),$$

where $y = x_0 + \tau(x - x_0)$ and $\tau \in (0, 1)$, $o(1) \in \mathbb{R}^N$ is a function and $o(1) \to 0$ uniformly as $|x - x_0| \to 0$.

Suppose $x \in \overline{B(x_0, \varepsilon)} \backslash B(x_0, \varepsilon, \delta, \theta)$. Denote $x^* = x_0 + \varepsilon \nabla p(x_0) / |\nabla p(x_0)|$.

Suppose $\frac{x-x_0}{|x-x_0|}\cdot\frac{\nabla p(x_0)}{|\nabla p(x_0)|}<\cos\theta$. When ε is small enough, we have

$$\begin{aligned} p(x)-p(x_0) &= (\nabla p(x_0)+o(1))\cdot(x-x_0) \\ &< |\nabla p(x_0)|\,|x-x_0|\cos\theta+o(\varepsilon) \\ &\leqslant (\nabla p(x_0)+o(1))\cdot\varepsilon\nabla p(x_0)/\,|\nabla p(x_0)| \\ &= p(x^*)-p(x_0), \end{aligned}$$

where $o(1)\in\mathbb{R}^N$ is a function and $o(1)\to 0$ as $\varepsilon\to 0$.

Suppose $|x-x_0|<\delta$. When ε is small enough, we have

$$\begin{aligned} p(x)-p(x_0) &= (\nabla p(x_0)+o(1))\cdot(x-x_0) \\ &\leqslant |\nabla p(x_0)|\,|x-x_0|+o(\varepsilon) \\ &< (\nabla p(x_0)+o(1))\cdot\varepsilon\nabla p(x_0)/\,|\nabla p(x_0)| \\ &= p(x^*)-p(x_0), \end{aligned}$$

where $o(1)\in\mathbb{R}^N$ is a function and $o(1)\to 0$ as $\varepsilon\to 0$. Thus

$$\max\{p(x)\mid x\in\overline{B(x_0,\varepsilon)}\}=\max\{p(x)\mid x\in B(x_0,\varepsilon,\delta,\theta)\}. \tag{2.19}$$

It follows from (2.17) and (2.19) that (2.18) is valid. Proof of Lemma 2.9 is complete. □

Lemma 2.10. *Suppose that $F(x,u,v)$ satisfies the following inequality*

$C_1|u|^{p(x)}[\ln(e+|u|)]^{a(x)}+C_1|v|^{q(x)}[\ln(e+|v|)]^{b(x)}\leqslant F(x,u,v),\forall\,|u|+|v|\geqslant M,\forall x\in\Omega,$

where $a(\cdot)>p(\cdot)$ and $b(\cdot)>q(\cdot)$ on $\overline{\Omega}$, and $x_1,x_2\in\Omega$ are two different points such that $\nabla p(x_1)\neq 0$ and $\nabla q(x_2)\neq 0$. Let

$$h_1(x)=\begin{cases} 0, & |x-x_1|>\varepsilon \\ \varepsilon-|x-x_1|, & |x-x_1|\leqslant\varepsilon \end{cases},$$

and

$$h_2(x)=\begin{cases} 0, & |x-x_2|>\varepsilon \\ \varepsilon-|x-x_2|, & |x-x_2|\leqslant\varepsilon \end{cases},$$

where ε is as defined in Lemma 2.9 and small enough such that $\varepsilon<|x_2-x_1|$. Then there holds

$$\int_\Omega|\nabla t\,h_1|^{p(x)}dx+\int_\Omega|\nabla t\,h_2|^{q(x)}dx-\int_\Omega\lambda\,|t\,h_1|^{\alpha(x)}\,|t\,h_2|^{\beta(x)}+F(x,t\,h_1,t\,h_2)dx\to-\infty,$$

as $t\to+\infty$.

Proof. According to $(H_{\alpha,\beta})$, there is a constant $\theta\in(0,1)$ such that

$$\frac{\alpha(x)}{\theta p(x)}+\frac{\beta(x)}{\theta q(x)}\leqslant 1,\forall x\in\overline{\Omega}.$$

Therefore, we have

$$|u|^{\alpha(x)}\,|v|^{\beta(x)}\leqslant\frac{\alpha(x)}{\theta p(x)}\,|u|^{\alpha(x)\frac{\theta p(x)}{\alpha(x)}}+(\frac{\theta p(x)}{\alpha(x)})^0\,|v|^{\beta(x)(\frac{\theta p(x)}{\alpha(x)})^0}\leqslant|u|^{\theta p(x)}+|v|^{\theta q(x)}+1.$$

To complete the proof of this lemma, it is sufficient to show that

$$G(th_1) := \int_\Omega \frac{1}{p(x)} |\nabla th_1|^{p(x)} - \int_\Omega C_1 |th_1|^{p(x)} [\ln(e+|th_1|)]^{a(x)} dx \to -\infty \text{ as } t \to +\infty.$$

It is easy to see the following two inequalities hold:

$$\int_\Omega \frac{1}{p(x)} |\nabla th_1|^{p(x)} dx \leqslant C_2 \int_{B(x_0,\varepsilon,\delta,\theta)} |\nabla th_1|^{p(x)} dx,$$

$$\int_\Omega C_1 |th_1|^{p(x)} [\ln(e+|th_1|)]^{a(x)} dx \geqslant \int_{B(x_0,\varepsilon,\delta,\theta)} C_1 |th_1|^{p(x)} [\ln(e+|th_1|)]^{a(x)} dx.$$

To proceed, we shall use polar coordinates. Let $r = |x - x_0|$. Since $p \in C^1(\overline{\Omega})$, it follows from (2.17) that there exist positive constants c_1 and c_2 such that

$$p(\varepsilon,\omega) - c_2(\varepsilon - r) \leqslant p(r,\omega) \leqslant p(\varepsilon,\omega) - c_1(\varepsilon - r), \forall (r,\omega) \in B(x_0,\varepsilon,\delta,\theta).$$

Therefore, we have

$$\begin{aligned}
\int_{B(x_0,\varepsilon,\delta,\theta)} |\nabla th_1|^{p(x)} dx &= \int_{B(x_0,\varepsilon,\delta,\theta)} t^{p(r,\omega)} r^{N-1} dr d\omega \\
&\leqslant \int_{B(x_0,\varepsilon,\delta,\theta)} t^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} r^{N-1} dr d\omega \\
&\leqslant \varepsilon^{N-1} \int_{B(x_0,\varepsilon,\delta,\theta)} t^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} dr d\omega \\
&\leqslant \varepsilon^{N-1} \int_{B(x_0,1,1,\theta)} \frac{t^{p(\varepsilon,\omega)}}{c_1 \ln t} d\omega. \qquad (2.20)
\end{aligned}$$

