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Controllability, Observability and Realizability

Inna N. Smith

CONTROLLABILITY, OBSERVABILITY AND REALIZABILITY

by

INNA SMITH

(Under the Direction of Donald W. Fausett)

ABSTRACT

In this research project we analyze three important concepts of the control theory, including controllability, observability and realizability. In particular, we analyze the theory of and solutions to linear time systems, continuous and discrete time systems, time-varying systems, and first-order matrix Sylvester systems. Examples and sample computed calculations are provided for both uncontrolled and controlled systems. We also develop a new method for computing the approximate values of matrix exponentials, and use this to estimate solutions to initial value problems.

INDEX WORDS: Control Theory, Controllability, Observability, Realizability, Duality, Transfer Function, Matrix Exponential Function, Fundamental Matrix Solution, Sylvester Systems.

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INTRODUCTION

1.1 Overview of Project

The aim of this research report is to present some fundamental theoretical and practical (computational) results concerning the controllability, observability and realizability criteria associated with linear time invariant systems, continuous-time systems, discrete-time systems, time-varying systems, and first-order matrix Sylvester systems. This paper is organized as follows.

In Chapter 2 we introduce classical (fundamental) control theory. We will restrict ourselves to the case of linear time invariant systems having scalar input $x(t)$ and scalar output $y(t)$. The analysis problem is to study the behavior of a given system in specified circumstances. We present the standard techniques of the Laplace transform for continuous time systems to solve n-th order differential equations. We introduce the transfer function and give examples of calculating the poles of the transfer function, or eigenvalues. For discrete time systems we will describe the z-transform in order to solve the difference equation. For higher order systems Laplace and z-transform methods require heavy algebraic manipulation, so it is preferable to discuss the techniques of matrix operations and introduce the state transition matrix and investigate its properties. We apply the described techniques to determine solutions of interesting uncontrolled and controlled systems based on real-life examples.

Next, it is reasonable to study the important theoretical properties of basic systems that one would like to control: namely controllability, observability and realizability. In Chapter 3 we present controllability, which is about how to find controls $\mathbf{u}$ that allow one to attain a desired state $\mathbf{x}$.

Chapter 4 contains the theory and examples about observability, which means that all information about the state $\mathbf{x}$ could possibly be estimated from the output $\mathbf{y}$. Since the controllability and observability are dual properties of the system to be controlled, it enables all discussion for controllability to be applied to observability in a similar way. For a system that is not controllable, which means not all models of the system are controllable, one can perform a decomposition to separate controllable and uncontrollable models, which we will present at the end of Chapter 4.

Later in this project we investigate another important theoretic property of a system with observers and dynamic feedback, realizability, which is a mathematical representation of a system with observers and dynamic feedback. In Chapter 5 we present some realization techniques to compute state representations, and hence also construct a physical system that satisfies given input/output specifications. Following the same logic, we will discuss realizability for constant systems, for discrete systems and time varying systems.

Focusing on solving first order matrix systems leads this paper to discuss some major topics of Lyapunov and Sylvester systems presented in Chapter 6. We will address questions related to input-output (zero-state) behavior on the Sylvester systems, and then present certain sufficient conditions for controllability, observability and realizability. Also, we will establish the general solution of the first order matrix Sylvester system in terms of the fundamental matrix solution of differential equations.

Furthermore, in Chapter 7 we develop a new method of computing the matrix exponential function to solve initial value problems. Some interesting examples serve to illustrate the theoretical part.

The main conclusions are presented in Chapter 8.

1.2 Preliminaries

In the 19th and 20th centuries some branches of applied mathematics were extensively developed. The 'classical' areas of applied mathematics focus on analyzing real-life situations by constructing mathematical descriptions, or *models*, of actual situations. Mathematicians then use mathematical methods to investigate properties of the models and form conclusions about the real-world problems that depend on assumptions made to describe given situations in the formulation of models. If one wants to have a realistic model, one often needs to use complicated mathematical equations, and consequently, it will be more difficult to solve the resulting equations.

The famous branches of applied mathematics such as hydromechanics, solid mechanics, thermodynamics and electromagnetic theory historically emphasized the analysis of a given physical problem. During last few decades, the contemporary world has come to require a quite different approach, the goal being to control the behavior of a physical system in a prescribed (desired) manner.

What is the meaning of the word 'system' here? Its role is somewhat analogous to that played in mathematics by the definition of 'function' as a set of ordered pairs: not itself the object of study, but a necessary foundation upon which the entire theoretical development will rest. A **system** is a collection of objects which are related by interactions and produce various outputs in response to different inputs. Obviously, this statement gives us a very wide range of situations including biological systems such as the human body, or high technology machines such as aircraft, spaceships, or industrial complexes, or economical structures such as countries or regions.

What are some **examples** of control problems associated with such systems? For instance, regulation of body functions such as temperature, blood pressure, heartbeat; automatic landing of aircraft, rendezvous with an artificial satellite, or soft landings

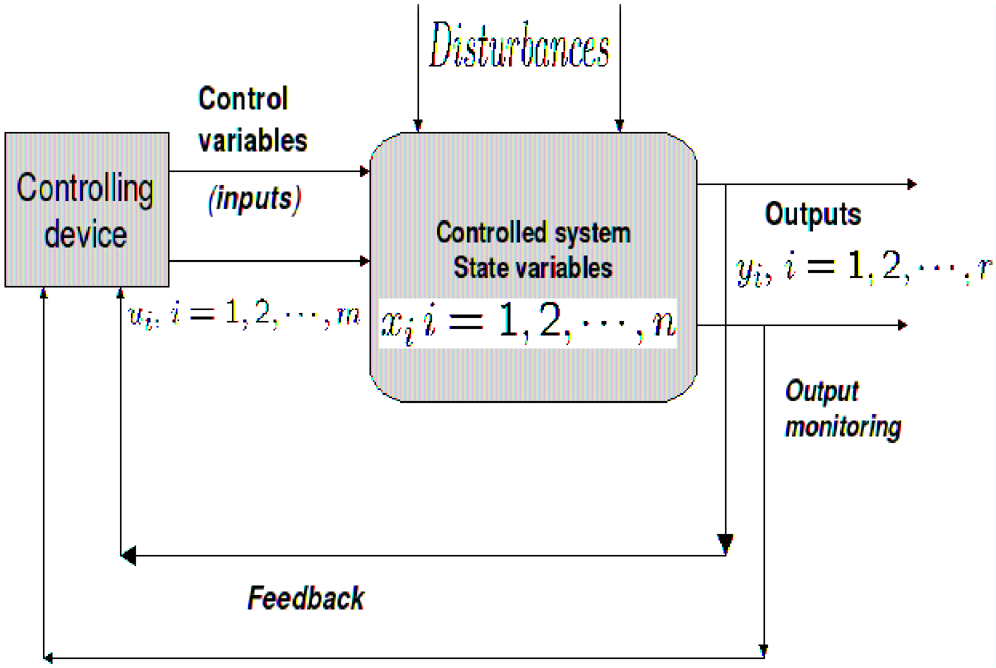

Figure 1.1: Block Diagram of a Control System

on the moon, and problems caused by economic inflation or political elections.

An example of a model for a control system can be represented as in Figure 1.1.

1.3 Notation

State variables x_j $(j = 1, 2, 3, \ldots, n)$ are variables used to describe the condition or state of a system, and can be used to calculate its future behavior from a knowledge of the inputs. **For example,** the state of the effectiveness of a university can be described by many variables, but it is only practical to measure a few of these, such as retention rate, profit of the university, supplement of funding, percentage of graduates finding good jobs and so on.

The **state space** is the n-dimensional vector space containing the state variables x_j. In practice, it is often not possible to determine the values of the state variables directly; therefore, a set of **output variables** y_i $(i = 1, 2, 3, \ldots, m)$ which depend in some way on the x_j, are measured (almost invariably with $m \leq n$). Practically, any system is described by a large number of variables but only a few of these can be observed or measured.

To make a given system perform in a desirable way one needs some controlling device, or 'controller', which in general, manipulates a set of **control variables** u_i $(i = 1, 2, 3 \ldots, m)$, where m is not necessary the same as the number of output variables.

Assumption 1.3.1. *All systems herein considered have the property that, given an initial state and any input, the resulting state and output at a specified later time are uniquely determined.*

Classical control theory considers a **linear time invariant system** having scalar input $x(t)$ and scalar output $y(t)$. Generally, a system is **linear** if, when the response to some input $\mathbf{u}(t)$ is $\mathbf{y}(t) = L\mathbf{u}(t)$, then $c_1 L(\mathbf{u}_1) + c_2 L(\mathbf{u}_2)$ is the response to the input $c_1 \mathbf{u_1} + c_2 \mathbf{u}_2$. Here L is some operator (differential, integral, etc.), the c_i are constants, and the $\mathbf{u}_i$ are vectors (in general). The system is called **time-invariant** if the shape of the output does not change with a delay of the input. Thus the response to $\mathbf{u}(t - \tau)$ is $\mathbf{y}(t - \tau)$, i.e. $L[\mathbf{u}(t - \tau)] = \mathbf{y}(t - \tau)$ for any fixed τ. Since linear mathematics is very well developed, linear time invariant sets of state equations are the easiest to manage analytically and numerically, and it often gives a good reason to construct the first model of a situation to be linear.

