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GHOST SERIES AND A MOTIVATED PROOF OF THE ANDREWS-BRESSOUD IDENTITIES

SHASHANK KANADE, JAMES LEPOWSKY, MATTHEW C. RUSSELL AND ANDREW V. SILLS

ABSTRACT. We present what we call a “motivated proof” of the Andrews-Bressoud partition identities for even moduli. A “motivated proof” of the Rogers-Ramanujan identities was given by G. E. Andrews and R. J. Baxter, and this proof was generalized to the odd-moduli case of Gordon’s identities by J. Lepowsky and M. Zhu. Recently, a “motivated proof” of the somewhat analogous Göllnitz-Gordon-Andrews identities has been found. In the present work, we introduce “shelves” of formal series incorporating what we call “ghost series,” which allow us to pass from one shelf to the next via natural recursions, leading to our motivated proof. We anticipate that these new series will provide insight into the ongoing program of vertex-algebraic categorification of the various “motivated proofs.”

1. INTRODUCTION

The classical Rogers-Ramanujan partition identities have numerous generalizations in various directions, notable among which are the generalizations by Gordon-Andrews for odd moduli, extended to even moduli by Andrews-Bressoud. The product sides of the Rogers-Ramanujan identities enumerate the partitions whose parts obey certain restrictions modulo 5, and the sum sides enumerate the partitions with certain difference-two and initial conditions. Generalizations of the Rogers-Ramanujan identities for all odd moduli were discovered by B. Gordon [G] and G. E. Andrews [A1]. Analogous identities for the even moduli of the form $4k + 2$ were discovered by Andrews in [A2] and [A3], and subsequently, for all the even moduli, by D. M. Bressoud in [Br]. The Andrews-Bressoud identities state that for any $k \geq 2$ and $i \in \{1, \dots, k\}$,

$$\frac{\prod_{m \geq 1} (1 - q^{2km})(1 - q^{2km - k - i + 1})(1 - q^{2km - k + i - 1})}{\prod_{m \geq 1} (1 - q^m)} = \sum_{n \geq 0} b_{k,i}(n) q^n,$$

where $b_{k,i}(n)$ is the number of partitions $\pi = (\pi_1, \dots, \pi_s)$ of n (with $\pi_t \geq \pi_{t+1}$) such that

- (1) $\pi_t - \pi_{t+k-1} \geq 2$,
- (2) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv i + k \pmod{2}$,
- (3) at most $k - i$ parts of π equal 1.

Here we have replaced r by $k - i + 1$ in the statement of the main theorem of [Br]. Also, here and below, q is a formal variable.

The product side above is the generating function for partitions not congruent to 0 or $\pm(k - i + 1)$ modulo $2k$, except in the case $i = 1$. The statement of the main theorem in [Br] excluded this exceptional case, simply because no natural combinatorial interpretation of the corresponding product side was known at the time, but the proof of the main theorem in [Br], in particular, Lemma 3, certainly did cover this case. Building on work of Andrews-Lewis [AL], an elegant combinatorial interpretation of the product in the case $i = 1$ was discovered in [S].

As we will recall below, the Gordon-Andrews-Bressoud identities, as well as more general families of such identities, also arise very naturally from the representation theory of vertex operator algebras. In [LW2]–[LW4], the Rogers-Ramanujan identities were proved, and the more general Gordon-Andrews-Bressoud identities were interpreted (including the case $i = 1$ for even moduli),

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using the theory of Z -algebras, which were invented for this very purpose. In [MP], this interpretation of the more general Gordon-Andrews-Bressoud identities was strengthened to a vertex-operator-theoretic proof of these identities. The Z -algebra structures came to be understood in retrospect as the natural generating substructures of certain twisted modules for certain generalized vertex operator algebras, once the theory of generalized vertex operator algebras and their modules and twisted modules was later developed.

As is recalled in the Introduction of [CKLMQRS], the Rogers-Ramanujan identities also arose in R. J. Baxter's work [Ba] in statistical mechanics. Since then, advances in this area have been made by a large number of authors, including A. Berkovich, P. J. Forrester, B. McCoy, A. Schilling and S. O. Warnaar among many others; see the references in [GOW] for a variety of relevant works. In [GOW], M. Griffin, K. Ono and Warnaar have illuminated certain arithmetic properties of Rogers-Ramanujan-type expressions using a framework incorporating the Hall-Littlewood polynomials.

In the present paper we shall present what we call a “motivated proof” of the Andrews-Bressoud even-modulus identities. The first “motivated proof” in this spirit was given by G. E. Andrews and Baxter for the Rogers-Ramanujan identities in [AB], and that proof was observed in that paper to be essentially the same as a Rogers-Ramanujan proof in [RR] and as Baxter's proof in [Ba]. This proof was generalized to the case of the Gordon-Andrews identities in [LZ] and recently, a “motivated proof” of the somewhat analogous Göllnitz-Gordon-Andrews identities was presented in [CKLMQRS].

However, the problem, solved in this paper, of constructing an analogous “motivated proof” of the even-modulus Andrews-Bressoud identities turned out to be much more subtle than for the Gordon-Andrews and Göllnitz-Gordon-Andrews identities. Interesting new phenomena emerged, as we shall see in a moment.

For discussions of the structure of “motivated proofs” (this designation has now become a technical term in our work, and we shall drop the quotation marks) and relevant terminology, and of the likely importance of such proofs from the standpoint of vertex operator algebra theory, we refer the reader to the Introductions in [LZ] and [CKLMQRS], which also include some references touching briefly on some of the large amount of work that has been done relating partition identities, old and new, with vertex operator algebra theory. This importance can be summarized as follows: It is expected that vertex-algebraic “categorifications” of the steps in the motivated proofs will provide useful insights into the representation theory of vertex operator algebras and will also facilitate the discovery of further families of natural identities in this spirit.

As is explained in the Introduction of [CKLMQRS], motivated proofs, as now understood, proceed by the creation of successively higher “shelves” of formal power series in a formal variable q (involving only nonnegative integral powers of q). For a fixed integer k , the given product expressions form the 0th shelf, and each new shelf is formed by taking linear combinations of the q -series on the previous shelf, over the field of fractions of the ring of polynomials in the formal variable q . The exact formulas for these linear combinations constitute a crucial ingredient of the motivated proofs. The main point is that for each constructed series, when the leading term 1 is subtracted, the result is a series divisible by a high power of q ; this mechanism was called the “Empirical Hypothesis” in [AB], and in all these motivated proofs, such an Empirical Hypothesis readily leads to a proof of the desired identities. As an example, consider the simplest nontrivial case—that of the Rogers-Ramanujan identities: Here, the 0th shelf is formed by the two Rogers-Ramanujan products, say, G_1 and G_2 , and the two q -series forming the 1st shelf are a copy of G_2 itself and $G_3 = (G_1 - G_2)/q$, which indeed turns out to be a formal power series in q . The higher shelves are then formed analogously. The motivated proof hinges on the Empirical Hypothesis that the j^{th} series G_j , again a formal power series, is such that $G_j - 1$ is divisible by q^j .

The source of the deep connection with the representation theory of vertex operator algebras is that the products G_1 and G_2 are both the graded dimensions (i.e., the generating functions for the dimensions of the finite-dimensional homogeneous subspaces) of certain twisted modules for a

certain generalized operator vertex operator algebra, and formulas of the type $G_3 = (G_1 - G_2)/q$ are expected to be “categorified” by means of exact sequences among these modules, where the maps between the modules are formed using vertex-algebraic intertwining operators. This phenomenon is expected to arise in considerable generality. The broad program of such categorifications was initiated by J. Lepowsky and is an active area of research. That such categorifications can be found for motivated proofs using “twisted” and “relativized” intertwining operators was an idea of J. Lepowsky and A. Milas, and is currently under investigation. Similar categorifications of the Rogers-Ramanujan, Rogers-Selberg and Euler recursions, as well as of new recursions, using untwisted and unrelativized intertwining operators, have already been successfully carried out in [CLM1], [CLM2], [Ca1], [Ca2], [CaLM1]–[CaLM4] and [Sa].

The products in the Gordon-Andrews identities are the graded dimensions of twisted modules for certain generalized vertex operator algebras corresponding to the affine Lie algebra $A_1^{(1)}$ at odd levels, and the Andrews-Bressoud identities similarly arise from the even levels (see [LM] — where the vertex-algebraic structure had not yet been discovered — and [LW1]–[LW4]). Lepowsky-Wilson’s Z -algebraic interpretation of the sum sides of these identities treats all the levels (and hence both families of identities) on an equal footing ([LW1]–[LW4]). Correspondingly, having motivated proofs of both families of identities, not just of the odd-modulus identities, would provide important insight. For the full family of Gordon-Andrews identities, the aforementioned linear combinations have the same “shape” as for the Rogers-Ramanujan special case. Namely, they involve appropriate subtractions in the numerator and a pure power of q in the denominator. However, the case of the even-modulus Andrews-Bressoud identities is starkly different, in that the denominator now involves more complicated expressions such as a sum of two pure powers of q . The source of this substantial subtlety is the parity conditions on the sum sides of the identities (see (2) above). Such a division is expected to be difficult to categorify (and is not “motivated,” either), and the problem now was to try to invent a natural mechanism that would restore one’s ability to divide by pure powers of q , and to thereby arrive at a suitable “Empirical Hypothesis,” for the even moduli.

We achieve this in the present paper by introducing shelves of what we call “ghost series.” Given the j^{th} shelf of “official” Bressoud series, we introduce relations which simultaneously define a j^{th} shelf of “ghost series” and also facilitate the passage to a $(j+1)^{\text{st}}$ shelf using pure powers of q dividing the subtractions of the official and the ghost series. In this way, we are able to obtain an Empirical Hypothesis that yields the Andrews-Bressoud identities. After our main theorem establishing these identities has been proved, we present a combinatorial interpretation of the ghost series, and it turns out that they differ from the official series only in the parity condition.