Since $p \in C^1(\overline{\Omega})$ and $a(\cdot) > p(\cdot)$ on $\overline{\Omega}$, we conclude that, for ε small enough, there exists a $\epsilon_1 > 0$ such that

$$a(x) \geqslant \max\{p(x) + \epsilon_1 \mid x \in B(x_0,\varepsilon,\delta,\theta)\}.$$

Thus, when t is large enough, we have

$$\begin{aligned}
&\int_{B(x_0,\varepsilon,\delta,\theta)} C_1 |th_1|^{p(x)} [\ln(e+|th_1|)]^{a(x)} dx \\
= \ & \int_{B(x_0,\varepsilon,\delta,\theta)} C_1 |t(\varepsilon-r)|^{p(r,\omega)} r^{N-1} [\ln(e+|t(\varepsilon-r)|)]^{a(r,\omega)} dr d\omega \\
\geqslant \ & C_1\delta^{N-1} \int_{B(x_0,\varepsilon,\delta,\theta)} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} |\varepsilon-r|^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} [\ln(e+|t(\varepsilon-r)|)]^{a(r,\omega)} dr d\omega \\
\geqslant \ & C_1\delta^{N-1} \int_{B(x_0,1,1,\theta)} d\omega \int_{\delta}^{\varepsilon-\frac{1}{\ln t}} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} |\varepsilon-r|^{p(\varepsilon,\omega)-c_1(\varepsilon-r)} [\ln(e+|t(\varepsilon-r)|)]^{a(r,\omega)} dr \\
\geqslant \ & C_3\delta^{N-1} \int_{B(x_0,1,1,\theta)} (\frac{1}{\ln t})^{p(\varepsilon,\omega)} [\ln(e+\frac{t}{\ln t})]^{p(\varepsilon,\omega)+\epsilon_1} \int_{\delta}^{\varepsilon-\frac{1}{\ln t}} |t|^{p(\varepsilon,\omega)-c_2(\varepsilon-r)} dr d\omega \\
\geqslant \ & C_4\delta^{N-1} \int_{B(x_0,1,1,\theta)} (\ln t)^{\epsilon_1} \frac{|t|^{p(\varepsilon,\omega)-\frac{c_2}{\ln t}}}{c_2 \ln t} d\omega \\
\geqslant \ & (\ln t)^{\epsilon_1} C_5 \delta^{N-1} \int_{B(x_0,1,1,\theta)} \frac{|t|^{p(\varepsilon,\omega)}}{c_2 \ln t} d\omega.
\end{aligned}$$

Hence, we have

$$\int_{B(x_0,\varepsilon,\delta,\theta)} C_1 |th_1|^{p(x)} [\ln(e+|t\,h_1|)]^{a(x)} dx \geqslant (\ln t)^{\epsilon_1} C_5 \int_{B(x_0,1,1,\theta)} \frac{|t|^{p(\varepsilon,\omega)}}{\ln t} d\omega \text{ as } t \to +\infty. \tag{2.21}$$

It follows from (2.20) and (2.21) that $G(t\,h_1) \to -\infty$. The proof of Lemma 2.10 is complete. $\square$

We have the following simple proposition concerning the growth rate of the nonlinearity $F(\cdot,\cdot,\cdot)$.

Proposition 2.11. *(see* [15]*) (i) If F satisfies*

$$0 < F(x,s,t) \leqslant \frac{1}{\theta_1} sF_s(x,s,t) + \frac{1}{\theta_2} tF_t(x,s,t) \text{ for } x \in \overline{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geqslant 2M,$$

then $F(x,s,t) \geqslant c_1[(|s|^{\theta_1} + |t|^{\theta_2}) - 1], \forall (x,s,t) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}.$

3. Proofs of main results

With the preparations in the last section, we will in this section give our proofs of the main results. To be rigorous, we first give the definition of a weak solution to the problem (P).

Definition 3.1. (i) We call $(u,v) \in X$ is a weak solution of (P) if

$$\begin{aligned}
\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi dx &= \int_\Omega (\lambda \alpha(x) |u|^{\alpha(x)-2} u |v|^{\beta(x)} + F_u(x,u,v))\phi dx, \forall \phi \in W_0^{1,p(\cdot)}(\Omega), \\
\int_\Omega |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \psi dx &= \int_\Omega (\lambda \beta(x) |u|^{\alpha(x)} |v|^{\beta(x)-2} v + F_v(x,u,v))\psi dx, \forall \psi \in W_0^{1,q(\cdot)}(\Omega).
\end{aligned}$$

The corresponding functional of (P) is given by $\varphi = \varphi(u,v)$ defined below on X:

$$\begin{aligned}
\varphi(u,v) &= \Phi(u,v) - \Psi(u,v) \\
&= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla v|^{q(x)} dx - \int_\Omega \lambda |u|^{\alpha(x)} |v|^{\beta(x)} + F(x,u,v) dx, \forall (u,v) \in X.
\end{aligned}$$

As compactness is crucial in showing the existence of weak solutions via critical point theory. We shall introduce the type of compactness which we shall use in the current study, i.e., the Cerami compactness condition.

Definition 3.2. We say φ satisfies Cerami condition in X, if any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n, v_n)\}$ is bounded and $\|\varphi'(u_n,v_n)\|(1 + \|(u_n,v_n)\|) \to 0$ as $n \to +\infty$ has a convergent subsequence.

It is well-known that the Cerami condition is weaker than the usual Palais-Samle condition. Under our new growth condition for the system under investigation, we manage to show that the corresponding functional φ satisfies the above Cerami type compactness condition which is sufficient to yield critical points. More specifically, we have the following lemma.

Lemma 3.3. *If $(H_{\alpha,\beta})$, (H_0) and (H_1) hold, then φ satisfies the Cerami condition.*

Proof. Let $\{(u_n,v_n)\} \subset X$ be a Cerami sequence such that $\varphi(u_n,v_n) \to c$. From Definition 3.2, we know that $\|\varphi'(u_n,v_n)\|(1 + \|(u_n,v_n)\|) \to 0$ as $n \to +\infty$. We first *claim* that to show φ satisfies the Cerami condition, it is sufficient to show that the Cerami sequence $\{(u_n,v_n)\}$ is bounded in X. Indeed, suppose $\{(u_n,v_n)\}$ is bounded, then $\{(u_n,v_n)\}$ admits a weakly convergent subsequence in X. Without loss of generality, we assume that $(u_n,v_n) \rightharpoonup (u,v)$ in X, then $\Psi'(u_n,v_n) \to \Psi'(u,v)$ in X^*. Since $\varphi'(u_n,v_n) = \Phi'(u_n,v_n) - \Psi'(u_n,v_n) \to 0$ in X^*, we have $\Phi'(u_n,v_n) \to \Psi'(u,v)$ in X^*. Since Φ' is a homeomorphism, we have $(u_n,v_n) \to (u,v)$, hence φ satisfies Cerami condition. Therefore, our claim holds.