The following question naturally arises: is it possible to give an explicit expression for the solution of time varying systems in which the elements of the matrices are continuous function of time? Discussion about time-varying systems will be presented in Chapter 2 and Chapter 5 of this project report.

If a scalar output $y(t)$ is required to be as close as possible to some given **reference signal** $r(t)$, then the control system is called a **servomechanism**. In case the reference signal r is constant, the control system is called a **regulator**. **For example**, a central-heating system is a regulator because it is designed to keep room temperature close to a predetermined value.

An essential first step in many control problems is to determine whether a desired objective can be achieved by manipulating the chosen control variables. If not, then either the objective will have to be modified or else control will have to be applied in some different fashion. **For example,** economists would dearly like to know whether the rate of inflation can be controlled by adjusting taxes, the money supply, bank lending rate, and so on.

The fundamental concepts of controllability, observability and realizability play very important roles in linear state space models. **Controllability** is determining whether or not the states in a state space model can be controlled by input, while **observability** deals with whether or not the initial state can be estimated by observing the output. **Realizability** involves the question of the existence of a corresponding linear state equation.

CHAPTER 2

FUNDAMENTAL CONCEPTS OF CONTROL THEORY

In this chapter we present the fundamental foundation of classical control theory. At the beginning we restrict ourselves to the case of linear time invariant systems having scalar input $x(t)$ and scalar output $y(t)$. We present the standard techniques of the Laplace transform for continuous time systems to solve the n-th order differential equation. Later in this Chapter we introduce the transfer function and give examples of calculating the poles, or eigenvalues, of the transfer function. For discrete time systems we describe the z-transform that is used to solve difference equations. For higher order systems we use the techniques of matrix operations, introduce the state transition matrix, and investigate its properties. Next, we apply the described techniques to determine solutions of uncontrolled and controlled systems for real-life examples. Lastly, we discuss the possibility of obtaining an explicit expression for the solutions of time varying systems in which the elements of the matrices are continuous functions of time.

2.1 Continuous-Time Systems: The Laplace Transform

Consider the case where input x and output y are continuous functions of time. Then the general model of classical linear control theory is the n-th order differential equation

$$y^{(n)} + k_1 y^{(n-1)} + ... + k_{n-1} y^{(1)} + k_n y = \beta_0 x^{(m)} + \beta_1 x^{(m-1)} + ... + \beta_m x \, , \quad (2.1)$$

where k_i and $\beta_{\mathbf{i}}$ are constants, superscript (j) indicates the j-th derivative with respect to time t, n is number of state variables, and m is number of control variables.

Assumption 2.1.1. *We shall assume that:*

(1) $m < n$ *(number of output variables is less then number of state variables);*

(2) the $y^{(i)}(0)$ *and* $x^{(j)}(0)$ *are zero for* $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *(because* x *and* y *are continuous functions of time).*

Using the standard techniques of Laplace transforms, we have

$$k(s)\bar{y}(s) = \beta(s)\bar{x}(s), \tag{2.2}$$

where

$$\bar{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty y(t)e^{-st}dt.$$

The **transfer function** is the ratio of the Laplace transform of the output to that of the input:

$$g(s) = \frac{\beta(s)}{k(s)}\,. \tag{2.3}$$

Typically, it is a rational function, i.e.

$$k(s) = s^n + k_1 s^{n-1} + \dots\dots + k_{n-1}s + k_n$$

and

$$\beta(s) = \beta_0 s^m + \beta_1 s^{m-1} + \dots\dots + \beta_{m-1}s + \beta_m$$

are polynomials. Now, we can rewrite (2.2) as

$$\bar{y}(s) = g(s)\bar{x}(s). \tag{2.4}$$

The zeros of the characteristic polynomial $k(s)$ are called the *poles* of the transfer function, or eigenvalues, and the zeros of $\beta(s)$ are *zeros* of $g(s)$.

Figure 2.1 visualizes the equation (2.4).

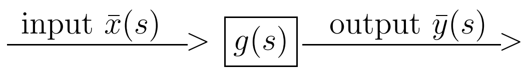

Figure 2.1: Transfer Function Diagram

Note: When we know $x(t)$ (and hence $\bar{x}(s)$), then solution for $y(t)$ requires expansion of the right hand side of equation (2.4) into partial fractions. If the poles of $g(s)$ (eigenvalues) are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, then this expansion involves terms of the form $\dfrac{c_i}{s-\lambda_i}$, where the c_i are constants. These correspond to terms of the form $c_i e^{\lambda_i t}$ in $y(t)$.

2.2 Discrete-Time Systems: z - Transform

In situations where the variables are measured (or sampled) only at discrete intervals of time rather than continuously, we have what are known as discrete-time systems. Real life gives a lot of **examples** of such systems: the temperature of a hospital patient is recorded each hour, the interest on savings accounts is calculated daily or monthly, and so on.

Suppose that the variables are defined at fixed intervals of time 0, T, $2T$, $3T$, ... , where T is a constant. Let use $\mathbf{X}(k)$ and $\mathbf{u}(k)$ to denote the values of the state variables $\mathbf{X}(kT)$ and the control $\mathbf{u}(kT)$, respectively ($k = 0, 1, 2, 3, \ldots$). For linear discrete systems the differential equation (2.1) is replaced by the **difference equation**:

$$\begin{aligned} X(k+n) + k_1 X(k+n-1) + \cdots + k_{n-1} X(k+1) + k_n X(k) \\ = \beta_0 u(k+m) + \beta_1 u(k+m-1) + \cdots + \beta_m u(k). \end{aligned} \tag{2.5}$$

The $\mathbf{z}-$ **transform** of a scalar (or indeed a vector) function $X(k)$ is defined by

$$\tilde{X}(z) := \mathbf{z}\{X(k)\} = \sum_{k=0}^{\infty} X(k) z^{-k} = X(0) + \frac{X(1)}{z} + \frac{X(2)}{z^2} + \cdots$$

for $(k = 0, 1, 2, 3, \ldots)$.

The general solution of (2.5) involves terms of the form $c_i \lambda_i^k$, where c_i is a constant. Application of **z**-transforms to (2.5) gives

$$\tilde{X}(z) = g(z)\bar{u}(z),$$

where the transfer function g is defined by (2.3).

2.3 Spectral form of solution of an uncontrolled system

Laplace and **z** - transform methods for solving continuous-time and discrete-time equations rely heavily on algebraic manipulation, so for higher order systems, it is preferable to apply the techniques of matrix operations. This approach requires one to express the equations in matrix form.

Assumption 2.3.1. *First we consider systems that are free of control variables.*

Such n-th order systems may be described by the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{2.6}$$

where $\mathbf{x}(t)$ is a state vector, $\mathbf{A}$ is a $n \times n$ matrix with constant elements; subject to an initial condition

$$\mathbf{x}(0) = \mathbf{x}_0. \tag{2.7}$$

Assumption 2.3.2. *All the eigenvalues* $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ *of* $\mathbf{A}$ *are distinct.*

Let $\mathbf{w}_i$ be an eigenvector corresponding to λ_i for $1 \leq i \leq n$, then $\mathbf{w}_1, \mathbf{w}_2,\ \ldots\ , \mathbf{w}_n$ are linearly independent, and the solution of (2.6) can be expressed as

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} c_i \mathbf{w}_i, \tag{2.8}$$

where the $c_i(t)$ are scalar functions of time. Differentiation of (2.8) and substitution into (2.6) gives

$$c_i(t) = e^{\lambda_i t} c_i(0), \tag{2.9}$$

and therefore

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i(0) e^{\lambda_i t} \mathbf{w}_i.$$

Suppose $\mathbf{W}$ denotes the matrix with columns

$$\mathbf{w}_1, \mathbf{w}_2, \ldots.., \mathbf{w}_n\,,$$

then the rows

$$\mathbf{v}_1, \mathbf{v}_2, \ldots..., \mathbf{v}_n$$

of $\mathbf{W^{-1}}$ are left eigenvectors of $\mathbf{A}$. If we multiply (2.9) on the left by $\mathbf{v}_i$, and utilize $\mathbf{v}_i\mathbf{w}_i = 1$ and $\mathbf{v}_i\mathbf{w}_j = 0$ if $i \neq j$, then setting $t = 0$ in the resulting expression gives

$$\mathbf{v}_i \mathbf{x}(0) = c_i(0)\,.$$

Thus we have the solution of (2.6) satisfying (2.7) is

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{x}(0) e^{\lambda_i t} \mathbf{w}_i\,.$$

This solution depends only on the initial conditions, and the eigenvalues, and eigenvectors of $\mathbf{A}$. It is known as the **spectral form solution** (where the **spectrum** of $\mathbf{A}$ is the set of eigenvalues $\{\lambda_i\}$).