In addition to the appearance of ghost series, there is another significant way in which this work differs from the earlier works [AB], [LZ] and [CKLMQRS]. In those works, the recursions defining the successively higher shelves could be, or could have been, predicted by appropriately specializing the already-known (a, x, q) -recursions, proved by Andrews in Chapter 7 of [A3], satisfied by the corresponding Rogers-Andrews expressions given in (7.2.1) and (7.2.2) of [A3], and the recursions for proving the Göllnitz-Gordon-Andrews identities were indeed “discovered” in this way in [CKLMQRS] (cf. Appendices A and C of [CKLMQRS]). But in the case of the Andrews-Bressoud identities, unlike in [CKLMQRS], the interplay between the various formal power series, including the ghost series, was discovered by purely empirical means. In particular, the “Empirical Hypothesis” in the present paper was *indeed* empirical; in [CKLMQRS], the “Empirical Hypothesis” was “secretly known in advance” because of its source in Andrews’s just-mentioned (a, x, q) -recursions. Chronologically, the decision was first made to examine these ghost series (with the “wrong” parity conditions), using the combinatorial interpretations stated in Section 7, in our search for a proper “Empirical Hypothesis.” Then, the relations which enable one to simultaneously define the ghost series and move to the next shelf via a division by a pure power of q were experimentally determined. It is also interesting to note that the proof of our main theorem requires the Empirical Hypothesis only for the official series and not for the ghost series.

The discovery of ghost series raises many interesting questions. These series have not yet been observed to come from vertex-algebraic structures, but they seem extremely natural from the point of view of the categorification mentioned above. Therefore, it is natural to ask where they reside in the theory of vertex operator algebras. This question is being examined. Their modularity properties and representations as multisums and weighted q -sums of infinite products are under investigation as well. It is hoped that the new insight of ghost series will prove fruitful in providing motivated proofs of other Rogers-Ramanujan-style partition identities, and such a program is also underway.

This paper is organized as follows: In Section 2 we recall basic notations regarding q -series and partitions. Then, after using the Jacobi triple product identity to re-express the Andrews-Bressoud products in the usual more amenable form, we present our recursions for building the official and ghost series comprising the higher shelves. In Section 3, we derive closed-form expressions for the 0th-shelf ghosts. These expressions are used as the base case of the induction in the proof of Theorem 4.1 in Section 4, giving closed-form expressions for the official series and the ghost series on all the shelves. In Section 5, using Theorem 4.1, we derive our Empirical Hypothesis in a number of different strengths; the weakest of these is enough to prove our main theorem. We put various recursions and relations into an elegant matrix formulation in Section 6, and as a consequence we derive one form of our main theorem, Theorem 6.3. In Section 7 we complete our motivated proof of the Andrews-Bressoud identities. Section 8 is concerned with comparing our closed-form expressions with various (x, q) -series contained in the paper [CoLoMa] of S. Corteel, J. Lovejoy and O. Mallet — we provide a “dictionary” between our closed-form expressions and appropriate specializations of the formal series \tilde{J} appearing in [CoLoMa]. Finally, in Section 9, we “reverse-engineer” our method and appropriately replace certain powers of q with x in our closed-form expressions to arrive at (new) (x, q) -expressions that govern the ghosts.

Just as in [LZ] and [CKLMQRS], throughout this paper we treat power series as purely formal series rather than as convergent series (in suitable domains) in complex variables.

2. THE SETTING

First we recall the standard notation

$$(a)_n = \prod_{s=0}^{n-1} (1 - aq^s) \quad (2.1)$$

$$(q)_n = \prod_{s=1}^n (1 - q^s) \quad (2.2)$$

$$(q)_\infty = \prod_{s \geq 1} (1 - q^s), \quad (2.3)$$

where a is any formal variable or complex number.

Fix an integer $k \geq 2$. For each $i \in \{1, \dots, k\}$, define

$$B_i = \frac{\prod_{m \geq 1} (1 - q^{2km})(1 - q^{2km-k+i-1})(1 - q^{2km-k-i+1})}{(q)_\infty}. \quad (2.4)$$

Recalling the Jacobi triple product identity,

$$\sum_{\lambda \in \mathbb{Z}} (-1)^\lambda z^\lambda q^{\lambda^2} = \prod_{m \geq 0} (1 - q^{2m+2})(1 - zq^{2m+1})(1 - z^{-1}q^{2m+1}),$$

and replacing q by q^k and z by q^{i-1} , we have

$$B_i = \frac{\sum_{n \in \mathbb{Z}} (-1)^n q^{n(i-1)} q^{kn^2}}{(q)_\infty}$$

$$= \frac{\sum_{n \geq 0} (-1)^n q^{kn^2 + n(i-1)} (1 - q^{(k-i+1)(2n+1)})}{(q)_\infty}. \quad (2.5)$$

We add to our list the next $k-1$ “official” series B_l for $l \in \{k+1, k+2, \dots, 2k-1\}$, along with what we shall call “ghost series” counterparts \tilde{B}_l for $l \in \{2, 3, \dots, k\}$, via the relations

$$B_{k+1} = \frac{B_{k-1} - \tilde{B}_k}{q} = \tilde{B}_k, \quad (2.6)$$

and

$$B_{k+h} = \frac{B_{k-h} - \tilde{B}_{k-h+1}}{q^h} = \frac{\tilde{B}_{k-h+1} - B_{k-h+2}}{q^{h-1}} \quad (2.7)$$

for $h \in \{2, 3, \dots, k-1\}$.

Remark 2.1. Note that we have not defined \tilde{B}_1 . Indeed, \tilde{B}_1 will not be necessary in proving our main theorem. However, once we get the combinatorial interpretation of the ghosts in Theorem 7.4, we will be able to attach meaning to \tilde{B}_1 ; see Remark 7.5.

Remark 2.2. The right-hand equalities in equations (2.6) and (2.7) serve to define the zeroth-shelf ghosts. After these ghosts are defined, (2.6) and (2.7) give the definition of the official series on the first shelf.

Continuing similarly, for $j \geq 0$ we define ghosts and official series on all shelves by

$$B_{(k-1)(j+1)+2} = \frac{B_{(k-1)j+k-1} - \tilde{B}_{(k-1)j+k}}{q^{j+1}} = \tilde{B}_{(k-1)j+k}, \quad (2.8)$$

and

$$B_{(k-1)(j+1)+i} = \frac{B_{(k-1)j+k-i+1} - \tilde{B}_{(k-1)j+k-i+2}}{q^{(j+1)(i-1)}} = \frac{\tilde{B}_{(k-1)j+k-i+2} - B_{(k-1)j+k-i+3}}{q^{(j+1)(i-2)}} \quad (2.9)$$

for $i \in \{3, 4, \dots, k\}$, with Remark 2.2 extending to all j .

Remark 2.3. As previewed in the Introduction, one clearly sees that ghost series yield the “correctly” shaped relations (2.8) and (2.9), i.e., these relations involve pure powers of q dividing the subtractions of the appropriate series.

As mentioned above in Remark 2.2, using the right-hand equalities in equations (2.8) and (2.9), one can define the ghosts explicitly for $j \geq 0$:

$$\tilde{B}_{(k-1)j+i} = \frac{B_{(k-1)j+i-1} + q^{j+1} B_{(k-1)j+i+1}}{1 + q^{j+1}} \quad \text{for } i \in \{2, \dots, k-1\}, \quad (2.10)$$

$$\tilde{B}_{(k-1)j+k} = \frac{B_{(k-1)j+k-1}}{1 + q^{j+1}}. \quad (2.11)$$

Remark 2.4. In Theorem 4.1, we will provide closed-form expressions for the various official and ghost series. In order to obtain the closed-form expressions for the official series, we shall use the left-hand equalities in (2.8) and (2.9), and for the ghosts, we will use (2.10) and (2.11).

Remark 2.5. The case $k = 2$ corresponds to a pair of Euler identities. In this case, several relations, namely, (2.7), (2.9) and (2.10), become vacuous, and only the “edge-matching” relations, namely, (2.6), (2.8) and (2.11), survive. As a result of this, the official series on the various shelves also manifest themselves as ghost series.

Now we turn to our notation regarding partitions. As usual, a *partition* π of n is a finite nonincreasing sequence of positive integers $(\pi_1, \pi_2, \dots, \pi_t)$ such that $\pi_1 + \pi_2 + \dots + \pi_t = n$. Each π_s is called a *part* of π . The *length* $\ell(\pi)$ of π is the number of parts in π . For any positive integer p , the *multiplicity* of p in π , denoted $m_p(\pi)$, is the number of parts in π equal to p .

We will prove two analogues of [LZ, Theorem 4.1] — for the official series (recovering the Andrews-Bressoud identities as a special case) and for the ghosts: For $r = (k-1)j + i$, with $j \geq 0$ and $1 \leq i \leq k$,

$$B_r = \sum_{n \geq 0} b_{k,r}(n)q^n,$$

where $b_{k,r}(n)$ denotes the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that

- (1) $\pi_t - \pi_{t+k-1} \geq 2$,
- (2) $\pi_{\ell(\pi)} \geq j + 1$,
- (3) $m_{j+1}(\pi) \leq k - i$,
- (4) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv r + k \pmod{2}$.

Moreover, for $r = (k-1)j + i$, with $j \geq 0$ and $2 \leq i \leq k$,

$$\tilde{B}_r = \sum_{n \geq 0} \tilde{b}_{k,r}(n)q^n,$$

where $\tilde{b}_{k,r}(n)$ denotes the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that

- (1) $\pi_t - \pi_{t+k-1} \geq 2$,
- (2) $\pi_{\ell(\pi)} \geq j + 1$,
- (3) $m_{j+1}(\pi) \leq k - i$,
- (4) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv r + k + 1 \pmod{2}$.

Note that for the ghosts, the parity condition is the “wrong” one; this is the key idea.