Now we show that each Cerami sequence $\{(u_n,v_n)\}$ is bounded in X. We argue by contradiction. Suppose not, then up to a subsequence (still denoted by $\{(u_n,v_n)\}$), we have

$$\varphi(u_n,v_n) \to c, \ \|\varphi'(u_n,v_n)\|(1 + \|(u_n,v_n)\|) \to 0, \ \|(u_n,v_n)\| \to +\infty. \tag{3.1}$$

Obviously, we have

$$\begin{aligned}
\left|\frac{1}{p(x)}u_n\right|_{p(\cdot)} &\leqslant \frac{1}{p^-}\left|u_n\right|_{p(\cdot)}, \left|\nabla\frac{1}{p(x)}u_n\right|_{p(\cdot)} \leqslant \frac{1}{p^-}\left|\nabla u_n\right|_{p(\cdot)} + C\left|u_n\right|_{p(\cdot)}, \\
\left|\frac{1}{q(x)}v_n\right|_{q(\cdot)} &\leqslant \frac{1}{q^-}\left|v_n\right|_{q(\cdot)}, \left|\nabla\frac{1}{q(x)}v_n\right|_{q(\cdot)} \leqslant \frac{1}{q^-}\left|\nabla v_n\right|_{q(\cdot)} + C\left|v_n\right|_{q(\cdot)}.
\end{aligned}$$

From the above inequalities, we easily see that $\left\|(\frac{1}{p(x)}u_n, \frac{1}{q(x)}v_n)\right\| \leqslant C\left\|(u_n, v_n)\right\|$. Therefore, we have $(\varphi'(u_n, v_n), (\frac{1}{p(x)}u_n, \frac{1}{q(x)}v_n)) \to 0$. We may assume that

$$\begin{aligned}
c+1 &\geqslant \varphi(u_n, v_n) - (\varphi'(u_n, v_n), (\frac{1}{p(x)}u_n, \frac{1}{q(x)}v_n)) \\
&= \int_\Omega \left(\frac{1}{p(x)}\left|\nabla u_n\right|^{p(x)} + \frac{1}{q(x)}\left|\nabla v_n\right|^{q(x)}\right)dx - \int_\Omega F(x, u_n, v_n)dx \\
&\quad -\{\int_\Omega \frac{1}{p(x)}\left|\nabla u_n\right|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}F_u(x, u_n, v_n)u_n dx - \int_\Omega \frac{1}{p^2(x)}u_n\left|\nabla u_n\right|^{p(x)-2}\nabla u_n \nabla p dx\} \\
&\quad -\{\int_\Omega \frac{1}{q(x)}\left|\nabla v_n\right|^{q(x)}dx - \int_\Omega \frac{1}{q(x)}F_v(x, u_n, v_n)v_n dx - \int_\Omega \frac{1}{q^2(x)}v_n\left|\nabla v_n\right|^{q(x)-2}\nabla v_n \nabla q dx\} \\
&\quad +\int_\Omega \lambda(\frac{\alpha(x)}{p(x)} + \frac{\beta(x)}{q(x)} - 1)\left|u_n\right|^{\alpha(x)}\left|v_n\right|^{\beta(x)}dx \\
&= \int_\Omega \frac{1}{p^2(x)}u_n\left|\nabla u_n\right|^{p(x)-2}\nabla u_n \nabla p dx + \int_\Omega \frac{1}{q^2(x)}v_n\left|\nabla v_n\right|^{q(x)-2}\nabla v_n \nabla q dx \\
&\quad +\int_\Omega \{\frac{1}{p(x)}F_u(x, u_n, v_n)u_n + \frac{1}{q(x)}F_v(x, u_n, v_n)v_n dx - F(x, u_n, v_n)\}dx \\
&\quad +\int_\Omega \lambda(\frac{\alpha(x)}{p(x)} + \frac{\beta(x)}{q(x)} - 1)\left|u_n\right|^{\alpha(x)}\left|v_n\right|^{\beta(x)}dx.
\end{aligned}$$

Hence, there holds

$$\begin{aligned}
&\int_\Omega \{\frac{1}{p(x)}F_u(x, u_n, v_n)u_n + \frac{1}{q(x)}F_v(x, u_n, v_n)v_n dx - F(x, u_n, v_n)\}dx \\
\leqslant\ & C_1(\int_\Omega \left|u_n\right|\left|\nabla u_n\right|^{p(x)-1}dx + \int_\Omega \left|v_n\right|\left|\nabla v_n\right|^{q(x)-1}dx + 1) \\
&+\int_\Omega \lambda(1 - \frac{\alpha(x)}{p(x)} - \frac{\beta(x)}{q(x)})\left|u_n\right|^{\alpha(x)}\left|v_n\right|^{\beta(x)}dx \\
\leqslant\ & \sigma\int_\Omega \frac{\left|\nabla u_n\right|^{p(x)}}{\ln(e+\left|u_n\right|)} + \frac{\left|\nabla v_n\right|^{q(x)}}{\ln(e+\left|v_n\right|)}dx + \int_\Omega \lambda(1 - \frac{\alpha(x)}{p(x)} - \frac{\beta(x)}{q(x)})\left|u_n\right|^{\alpha(x)}\left|v_n\right|^{\beta(x)}dx \\
&+C(\sigma)\int_\Omega \left|u_n\right|^{p(x)}\left[\ln(e+\left|u_n\right|)\right]^{p(x)-1} + \left|v_n\right|^{q(x)}\left[\ln(e+\left|v_n\right|)\right]^{q(x)-1}dx + C_1, \quad (3.2)
\end{aligned}$$

where $\sigma > 0$ is a sufficiently small constant.

Note that $\frac{u_n}{\ln(e+|u_n|)} \in W_0^{1,p(\cdot)}(\Omega)$, and $\left\|\frac{u_n}{\ln(e+|u_n|)}\right\|_{p(\cdot)} \leqslant C_2 \|u_n\|_{p(\cdot)}$. Choosing $\frac{u_n}{\ln(e+|u_n|)}$ as a test function, we have

$$\begin{aligned}
&\int_\Omega F_u(x,u_n,v_n)\frac{u_n}{\ln(e+|u_n|)} + \lambda \frac{\alpha(x)}{\ln(e+|u_n|)} |u_n|^{\alpha(x)} |v_n|^{\beta(x)} dx \\
= &\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \frac{u_n}{\ln(e+|u_n|)} dx + o(1) \\
= &\int_\Omega \frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)} dx - \int_\Omega \frac{|u_n| |\nabla u_n|^{p(x)}}{(e+|u_n|)[\ln(e+|u_n|)]^2} dx + o(1).
\end{aligned}$$