Example 2.3.3. *The Spectral Form Solution.*

Find the general solution of (2.6) subject to the initial condition (2.7) if matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution:

a) find the spectrum of **A**

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= (1-\lambda)\begin{bmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} + (-1)\begin{bmatrix} 1 & 2-\lambda \\ 2 & 2 \end{bmatrix} \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 4) - (2\lambda - 2) \\
&= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \\
&= -(\lambda_1 - 1)(\lambda - 2 - 2)(\lambda_3 - 3) = 0\,.
\end{aligned}$$

thus the eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3\,,$$

and the spectrum of **A** is $\{\,1\,,2\,,3\,\}$.

b) find the corresponding eigenvectors

for $\lambda_1 = 1$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix},$$

then the eigenvector is

$$\mathbf{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda_2 = 2$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix},$$

then the eigenvector is

$$\mathbf{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

for $\lambda_3 = 3$:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix},$$

then the eigenvector is

$$\mathbf{w}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

c) find the spectral form solution. The general solution is

$$\mathbf{x}(t) = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{x}(0) e^{\lambda_i t} \mathbf{w}_i\,,$$

where matrix $\mathbf{W}$ is

$$\mathbf{W} = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix},$$

then the inverse

$$\mathbf{W}^{-1} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}.$$

Now vectors $\mathbf{v}_i$ are

$$\mathbf{v}_1 = [\,0\ 1\ -\frac{1}{2}\,]\,, \mathbf{v}_2 = [\,1\ 1\ 0\,]\,, \mathbf{v}_3 = [\,1\ 1\ \frac{1}{2}\,]\,.$$

Therefore the spectral form solution of $\mathbf{A}$ is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (\mathbf{x}_2(0) - \frac{1}{2}\mathbf{x}_3(0))\, e^{t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (\mathbf{x}_1(0) + \mathbf{x}_2(0))\, e^{2t} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$
$$+ (\mathbf{x}_1(0) + \mathbf{x}_2(0) + \frac{1}{2}\mathbf{x}_3(0))\, e^{3t} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

2.4 Exponential matrix form of solution of an uncontrolled system

The **exponential matrix** is defined by

$$e^{\mathbf{A}t} := \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots + \frac{\mathbf{A}^{n-1}}{(n-1)!}t^{n-1} + \cdots. \tag{2.10}$$

Use of the exponential matrix provides an alternative to the calculation of the eigenvectors presented in preceding Section 2.4. This approach is based on generalization of the fact that when $n = 1$, i.e. $\mathbf{A}$ is a scalar, the solution of (2.6) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{2.11}$$

The series on the right in (2.10) converges for all finite t and all $n \times n$ matrices $\mathbf{A}$ having finite elements [1]. Clearly

$$e^{\mathbf{A}t}|_{t=0} = e^{\mathbf{0}} = I\,,$$

where $\mathbf{0}$ is the zero matrix, and

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}\, e^{\mathbf{A}t}.$$

Therefore

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

represents a solution of (2.6). Suppose that the initial condition (2.7) in more general form is given by

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{2.12}$$

The solution of (2.6) satisfying (2.12) is frequently written in control theory literature as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)x_0, \tag{2.13}$$

where

$$\mathbf{\Phi}(t, t_0) := e^{\mathbf{A}(t-t_0)} \tag{2.14}$$

is called the **state transition matrix**, since it relates the state at any time t to the state at an initial time t_0 . Furthermore, we verify that $\mathbf{\Phi}(t, t_0)$ has the following properties.

Property I:

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{A}\mathbf{\Phi}(t, t_0) \tag{2.15}$$

Proof. Consider

$$\begin{aligned}
\frac{d}{dt}\mathbf{\Phi}(t, t_0) &= \frac{d}{dt}e^{\mathbf{A}(t-t_0)} \\
&= e^{-\mathbf{A}t_0}\frac{d}{dt}e^{\mathbf{A}t} \\
&= e^{-\mathbf{A}t_0}\mathbf{A}e^{\mathbf{A}t} \\
&= \mathbf{A}e^{\mathbf{A}(t-t_0)} \\
&= \mathbf{A}\mathbf{\Phi}(t, t_0)\,.
\end{aligned}$$

□

Property II:

$$\mathbf{\Phi}(t, t) = \mathbf{I}$$

Proof. Consider

$$\begin{aligned}\mathbf{\Phi}(t,t) &= e^{\mathbf{A}(t-t)} \\ &= e^{\mathbf{A}0} \\ &= \mathbf{I}\,.\end{aligned}$$

□

Property III:

$$\mathbf{\Phi}(t_0,t) = \mathbf{\Phi}^{-1}(t,t_0) \tag{2.16}$$

Proof. Consider

$$\begin{aligned}\mathbf{\Phi}(t,t_0)\mathbf{\Phi}(t_0,t) &= e^{\mathbf{A}(t-t_0)}e^{\mathbf{A}(t_0-t)} \\ &= e^{\mathbf{A}(t-t_0+t_0-t)} \\ &= e^{\mathbf{A0}} = \mathbf{I}\,,\end{aligned}$$

thus

$$\mathbf{\Phi}(t,t_0) = \mathbf{\Phi}^{-1}(t_0,t)$$

□

Property IV:

$$\mathbf{\Phi}(t,t_0) = \mathbf{\Phi}(t,t_1)\mathbf{\Phi}(t_1,t_0)\,. \tag{2.17}$$

Proof. Consider

$$\begin{aligned}\mathbf{\Phi}(t,t_1)\mathbf{\Phi}(t_1,t_0) &= e^{\mathbf{A}(t-t_1)}e^{\mathbf{A}(t_1-t_0)} \\ &= e^{\mathbf{A}t-\mathbf{A}t_1+\mathbf{A}t_1-\mathbf{A}t_0} \\ &= e^{\mathbf{A}(t-t_0)} \\ &= \mathbf{\Phi}(t_1-t_0)\end{aligned}$$

□

We use these properties later in Chapter 3 (section 3.3) for the proof of the sufficiency of condition for controllability.

Next, to obtain one more interesting result, we consider (2.6) subject to the initial condition (2.7) again, and taking the Laplace transform of (2.6), we have

$$\mathcal{L}\{\dot{\mathbf{x}}\} = \mathcal{L}\{\mathbf{A}\mathbf{x}\}\,,$$

$$\mathcal{L}\{\dot{\mathbf{x}}\} = \mathbf{A}\mathcal{L}\{\mathbf{x}\}\,,$$

using standard property of Laplace transform

$$\mathcal{L}\{\dot{\mathbf{x}}\} = s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}_0\,.$$

Therefore

$$s\mathcal{L}\{\mathbf{x}\} - \mathbf{x}_0 = \mathbf{A}\mathcal{L}\{\mathbf{x}\}\,,$$

or after rearrangement

$$(s\mathbf{I} - \mathbf{A})\mathcal{L}\{\mathbf{x}\} = \mathbf{x}_0\,.$$

Thus

$$\mathcal{L}\{\mathbf{x}\} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x_0}\,.$$

Taking the inverse transform on both sides, we have that

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0\} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}\mathbf{x}_0\,.$$

By comparison with the solution given in exponential matrix defined in (2.11), we have

$$\mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = e^{\mathbf{A}t}\,,$$

which is a generalization to matrix form of the scalar result when $n = 1$,

$$\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}\,.$$

2.5 Solution of a controlled system

Consider the constant linear system with multiple inputs and outputs described by the vector-matrix equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\,, \tag{2.18}$$

with $\mathbf{x}(t)$ a column vector of n state variables, and $\mathbf{u}(t)$ a column vector of m control variables. Hence, $\mathbf{A}$ and $\mathbf{B}$ are constant matrices of dimensions $n \times n$ and $n \times m$, respectively. In practice, we generally have that $m \leq n$.

Lemma 2.5.1. *A solution to (2.18)*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

for initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\,[\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t_0, \tau)\mathbf{B}\,\mathbf{u}(\tau)d\,\tau\,]\,. \tag{2.19}$$

Proof. We multiply both sides of (2.18) by $e^{-\mathbf{A}t}$ to obtain

$$e^{-\mathbf{A}t}\,\dot{\mathbf{x}} = e^{-\mathbf{A}t}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})\,,$$

or after rearrangement

$$e^{-\mathbf{A}t}\,\dot{\mathbf{x}} - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x} = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}\,.$$

Notice that

$$-\mathbf{A}e^{-\mathbf{A}t} = -e^{-\mathbf{A}t}\mathbf{A}$$

and observing that by the product rule

$$\frac{d}{dt}(e^{-\mathbf{A}t}\mathbf{x}) = -\mathbf{A}e^{-\mathbf{A}t}\,\mathbf{x} + e^{-\mathbf{A}t}\,\dot{\mathbf{x}}\,.$$

Thus we have that

$$\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x} = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}\,.$$

Now we can integrate on both sides to obtain

$$e^{-\mathbf{A}t}\,\mathbf{x}(t) - \mathbf{x}_0 = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\,\tau\,,$$

therefore,

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,[\,\mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\,\tau\,]\,. \tag{2.20}$$

Recalling that $\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}$, the solution can be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\,[\,\mathbf{x}_0 + \int_{t_0}^t \mathbf{\Phi}(t_0, \tau)\mathbf{B}\,\mathbf{u}(\tau)d\,\tau\,]\,.$$

□

Therefore, if control $\mathbf{u}(t)$ is known for $t_0 \leq t$, state variables $\mathbf{x}(t)$ can be determined by finding the state transition matrix $\mathbf{\Phi}$ and carrying out the integration in (2.19).