3. THE ZEROth SHELF

In this section we derive closed-form expressions for the ghosts on the zeroth shelf. These closed-form expressions will be used later as the base case of the induction in the proof of Theorem 4.1. Recall equation (2.5): For $i \in \{1, \dots, k\}$,

$$B_i = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+n(i-1)}(1 - q^{(k-i+1)(2n+1)})}{(q)_\infty}. \quad (3.1)$$

Solving (2.6) for \tilde{B}_k , we get that

$$\tilde{B}_k = \frac{B_{k-1}}{1+q}. \quad (3.2)$$

Next, solving (2.7) for \tilde{B}_{k-i+1} ,

$$\tilde{B}_{k-i+1} = \frac{B_{k-i} + qB_{k-i+2}}{1+q} \quad (3.3)$$

for $i \in \{2, 3, \dots, k-1\}$. Equations (3.2) and (3.3) express the ghosts on the zeroth shelf in terms of the official series on the zeroth shelf. Re-indexing (3.3), we get, for $i \in \{2, 3, \dots, k-1\}$,

$$\tilde{B}_i = \frac{B_{i-1} + qB_{i+1}}{1+q}. \quad (3.4)$$

From (2.5) (as recalled above), we gather that for $i \in \{2, 3, \dots, k-1\}$,

$$B_{i-1} = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+n(i-2)}(1 - q^{(k-i+2)(2n+1)})}{(q)_\infty}, \quad (3.5)$$

$$B_{i+1} = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+ni}(1 - q^{(k-i)(2n+1)})}{(q)_\infty}. \quad (3.6)$$

Hence

$$(q)_\infty(B_{i-1} + qB_{i+1})$$

$$\begin{aligned}
&= \sum_{n \geq 0} (-1)^n \left(q^{kn^2+n(i-2)} \left(1 - q^{(k-i+2)(2n+1)} \right) + q^{kn^2+ni+1} \left(1 - q^{(k-i)(2n+1)} \right) \right) \\
&= \sum_{n \geq 0} (-1)^n q^{kn^2+n(i-2)} \left(1 - q^{(k-i+2)(2n+1)} + q^{2n+1} \left(1 - q^{(k-i)(2n+1)} \right) \right) \\
&= \sum_{n \geq 0} (-1)^n q^{kn^2+n(i-2)} \left(1 - q^{(k-i+1)(2n+1)+2n+1} + q^{2n+1} - q^{(k-i+1)(2n+1)} \right) \\
&= \sum_{n \geq 0} (-1)^n q^{kn^2+n(i-2)} \left(1 - q^{(k-i+1)(2n+1)} \right) (1 + q^{2n+1}).
\end{aligned}$$

Therefore, for $i \in \{2, 3, \dots, k-1\}$, from (3.4),

$$\tilde{B}_i = \frac{B_{i-1} + qB_{i+1}}{1+q} = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+n(i-2)} (1 - q^{(k-i+1)(2n+1)}) (1 + q^{2n+1})}{(q)_\infty (1+q)}. \quad (3.7)$$

Now we find \tilde{B}_k in the same way:

$$\tilde{B}_k = \frac{B_{k-1}}{1+q} = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+n(k-2)} (1 - q^{2(2n+1)})}{(q)_\infty (1+q)},$$

and it is easy to see that this is the same expression we get by setting $i = k$ in (3.7). Therefore, (3.7) holds for $i \in \{2, 3, \dots, k\}$.

4. DETERMINATION OF CLOSED-FORM EXPRESSIONS FOR HIGHER SHELVES AND GHOSTS

The aim of this section is to provide explicit closed-form expressions for all of the ghosts and the official series. These closed-form expressions will enable us to provide a proof of our Empirical Hypothesis in Section 5.

Theorem 4.1. For $j \geq 0$ and $i \in \{1, 2, \dots, k\}$,

$$B_{(k-1)j+i} \in \mathbb{C}[[q]]$$

and in fact,

$$\begin{aligned}
&B_{(k-1)j+i} \\
&= \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+((k-1)j+i-1)n} (1 - q^{(2n+j+1)(k-i+1)}) (1 - q^{2(n+1)}) \dots (1 - q^{2(n+j)})}{(q)_\infty (1+q) \dots (1+q^j)}.
\end{aligned} \quad (4.1)$$

Denoting the right-hand side of (4.1) by $RHS_{j,i}$, we have that for each $j \geq 1$, the two expressions for $B_{(k-1)j+i}$ match:

$$RHS_{j-1,k} = RHS_{j,1}. \quad (4.2)$$

Moreover, for $j \geq 0$ and $i \in \{2, \dots, k\}$,

$$\tilde{B}_{(k-1)j+i} \in \mathbb{C}[[q]]$$

and in fact,

$$\begin{aligned}
&\tilde{B}_{(k-1)j+i} \\
&= \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+((k-1)j+i-2)n} (1 - q^{2(n+1)}) \dots (1 - q^{2(n+j)}) (1 + q^{2n+j+1}) (1 - q^{(2n+j+1)(k-i+1)})}{(q)_\infty (1+q) \dots (1+q^{j+1})}.
\end{aligned} \quad (4.3)$$

Proof. From the previous sections, we see that the conclusions hold if $j = 0$. We proceed by induction. Assume that the assertions in the theorem are true for all $i \in \{1, 2, \dots, k\}$ for a certain index j .

For $i \in \{2, \dots, k\}$,

$$B_{(k-1)(j+1)+i} = \frac{B_{(k-1)j+k-i+1} - \tilde{B}_{(k-1)j+k-i+2}}{q^{(j+1)(i-1)}}.$$

Therefore,

$$\begin{aligned} & (q)_\infty (1+q) \cdots (1+q^{j+1}) q^{(j+1)(i-1)} B_{(k-1)(j+1)+i} \\ &= \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{(2n+j+1)i}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) (1 + q^{j+1}) \\ & \quad - \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) (1 + q^{2n+j+1}) (1 - q^{(2n+j+1)(i-1)}) \\ &= \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) \\ & \quad \cdot \left(1 - q^{(2n+j+1)i} + q^{j+1} - q^{(2n+j+1)i+j+1} - 1 - q^{2n+j+1} + q^{(2n+j+1)(i-1)} + q^{(2n+j+1)i} \right) \\ &= \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) \\ & \quad \cdot \left(q^{j+1} - q^{(2n+j+1)i+j+1} - q^{2n+j+1} + q^{(2n+j+1)(i-1)} \right) \\ &= \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) \\ & \quad \left(q^{j+1} (1 - q^{2n}) + q^{(2n+j+1)(i-1)} (1 - q^{2(n+j+1)}) \right) \\ &= \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n+j+1} (1 - q^{2n}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) \end{aligned} \tag{4.4}$$

$$+ \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n + (2n+j+1)(i-1)} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}). \tag{4.5}$$

In (4.4), the term corresponding to $n = 0$ is 0, and hence, making the index change $n \mapsto n + 1$, we get

$$\begin{aligned} & \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n+j+1} (1 - q^{2n}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) \\ &= \sum_{n \geq 0} (-1)^{n+1} q^{kn^2 + 2kn + k + ((k-1)j+k-i)(n+1)+j+1} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}) \\ &= \sum_{n \geq 0} (-1)^{n+1} q^{kn^2 + ((k-1)j+k-i)n + (2n+j+1)(i-1) + (2n+j+2)(k-i+1)} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}). \end{aligned} \tag{4.6}$$

Combining (4.5) with (4.6), we arrive at

$$\sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n + (2n+j+1)(i-1)} (1 - q^{(2n+j+2)(k-i+1)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}),$$

which equals

$$q^{(j+1)(i-1)} \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+i-1)n} (1 - q^{(2n+j+2)(k-i+1)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}).$$

It is easy to see that the proof above of the equality

$$\begin{aligned}
& \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{(2n+j+1)i}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) (1 + q^{j+1}) \\
& - \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)j+k-i)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) (1 + q^{2n+j+1}) (1 - q^{(2n+j+1)(i-1)}) \\
& = q^{(j+1)(i-1)} \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+i-1)n} (1 - q^{(2n+j+2)(k-i+1)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)})
\end{aligned} \tag{4.7}$$

in fact works even if $i = 1$. For the $i = 1$ case, the second term on the left-hand side of (4.7) is equal to 0. Therefore, dividing through by $(q)_\infty (1 + q) \cdots (1 + q^{j+1})$ precisely gives the edge-matching, i.e.,

$$RHS_{j,k} = RHS_{(j+1),1}.$$

Finally, we use the formulas for $B_{(k-1)(j+1)+i}$ along with the edge-matching phenomenon in order to prove the required formulas for \tilde{B} . For $i \in \{2, \dots, k-1\}$,

$$\tilde{B}_{(k-1)(j+1)+i} = \frac{B_{(k-1)(j+1)+i-1} + q^{j+2} B_{(k-1)(j+1)+i+1}}{1 + q^{j+2}}.$$

Therefore,

$$\begin{aligned}
& (q)_\infty (1 + q) \cdots (1 + q^{j+2}) \tilde{B}_{(k-1)(j+1)+i} \\
& = \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+i-2)n} (1 - q^{(2n+j+2)(k-i+2)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}) \\
& \quad + \sum_{n \geq 0} (-1)^n q^{j+2+kn^2 + ((k-1)(j+1)+i)n} (1 - q^{(2n+j+2)(k-i)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}) \\
& = \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+i-2)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}) \\
& \quad \cdot \left(1 - q^{(2n+j+2)(k-i+2)} + q^{2n+j+2} - q^{(2n+j+2)(k-i+1)} \right) \\
& = \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+i-2)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j+1)}) \\
& \quad \cdot (1 + q^{2n+j+2}) (1 - q^{(2n+j+2)(k-i+1)}).
\end{aligned} \tag{4.8}$$

For $i = k$,

$$\tilde{B}_{(k-1)(j+1)+k} = \frac{B_{(k-1)(j+1)+k-1}}{1 + q^{j+2}}.$$

Therefore,

$$\begin{aligned}
& (q)_\infty (1 + q) \cdots (1 + q^{j+2}) \tilde{B}_{(k-1)(j+1)+k} \\
& = \sum_{n \geq 0} (-1)^n q^{kn^2 + ((k-1)(j+1)+k-2)n} (1 - q^{2(2n+j+2)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)}) (1 - q^{2(n+j+1)}),
\end{aligned}$$

which equals (4.8) with $i = k$. □

Remark 4.2. Note that the factor $(q)_\infty$ played no role whatsoever in the proof above, except for the identification of the closed forms on the zeroth shelf with the products in the Andrews-Bressoud identities.