It is easy to check that $\frac{|u_n||\nabla u_n|^{p(x)}}{(e+|u_n|)[\ln(e+|u_n|)]^2} \leqslant \frac{1}{2}\frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)}$. Therefore, we have

$$\begin{aligned}
C_3 \int_\Omega \frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)} dx - C_4 \ &\leqslant \ \int_\Omega \frac{F_u(x,u_n,v_n)u_n}{\ln(e+|u_n|)} + \lambda \frac{\alpha(x)}{\ln(e+|u_n|)} |u_n|^{\alpha(x)} |v_n|^{\beta(x)} dx \\
&\leqslant \ C_5 \int_\Omega \frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)} dx + C_6. \qquad (3.3)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
C_7 \int_\Omega \frac{|\nabla v_n|^{q(x)}}{\ln(e+|v_n|)} dx - C_8 \ &\leqslant \ \int_\Omega \frac{F_v(x,u_n,v_n)v_n}{\ln(e+|v_n|)} + \lambda \frac{\beta(x)}{\ln(e+|v_n|)} |u_n|^{\alpha(x)} |v_n|^{\beta(x)} dx \\
&\leqslant \ C_9 \int_\Omega \frac{|\nabla v_n|^{q(x)}}{\ln(e+|v_n|)} dx + C_{10}. \qquad (3.4)
\end{aligned}$$

From (3.2), (3.3), (3.4) and condition (H_1), we obtain that

$$\begin{aligned}
&\int_\Omega \frac{F_u(x,u_n,v_n)u_n}{\ln(e+|u_n|)} + \frac{F_v(x,u_n,v_n)v_n}{\ln(e+|v_n|)} \\
&\overset{(\mathrm{H}_1)}{\leqslant} C_7 \int_\Omega \{\frac{F_u(x,u_n,v_n)u_n}{p(x)} + \frac{F_v(x,u_n,v_n)v_n}{q(x)} - F(x,u_n)\} dx \\
\leqslant \ & C_7\{\sigma \int_\Omega \frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)} + \frac{|\nabla v_n|^{q(x)}}{\ln(e+|v_n|)} dx + \int_\Omega \lambda(1 - \frac{\alpha(x)}{p(x)} - \frac{\beta(x)}{q(x)}) |u_n|^{\alpha(x)} |v_n|^{\beta(x)} dx + C_8 \\
& + C(\sigma) \int_\Omega |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} + |v_n|^{q(x)} [\ln(e+|v_n|)]^{q(x)-1} dx\} \\
\leqslant \ & C_7\sigma \int_\Omega \frac{|\nabla u_n|^{p(x)}}{\ln(e+|u_n|)} + \frac{|\nabla v_n|^{q(x)}}{\ln(e+|v_n|)} dx + C_7 \int_\Omega \lambda(1 - \frac{\alpha(x)}{p(x)} - \frac{\beta(x)}{q(x)}) |u_n|^{\alpha(x)} |v_n|^{\beta(x)} dx \\
& + C_7 C(\sigma) \int_\Omega |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} + |v_n|^{q(x)} [\ln(e+|v_n|)]^{q(x)-1} dx + C_9 \\
\leqslant \ & \frac{1}{2} \int_\Omega \frac{F_u(x,u_n,v_n)u_n}{\ln(e+|u_n|)} + \frac{F_v(x,u_n,v_n)v_n}{\ln(e+|v_n|)} dx + C_7 \int_\Omega \lambda(1 - \frac{\alpha(x)}{p(x)} - \frac{\beta(x)}{q(x)}) |u_n|^{\alpha} |v_n|^{\beta} dx \\
& + C_7 C(\sigma) \int_\Omega |u_n|^{p(x)} [\ln(e+|u_n|)]^{p(x)-1} + |v_n|^{q(x)} [\ln(e+|v_n|)]^{q(x)-1} dx + C_{10}.
\end{aligned}$$

In view of the assumptions $(H_{\alpha,\beta})$, (H_1) and the above inequality, we conclude that

$$\begin{aligned}
&\int_\Omega |u_n|^{p(x)}\left[\ln(e+|u_n|)\right]^{a(x)-1}+|v_n|^{q(x)}\left[\ln(e+|v_n|)\right]^{b(x)-1}dx\\
\leqslant\ & C_{11}\int_\Omega F_u(x,u_n,v_n)\frac{u_n}{\ln(e+|u_n|)}+F_v(x,u_n,v_n)\frac{v_n}{\ln(e+|v_n|)}dx\\
\leqslant\ & C_{12}\int_\Omega |u_n|^{p(x)}\left[\ln(e+|u_n|)\right]^{p(x)-1}+|v_n|^{q(x)}\left[\ln(e+|v_n|)\right]^{q(x)-1}dx+C_{12}.
\end{aligned}$$

Noticing that $a(\cdot)>p(\cdot)$ and $b(\cdot)>q(\cdot)$ on $\overline{\Omega}$, we can conclude that

$$\left\{\int_\Omega |u_n|^{p(x)}\left[\ln(e+|u_n|)\right]^{a(x)-1}+|v_n|^{q(x)}\left[\ln(e+|v_n|)\right]^{b(x)-1}dx\right\}$$

is bounded, which further yields that

$$\left\{\int_\Omega F_u(x,u_n,v_n)\frac{u_n}{\ln(e+|u_n|)}+F_v(x,u_n,v_n)\frac{v_n}{\ln(e+|v_n|)}dx\right\}$$

and

$$\left\{\int_\Omega \lambda(\alpha(x)+\beta(x))\,|u_n|^{\alpha(x)}\,|v_n|^{\beta(x)}\,dx\right\}$$

are bounded. Now, it is easy to see that $\left\{\int_\Omega \frac{|F_u(x,u_n,v_n)u_n|}{\ln(e+|u_n|)}+\frac{|F_v(x,u_n,v_n)v_n|}{\ln(e+|v_n|)}dx\right\}$ is bounded.