Example 2.5.2. *Unit Mass Object.*

Consider an object of a unit mass moving in a straght line. The equation of motion

$$\frac{d^2\mathbf{z}}{d\,t^2} = \mathbf{u}(t)\,,$$

where $\mathbf{u}(t)$ is an external force, and $\mathbf{z}$ is the displacement from some fixed point. In state-space form, we need to determine state variables. Taking $x_1 = \mathbf{z}$ and $x_2 = \dot{\mathbf{z}}$, we have

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}(t)$$

$$= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,.$$

The solution is given by (2.20). The problem is how to compute $e^{\mathbf{A}t}$. Using its definition (2.10), and since $\mathbf{A}^2 = 0$, we only have

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t\,,$$

So (2.20) gives

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(\tau)\,d\tau\,.$$

Solving for $x_1(t)$ leads to

$$\mathbf{z}(t) = \mathbf{z}(0) + t\,\dot{\mathbf{z}}(0) + \int_0^t (t-\tau)\mathbf{u}(\tau)\,d\tau\,,$$

where $\dot{\mathbf{z}}(0)$ denotes the initial velocity of the mass. Considering the case when the control $\mathbf{u}(t)$ is equal to a constant for all $t \geq 0$, we obtain the familiar formula for displacement along a straight line under constant acceleration.

$$\mathbf{z}(t) = \mathbf{z}(0) + t\,\dot{\mathbf{z}}(0) + \frac{1}{2}\mathbf{u}\,t^2\,,$$

or

$$\mathbf{z}(t) - \mathbf{z}(0) = t\,\dot{\mathbf{z}}(0) + \frac{1}{2}\mathbf{u}\,t^2\,.$$

2.6 Solution of time varying systems

The general form for the controlled time-varying system considered next is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)\mathbf{u}(t)\,,$$

where $\mathbf{A}$ and $\mathbf{B}$ are continuous matrix functions of time, $t \geq 0$, of dimension $n \times n$ and $n \times m$, respectively. In general it is not possible to express solutions explicitly, hence, our goal here is to obtain some general properties. We will consider both the controllable and uncontrollable cases here.

Case I: We first consider the uncontrolled problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)\,, \tag{2.21}$$

subject to the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0\,, \tag{2.22}$$

and we state an existence and uniqueness result.

Theorem 2.6.1. *Suppose that* $\mathbf{A}(t)$ *is continuous for* $t \geq 0$*. Then (2.21) with initial condition (2.22) has a unique solution of the form*

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{x}_0 \tag{2.23}$$

for $t \geq t_0$*, for some* $n \times n$ *matrix function* $\mathbf{X}(t)$ *that satisfies*

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)\,, \tag{2.24}$$

and such that

$$\mathbf{X}(0) = \mathbf{I}\,.$$

Proof. In the literature we found several ways to establish that result. The most interesting is presented in ([5], p.167). Richard Bellman based his method upon the following identity of Jacobi:

$$|\mathbf{X}(t)| = e^{\int_0^t tr(\mathbf{A}(s)\,d\,s}\,.$$

In order to simplify the notations, he considers the two-dimensional case and takes the derivatives of scalar function $|X(t)|$.

$$|\mathbf{X}t)| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Thus we have

$$\frac{d\,x_1}{d\,t} = a_{11}\,x_1 + a_{12}\,x_2 \qquad \frac{d\,y_1}{d\,t} = a_{11}\,y_1 + a_{12}\,y_2$$
$$\frac{d\,x_2}{d\,t} = a_{21}\,x_1 + a_{22}\,x_2 \qquad \frac{d\,y_2}{d\,t} = a_{21}\,y_1 + a_{22}\,y_2$$

for boundary conditions

$$x_1(0) = 1 \qquad y_1(0) = 0$$
$$x_2(0) = 0 \qquad y_2(0) = 1\,.$$

Then

$$\begin{aligned}
\frac{d}{dt}\left|\mathbf{X}(t)\right| &= \begin{vmatrix} \frac{dx_1}{dt} & \frac{dy_1}{dt} \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ \frac{dx_2}{dt} & \frac{dy_2}{dt} \end{vmatrix} \\
&= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{11}y_1 + a_{12}y_2 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ a_{21}x_1 + a_{22}x_2 & a_{21}y_1 + a_{22}y_2 \end{vmatrix} \\
&= a_{11}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + a_{22}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
&= (\mathrm{tr}\mathbf{A}(t))\left|\mathbf{X}(t)\right|\,.
\end{aligned}$$

Thus

$$\left|\mathbf{X}(t)\right| = e^{\int_0^t tr(\mathbf{A}(s)\,d\,s}$$

since $|\mathbf{X}(0)| = 1$. That completed the proof of nonsingularity of solution. □

For time-varying systems we can no longer define a matrix exponential, but there is a result corresponding to the fact that $e^{\mathbf{A}t}$ is nonsingular when $\mathbf{A}$ is constant:

Theorem 2.6.2. *The matrix function* $\mathbf{X}(t)$ *in (2.23) is nonsingular for all* $t \geq t_0$.

Proof. Define an $n \times n$ matrix function $\mathbf{Y}(t)$ as the solution to

$$\dot{\mathbf{Y}}(t) = -\mathbf{Y}(t)\mathbf{A}(t)\,,$$

subject to initial condition

$$\mathbf{Y}(t_0) = I\,.$$

By an argument similar to that in preceding proof, there exists a unique solution $\mathbf{Y}(t)$. Moreover, since

$$\frac{d}{dt}(\mathbf{YX}) = \dot{\mathbf{Y}}\mathbf{X} + \mathbf{Y}\dot{\mathbf{X}} = -\mathbf{YAX} + \mathbf{YAX} = 0\,, \tag{2.25}$$

it follows that that $\mathbf{Y}(t)\mathbf{X}(t)$ is a constant matrix, and moreover

$$\mathbf{Y}(t_0)\mathbf{X}(t_0) = \mathbf{I}\cdot\mathbf{I} = \mathbf{I}\,,$$

and so that constant matrix is the identity, i.e. for all $t \geq t_0$,

$$\mathbf{Y}(t)\mathbf{X}(t) = \mathbf{I}\,.$$

Hence, the rank of $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ is n for all t, and moreover, the inverse of $\mathbf{X}(t)$ is $\mathbf{Y}(t)$. In particular, $\mathbf{X}(t)$ is nonsingular. □

Recalling that the solution to the time invariant (constant) system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0\,,$$

can be expressed as

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0$$

with

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)}\,,$$

the state transition matrix. This idea can be generalized to time-varying systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$$

with

$$\mathbf{\Phi}(t, t_0) := \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\,. \tag{2.26}$$

In this case,

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0$$

solves (2.21) since

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \dot{\mathbf{\Phi}}(\mathbf{t}, \mathbf{t_0})\, \mathbf{x}_0 \\
&= \underbrace{\dot{\mathbf{X}}(t)(\mathbf{X}(t_0))^{-1}}\, \mathbf{x}_0 \\
&= \mathbf{A}(t)\, \underbrace{\mathbf{X}(t)(\mathbf{X}(t_0))^{-1}}\, \mathbf{x}_0 \\
&= \mathbf{A}(t) \underbrace{\Phi(t, t_0)\, \mathbf{x}_0} \\
&= \mathbf{A}(t)\, \mathbf{x}(t)\,.
\end{aligned}$$

Check initial conditions:

$$\begin{aligned}
\mathbf{X}(t_0) &= \mathbf{\Phi}(t_0, t_0)\mathbf{x}_0 \\
&= \mathbf{X}(t_0)(\mathbf{X}(t_0))^{-1}\mathbf{x}_0 \\
&= \mathbf{x}_0\,.
\end{aligned} \tag{2.27}$$

Conclusion I. To summarize, although it may not be possible to find an explicit expression for the solution to (2.23), and hence for $\Phi(t, t_0)$ in (2.25); however, this matrix equation has the same properties as for the constant case given in equations (2.15 - 2.17).

Case II: Second, we consider the controlled case.