5. THE EMPIRICAL HYPOTHESIS

Using the formula for $B_{(k-1)j+i}$ in Theorem 4.1, we see that the $n = 0$ term of the sum is

$$\begin{aligned} \frac{(1 - q^{(j+1)(k-i+1)})(1 - q^2) \cdots (1 - q^{2j})}{(q)_\infty(1 + q) \cdots (1 + q^j)} &= \frac{(1 - q^{(j+1)(k-i+1)})(1 - q) \cdots (1 - q^j)}{(q)_\infty} \\ &= \frac{1 - q^{(j+1)(k-i+1)}}{(1 - q^{j+1})(1 - q^{j+2}) \cdots}, \end{aligned}$$

and so

$$\begin{aligned} B_{(k-1)j+i} &= \frac{(1 - q^{(j+1)(k-i+1)})(1 - q^2) \cdots (1 - q^{2j})}{(q)_\infty(1 + q) \cdots (1 + q^j)} \\ &\quad + \sum_{n \geq 1} (-1)^n \frac{q^{kn^2 + ((k-1)j+i-1)n} (1 - q^{(2n+j+1)(k-i+1)})(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(q)_\infty(1 + q) \cdots (1 + q^j)}. \end{aligned}$$

For any $j \geq 0$, $i \in \{1, \dots, k\}$ and for any $n \geq 1$, we see that

$$kn^2 + ((k-1)j + i - 1)n \geq j + 2,$$

since $k \geq 2$. Analyzing the first summand above, we get that

$$B_{(k-1)j+i} = \begin{cases} 1 + q^{j+1} \gamma_i^{(j+1)}(q) & \text{if } 1 \leq i \leq k-1 \\ 1 + q^{j+2} \gamma_k^{(j+2)}(q) & \text{if } i = k \end{cases} \quad (5.1)$$

where

$$\gamma_i^{(j+1)}(q) \in \mathbb{C}[[q]]. \quad (5.2)$$

Similarly,

$$\begin{aligned} \tilde{B}_{(k-1)j+i} &= \frac{1 - q^{(j+1)(k-i+1)}}{(1 - q^{j+1})(1 - q^{j+2}) \cdots} \\ &\quad + \sum_{n \geq 1} (-1)^n \frac{q^{kn^2 + ((k-1)j+i-2)n} (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})(1 + q^{2n+j+1})(1 - q^{(2n+j+1)(k-i+1)})}{(q)_\infty(1 + q) \cdots (1 + q^{j+1})}. \end{aligned}$$

Hence

$$\tilde{B}_{(k-1)j+i} = \begin{cases} 1 + q^{j+1} \tilde{\gamma}_i^{(j+1)}(q) & \text{if } 2 \leq i \leq k-1 \\ 1 + q^{j+2} \tilde{\gamma}_k^{(j+2)}(q) & \text{if } i = k \end{cases} \quad (5.3)$$

where

$$\tilde{\gamma}_i^{(j+1)}(q) \in \mathbb{C}[[q]]. \quad (5.4)$$

Equations (5.1)–(5.4) form our Empirical Hypothesis.

Remark 5.1. Carefully analyzing the various γ 's and $\tilde{\gamma}$'s appearing above, we can make our Empirical Hypothesis stronger, as follows:

$$B_{(k-1)j+i} = \begin{cases} 1 + q^{j+1} + \cdots & \text{if } 1 \leq i \leq k-1 \\ 1 + q^{j+2} + \cdots & \text{if } i = k \end{cases} \quad (5.5)$$

and

$$\tilde{B}_{(k-1)j+i} = \begin{cases} 1 + q^{j+1} + \cdots & \text{if } 2 \leq i \leq k-1 \\ 1 + q^{j+2} + \cdots & \text{if } i = k. \end{cases} \quad (5.6)$$

Equations (5.5) and (5.6) will be collectively called the Strong Empirical Hypothesis.

Remark 5.2. The way we have deduced our Empirical Hypothesis highlights the importance of the factor $(q)_\infty$ appearing in the denominator of our closed-form expressions — it is the unique such factor that can yield the Empirical Hypothesis. Recall that by contrast, this factor played no significant role in the proof of Theorem 4.1 (see Remark 4.2).

Remark 5.3. As will be clear from the proofs of our main theorems, Theorem 6.3 and Theorem 7.3, we will only need the information related to the series B_r (and not the series \tilde{B}_r) from our Empirical Hypotheses. Moreover, the only form of the Empirical Hypothesis that is logically needed to prove the Andrews-Bressoud identities is a weaker one, which states that for any positive integer r there exists a positive integer $f(r)$ with

$$B_r \in 1 + q^{f(r)}\mathbb{C}[[q]], \quad (5.7)$$

such that

$$\lim_{r \rightarrow \infty} f(r) = \infty. \quad (5.8)$$

We will refer to this form as the Weak Empirical Hypothesis.

6. MATRIX INTERPRETATION

The aim of this section is to give a matrix formulation of the shelf picture obtained so far. Using this formulation, we present Theorem 6.3, which is one form of our main theorem.

Eliminating the \tilde{B}_r 's from equations (2.6) – (2.9), we arrive at the following recursions:

Proposition 6.1. For $j \geq 0$,

$$B_{(k-1)(j+1)+1} = B_{(k-1)j+k} \quad (6.1)$$

$$B_{(k-1)(j+1)+2} = \frac{B_{(k-1)j+k-1}}{1 + q^{j+1}} \quad (6.2)$$

$$B_{(k-1)(j+1)+i} = \frac{B_{(k-1)j+k-i+1} - B_{(k-1)j+k-i+3}}{q^{(j+1)(i-2)}(1 + q^{j+1})} \text{ for } i \in \{3, \dots, k\}. \quad (6.3)$$

Note that equation (6.1) gives the edge-matching. Collectively, equations (6.1)–(6.3) provide recursions defining the higher-shelf B_r 's in terms of only lower-shelf B_r 's. Now we reverse this procedure. We can rewrite (6.1)–(6.3) as follows:

$$B_{(k-1)j+k} = B_{(k-1)(j+1)+1} \quad (6.4)$$

$$B_{(k-1)j+k-1} = B_{(k-1)(j+1)+2}(1 + q^{j+1}) \quad (6.5)$$

$$B_{(k-1)j+i} = B_{(k-1)(j+1)+k-i+1}q^{(j+1)(k-i-1)}(1 + q^{j+1}) + B_{(k-1)j+i+2} \text{ for } i \in \{1, \dots, k-2\}. \quad (6.6)$$

Fix an integer

$$J \geq 0.$$

This will denote a “starting” shelf, which need not be the zeroth shelf. Using (6.4)–(6.6), for any $j \geq J$ and $i \in \{1, \dots, k\}$, we can write

$$B_{(k-1)J+i} = {}_i^J h_1^{(j)} B_{(k-1)j+1} + \dots + {}_i^J h_k^{(j)} B_{(k-1)j+k}. \quad (6.7)$$

From the form of (6.4)–(6.6), we see that the coefficients ${}_i^J h_l^{(j)}$ are polynomials in q with nonnegative integral coefficients, and moreover, it is not hard to see that

$$\begin{aligned} k \equiv 1 \pmod{2} & \text{ implies that } {}_i^J h_l^{(j)} = 0 \text{ for } l \not\equiv i \pmod{2}, \\ k \equiv 0 \pmod{2} & \text{ implies that } {}_i^J h_l^{(j)} = 0 \text{ for } j - J + l \not\equiv i \pmod{2}. \end{aligned} \quad (6.8)$$

For each $j \geq J$, the coefficients ${}^J h_l^{(j)}$ can be assembled to form a $k \times k$ matrix ${}^J \mathbf{h}^{(j)}$:

$${}^J \mathbf{h}^{(j)} = \begin{bmatrix} {}^J \mathbf{h}^{(j)} \\ \vdots \\ {}^J \mathbf{h}^{(j)} \end{bmatrix} = \begin{bmatrix} {}^J h_1^{(j)} & \cdots & {}^J h_k^{(j)} \\ \vdots & \ddots & \vdots \\ {}^J h_1^{(j)} & \cdots & {}^J h_k^{(j)} \end{bmatrix}. \quad (6.9)$$

The row vectors of ${}^J \mathbf{h}^{(j)}$ are

$${}^J \mathbf{h}^{(j)} = [{}^J h_1^{(j)}, \dots, {}^J h_k^{(j)}]. \quad (6.10)$$

For $j = J$,

$${}^J \mathbf{h}^{(J)} = [0, \dots, 0, 1, 0, \dots, 0], \quad (6.11)$$

where the 1 is in the i^{th} position, and therefore,

$${}^J \mathbf{h}^{(J)} = \mathbf{I}_{k \times k}. \quad (6.12)$$

Now, independently of the left subscripts i and superscripts J , the ${}^J h_l^{(j)}$ satisfy the same recursions with respect to j , as given in the following proposition:

Proposition 6.2. (1) If k is odd,

$$\begin{aligned} {}^J h_1^{(j+1)} &= ({}^J h_1^{(j)} + {}^J h_3^{(j)} + \cdots + {}^J h_{k-2}^{(j)} + {}^J h_k^{(j)}) \\ {}^J h_3^{(j+1)} &= ({}^J h_1^{(j)} + {}^J h_3^{(j)} + \cdots + {}^J h_{k-2}^{(j)})(1 + q^{j+1})q^{j+1} \\ &\vdots \\ {}^J h_{k-2}^{(j+1)} &= ({}^J h_1^{(j)} + {}^J h_3^{(j)})(1 + q^{j+1})q^{(k-4)(j+1)} \\ {}^J h_k^{(j+1)} &= {}^J h_1^{(j)}(1 + q^{j+1})q^{(k-2)(j+1)} \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} {}^J h_2^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)} + \cdots + {}^J h_{k-3}^{(j)} + {}^J h_{k-1}^{(j)})(1 + q^{j+1}) \\ {}^J h_4^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)} + \cdots + {}^J h_{k-3}^{(j)})(1 + q^{j+1})q^{2(j+1)} \\ &\vdots \\ {}^J h_{k-3}^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)})(1 + q^{j+1})q^{(k-5)(j+1)} \\ {}^J h_{k-1}^{(j+1)} &= {}^J h_2^{(j)}(1 + q^{j+1})q^{(k-3)(j+1)}. \end{aligned} \quad (6.14)$$

(2) If k is even,

$$\begin{aligned} {}^J h_1^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)} + \cdots + {}^J h_{k-2}^{(j)} + {}^J h_k^{(j)}) \\ {}^J h_3^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)} + \cdots + {}^J h_{k-2}^{(j)})(1 + q^{j+1})q^{j+1} \\ &\vdots \\ {}^J h_{k-3}^{(j+1)} &= ({}^J h_2^{(j)} + {}^J h_4^{(j)})(1 + q^{j+1})q^{(k-5)(j+1)} \\ {}^J h_{k-1}^{(j+1)} &= {}^J h_2^{(j)}(1 + q^{j+1})q^{(k-3)(j+1)}. \end{aligned} \quad (6.15)$$

and

$$\begin{aligned} {}^J h_2^{(j+1)} &= ({}^J h_1^{(j)} + {}^J h_3^{(j)} + \cdots + {}^J h_{k-3}^{(j)} + {}^J h_{k-1}^{(j)})(1 + q^{j+1}) \\ {}^J h_4^{(j+1)} &= ({}^J h_1^{(j)} + {}^J h_3^{(j)} + \cdots + {}^J h_{k-3}^{(j)})(1 + q^{j+1})q^{2(j+1)} \\ &\vdots \end{aligned} \quad (6.16)$$

$$\begin{aligned} {}_i^J h_{k-2}^{(j+1)} &= ({}_i^J h_1^{(j)} + {}_i^J h_3^{(j)})(1 + q^{j+1})q^{(k-4)(j+1)} \\ {}_i^J h_k^{(j+1)} &= {}_i^J h_1^{(j)}(1 + q^{j+1})q^{(k-2)(j+1)}. \end{aligned}$$

Proof. Immediate from (6.4)–(6.7). \square

We put the recursions for the B_r 's and h_l 's into a succinct matrix form. For each $j \geq 0$, define the vectors

$$\mathbf{B}_{(j)} = \begin{bmatrix} B_{(k-1)j+1} \\ B_{(k-1)j+2} \\ \vdots \\ B_{(k-1)j+k} \end{bmatrix}, \quad (6.17)$$

$$\tilde{\mathbf{B}}_{(j)} = \begin{bmatrix} B_{(k-1)j+2} \\ \vdots \\ B_{(k-1)j+k} \end{bmatrix}. \quad (6.18)$$

Set

$$\mathcal{B}_{(j)} = (1 + q^j)^{-1} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (1 + q^j) \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q^{-j} & 0 & -q^{-j} \\ 0 & 0 & 0 & \cdots & q^{-2j} & 0 & q^{-2j} & 0 \\ \vdots & \vdots & \vdots & \swarrow & \swarrow & \swarrow & \vdots & \vdots \\ q^{-(k-2)j} & 0 & -q^{-(k-2)j} & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (6.19)$$

Then for $j \geq 1$,

$$\mathbf{B}_{(j)} = \mathcal{B}_{(j)}\mathbf{B}_{(j-1)}. \quad (6.20)$$

Recall that we have fixed an integer $J \geq 0$. For any $j > J$,

$$\mathbf{B}_{(j)} = \mathcal{B}_{(j)}\mathcal{B}_{(j-1)} \cdots \mathcal{B}_{(J+1)}\mathbf{B}_{(J)}. \quad (6.21)$$

Recalling (6.7), (6.9)–(6.12), we immediately have

$$\mathbf{B}_{(J)} = {}^J \mathbf{h}^{(j)}\mathbf{B}_{(j)} \quad (6.22)$$

for any $j \geq J$.

If k is odd, define a $k \times k$ matrix

$$\mathcal{A}_{(j)} = (1 + q^j) \begin{bmatrix} (1 + q^j)^{-1} & 0 & q^j & 0 & \cdots & 0 & q^{(k-2)j} \\ 0 & 1 & 0 & q^{2j} & \cdots & q^{(k-3)j} & 0 \\ (1 + q^j)^{-1} & 0 & q^j & 0 & \swarrow & 0 & 0 \\ 0 & 1 & 0 & q^{2j} & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (1 + q^j)^{-1} & 0 & q^j & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ (1 + q^j)^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (6.23)$$

and if k is even, define a $k \times k$ matrix

$$\mathcal{A}_{(j)} = (1 + q^j) \begin{bmatrix} 0 & 1 & 0 & q^{2j} & \cdots & 0 & q^{(k-2)j} \\ (1 + q^j)^{-1} & 0 & q^j & 0 & \cdots & q^{(k-3)j} & 0 \\ 0 & 1 & 0 & q^{2j} & \swarrow & 0 & 0 \\ (1 + q^j)^{-1} & 0 & q^j & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (1 + q^j)^{-1} & 0 & q^j & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ (1 + q^j)^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (6.24)$$

With this, it is now clear that for any $j \geq J$,

$$J_{\mathbf{h}^{(j+1)}} = J_{\mathbf{h}^{(j)}} \mathcal{A}_{(j+1)} \quad (6.25)$$

and therefore,

$$J_{\mathbf{h}^{(j)}} = \mathcal{A}_{(j+1)} \mathcal{A}_{(j+2)} \cdots \mathcal{A}_{(j)}. \quad (6.26)$$

The recursion for the h_l 's is the “inverse” of the recursion for the B_r 's, in the precise sense that

$$\mathcal{B}_{(j)} = (\mathcal{A}_{(j)})^{-1}. \quad (6.27)$$

This gives

$$\mathcal{A}_{(j+1)} \mathbf{B}_{(j+1)} = \mathbf{B}_{(j)} \quad (6.28)$$

for $j \geq 0$ and

$$\mathbf{B}_{(J)} = \mathcal{A}_{(J+1)} \mathcal{A}_{(J+2)} \cdots \mathcal{A}_{(j)} \mathbf{B}_{(j)} \quad (6.29)$$

for any $j > J$.

Consider also the $(k-1) \times k$ matrix

$$\tilde{\mathcal{B}}_{(j)} = (1 + q^{j+2})^{-1} \begin{bmatrix} 1 & 0 & q^{j+2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & q^{j+2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \searrow & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & q^{j+2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (6.30)$$

Then for any $j \geq 0$,

$$\tilde{\mathbf{B}}_{(j)} = \tilde{\mathcal{B}}_{(j)} \mathbf{B}_{(j)}. \quad (6.31)$$

We say that a sequence of formal power series (or polynomials) $\{P_t(q) = \sum_{s \geq 0} p_{t,s} q^s\}_{t \geq 0}$, with $p_{t,s} \in \mathbb{Z}$, has a limit if each of the sequences $\{p_{t,s}\}_{t \geq 0}$ has a limit as $t \rightarrow \infty$ (that is, the coefficients stabilize). The limit of $\{P_t(q)\}$, if it exists, is denoted by $\lim_{t \rightarrow \infty} P_t(q)$, and is defined by:

$$\lim_{t \rightarrow \infty} P_t(q) = \sum_{s \geq 0} \left(\lim_{t \rightarrow \infty} p_{t,s} \right) q^s. \quad (6.32)$$

Due to the pure powers of q appearing in the right-hand sides of (6.13)–(6.16), it is clear that if $l > 2$, then

$$\lim_{j \rightarrow \infty} J_i h_l^{(j)} = 0. \quad (6.33)$$

For any integer i , let

$$\bar{i} \in \{1, 2\} \text{ be such that } i \equiv \bar{i} \pmod{2} \quad (6.34)$$

and let

$$\bar{i}' \in \{1, 2\} \text{ be such that } i \not\equiv \bar{i}' \pmod{2}. \quad (6.35)$$

Now we analyze the cases $l = 1$ and $l = 2$. First let k be odd. Let t be a nonnegative integer and let τ be any integer sufficiently larger than t . Then because of the Empirical Hypothesis, (6.7), (6.8) and (6.33), we conclude that the coefficient of q^t in $B_{(k-1)j+i}$ is the same as the coefficient of q^t in ${}_i^J h_i^{(\tau)}$. This shows that the coefficient of any power of q in ${}_i^J h_i^{(j)}$ stabilizes as $j \rightarrow \infty$. Hence

$$\lim_{j \rightarrow \infty} {}_i^J h_i^{(j)}(q) \quad (6.36)$$

exists, which we denote by

$${}_i^J h^{(\infty)}(q). \quad (6.37)$$

Now let k be even. Consider the sequence of polynomials

$${}_i^J h_i^{(J)}(q), {}_i^J h_{i'}^{(J+1)}(q), {}_i^J h_i^{(J+2)}(q), {}_i^J h_{i'}^{(J+3)}(q), \dots \quad (6.38)$$

The general term of this sequence is ${}_i^J h_{j-J+i}^{(j)}(q)$ for $j \geq J$. Similarly as above, because of the Empirical Hypothesis, (6.7), (6.8) and (6.33), the coefficient of q^t in $B_{(k-1)j+i}$ is the same as the coefficient of q^t in ${}_i^J h_{j'-J+i}^{(j')}$ for any j' sufficiently larger than t . This shows that the coefficient of any power of q in ${}_i^J h_{j-J+i}^{(j)}$ stabilizes as $j \rightarrow \infty$. Hence

$$\lim_{j \rightarrow \infty} {}_i^J h_{j-J+i}^{(j)}(q) \quad (6.39)$$

exists, which we again denote by

$${}_i^J h^{(\infty)}(q). \quad (6.40)$$

It is worth noting that when k is odd, $\left\{ {}_i^J h_{i'}^{(j)} \right\}_{j \geq J}$ is a sequence of zero polynomials, and when k is even, $\left\{ {}_i^J h_{j-J+i}^{(j)} \right\}_{j \geq J}$ is again a sequence of zero polynomials. With this discussion, we have in fact proved one form of our main theorem:

Theorem 6.3. For any $J \geq 0$ and $i \in \{1, \dots, k\}$,

$$B_{(k-1)J+i} = {}_i^J h^{(\infty)}(q). \quad (6.41)$$

Proof. Let k be odd. It follows from the Empirical Hypothesis, (6.7), (6.8), (6.33) and the preceding discussion that

$$\begin{aligned} B_{(k-1)J+i} &= \left(\lim_{j \rightarrow \infty} {}_i^J h_i^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+i} \right) + \left(\lim_{j \rightarrow \infty} {}_i^J h_{i'}^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+i'} \right) \\ &\quad + \left(\lim_{j \rightarrow \infty} {}_i^J h_3^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+3} \right) + \dots + \left(\lim_{j \rightarrow \infty} {}_i^J h_k^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+k} \right) \\ &= \left(\lim_{j \rightarrow \infty} {}_i^J h_i^{(j)} \right) \cdot 1 + 0 + \dots + 0 \\ &= {}_i^J h^{(\infty)}. \end{aligned}$$

Similarly, if k is even,

$$\begin{aligned} B_{(k-1)J+i} &= \left(\lim_{j \rightarrow \infty} {}_i^J h_{j-J+i}^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+j-J+i} \right) + \left(\lim_{j \rightarrow \infty} {}_i^J h_{j'-J+i}^{(j')} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+j'-J+i'} \right) \\ &\quad + \left(\lim_{j \rightarrow \infty} {}_i^J h_3^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+3} \right) + \dots + \left(\lim_{j \rightarrow \infty} {}_i^J h_k^{(j)} \right) \left(\lim_{j \rightarrow \infty} B_{(k-1)j+k} \right) \\ &= \left(\lim_{j \rightarrow \infty} {}_i^J h_{j-J+i}^{(j)} \right) \cdot 1 + 0 + \dots + 0 \end{aligned}$$

$$= {}_i^J h^{(\infty)}.$$

For both of the cases, in the first equalities, only the first two terms depend on the parity of i . The rest of the terms, corresponding to the indices $3, \dots, k$, appear in their natural order. \square

7. COMBINATORIAL INTERPRETATIONS

In this section, we will give combinatorial interpretations of various polynomials and formal power series discussed thus far. In particular, we will prove the Andrews-Bressoud identities in this section.

For $j \geq J+1$ and for $i, l \in \{1, \dots, k\}$, ${}_i^J h_l^{(j)}$ is a polynomial with nonnegative integral coefficients. Write

$${}_i^J h_l^{(j)} = \sum_{n \geq 0} {}_i^J h_l^{(j)}(n) q^n.$$

Theorem 7.1. If k is odd, then ${}_i^J h_l^{(j)} = 0$ for $l \not\equiv i \pmod{2}$, and if k is even, then ${}_i^J h_l^{(j)} = 0$ for $j - J + l \not\equiv i \pmod{2}$. Otherwise, ${}_i^J h_l^{(j)}(n)$ equals the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that:

- (1) $\pi_t - \pi_{t+k-1} \geq 2$ for all positive integers t for which $t + k - 1 \leq \ell(\pi)$,
- (2) $\pi_1 \leq j$,
- (3) $m_j(\pi) \in \{l - 2, l - 1\} \cap \mathbb{N}$,
- (4) $\pi_{\ell(\pi)} > J$,
- (5) $m_{J+1}(\pi) \leq k - i$,
- (6) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv (k - 1)j + l - k \pmod{2}$ for all positive integers t for which $t + k - 2 \leq \ell(\pi)$.

Proof. From Proposition 6.2 it is clear that:

- (i) If k is odd and if ${}_i^J h_{l'}^{(j)} = 0$ for all l' with $l' \not\equiv i \pmod{2}$, then ${}_i^J h_l^{(j+1)} = 0$ for all l with $l \not\equiv i \pmod{2}$.
- (ii) If k is even and if ${}_i^J h_{l'}^{(j)} = 0$ for all l' with $j - J + l' \not\equiv i \pmod{2}$, then ${}_i^J h_l^{(j+1)} = 0$ for all l with $j - J + l \equiv i \pmod{2}$, i.e., $(j + 1) - J + l \not\equiv i \pmod{2}$.

For the remaining cases, that is, when k is odd and $l \equiv i \pmod{2}$, or when k is even and $j - J + l \equiv i \pmod{2}$, it is now enough to show that the generating functions for the partitions described above have the same recursions and initial conditions as ${}_i^J h_l^{(j)}$.

We say that a partition π is of type $(n; k - 1, j, S, J, k - i, p)$ if it partitions n and satisfies conditions (1) – (6) above with the set $\{l - 2, l - 1\}$ in (3) replaced by the set S and the congruence in (5) replaced by $\equiv p \pmod{2}$. Also let $|\dots|$ be the number of partitions of type (\dots) .

First, suppose that $l \in \{1, \dots, k\}$ and that π is a partition of type $(n; k - 1, j + 1, \{l - 2, l - 1\} \cap \mathbb{N}, J, k - i, (k - 1)(j + 1) + l - k)$. We claim that for such a π , $m_j(\pi) \leq k - l$. Indeed, it is clear that condition (1) forces $m_{j+1}(\pi) + m_j(\pi) \leq k - 1$. Now, in the case that $m_{j+1}(\pi) = l - 1$ and $m_j(\pi) = k - l$, $(l - 1)(j + 1) + (k - l)j = (k - 1)(j + 1) + l - k$ (and hence, congruent mod 2). But when $l > 1$, $m_{j+1}(\pi) = l - 2$ and $m_j(\pi) = k - l + 1$, $(l - 2)(j + 1) + (k - l + 1)j = (k - 1)(j + 1) + l - k - 1$ making $(l - 2)(j + 1) + (k - l + 1)j$ and $(k - 1)(j + 1) + l - k$ noncongruent mod 2, thereby forcing $m_j(\pi) < k - l + 1$. With this analysis, it is clear that for such a π , $m_j(\pi) \leq k - l$, and in fact, $m_j(\pi)$ can assume any value from $\{0, \dots, k - l\}$.

Now we formulate several cases. The proofs in all the cases follow the same pattern, but for ease of comprehension, we spell out each case in detail.

- (a) k is odd and $l > 1$:

In this case, the parity condition (5) is independent of j since k is odd, i.e.,

$$(k - 1)(j + 1) + l - k \equiv (k - 1)j + l - k \pmod{2}.$$

Also, the disjoint union of the sets $\{t-2, t-1\} \cap \mathbb{N}$ as t varies over $1 \leq t \leq k-l+1$ with the restriction that $t \equiv l \pmod{2}$ is exactly $\{0, \dots, k-l\}$. Therefore,

$$\begin{aligned}
& |(n; k-1, j+1, \{l-2, l-1\}, J, k-i, (k-1)(j+1) + l - k)| \\
&= |(n - (j+1)(l-1); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)(j+1) + l - k)| \\
&\quad + |(n - (j+1)(l-2); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)(j+1) + l - k)| \\
&= |(n - (j+1)(l-1); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)j + l - k)| \\
&\quad + |(n - (j+1)(l-2); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)j + l - k)| \\
&= \sum_{\substack{1 \leq t \leq k-l+1 \\ t \equiv l \pmod{2}}} |(n - (j+1)(l-1); k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j + t - k)| \\
&\quad + \sum_{\substack{1 \leq t \leq k-l+1 \\ t \equiv l \pmod{2}}} |(n - (j+1)(l-2); k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j + t - k)|.
\end{aligned}$$

(b) k is odd and l is 1:

Note again that, $(k-1)(j+1) + 1 - k \equiv (k-1)j + 1 - k \pmod{2}$ since k is odd and the disjoint union of the sets $\{t-2, t-1\} \cap \mathbb{N}$ as t varies over $1 \leq t \leq k$ with $t \equiv 1 \pmod{2}$ is $\{0, \dots, k-1\}$. Therefore,

$$\begin{aligned}
& |(n, k-1, j+1, \{0\}, J, k-i, (k-1)(j+1) + 1 - k)| \\
&= |(n, k-1, j, \{0, \dots, k-1\}, J, k-i, (k-1)j + 1 - k)| \\
&= \sum_{\substack{1 \leq t \leq k \\ t \equiv 1 \pmod{2}}} |(n; k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j + t - k)|.
\end{aligned}$$

(c) k is even and $l > 1$:

In this case, $(k-1)(j+1) + l - k \not\equiv (k-1)j + l - k \pmod{2}$, or $(k-1)(j+1) + l - k \equiv (k-1)j + (l+1) - k \pmod{2}$. Also, the disjoint union of the sets $\{t-2, t-1\} \cap \mathbb{N}$ as t varies over $1 \leq t \leq k-l+1$ with $t \not\equiv l \pmod{2}$ is $\{0, \dots, k-l\}$. Therefore,

$$\begin{aligned}
& |(n; k-1, j+1, \{l-2, l-1\}, J, k-i, (k-1)(j+1) + l - k)| \\
&= |(n - (j+1)(l-1); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)(j+1) + l - k)| \\
&\quad + |(n - (j+1)(l-2); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)(j+1) + l - k)| \\
&= |(n - (j+1)(l-1); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)j + (l+1) - k)| \\
&\quad + |(n - (j+1)(l-2); k-1, j, \{0, \dots, k-l\}, J, k-i, (k-1)j + (l+1) - k)| \\
&= \sum_{\substack{1 \leq t \leq k-l+1 \\ t \not\equiv l \pmod{2}}} |(n - (j+1)(l-1); k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j + t - k)| \\
&\quad + \sum_{\substack{1 \leq t \leq k-l+1 \\ t \not\equiv l \pmod{2}}} |(n - (j+1)(l-2); k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j + t - k)|.
\end{aligned}$$

(d) k is even and l is 1:

Similarly as in the previous case, $(k-1)(j+1) + 1 - k \equiv (k-1)j - k \pmod{2}$ since k is even and the disjoint union of the sets $\{t-2, t-1\} \cap \mathbb{N}$ as t varies over $1 \leq t \leq k$ with $t \not\equiv 1 \pmod{2}$ is $\{0, \dots, k-1\}$. Therefore,

$$\begin{aligned}
& |(n, k-1, (j+1), \{0\}, J, k-i, (k-1)(j+1) + 1 - k)| \\
&= |(n, k-1, j, \{0, \dots, k-1\}, J, k-i, (k-1)j - k)|
\end{aligned}$$

$$= \sum_{\substack{1 \leq t \leq k \\ t \not\equiv 1 \pmod{2}}} |(n; k-1, j, \{t-2, t-1\} \cap \mathbb{N}, J, k-i, (k-1)j+t-k)|.$$

The discussion above leads us to the conclusion that the generating functions for the partitions satisfying conditions (1) – (6) above have the same recursions as the polynomials ${}_i^J h_l^{(j)}$.