Let $\varepsilon>0$ satisfy $\varepsilon<\min\{1,p^--1,q^--1,\frac{1}{p^{*+}},\frac{1}{q^{*+}},(\frac{p^*}{\gamma})^--1,(\frac{q^*}{\delta})^--1\}$. Since $\|\varphi'(u_n,v_n)\|\,\|(u_n,v_n)\|\to 0$, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)}+|\nabla v_n|^{q(x)}\,dx\\
=\ & \int_\Omega F_u(x,u_n,v_n)u_n+F_v(x,u_n,v_n)v_ndx+\int_\Omega \lambda(\alpha(x)+\beta(x))\,|u_n|^{\alpha(x)}\,|v_n|^{\beta(x)}\,dx+o(1)\\
=\ & \int_\Omega |F_u(x,u_n,v_n)u_n|^{\varepsilon}\left[\ln(e+|u_n|)\right]^{1-\varepsilon}\left|F_u(x,u_n,v_n)\frac{u_n}{\ln(e+|u_n|)}\right|^{1-\varepsilon}dx\\
&+\int_\Omega |F_v(x,u_n,v_n)v_n|^{\varepsilon}\left[\ln(e+|v_n|)\right]^{1-\varepsilon}\left|F_v(x,u_n,v_n)\frac{v_n}{\ln(e+|v_n|)}\right|^{1-\varepsilon}dx+C\\
\leqslant\ & C_9(1+\|u_n\|)^{1+\varepsilon}\int_\Omega\left[\frac{[|F_u(x,u_n,v_n)u_n|]^{1+\varepsilon}+C_{10}}{(1+\|(u_n,v_n)\|)^{\frac{1+\varepsilon}{\varepsilon}}}\right]^{\varepsilon}\left[\frac{|F_u(x,u_n,v_n)u_n|}{\ln(e+|u_n|)}\right]^{1-\varepsilon}dx\\
&+C_9(1+\|v_n\|)^{1+\varepsilon}\int_\Omega\left[\frac{[|F_v(x,u_n,v_n)v_n|]^{1+\varepsilon}+C_{10}}{(1+\|(u_n,v_n)\|)^{\frac{1+\varepsilon}{\varepsilon}}}\right]^{\varepsilon}\left[\frac{|F_v(x,u_n,v_n)v_n|}{\ln(e+|v_n|)}\right]^{1-\varepsilon}dx+C\\
\leqslant\ & C_{11}(1+\|(u_n,v_n)\|)^{1+\varepsilon}+C_{12}.
\end{aligned}$$

The above inequality contradicts with (3.1). Therefore, we can conclude that $\{(u_n,v_n)\}$ is bounded, and the proof of Lemma 3.3 is complete. $\square$

Now we are in a position to give a proof of Theorem 1.1.

Denote $F^{++}(x,u,v) = F(x,S(u),S(v))$, where $S(t) = \max\{0,t\}$. For any $(u,v) \in X$, we say (u,v) belong to the first, the second, the third or the fourth quadrant of X, if $u \geqslant 0$ and $v \geqslant 0$, $u \leqslant 0$ and $v \geqslant 0$, $u \leqslant 0$ and $v \leqslant 0$, $u \geqslant 0$ and $v \leqslant 0$, respectively.

Proof of Theorem 1.1.

It is easy to check that $F^{++}(x,s,t) \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, and

$$F_u^{++}(x,u,v) = F_u(x,S(u),S(v)), F_v^{++}(x,u,v) = F_v(x,S(u),S(v)).$$

Let's consider the following auxiliary problem

$$(P^{++}) \begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = \lambda\alpha(x)\,|S(u)|^{\alpha(x)-2}S(u)\,|S(v)|^{\beta(x)} + F_u^{++}(x,u,v) \text{ in } \Omega, \\ -div(|\nabla v|^{q(x)-2}\nabla v) = \lambda\beta(x)\,|S(u)|^{\alpha(x)}\,|S(v)|^{\beta(x)-2}S(v) + F_v^{++}(x,u,v) \text{ in } \Omega, \\ u = 0 = v \quad \text{on } \partial\Omega. \end{cases}$$

The corresponding functional of problem (P^{++}) is

$$\varphi^{++}(u,v) = \Phi(u,v) - \Psi^{++}(u,v), \forall (u,v) \in X,$$

where

$$\Psi^{++}(u,v) = \int_\Omega \lambda\,|S(u)|^{\alpha(x)}\,|S(v)|^{\beta(x)} + F(x,S(u),S(v))dx, \forall (u,v) \in X.$$

Let $\sigma > 0$ be small enough such that $\sigma \leqslant \frac{1}{4}\min\{\lambda_{p(\cdot)}, \lambda_{q(\cdot)}\}$. Such a σ exits, as $\lambda_{p(\cdot)} > 0$ and $\lambda_{q(\cdot)} > 0$ due to Proposition 2.6. By the assumptions (H_0) and (H_2), we have

$$F(x,u,v) \leqslant \sigma(\frac{1}{p(x)}\,|u|^{p(x)} + \frac{1}{q(x)}\,|v|^{q(x)}) + C(\sigma)(|u|^{\gamma(x)} + |v|^{\delta(x)}),\ \forall (x,t) \in \Omega \times \mathbb{R}.$$

As noticed above, $\lambda_{p(\cdot)}, \lambda_{q(\cdot)} > 0$ and we have also by the choice of σ that

$$\begin{aligned} \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \sigma\int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,dx &\geqslant \frac{3}{4}\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}, \\ \int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx - \sigma\int_\Omega \frac{1}{q(x)}\,|v|^{q(x)}\,dx &\geqslant \frac{3}{4}\int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}. \end{aligned}$$

Next, we shall use spatial decomposition technique. We divide the underlying domain Ω into disjoint subsets $\Omega_1, \cdots, \Omega_{n_0}$ such that

$$\begin{aligned} \min_{x\in\overline{\Omega_j}} p^*(x) &> \max_{x\in\overline{\Omega_j}} \gamma(x) \geqslant \min_{x\in\overline{\Omega_j}} \gamma(x) > \max_{x\in\overline{\Omega_j}} p(x), j = 1,\cdots,n_0, \\ \min_{x\in\overline{\Omega_j}} q^*(x) &> \max_{x\in\overline{\Omega_j}} \delta(x) \geqslant \min_{x\in\overline{\Omega_j}} \delta(x) > \max_{x\in\overline{\Omega_j}} q(x), j = 1,\cdots,n_0. \end{aligned}$$

In the following, we denote

$$f_j^- = \min_{x\in\overline{\Omega_j}} f(x),\ f_j^+ = \max_{x\in\overline{\Omega_j}} f(x),\ j=1,\cdots,n_0,\ \forall f\in C(\overline{\Omega}),$$

and

$$\Phi_{\Omega_j}(u,v) = \int_{\Omega_j} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx + \int_{\Omega_j} \frac{1}{q(x)} |\nabla v|^{q(x)}\,dx, \forall (u,v)\in X.$$

Denote

$$\epsilon = \min_{1\leqslant i\leqslant n_0} \{\inf_{\Omega_i} \gamma(x) - \sup_{\Omega_i} p(x), \inf_{\Omega_i} \delta(x) - \sup_{\Omega_i} q(x)\}.$$

Denote also $\|u\|_{p(\cdot),\Omega_i}$ the norm of u on Ω_i, i.e.

$$\int_{\Omega_i} \frac{1}{p(x)} \left|\nabla \frac{u}{\|u\|_{\Omega_i}}\right|^{p(x)} dx = 1.$$

It is easy to see that $\|u\|_{p(\cdot),\Omega_i} \leqslant C\|u\|_{p(\cdot)}$, and there exist $\xi_i, \eta_i \in \overline{\Omega_i}$ such that

$$\begin{aligned} |u|_{\gamma(\cdot),\Omega_i}^{\gamma(\xi_i)} &= \int_{\Omega_i} |u|^{\gamma(x)}\,dx, \\ \|u\|_{p(\cdot),\Omega_i}^{p(\eta_i)} &= \int_{\Omega_i} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx. \end{aligned}$$