Theorem 2.6.3. *The solution of*

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\,, \tag{2.28}$$

subject to the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$*, is*

$$\mathbf{x}(t) = \Phi(t, t_0)\left[\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\right]\,, \tag{2.29}$$

where Φ *is defined in (2.26).*

Proof. Using the standard method of variation of parameters, assume the solution has the form

$$\mathbf{x} = \mathbf{X}(t)\mathbf{z}(t)\,, \tag{2.30}$$

where $\mathbf{X}$ is the unique $n \times n$ matrix defined by (2.22). From (2.24) we see that

$$\frac{dx}{dt} = \dot{\mathbf{X}}\mathbf{z} + \mathbf{X}\frac{dz}{dt} = \mathbf{A}\mathbf{X}\mathbf{z} + \mathbf{X}\frac{dz}{dt}\,.$$

Therefore, by (2.28)

$$\dot{\mathbf{X}}\mathbf{z} + \mathbf{X}\frac{dz}{dt} = \mathbf{A}\mathbf{X}\mathbf{z} + \mathbf{B}\mathbf{u}\,,$$

so we have

$$\mathbf{X}\frac{d\mathbf{z}}{dt} = \mathbf{B}\mathbf{u}\,.$$

and

$$\frac{d\mathbf{z}}{dt} = \mathbf{X}^{-1}\mathbf{B}\mathbf{u}\,.$$

Integrating on both sides yells

$$\mathbf{z}(t) = \mathbf{z}(t_0) + \int_{t_0}^{t} \mathbf{X}^{-1}(\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau\,. \tag{2.31}$$

Evaluating at $t = t_0$, we have

$$\mathbf{x}(t_0) = \mathbf{X}(t_0)\mathbf{z}(t_0) = \mathbf{x}_0\,,$$

therefore we obtain

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 = \mathbf{\Phi}(t, t_0)\mathbf{X}(t_0)\mathbf{z}(t_0)\,.$$

Multiply both sides of (2.31) by $\mathbf{X}(t)$:

$$\mathbf{X}(t)\mathbf{z}(t) = \mathbf{X}(t)\mathbf{z}(t_0) + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau$$

so

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau\,.$$

Notice that

$$\mathbf{X}(t)\mathbf{X}^{-1}(\tau) = \mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,t_0)\mathbf{\Phi}(t_0,\tau)\,,$$

therefore

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\left[\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{u}\,d\tau\,\right].$$

□

Applying the initial conditions and using

$$x_0 = \mathbf{X}(t_0)\mathbf{z}(t_0)$$

and

$$\mathbf{\Phi}(t,t_0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0),$$

we have the desired expression. Since

$$\mathbf{z}(t_0) = \mathbf{X}^{-1}(t_0)\mathbf{x}(t_0)\,,$$

hence

$$\begin{aligned}\mathbf{X}(t)\mathbf{z}(t_0) &= \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}(t_0)\\ &= \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0)\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Phi}(t,t_0)\mathbf{\Phi}(t_0,\tau) &= \mathbf{X}(t)\,\underbrace{\mathbf{X}^{-1}(t_0)\mathbf{X}(t_0)}\,\mathbf{X}^{-1}(\tau)\\ &= \mathbf{X}(t)\cdot\mathbf{I}\cdot\mathbf{X}^{-1}(\tau)\\ &= \mathbf{X}(t)\mathbf{X}^{-1}(\tau)\\ &= \mathbf{\Phi}(t,\tau)\end{aligned}$$

Conclusion II. The material presented in section 2.6 shows that some of the results on linear systems carry over when the matrix elements are time varying. This is very a useful aspect of the state space approach, since transform methods can only be applied to equations with constant coefficients.

Example 2.6.4. *Homogeneous System.*

Find solution of the system described by the homogeneous equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \mathbf{x}(t)\,.$$

From this

$$\dot{x}_1(t) = 0$$

$$\dot{x}_2(t) = t x_1(t)\,.$$

In particular, since $\dot{x}_1(t) = 0$, $x_1(t) \equiv$ constant and so $x_1(t) = x_1(0)$ for all t.

Therefore $\dot{x}_2(t) = t x_1(t) = t x_1(0)$ and so

$$x_2(t) = \frac{1}{2}\, t^2 x_1(0) + x_2(0).$$

Thus the general solution is

$$\mathbf{X}(t) = \begin{bmatrix} x_1(0) \\ \frac{1}{2} t^2 x_1(0) + x_2(0) \end{bmatrix} = x_1(0) \begin{bmatrix} 1 \\ \frac{1}{2} t^2 \end{bmatrix} + x_2(0) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= x_1(0)\mathbf{v}_1(t) + x_2(0)\mathbf{v}_2(t)\,.$$

$\Rightarrow \{\mathbf{v}_1(t),\ \mathbf{v}_2(t)\}$ is the basis of the solution space for the given equation.

Hence

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\, t^2 & 1 \end{bmatrix}.$$

For the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} 1 \\ \frac{1}{2}\, t^2 \end{bmatrix}$$

If the initial condition is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} 1 \\ \frac{1}{2}\,t^2 + 2 \end{bmatrix},$$

and so

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 1 \\ \frac{1}{2}t^2 & \frac{1}{2}\,t^2 + 2 \end{bmatrix}$$

is also a fundamental matrix solution. Since we have two linearly independent columns in $\mathbf{X}(t)$ it should be nonsingular.

$$\det \mathbf{X}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} = 1 \neq 0$$

for any t. Since the fundamental matrix $\mathbf{X}(t)$ consists of solution space of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t)$, we have

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}(t).$$

In particular,

$$\dot{\mathbf{X}} = \frac{d}{dt}\begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} = \mathbf{A}(t)\mathbf{X}(t)\,.$$

The state transition matrix $\mathbf{\Phi}(t, t_0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$ is also the unique solution of

$$\frac{d}{d\,t}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$$

with the initial condition $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$. For our system in case I,

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} \Rightarrow \mathbf{X}^{-1}(t_0) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}\,t_0^2 & 1 \end{bmatrix}.$$

From this

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{2}\,t_0^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(t^2 - t_0^2) & 1 \end{bmatrix}.$$

In case II,

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 1 \\ \frac{1}{2}\,t^2 & \frac{1}{2}\,t^2 + 2 \end{bmatrix} \quad \Rightarrow \mathbf{X}^{-1}(t_0) = \begin{bmatrix} \frac{1}{4}\,t^2 + 1 & -\frac{1}{2} \\ -\frac{1}{4}\,t_0^2 & \frac{1}{2} \end{bmatrix}.$$

Therefore

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} 1 & 1 \\ \frac{1}{2}\,t^2 & \frac{1}{2}\,t^2 + 2 \end{bmatrix} \begin{bmatrix} \frac{1}{4}\,t_0^2 + 1 & -\frac{1}{2} \\ -\frac{1}{4}\,t_0^2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(t^2 - t_0^2) & 1 \end{bmatrix}.$$

If $\mathbf{A}(t)$ is continuous of t, then $\mathbf{X}(t,)$ and $\mathbf{\Phi}(t, t_0)$ are continuous of t.

Finally, show that the solution of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ is $\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0)$.

Indeed, in case I,

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}\,t^2 \end{bmatrix},$$

and for case II,

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\,t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2}\,t^2 + 2 \end{bmatrix}.$$

Note: The physical meaning of the state transition matrix $\mathbf{\Phi}(t, t_0)$ is that it governs the motion of the state vector in the time interval when input $\mathbf{u} = 0$. $\mathbf{\Phi}(t, t_0)$ is a linear transformation that maps the state $\mathbf{x}(t_0)$ into the state $\mathbf{x}(t)$.

CHAPTER 3

CONTROLLABILITY

In this chapter we present several basic results regarding controllability, i.e. how to find controls u that allow one to attain a desired state x. We restrict ourselves to the case of considering only the state equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$. We introduce definitions for controllability ('controllability to the origin' and 'controllability from the origin'). Later in this chapter, the controllability Grammian matrix $\mathbf{W}_c(t)$ and conditions for controllability are introduced. Furthermore, we give the proof of sufficiency of the condition of controllability and finding the control $\mathbf{u}(t)$. Several interesting examples of uncontrollable linearized models are given, and we test them for controllability using the definition and using the controllability matrix $\mathbf{W}_c(t)$.

3.1 Definitions

In preceding chapter it was stated that control theory has developed rapidly over the past four decades and it is now established as an important area of contemporary applied mathematics. The question of controllability is an important issue for a system that one would like to control. A linear controllable system may be defined as a system which can be steered to any desired state from the zero initial state. Here we discuss the general property of being able to transfer a system from an arbitrary given state to any other states by finding a suitable control function. **For example,** if the linear system is a circuit consisting of capacitors, inductors, and resistors controlled by an external voltage, then to be controllable means that by varying in time the external voltage, we can achieve at some point in time any combination of voltages on the capacitors and currents though inductors.

The controllability property plays an important role in many control problems, such as stabilization of an unstable system by feedback, or optimal control.

The basic representation for linear systems is the linear state equation, often written in the standard form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\,, \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\,, \end{aligned} \tag{3.1}$$

where the matrix functions are continuous, real valued functions defined for all $t \in \mathbb{R}$, $\mathbf{A}(t)$ is $n \times n$ (dynamical matrix), $\mathbf{B}(t)$ is $n \times m$ (input matrix), $\mathbf{C}(t)$ is $p \times n$ (output matrix), $\mathbf{D}(t)$ is $p \times m$, $\mathbf{u}(t)$, the control (or input signal) is an $m \times 1$ vector, $\mathbf{x}(t)$ the state vector, is an $n \times 1$ vector. Its components $(x_1, x_2, \cdots, x_n)^T$ are called the state variables, $\mathbf{y}(t)$ is the output signal. We assume that $p, m \leq n$, a sensible formulation in terms of consideration of the independence of the components of the vector input and output signals. The input signal $\mathbf{u}(t)$ is assumed to be defined for all $t \in \mathbb{R}$ and to be continuous.

Typically in engineering problems, there is a fixed initial time $t = t_0$, properties of the solution $\mathbf{x}(t)$ of a linear state equation are specified for a given initial state

$$\mathbf{x}(t_0) = \mathbf{x}_0\,,$$

and a suitable control (input) signal $\mathbf{u}(t)$, specified for $t \in [t_0, \infty)$, is of interest for $t \geq t_0$. However, from a mathematical point of view, there are occasions, when solutions 'backward in time' are of interest, and this is the reason that the interval of definition of the output signal and the coefficient matrix in the state equation is $(-\infty, \infty)$. That is the solution $\mathbf{x}(t)$ for $t \geq t_0$ and for $t < t_0$ are both mathematically valid.