It is easy to list the partitions satisfying the conditions (1) – (6) when $j = J+1$, and similarly, using Proposition 6.2 and equation (6.12), it is easy to find the polynomials ${}_i^J h_l^{(J+1)}$. The following cases show that the initial values match those of the generating functions:

(1) k odd, i odd:

$${}_i^J \mathbf{h}^{(J+1)} = [1, 0, q^{J+1} + q^{2(J+1)}, 0, q^{3(J+1)} + q^{4(J+1)}, 0, \dots, 0, q^{(k-i-1)(J+1)} + q^{(k-i)(J+1)}, 0, \dots, 0].$$

(2) k odd, i even:

$${}_i^J \mathbf{h}^{(J+1)} = [0, 1 + q^{J+1}, 0, q^{2(J+1)} + q^{3(J+1)}, 0, \dots, 0, q^{(k-i-1)(J+1)} + q^{(k-i)(J+1)}, 0, \dots, 0].$$

(3) k even, i odd:

$${}_i^J \mathbf{h}^{(J+1)} = [0, 1 + q^{J+1}, 0, q^{2(J+1)} + q^{3(J+1)}, 0, \dots, 0, q^{(k-i-1)(J+1)} + q^{(k-i)(J+1)}, 0, \dots, 0].$$

(4) k even, i even:

$${}_i^J \mathbf{h}^{(J+1)} = [1, 0, q^{J+1} + q^{2(J+1)}, 0, q^{3(J+1)} + q^{4(J+1)}, 0, \dots, 0, q^{(k-i-1)(J+1)} + q^{(k-i)(J+1)}, 0, \dots, 0].$$

□

With this, we are now ready to give combinatorial interpretations of all of the official series and the ghosts.

Theorem 7.2. For $r = (k-1)J + i$, with $i \in \{1, \dots, k\}$,

$$B_r = \sum_{n \geq 0} b_{k,r}(n) q^n,$$

where $b_{k,r}(n)$ denotes the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that:

- $\pi_t - \pi_{t+k-1} \geq 2$,
- $\pi_{\ell(\pi)} \geq J+1$,
- $m_{J+1}(\pi) \leq k-i$,
- $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv r+k \equiv (k-1)J+i+k \pmod{2}$.

Proof. Let k be odd. Recall the notation (6.34). Using Theorem 7.1, we see that ${}_i^J h_{\bar{i}}^{(j)}(q)$ is the generating function for partitions satisfying:

- (1) $\pi_t - \pi_{t+k-1} \geq 2$ for all positive integers t for which $t+k-1 \leq \ell(\pi)$,
- (2) $\pi_1 \leq j$,
- (3) $m_j(\pi) \in \{\bar{i}-2, \bar{i}-1\} \cap \mathbb{N}$,
- (4) $\pi_{\ell(\pi)} > J$,
- (5) $m_{J+1}(\pi) \leq k-i$,
- (6) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \dots + \pi_{t+k-2} \equiv (k-1)j + \bar{i} - k \pmod{2}$ for all positive integers t for which $t+k-2 \leq \ell(\pi)$.

Now, since $k-1$ is even, and since $\bar{i} \equiv i \pmod{2}$, we see that $(k-1)j + \bar{i} - k \equiv (k-1)J + i - k \pmod{2}$.

Also, regardless of the value of \bar{i} , ${}_i^J h_{\bar{i}}^{(j)}(q)$ equals, up to the coefficient of q^j , with the generating function for partitions satisfying conditions (1) – (6) above, but with condition (3) replaced by:

$$m_j(\pi) = 0.$$

We conclude that B_r , which equals ${}_i^J h^{(\infty)}$ by Theorem 6.3, is the generating function for partitions satisfying the required conditions.

Now let k be even. Using Theorem 7.1, we see that ${}_i^J h_{j-J+i}^{(j)}(q)$ is the generating function for partitions satisfying:

- (1) $\pi_t - \pi_{t+k-1} \geq 2$ for all positive integers t for which $t+k-1 \leq \ell(\pi)$,
- (2) $\pi_1 \leq j$,
- (3) $m_j(\pi) \in \{\overline{j-J+i-2}, \overline{j-J+i-1}\} \cap \mathbb{N}$,
- (4) $\pi_{\ell(\pi)} > J$,
- (5) $m_{J+1}(\pi) \leq k-i$,
- (6) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv (k-1)j + \overline{j-J+i} - k \pmod{2}$ for all positive integers t for which $t+k-2 \leq \ell(\pi)$.

Since k is even, $(k-1)j + \overline{j-J+i} - k \equiv -J+i \equiv (k-1)J+i-k \pmod{2}$. Just like before, regardless of the value of $\overline{j-J+i}$, ${}_i^J h_{j-J+i}^{(j)}(q)$ equals, up to the coefficient of q^j , with the generating function for partitions satisfying conditions (1) – (6) above, but with condition (3) replaced by:

$$m_j(\pi) = 0.$$

We conclude that B_r , which equals ${}_i^J h^{(\infty)}$ by Theorem 6.3, is the generating function for partitions satisfying the required conditions. \square

Specializing $J = 0$ in Theorem 7.2, we arrive at the Andrews-Bressoud identities:

Theorem 7.3. For $i \in \{1, \dots, k\}$,

$$B_i = \sum_{n \geq 0} b_{k,i}(n)q^n,$$

where $b_{k,i}(n)$ denotes the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that:

- (1) $\pi_t - \pi_{t+k-1} \geq 2$,
- (2) $m_1(\pi) \leq k-i$,
- (3) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv i+k \pmod{2}$.

We now present the combinatorial interpretation of the ghosts.

Theorem 7.4. For $r = (k-1)J+i$, with $i \in \{2, \dots, k\}$,

$$\tilde{B}_r = \sum_{n \geq 0} \tilde{b}_{k,r}(n)q^n,$$

where $\tilde{b}_{k,r}(n)$ denotes the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_{\ell(\pi)})$ of n such that:

- (1) $\pi_t - \pi_{t+k-1} \geq 2$,
- (2) $\pi_{\ell(\pi)} \geq J+1$,
- (3) $m_{J+1}(\pi) \leq k-i$,
- (4) $\pi_t - \pi_{t+k-2} \leq 1$ only if $\pi_t + \pi_{t+1} + \cdots + \pi_{t+k-2} \equiv r+k+1 \equiv (k-1)J+i+k+1 \pmod{2}$.

Proof. Using the left- and right-hand sides of (2.8) and (2.9) and making appropriate index changes, we find that for $J \geq 0$ and $i \in \{2, \dots, k-1\}$,

$$\tilde{B}_{(k-1)J+k} = B_{(k-1)(J+1)+2}, \quad (7.1)$$

$$\tilde{B}_{(k-1)J+i} = q^{(J+1)(k-i)} B_{(k-1)(J+1)+k-i+2} + B_{(k-1)J+i+1}. \quad (7.2)$$

Let us first consider the case of $\tilde{B}_{(k-1)J+k}$. Let π be a partition counted in $\tilde{b}_{k,(k-1)J+k}(n)$. Then, conditions (2) and (3) for π imply that $\pi_{\ell(\pi)} \geq J+2$. Moreover, if $m_{J+2}(\pi) = k-1$, then $(k-1)(J+2) \not\equiv (k-1)J+k+k+1 \pmod{2}$, and therefore condition (4) gets violated. Hence, $m_{J+2}(\pi) \leq k-2$, and π satisfies exactly the conditions satisfied by the partitions counted in $b_{k,(k-1)(J+1)+2}(n)$.

Now we consider the case of $\tilde{B}_{(k-1)J+i}$ for $i \in \{2, \dots, k-1\}$. Let π be a partition counted in $\tilde{b}_{k,(k-1)J+i}(n)$. If $m_{J+1}(\pi) = k-i$, then $m_{J+2}(\pi) < i-1$, because $m_{J+2}(\pi) = i-1$ implies that $(J+1)(k-i) + (J+2)(i-1) \equiv (k-1)J+i+k \pmod{2}$, thereby violating the condition (4) for π . Moreover, both $B_{(k-1)(J+1)+k-i+2}$ and $B_{(k-1)J+i+1}$ have the same ‘‘parity’’ condition as that of π , because $(k-1)(J+1) + k-i+2+k \equiv (k-1)J+k+i+1 \pmod{2}$. Therefore, if $m_J(\pi) = k-i$, it gets counted in $q^{(J+1)(k-i)}B_{(k-1)(J+1)+k-i+2}$, else if $m_J(\pi) < k-i$, it is counted in $B_{(k-1)J+i+1}$. \square

Remark 7.5. Recall from Remark 2.1 that it was not necessary to define \tilde{B}_1 . We may now define it as

$$\tilde{B}_1 = B_2, \quad (7.3)$$

in the light of equation (7.1) (which corresponds to the $i = k$ case of Theorem 7.4), with $J = -1$. With this, the statement of Theorem 7.4 can be seen to hold for \tilde{B}_1 as well.