When $\|u\|_{p(\cdot)}$ is small enough, we have

$$\begin{aligned} C(\sigma)\int_{\Omega} |u|^{\gamma(x)}\,dx &= C(\sigma)\sum_{i=1}^{n_0} \int_{\Omega_i} |u|^{\gamma(x)}\,dx \\ &= C(\sigma)\sum_{i=1}^{n_0} |u|_{\gamma(\cdot),\Omega_i}^{\gamma(\xi_i)} \ \text{(where } \xi_i\in\overline{\Omega_i}) \\ &\leqslant C_1\sum_{i=1}^{n_0} \|u\|_{p(\cdot),\Omega_i}^{\gamma(\xi_i)} \ \text{(by Proposition 2.5)} \\ &\leqslant C_2\|u\|_{p(\cdot)}^{\epsilon} \sum_{i=1}^{n_0} \|u\|_{p(\cdot),\Omega_i}^{p(\eta_i)} \ \text{(where } \eta_i\in\overline{\Omega_i}) \\ &= C_2\|u\|_{p(\cdot)}^{\epsilon} \sum_{i=1}^{n_0} \int_{\Omega_i} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx \\ &= C_2\|u\|_{p(\cdot)}^{\epsilon} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx \\ &\leqslant \frac{1}{4}\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx. \end{aligned}$$

Similarly, when $\|v\|_{q(\cdot)}$ is small enough, we have

$$C(\sigma)\int_\Omega |v|^{\delta(x)}\,dx \leqslant \frac{1}{4}\int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx.$$

Thus, when $\|(u,v)\|$ is small enough, we have

$$\Phi(u,v)-\int_\Omega F(x,u,v)dx \geqslant \frac{1}{2}\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx+\frac{1}{2}\int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx. \tag{3.5}$$

When λ is small enough, for any $(u,v)\in X$ with small enough norm, we have

$$\begin{aligned}\varphi^{++}(u,v) &= \Phi(u,v)-\Psi^{++}(u,v)\\ &\geqslant \frac{1}{2}\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx+\frac{1}{2}\int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx-\int_\Omega \lambda\,|u|^{\alpha(x)}\,|v|^{\beta(x)}\,dx,\\ &\geqslant \frac{1}{4}\int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx+\frac{1}{4}\int_\Omega \frac{1}{q(x)}\,|\nabla v|^{q(x)}\,dx.\end{aligned}$$

Therefore, when λ is small enough, there exist $r>0$ and $\varepsilon>0$ such that $\varphi(u,v)\geqslant\varepsilon>0$ for every $(u,v)\in X$ and $\|(u,v)\|=r$.

Let $\Omega_0\subset\Omega$ be an open ball with radius ε. Notice that $(H_{\alpha,\beta})$ holds. Let $\varepsilon>0$ be small enough such that

$$\begin{aligned}p^-_{\overline{\Omega}_0} &:= \min\{p(x)\mid\overline{\Omega}_0\}>\alpha^+_{\overline{\Omega}_0}:=\max\{\alpha(x)\mid\overline{\Omega}_0\},\\ q^-_{\overline{\Omega}_0} &:= \min\{q(x)\mid\overline{\Omega}_0\}>\beta^+_{\overline{\Omega}_0}:=\max\{\beta(x)\mid\overline{\Omega}_0\},\end{aligned}$$

and

$$\frac{\alpha^+_{\overline{\Omega}_0}}{p^-_{\overline{\Omega}_0}}+\frac{\beta^+_{\overline{\Omega}_0}}{q^-_{\overline{\Omega}_0}}<1.$$

Now we pick two functions $u_0,v_0\in C_0^2(\overline{\Omega}_0)$ that are positive in Ω_0. From $(H_{\alpha,\beta})$, it is easy to see that

$$\begin{aligned}&\varphi^{++}(t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0,t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0)\\ =\;&\Phi(t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0,t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0)-\Psi^{++}(t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0,t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0)\\ \leqslant\;&\Phi(t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0,t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0)+2\int_\Omega |t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0|^{p(x)}+|t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0|^{q(x)}dx\\ &-\lambda\int_\Omega |t^{\frac{1}{p^-_{\overline{\Omega}_0}}}u_0|^{\alpha(x)}\cdot|t^{\frac{1}{q^-_{\overline{\Omega}_0}}}v_0|^{\beta(x)}dx\\ \leqslant\;& t\Phi(u_0,v_0)+2t\int_\Omega |u_0|^{p(x)}+|v_0|^{q(x)}dx-\lambda t^{\frac{\alpha^+_{\overline{\Omega}_0}}{p^-_{\overline{\Omega}_0}}+\frac{\beta^+_{\overline{\Omega}_0}}{q^-_{\overline{\Omega}_0}}}\int_\Omega |u_0|^{\alpha(x)}|v_0|^{\beta(x)}dx<0 \text{ as } t\to0^+.\end{aligned}$$

Thus, $\varphi^{++}(u,v)$ has at least one nontrivial critical point (u_1^*,v_1^*) with $\varphi^{++}(u_1^*,v_1^*)<0$.

From assumption (H_3), it is easy to see that (u_1^*, v_1^*) lies in the first quadrant of X. It is easy to see that $S(-u_1^*) \in W_0^{1,p(\cdot)}(\Omega)$. Choosing $S(-u_1^*)$ as a test function, we have

$$\begin{aligned}
& \int_\Omega |\nabla u_1^*|^{p(x)-2} \nabla u_1^* S(-u_1^*) dx \\
= & \int_\Omega [\lambda\alpha(x) |S(u_1^*)|^{\alpha(x)-2} S(u_1^*) |S(v_1^*)|^{\beta(x)} + F_u(x, S(u_1^*), S(v_1^*))] S(-u_1^*) dx \\
= & \int_\Omega F_u(x, S(u_1^*), S(v_1^*)) S(-u_1^*) dx \overset{(H_3)}{=} 0.
\end{aligned}$$

Thus, $u_1^* \geqslant 0$. Similarly, we have $v_1^* \geqslant 0$. Therefore, (u_1^*, v_1^*) is a nontrivial solution with constant sign of (P) and such that $\varphi(u_1^*, v_1^*) < 0$. By the former discussion and (3.5), we can see that u_1^* and v_1^* are both nontrivial. Similarly, we can see that (P) has a nontrivial (u_i^*, v_i^*) with constant sign in the i-th quadrant of X, such that $\varphi(u_i^*, v_i^*) < 0$, $i = 2, 3, 4$. Thus (P) has at least four nontrivial solutions with constant sign. By now, we finished the proof of Theorem 1.1.