Definition 3.1.1. *The linear time varying system governed by*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

and

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\,,$$

is said to be **completely controllable** *if for any time t_0, any given initial state*

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

and any given final state x_f there exists a final time $t_1 \geq t_0$, and a continuous control signal $\mathbf{u}(t)$, $t_0 \leq t \leq t_1$ such that the corresponding solution of

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

at some time t_f is "the zero solution"

$$\mathbf{x}(t_f) = \mathbf{x}_f$$

In other words, the state of the system $\mathbf{x}$ can be transferred from $\mathbf{x}_0$ to the origin over some finite time interval $[t_0, t_f]$.

The qualifying term 'completely' implies that the definition holds for *all* x_0 and x_f, and several other types of controllability can be defined. **For example**, complete output controllability requires attainment of arbitrary final output. The control $\mathbf{u}(t)$ is assumed *piecewise continuous* in the interval t_0 to t_f, that is continuous except at a finite number of points in the interval.

From the expression (2.29) for the solution of (3.1) we have

$$\mathbf{x}(f) = \mathbf{\Phi}(t_1, t_0)\,[\,\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\,\mathbf{u}(\tau)d\,\tau\,]\,.$$

Using the Property III (2.16) of the state transition matrix and rearranging, we obtain

$$0 = \mathbf{\Phi}(t_1, t_0)\,\{\,[\,\mathbf{x}_0 - \mathbf{\Phi}(t_0, t_1)\mathbf{x}(f)\,]\, + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\,\mathbf{u}(\tau)\,d\,\tau\,\}$$

By comparison both equations and since $\mathbf{\Phi}$ is nonsingular it follows that if $\mathbf{u}(u(t)$ transfers $\mathbf{x}_0$ to $\mathbf{x}_f$ it also transfers $[\mathbf{x}_0 - \mathbf{\Phi}(t_0, t_1)\,\mathbf{x}_f]$ to the origin in the same time interval. We are choosing $\mathbf{x}_0$ and $\mathbf{x}_f$ arbitrary, therefore in the definition the given final state can be taken to be the null vector without loss of generality.

For time invariant systems the initial time t_0 in the controllability definition can be set equal to zero, and a general algebraic criterion will be derived in Section 3.3

Remarks:

(1) Control has the following interpretation: the level of control effort required to achieve a desired result is proportional to the difference between the desired terminal state $\mathbf{x}_f$ and the predicted endpoint of a free trajectory starting from $\mathbf{x}_0$.

(2) Definition (3.1.1) is sometimes called "*controllability to the origin*", where $\mathbf{x}_f = 0$. In the literature, there are two other definitions which are frequently used to describe the controllability of a system as follows:

  (i) The capability of the control input $\mathbf{u}(t)$ to transfer the system state variables from any given state to any specified state in the state space.

  (ii) The capability of the control input $\mathbf{u}(t)$ to transfer the system state variables from the zero initial state to any specified state in the state space. Sometimes this is called "controllability from the origin", or "reachability".

Generally speaking, for continuous-time systems, all of the above three definitions are equivalent.

(3) The above controllability definitions only require that the control input $\mathbf{u}(t)$ be able to move the system $\mathbf{x}$ from any state in the state space to the origin (or any other specified state) in a *finite* period of time. There is no specification of the state trajectory, nor are there any constraints on the input control.

(4) The above controllability definitions reflect the notion that the input affects *each* state variable *independently* over the specified time interval. This notion can help explain why a system is not controllable in some cases.

(5) For time invariant systems the initial time t_0 in the controllability definition can be set equal to zero without loss of generality.

Example 3.1.2. *Rotating wheel.*

Consider a wheel rotating on an axle with the total moment of inertia J and angular velocity $x(t)$. Find a braking torque $\mathbf{u}(t)$ which one needs to apply to bring the system to rest. The equation of motion is

$$J\dot{x}_1 = u\,.$$

After integrating on both parts, we obtain

$$x_1(t_1) = x_1(t_0) + \frac{1}{J}\int_{t_0}^{t_1} u(t)\,d\,t\,.$$

We shall choose $u(t)$ such that

$$\int_{t_0}^{t_1} u(t)\,d\,t = -Jx_1(t_0)$$

since $x_1(t_1) = 0$. Therefore, the system is controllable. Apparently, there are infinitely many suitable choices for $u(t)$, for example, a constant torque

$$u(t) = -\frac{Jx_1(t_0)}{t_1 - t_0}\,.$$

However, to make the control force unique we would have to add an additional requirement, such as bringing the wheel to rest as quickly as possible or with minimum expenditure of energy.

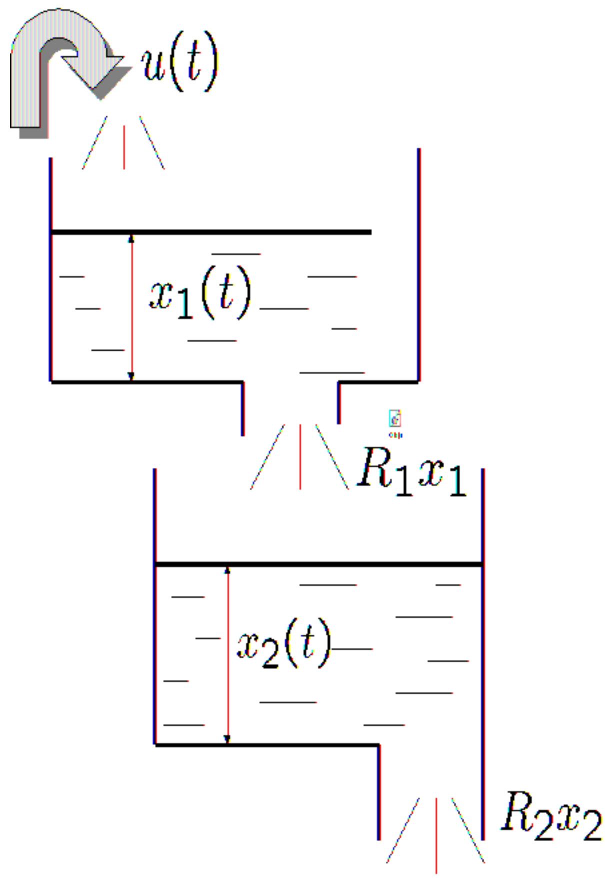

Figure 3.1: Two Tanks System

3.2 Examples of Noncontrollable Linearized Models

Example 3.2.1. *Two Tanks System.*

Consider the problem of controlling the water levels in a pair of connected water tanks as presented on Figure 3.1, where x_1 is water level in tank $T1$ and x_2 is water level in tank $T2$.

A linearized model of this type is given by the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -R_1 & 0 \\ R_1 & -R_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}\,,$$

$$\mathbf{y} = [1\ 0]\,\mathbf{x}(t)$$

where R_1 and R_2 could be any constant. Can we control x_1 and x_2 by varying $\mathbf{u}$? Since

$$\dot{x}_1 = -R_1\,x_1 + u$$

$$\dot{x}_2 = R_1\,x_1 - R_2\,x_2$$

the scalar input $u(t)$ has no direct influence on the state variable x_2, but the term $R_1 x_1$ is determined by the choice of input $u(t)$, hence x_2 is controllable; therefore, we say the system is controllable.

How to make two tanks system noncontrollable? Moving the input into second tank: a linearized model of this type is given by the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -R_1 & 0 \\ R_1 & -R_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}\,.$$

Since

$$\dot{x}_1 = -R_1\,x_1$$

$$\dot{x}_2 = R_1\,x_1 - R_2\,x_2 + u$$

the scalar input $u(t)$ has no influence on the state variable x_1, hence x_1 is not controllable; therefore, we say the system is not controllable (by the definition).

Example 3.2.2. *Three Parallel Tanks System.*

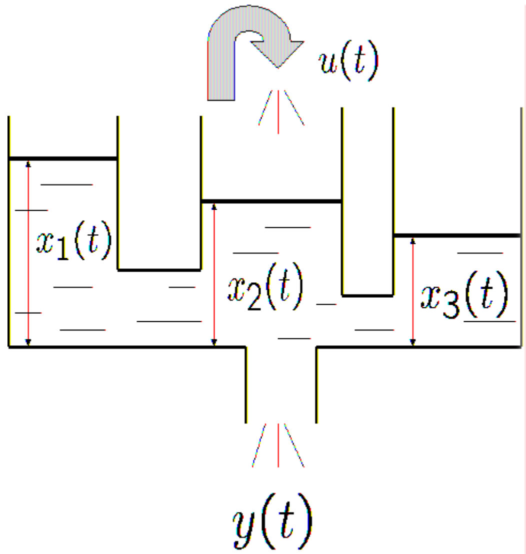

Figure 3.2: Three Parallel Tanks System

Consider the problem of controlling the water level in the three parallel connected water tanks with the input to the second tank as presented in Figure 3.2. A linearized model of this type is given by the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u},$$

and

$$\mathbf{y} = [0\ 1\ 0]\,\mathbf{x}(t)\,.$$

So we have

$$\dot{x_1} = -x_1 + x_2$$

$$\dot{x}_2 = x_1 - 3x_2 + x_3 + u$$

$$\dot{x}_3 = x_2 - x_3\,,$$

therefore:

$$\dot{x}_1 - \dot{x}_3 = -(x_1 - x_3).$$

Let

$$x = x_1 - x_3\,.$$

If

$$\frac{dx}{dt} = -x\,,$$

then

$$\frac{1}{x}\,d\,x = -d\,t\,.$$

Integrating on both parts gives

$$\int \frac{1}{x}\,d\,x = -\int d\,t\,,$$

or

$$\ln x = -t\,.$$

Thus

$$x = e^{-t}\,.$$

We observe that $x \longrightarrow 0$ as $t \longrightarrow \infty$. Or in other words,

$$x_1 - x_3 \longrightarrow 0$$

as time runs long enough. Therefore,

$$x_1 \approx x_3$$

after enough time. It means the liquid levels in tanks 1 and 3 cannot be *independently* controlled to different heights. Thus, the system is not controllable.