Remark 7.6. (Cf. Remark 2.1 in [LZ].) As discussed in [AB], [R] and [A4], we can give an alternate, shorter proof of Theorems 7.3 and 7.4 using only the Empirical Hypothesis and without using the combinatorial interpretation of the ${}_i^j h_i^{(j)}$ polynomials as given in Proposition 7.1. Let $D_1(q), D_2(q), \dots$ and $\tilde{D}_1(q), \tilde{D}_2(q), \dots$ be sequences of formal power series in q with constant term 1 satisfying the recursions (2.8) and (2.9) (with $D_p(q)$ ’s in place of the $B_p(q)$ ’s and $\tilde{D}_p(q)$ ’s in place of the $\tilde{B}_p(q)$ ’s), and suppose that the Empirical Hypothesis holds for $D_1(q), D_2(q), \dots$ and $\tilde{D}_1(q), \tilde{D}_2(q), \dots$. Then, in fact, $D_1(q), D_2(q), \dots, D_k(q)$ and $\tilde{D}_1(q), \tilde{D}_2(q), \dots$ are uniquely determined by these recursions and the Empirical Hypothesis. Now, on the one hand, it is easy to see that the official sum sides and the ghost sum sides, i.e., the combinatorial generating functions appearing in the statements of Theorems 7.2 and 7.4, respectively, trivially satisfy the recursions just mentioned and the Empirical Hypothesis as well. On the other hand, the sequences $B_1(q), B_2(q), \dots$ and $\tilde{B}_1(q), \tilde{B}_2(q), \dots$ satisfy the recursions by definition and we have proved in Section 5 that these sequences also satisfy the Empirical Hypothesis. Therefore, by the aforementioned uniqueness, it must be that $B_r(q)$ and $\tilde{B}_r(q)$ equal the combinatorial generating functions appearing in the statements of Theorems 7.2 and 7.4 respectively.

8. AN (x, q) -DICTIONARY

In this section, we obtain a dictionary between our official expressions on the various shelves with relevantly specialized \tilde{J} expressions appearing in [CoLoMa]. Note that this section is concerned only with the official series. First we let $a \mapsto 0$ in [CoLoMa]. Thus we get: For $i \in \{1, \dots, k\}$,

$$\tilde{H}_{k,i}(x, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+n-in} x^{(k-1)n} (1-x^i q^{2ni}) (-x)_n}{(-q)_n (q)_n (xq^n)_\infty}, \quad (8.1)$$

from (1.6) in [CoLoMa]. Therefore, using (1.5) in [CoLoMa],

$$\tilde{J}_{k,i}(x, q) = \tilde{H}_{k,i}(xq, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+(k-i)n} x^{(k-1)n} (1-x^i q^{(2n+1)i}) (-xq)_n}{(-q)_n (q)_n (xq^{n+1})_\infty} \quad (8.2)$$

$$\tilde{J}_{k,i}(q^j, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2+((k-1)j+k-i)n} (1-q^{(2n+j+1)i}) (-q^{j+1})_n}{(-q)_n (q)_n (q^{n+j+1})_\infty}. \quad (8.3)$$

Now we fill the ‘‘gap’’ in the denominator by multiplying and dividing the n -th summand by $(1-q^{n+1}) \dots (1-q^{n+j})$:

$$\frac{(-q^{j+1})_n}{(-q)_n (q)_n (q^{n+j+1})_\infty} = \frac{(1+q^{j+1}) \dots (1+q^{j+n}) (1-q^{n+1}) \dots (1-q^{n+j})}{(1+q) \dots (1+q^n) (q)_\infty}. \quad (8.4)$$

If $n > j$, we cancel the extra terms in the denominator in the right-hand side, arriving at

$$\begin{aligned} \frac{(-q^{j+1})_n}{(-q)_n(q)_n(q^{n+j+1})_\infty} &= \frac{(1+q^{n+1}) \cdots (1+q^{j+n})(1-q^{n+1}) \cdots (1-q^{n+j})}{(1+q) \cdots (1+q^j)(q)_\infty} \\ &= \frac{(1-q^{2(n+1)}) \cdots (1-q^{2(n+j)})}{(1+q) \cdots (1+q^j)(q)_\infty}. \end{aligned}$$

Otherwise, if $n \leq j$, we multiply and divide by $(1+q^{n+1}) \cdots (1+q^j)$ to arrive at

$$\begin{aligned} \frac{(-q^{j+1})_n}{(-q)_n(q)_n(q^{n+j+1})_\infty} &= \frac{(1+q^{n+1}) \cdots (1+q^j)(1+q^{j+1}) \cdots (1+q^{j+n})(1-q^{n+1}) \cdots (1-q^{n+j})}{(1+q) \cdots (1+q^n)(1+q^{n+1}) \cdots (1+q^j)(q)_\infty} \\ &= \frac{(1-q^{2(n+1)}) \cdots (1-q^{2(n+j)})}{(1+q) \cdots (1+q^j)(q)_\infty}. \end{aligned}$$

Putting these cases together and making the change $i \mapsto k - i + 1$, we conclude that

$$\tilde{J}_{k,k-i+1}(q^j, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + ((k-1)j+i-1)n} (1 - q^{2(n+j+1)(k-i+1)}) (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(q)_\infty (1+q) \cdots (1+q^j)}.$$

Theorem 4.1 now yields the following ‘‘dictionary’’:

$$B_{(k-1)j+i} = \tilde{J}_{k,k-i+1}(q^j, q) \quad (8.5)$$

for $j \geq 0$ and $i \in \{1, \dots, k\}$.

9. AN (x, q) -EXPRESSION GOVERNING THE GHOSTS

In this section, we derive a (new) (x, q) -expression governing the ghost series. Specializing x to successively higher powers of q in this expression gives us back the closed-form expressions for the ghosts already obtained in Theorem 4.1.

For $i \in \{2, \dots, k-1\}$, let

$$\tilde{\tilde{J}}_{k,i}(x, q) = \frac{\tilde{\tilde{J}}_{k,i+1}(x, q) + xq\tilde{\tilde{J}}_{k,i-1}(x, q)}{1+xq}, \quad (9.1)$$

and for $i = 1$, let

$$\tilde{\tilde{J}}_{k,1}(x, q) = \frac{\tilde{\tilde{J}}_{k,2}(x, q)}{1+xq}. \quad (9.2)$$

Equations (8.5), (2.10), (2.11) immediately yield

$$\tilde{B}_{(k-1)j+i} = \tilde{\tilde{J}}_{k,k-i+1}(q^j, q). \quad (9.3)$$

Hence for $i \in \{2, \dots, k-1\}$, from (8.2),

$$\begin{aligned} \tilde{\tilde{J}}_{k,i}(x, q) &= \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + (k-i-1)n} x^{(k-1)n} (1 - x^{i+1} q^{(2n+1)(i+1)}) (-xq)_n}{(-q)_n (q)_n (xq^{n+1})_\infty (1+xq)} \\ &\quad + \sum_{n \geq 0} (-1)^n \frac{q^{1+kn^2 + (k-i+1)n} x^{1+(k-1)n} (1 - x^{i-1} q^{(2n+1)(i-1)}) (-xq)_n}{(-q)_n (q)_n (xq^{n+1})_\infty (1+xq)} \\ &= \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + (k-i-1)n} x^{(k-1)n} (-xq)_n}{(-q)_n (q)_n (xq^{n+1})_\infty (1+xq)} \\ &\quad \cdot \left(1 - x^{i+1} q^{(2n+1)(i+1)} + xq^{2n+1} (1 - x^{i-1} q^{(2n+1)(i-1)}) \right). \end{aligned}$$

Noting that

$$1 - x^{i+1} q^{(2n+1)(i+1)} + xq^{2n+1} (1 - x^{i-1} q^{(2n+1)(i-1)}) = 1 - x^{i+1} q^{(2n+1)(i+1)} + xq^{2n+1} - x^i q^{(2n+1)i}$$

$$= (1 - x^i q^{(2n+1)i})(1 + xq^{2n+1}),$$

we arrive at

$$\tilde{J}_{k,i}(x, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + (k-i-1)n} x^{(k-1)n} (1 - x^i q^{(2n+1)i}) (-xq)_n (1 + xq^{2n+1})}{(-q)_n (q)_n (xq^{n+1})_\infty (1 + xq)} \quad (9.4)$$

for $i \in \{2, \dots, k-1\}$. As for $i = 1$, (9.2) gives

$$\tilde{J}_{k,1}(x, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + (k-2)n} x^{(k-1)n} (1 - x^2 q^{2(2n+1)}) (-xq)_n}{(-q)_n (q)_n (xq^{n+1})_\infty (1 + xq)},$$

which equals (9.4) with $i = 1$.

We conclude that

$$\tilde{J}_{k,i}(x, q) = \sum_{n \geq 0} (-1)^n \frac{q^{kn^2 + (k-i-1)n} x^{(k-1)n} (1 - x^i q^{(2n+1)i}) (-xq)_n (1 + xq^{2n+1})}{(-q)_n (q)_n (xq^{n+1})_\infty (1 + xq)} \quad (9.5)$$

for $i \in \{1, \dots, k-1\}$.

Remark 9.1. It is quite straightforward to generalize the reverse-engineering procedure of this section, i.e., replacing judiciously chosen instances of pure powers of q with x . This can be done throughout the proof of Theorem 4.1, and in this way our motivated proof would yield an “ (x, q) -proof” similar in spirit to the ones in Chapter 7 of [A3], but this time involving two “ J ” expressions, namely, \tilde{J} and \tilde{J} . A similar observation (without ghosts, of course) can be found in Section 5 of [AB], for the original case of the Rogers-Ramanujan identities.

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