Proof of Theorem 1.2.

According to the proof of Theorem 1.1, when λ is small enough, there exist $r > 0$ and $\varepsilon > 0$ such that $\varphi^{++}(u, v) \geqslant \varepsilon > 0$ for every $(u, v) \in X$ and $\|(u, v)\| = r$.

From (H_1), for any $(x, u, v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$, we have

$$F(x, u, v) \geqslant C_1 |u|^{p(x)} [\ln(1 + |u|)]^{a(x)} + C_1 |v|^{q(x)} [\ln(1 + |v|)]^{b(x)} - c_2.$$

We may assume that there exists two different points $x_1, x_2 \in \Omega$ such that $\nabla p(x_1) \neq 0, \nabla q(x_2) \neq 0$.

Now we define $h_1 \in C_0(\overline{B(x_1, \varepsilon)})$ $h_2 \in C_0(\overline{B(x_2, \varepsilon)})$ as follows:

$$h_1(x) = \begin{cases} 0, & |x - x_1| \geqslant \varepsilon \\ \varepsilon - |x - x_1|, & |x - x_1| < \varepsilon \end{cases},$$

$$h_2(x) = \begin{cases} 0, & |x - x_2| \geqslant \varepsilon \\ \varepsilon - |x - x_2|, & |x - x_2| < \varepsilon \end{cases}.$$

From Lemma 2.10, we may let $\varepsilon > 0$ be small enough such that $\varepsilon < \frac{1}{2} |x_2 - x_1|$ and

$$\begin{aligned}
\int_\Omega \frac{1}{p(x)} |\nabla t h_1|^{p(x)} - \int_\Omega C_1 |t h_1|^{p(x)} [\ln(1 + |t h_1|)]^{a(x)} dx &\to -\infty \text{ as } t \to +\infty \\
\int_\Omega \frac{1}{q(x)} |\nabla t h_2|^{q(x)} - \int_\Omega C_1 |t h_2|^{q(x)} [\ln(1 + |t h_2|)]^{b(x)} dx &\to -\infty \text{ as } t \to +\infty.
\end{aligned}$$

which imply that $\varphi^{++}(t h_1, t h_2) \to -\infty$ (as $t \to +\infty$). Since $\varphi^{++}(0, 0) = 0$, φ^{++} satisfies the conditions of the Mountain Pass Lemma. From Lemma 3.3, we know that φ^{++} satisfies Cerami condition. Therefore, we conclude that φ^{++} admits at least one nontrivial critical point (u_1, v_1) with $\varphi^{++}(u_1, v_1) > 0$. From assumption (H_3) , we can easily see that (u_1, v_1) lies in the first quadrant of X. Thus, (u_1, v_1) is a nontrivial solution with constant sign to the problem (P) in the first quadrant of X satisfying $\varphi(u_1, v_1) > 0$.

Similarly, we can see that (P) has a nontrivial solution (u_2, v_2) in the third quadrant in X, which satisfy $\varphi(u_2, v_2) > 0$, and (u_1, v_1), (u_2, v_2) are all nontrivial. From Theorem 1.1, (P) has nontrivial solutions with constant sign (u_i^*, v_i^*) in the i-th quadrant of X $(i = 1, 2, 3, 4)$, which satisfies $\varphi(u_i^*, v_i^*) < 0$. Thus, (P) has at least six nontrivial solutions with constant sign. By far, we have finished the proof of Theorem 1.2.

Now we proceed to prove Theorem 1.3. For this purpose, we need to do some preparations. Noticing that X is a reflexive and separable Banach space (see [38], Section 17, Theorem 2-3), then there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span}\{e_j,\, j = 1, 2, \cdots\},\ X^* = \overline{span}^{W^*}\{e_j^*,\, j = 1, 2, \cdots\},$$

and

$$< e_j^*, e_j >= \begin{cases} 1, i = j, \\ 0, i \neq j. \end{cases}$$

For convenience, we write $X_j = span\{e_j\}$, $Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j$, $Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}$.

Lemma 3.4. *If $\gamma, \delta \in C_+\left(\overline{\Omega}\right)$, $\gamma(x) < p^*(x)$ and $\delta(x) < q^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\left\{ |u|_{\gamma(\cdot)} + |u|_{\delta(\cdot)}\, |\|(u, v)\| = 1, (u, v) \in Z_k \right\},$$

then $\lim\limits_{k\to\infty} \beta_k = 0$.

Proof. Obviously, $0 < \beta_{k+1} \leqslant \beta_k$, so $\beta_k \to \beta \geqslant 0$. Let $u_k \in Z_k$ satisfy

$$\|(u_k, v_k)\| = 1,\ 0 \leqslant \beta_k - |u_k|_{\gamma(\cdot)} - |v_k|_{\delta(\cdot)} < \frac{1}{k}.$$

Then there exists a subsequence of $\{(u_k, v_k)\}$ (which we still denote by (u_k, v_k)) such that $(u_k, v_k) \rightharpoonup (u, v)$, and

$$< e_j^*, (u, v) >= \lim_{k\to\infty} \left\langle e_j^*, (u_k, v_k) \right\rangle = 0,\ \forall e_j^*,$$

which implies that $(u, v) = (0, 0)$, and so $(u_k, v_k) \rightharpoonup (0, 0)$. Since the imbedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{\gamma(\cdot)}(\Omega)$ is compact, then $u_k \to 0$ in $L^{\gamma(\cdot)}(\Omega)$. Similarly, we have $v_k \to 0$ in $L^{\delta(\cdot)}(\Omega)$. Hence we get $\beta_k \to 0$ as $k \to \infty$. Proof of Lemma 3.4 is complete. □

In odder to prove Theorem 1.3, we need the following lemma (see in particular, [40, Theorem 4.7]). For a version of this lemma with the Palais-Samle condition, the (P.S.)-condition, see [4, P 221, Theorem 3.6].

Lemma 3.5. *Suppose $\varphi \in C^1(X, \mathbb{R})$ is even, and satisfies the Cerami condition. Let V^+, $V^- \subset X$ be closed subspaces of X with $codimV^+ + 1 =dim\, V^-$, and suppose there holds*

(1^0) $\varphi(0,0) = 0$.

(2^0) $\exists \tau > 0,\ \gamma > 0$ *such that* $\forall (u,v) \in V^+ : \|(u,v)\| = \gamma \Rightarrow \varphi(u,v) \geqslant \tau$.

(3^0) $\exists \rho > 0$ *such that* $\forall (u,v) \in V^- : \|(u,v)\| \geqslant \rho \Rightarrow \varphi(u,v) \leqslant 0$.