Example 3.2.3. *Model.*

Consider the system described by

$$\dot{x}_1 = a_1 x_1 + a_2 x_2 + b_1 u$$
$$\dot{x}_2 = a_3 x_2 \, .$$

Since $x_2(t)$ is entirely determined by the second equation and $x_2(0)$ and $u(t)$ does not have influence on $x_2(t)$, the system is not completely controllable.

3.3 Conditions of Controllability

A dynamic system is guided from an initial state to a desired state by manipulating the control variables. Therefore, when addressing the controllability problem, one needs to consider only the state equation (3.1), often represented by $(\mathbf{A}, \mathbf{B})$. The conditions of controllability are imposed on $(\mathbf{A}, \mathbf{B})$. For time invariant systems the initial time t_0 in the controllability definition can be set equal to zero, and a general algebraic criterion can be derived. The necessary and sufficient conditions for testing the controllability of linear time invariant systems are given as follows:

Theorem 3.3.1. *The n-dimensional system* $(\mathbf{A}, \mathbf{B})$ *is completely controllable if and only if the following equivalent conditions are satisfied:*

*(I) The $n \times n$ symmetric matrix (***controllability Grammian*** matrix)*

$$\mathbf{W}_c(t_0, t_1) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_o, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0, \tau)\, d\tau \, , \tag{3.2}$$

where $\mathbf{\Phi}$ is defined in (2.14), is positive definite for any $t > 0$.

In this case the control

$$\mathbf{u}(t) = -\mathbf{B}^T(t)\mathbf{\Phi}^T(t_0, t)\mathbf{W}_c^{-1}(t_0, t_1)[\, \mathbf{x}_0 - \mathbf{\Phi}(t_0, t_1)\, \mathbf{x}_f \,] \, , \tag{3.3}$$

defined on $t_0 \leq t \leq t_1$, *transfers*

$$\mathbf{x}(t_0) = \mathbf{x_0}$$

to

$$\mathbf{x}(t_1) = \mathbf{x}_f \,.$$

(II) The $n \times nm$ *controllability matrix*

$$\mathbb{C} = \begin{bmatrix}\mathbf{B} \ \mathbf{AB} \ \mathbf{A^2B} \ \ldots \ \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \tag{3.4}$$

is such that rank $\mathbb{C} = n$; *i.e.* $\mathbb{C}$ *has full row rank.*

(III) The $n \times (n+p)$ *matrix* $[\mathbf{A} - \lambda\mathbf{I}\mathbf{B}]$ *has full row rank at each eigenvalue* λ *of* $\mathbf{A}$*, i.e.*

$$\text{rank}\,[\mathbf{A} - \lambda\mathbf{I}\ \mathbf{B}] = n\,,$$

(IV) For a matrix $\mathbf{A}$ *whose eigenvalues are located in the open left - half of the s-plane, the Lyapunov equation*

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T$$

has the unique positive definite (hence non-singular) solution

$$\mathbf{W}_c = \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\,\tau$$

The proof of the conditions for controllability is well-presented in the literature (for example, a method of contradiction is used to prove the necessity of condition (II) in [1]. Thus, we restrict ourself to the **proof of sufficiency of condition (I)** for illustration.

Proof. Assume $\mathbf{W}_c$ is nonsingular, then the control, defined by (3.3), exists.

Next, we need to show that system is completely controllable, indeed, because of the definition (3.1.1), we need to show that

$$\mathbf{x}(t_1) = \mathbf{x}_f \,.$$

Substituting

$$\mathbf{u}(t) = -\mathbf{B}^T(t)\mathbf{\Phi}^T(t_0,t)\mathbf{W}_c^{-1}(t_0,t_1)\,[\,\mathbf{x}_0 - \mathbf{\Phi}(t_0,t_1)\,\mathbf{x}_f\,]$$

into the general form of the solution of the linear state equation (2.19) for $\mathbf{x}(t_1)$,

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1,t_0)[\,\mathbf{x}_0 + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\,\tau\,]\,,$$

we have

$$\begin{aligned}\mathbf{x}(t_1) &= \mathbf{\Phi}(t_1,t_0)[\,\mathbf{x}_0 + \\ &+ \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\{-\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\mathbf{W}_c^{-1}(t_0,t_1)\,[\,\mathbf{x}_0 - \mathbf{\Phi}(t_0,t_1)\,\mathbf{x}_f\,]\}\,d\,\tau\,] \\ &= \mathbf{\Phi}(t_1,t_0)[\,\mathbf{x}_0 - \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\mathbf{W}_c^{-1}(t_0,t_1)\,[\,\mathbf{x}_0 + \mathbf{\Phi}(t_0,t_1)\,\mathbf{x}_f\,]d\,\tau\,]\,.\end{aligned}$$

We divide the right hand side into two integrals,

$$\begin{aligned}&\mathbf{\Phi}(t_1,t_0)\,[\,\mathbf{x}_0 - \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\mathbf{W}_c^{-1}(t_0,t_1)\,\mathbf{x}_0\,d\,\tau \\ &\quad + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\mathbf{W}_c^{-1}(t_0,t_1)\mathbf{\Phi}(t_0,t_1)\,\mathbf{x}_f\,d\,\tau\,]\,.\end{aligned}$$

We observe that some terms in the right hand side do not depend on τ, so they can be taken outside of the integrals to obtain the expression

$$\begin{aligned}&\mathbf{\Phi}(t_1,t_0)\,[\,\mathbf{x}_0 - \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\,d\,\tau\;\mathbf{W}_c^{-1}(t_0,t_1)\,\mathbf{x}_0 \\ &\quad + \int_{t_0}^{t_1} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\,d\,\tau\;\mathbf{W}_c^{-1}(t_0,t_1)\mathbf{\Phi}(t_0,t_1)\,\mathbf{x}_f\,.\end{aligned}$$

From the expression for the controllability Grammian matrix (3.2)

$$\mathbf{W}_c(t_0, t) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0, \tau)\, d\tau\,,$$

the preceding expression can be written as

$$\mathbf{\Phi}(t_1, t_0)\,[\,\mathbf{x}_0 - \underbrace{\mathbf{W}_c(t_0, t_1)\,(\mathbf{W}_c(t_0, t_1))^{-1}}\,\mathbf{x}_0$$

$$- \underbrace{\mathbf{W}_c(t_0, t_1)\,(\mathbf{W}_c(t_0, t_1))^{-1}}\,\mathbf{\Phi}(t_0, t_1)\,\mathbf{x}_f\ ]\,.$$

Using one of the general properties of the matrices: $\mathbf{W}_c(t_0, t_1)(\mathbf{W}_c(t_0, t_1))^{-1} = \mathbf{I}$. Therefore,

$$\mathbf{x}(t_1) = \mathbf{\Phi}(t_1, t_0)\,[\,\mathbf{x}_0 - \mathbf{I}\,\mathbf{x}_0 + \mathbf{I}\mathbf{\Phi}(t_0, t_1)\,\mathbf{x}_f\,]\,,$$

or

$$\mathbf{x}(t_1) = \underbrace{\mathbf{\Phi}(t_1, t_0)\mathbf{\Phi}(t_0, t_1)}\,\mathbf{x}_f\,.$$

Applying the Property III for the state transition matrix, verified earlier in Section 2.5 (2.16)

$$\mathbf{\Phi}(t_1, t_0)\mathbf{\Phi}(t_0, t_1) = \mathbf{I}$$

it does indeed give us

$$\mathbf{x}(t_1) = \mathbf{x}_f\,,$$

as required. □

That means the system is completely controllable and the control $\mathbf{u}(t)$, determined by (3.3) transfers a system from any given state to any other state.

Remarks:

(1) For constant systems, the property of controllability is independent of the time interval $[0, t_1]$ or $[t_0, t_1]$. Therefore, the term *controllable* for linear time invariant systems is used without referring to a specific time interval.

(2) Condition (II) in Theorem 3.3.1 gives a criterion to determine whether a constant linear system is completely controllable, but gives no help in determining a control vector which will carry out a required alteration of states.

(3) Condition (I) in Theorem 3.3.1 contains an explicit expression for a control vector for both constant and time varying systems.