Consider the following set:

$$\Gamma = \{g \in C^0(X,X) \mid g \textit{ is odd}, \ g(u,v) = (u,v) \textit{ if } (u,v) \in V^- \textit{ and } \|(u,v)\| \geqslant \rho\},$$

then

(a) $\forall \delta > 0,\ g \in \Gamma,\ S_\delta^+ \cap g(V^-) \neq \varnothing$, *here* $S_\delta^+ = \{(u,v) \in V^+ \mid \|(u,v)\| = \delta\}$;

(b) the number $\varpi := \inf\limits_{g\in\Gamma} \sup\limits_{(u,v)\in V^-} \varphi(g(u,v)) \geqslant \tau > 0$ *is a critical value for* φ.

Proof of Theorem 1.3.

According to $(H_{\alpha,\beta})$, (H_0), (H_1) and (H_4), we know that φ is an even functional and satisfies the Cerami condition. Let $V_k^+ = Z_k$, it is a closed linear subspace of X and $V_k^+ \oplus Y_{k-1} = X$.

We may assume that there exists different points $x_n, y_n \in \Omega$ such that $\nabla p(x_n) \neq 0, \nabla q(y_n) \neq 0$. We then define $h_n \in C_0(\overline{B(x_n,\varepsilon_n)})$ and $h_n^* \in C_0(\overline{B(y_n,\varepsilon_n)})$ as follows:

$$h_n(x) = \left\{ \begin{array}{cc} 0, & |x - x_n| \geqslant \varepsilon_n \\ \varepsilon_n - |x - x_n|, & |x - x_n| < \varepsilon_n \end{array} \right. ,$$

$$h_n^*(x) = \left\{ \begin{array}{cc} 0, & |x - y_n| \geqslant \varepsilon_n \\ \varepsilon_n - |x - y_n|, & |x - y_n| < \varepsilon_n \end{array} \right. .$$

Without loss of generality, we may assume that

$$\textit{supp}\, h_i \cap \textit{supp}\, h_i^* = \varnothing \text{ for } i = 1, 2, \cdots$$

and

$$\textit{supp}\, h_i \cap \textit{supp}\, h_j = \varnothing,\ \textit{supp}\, h_i^* \cap \textit{supp}\, h_j^* = \varnothing,\ \forall i \neq j.$$

From Lemma 2.9, we may let $\varepsilon_n > 0$ be small enough such that

$$\begin{aligned} \int_\Omega \frac{1}{p(x)} |\nabla t\, h_n|^{p(x)} - \int_\Omega C_1 |t h_n|^{p(x)} [\ln(1 + |t\, h_n|)]^{a(x)} dx &\to -\infty \text{ as } t \to +\infty, \\ \int_\Omega \frac{1}{q(x)} |\nabla t\, h_n^*|^{q(x)} - \int_\Omega C_1 |t\, h_n^*|^{q(x)} [\ln(1 + |t\, h_n^*|)]^{b(x)} dx &\to -\infty \text{ as } t \to +\infty, \end{aligned}$$

which imply that $\varphi(t\, h_n, t\, h_n^*) \to -\infty$ (as $t \to +\infty$).

Set $V_k^- = \textit{span}\{(h_1, h_1^*), \cdots, (h_k, h_k^*)\}$. We will prove that there are infinitely many pairs of V_k^+ and V_k^-, such that φ satisfies the conditions of Lemma 3.5 and that the corresponding critical value $\varpi_k := \inf_{g\in\Gamma} \sup_{(u,v)\in V_k^-} \varphi(g(u,v)) \to +\infty$ when $k \to +\infty$, which implies that there are infinitely many pairs of solutions to the problem (P).

For any $k = 1, 2, \cdots$, we shall show that there exist $\rho_k > \gamma_k > 0$ and large enough k such that

$$\begin{aligned}
(A_1)\quad b_k \quad &: \; = \inf\left\{\varphi(u,v) \mid (u,v) \in V_k^+, \|(u,v)\| = \gamma_k\right\} \to +\infty \ (k \to +\infty);\\
(A_2)\quad a_k \quad &: \; = \max\left\{\varphi(u,v) \mid (u,v) \in V_k^-, \|(u,v)\| = \rho_k\right\} \leqslant 0.
\end{aligned}$$

First, we prove that (A_1) holds. By direct computations, we have, for any $(u,v) \in Z_k$ with $\|(u,v)\| = \gamma_k = (2p^+q^+C\beta_k)^{1/(\min\{p^-,q^-\}-\max\{\gamma^+,\delta^+\})}$, we have

$$\begin{aligned}
&\varphi(u,v) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)}\, dx\\
&-\int_\Omega \lambda |u|^{\alpha(x)} |v|^{\beta(x)}\, dx - \int_\Omega F(x,u,v)dx\\
&\geqslant \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\, dx + \frac{1}{q^+}\int_\Omega |\nabla v|^{q(x)}\, dx\\
&-C\int_\Omega |u|^{\gamma(x)}\, dx - C\int_\Omega |v|^{\delta(x)}\, dx - C_1\\
&\geqslant \frac{1}{p^+}\|u\|_{p(\cdot)}^{p^-} - C|u|_{\gamma(\cdot)}^{\gamma(\xi)} + \frac{1}{q^+}\|v\|_{q(\cdot)}^{q^-} - C|v|_{\delta(\cdot)}^{\delta(\eta)} - C_1 \ (\text{where } \xi,\eta \in \Omega)\\
&\geqslant \frac{1}{p^+}\|u\|_{p(\cdot)}^{p^-} - C\beta_k^{\gamma^+}\|u\|_{p(\cdot)}^{\gamma^+} + \frac{1}{q^+}\|v\|_{q(\cdot)}^{q^-} - C\beta_k^{\delta^-}\|v\|_{q(\cdot)}^{\delta^+} - C_2\\
&\geqslant \frac{1}{p^+q^+}\|(u,v)\|^{\min\{p^-,q^-\}} - C\beta_k\|(u,v)\|^{\max\{\gamma^+,\delta^+\}} - C_2\\
&= \frac{1}{2p^+q^+}(2p^+q^+C\beta_k)^{\min\{p^-,q^-\}/(\min\{p^-,q^-\}-\max\{\gamma^+,\delta^+\})} - C_2.
\end{aligned}$$

Therefore $\varphi(u,v) \geqslant \frac{1}{2p^+q^+}\gamma_k^{\min\{p^-,q^-\}} - C_2$, $\forall (u,v) \in Z_k$ with $\|(u,v)\| = \gamma_k$, then $b_k \to +\infty, (k \to \infty)$. So we have shown that (A_1) holds.

Next we show that (A_2) holds. From the definition of (h_n, h_n^*), it is easy to see that

$$\varphi(th, th^*) \to -\infty \text{ as } t \to +\infty,$$

for any $(h,h^*) \in V_k^- = span\{(h_1,h_1^*),\cdots,(h_k,h_k^*)\}$ with $\| (h,h^*) \| = 1$. Therefore, (A_2) also holds.

Now, applying Lemma 3.5, we finish the proof of Theorem 1.3.

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