(4) Since Theorem 3.3.1 shows that controllability of the system is independent of the matrix $\mathbf{C}$, we shall often refer to the *controllability of the pair* $[\mathbf{A}, \mathbf{B}]$ instead of that of the system.

The control function (3.3) which transfers the system from $(\mathbf{x}_0, t_0)$ to $(\mathbf{x}_f, t_1)$ requires calculation of the transition matrix $\mathbf{\Phi}$ and the controllability matrix $\mathbf{W}$ which is not too difficult for constant linear systems as shown in Example 3.4.4.

(5) Using exponential matrix form introduced in section (2.4), we may express the state transition matrix $\mathbf{\Phi}$ as

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} ,$$

so that the controllability Grammian matrix ($n \times n$ symmetric matrix) then has the form

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\,\tau = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t-\tau)}\, d\,\tau . \qquad (3.5)$$

Furthermore, it also generates an input function that can transfer the state from its initial position to zero within the time interval $[0, t_1]$ (Example 3.4.4)

$$\mathbf{u}_{me}(t) = -\mathbf{B}^T e^{\mathbf{A}^T(t_1-t)}\mathbf{W}_c^{-1}(t_1)e^{\mathbf{A}t_1}\mathbf{x}_0 .$$

In the control literature, the above input is called **the minimal energy control** in the sense that for any other input $\mathbf{u}(\mathrm{t})$ we have

$$\int_0^{t_1} \mathbf{u}^T(t)\mathbf{u}(t)dt \geq \int_0^{t_1} \mathbf{u}_{me}^T(t)\mathbf{u}_{me}(t)dt\,.$$

(6) It has been proved that the property of controllability is invariant under an (algebraic) equivalence transformation. [1]

3.4 Testing For Controllability

Example 3.4.1. *Three Tanks System.*

Consider again the three tanks model as shown in Figure 3.2 with input to the the second tank:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{u}\,;$$

and now use conditions from Theorem 3.3.1 and compute the rank of the controllability matrix $\mathbf{C}$:

$$\text{rank}\begin{bmatrix}\mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B}\end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 & -4 \\ 1 & -3 & 11 \\ 0 & 1 & -4 \end{bmatrix} = 2 < 3,$$

therefore, the system is not controllable.

Example 3.4.2. *Modified-1 Three Tanks System.*

Consider the same three tanks system as above but now moving the input to the first tank, as in figure 3.3. A linearized model in this case is:

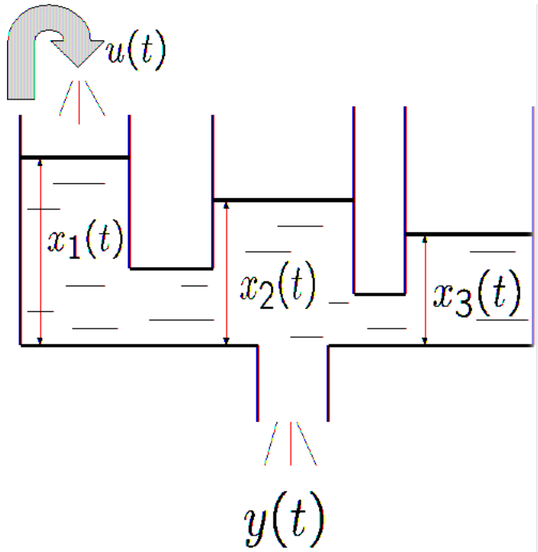

Figure 3.3: Three Tanks, Controlled

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}\,;$$

and checking the rank of the controllability matrix:

$$\text{rank}\left[\mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B}\ \right] = \text{rank} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = 3,$$

thus, the system is controllable (by conditions of Controllability from (3.5).

Example 3.4.3. *Modified-2 Three Tanks System. Now consider the three-tank system with linearized model such that:*

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}\,.$$

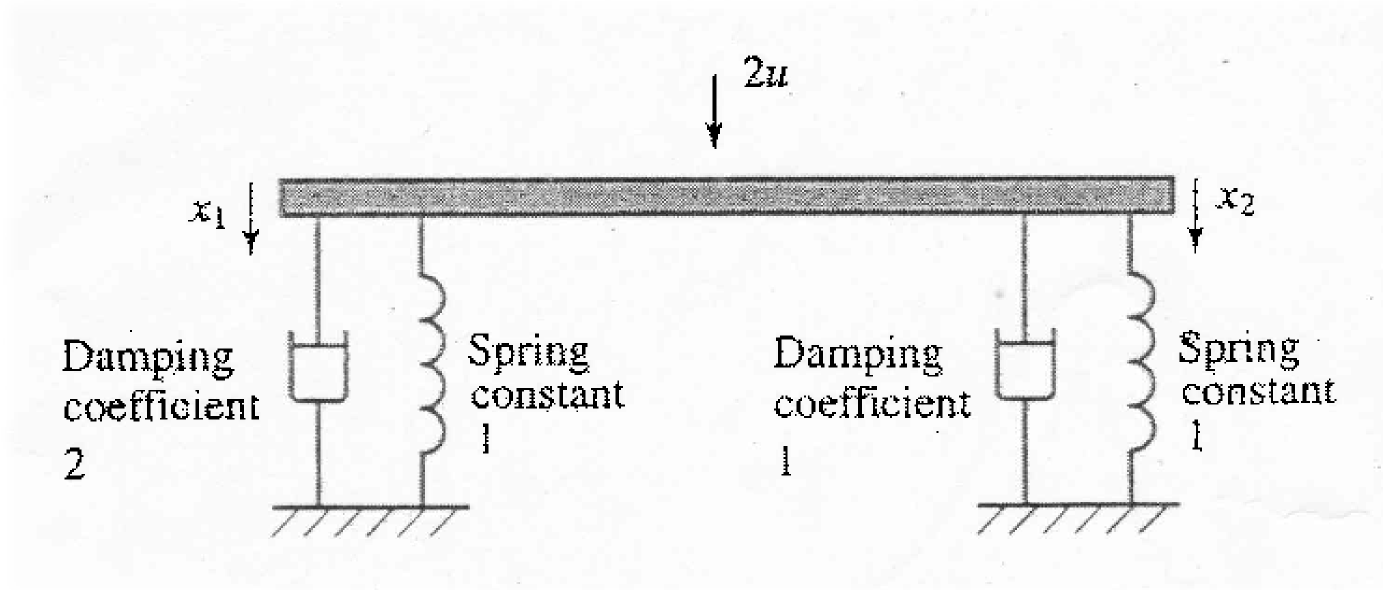

Figure 3.4: Platform

Constructing the controllability matrix and testing conditions (3.5), we have:

$$\text{rank}\left[\mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B}\right] = \text{rank}\begin{bmatrix} 1 & 0 & -2 & -2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & -4 \\ 1 & 1 & -4 & -7 & 13 & 25 \end{bmatrix} = 3.$$

Since this matrix has three independent columns, therefore, the rank is 3. Thus, the system is controllable.

Example 3.4.4. *The Platform System.*

Consider the platform system shown in Figure 3.4; it can be used to study suspension system of automobiles. The system consists of one platform; both ends of the platform are supported on the ground by means of springs and dashpots, which provide viscous friction. The mass of the platform is assumed to be zero; thus the movements of the two spring systems are independent and half of the force is applied to each spring system. Choose the displacement of the two spring systems from equilibrium as state variables x_1 and x_2, then we have

$$x_1 + 2\dot{x}_1 = u \quad \text{and} \quad x_2 + \dot{x}_2 = u$$

or

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u\,.$$

If $u = 0$ and $\mathbf{x}(0) \neq 0$, then

$$\mathbf{x}(t) = \begin{bmatrix} e^{\frac{1}{2}t} \\ e^{-t} \end{bmatrix} \mathbf{x}(0)\,,$$

and the platform will return to zero exponentially.

$$\text{rank}(\mathcal{C}) = \text{rank}[\mathbf{B}\ \ \mathbf{AB}] = \text{rank} \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & -1 \end{bmatrix} = 2 \ \Rightarrow$$

The platform is controllable.

Case I: Assume

$$\mathbf{x}(0) = \begin{bmatrix} 10 \\ -1 \end{bmatrix}.$$

We want to construct the control $u_1(t)$ such that the state $\mathbf{x}(2) = 0$. From (3.2) for controllability Grammian matrix we obtain

$$\mathbf{W}_C(0, t_1) = \int_0^2 e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}\, \mathrm{d}\tau\,,$$

and from (3.3) for the control function we have

$$\mathbf{u}(t) = -\mathbf{B}^T e^{\mathbf{A}^T(t_1 - t_0)} \mathbf{W}_C^{-1}(0, t_1)\, [\, e^{\mathbf{A}_1 t_1}\, \mathbf{x_0} - \mathbf{x_1}\,]\,.$$

In particular,

$$\mathbf{W}_C(2) = \int_0^2 \left(\begin{bmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} [\,\frac{1}{2} \quad 1\,] \begin{bmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) \mathrm{d}\tau\,,$$

and

$$u(t) = -[\,\frac{1}{2} \;\; 1\,] \begin{bmatrix} e^{-\frac{1}{2}(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} \mathbf{W}_C^{-1}(2) \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix}.$$

From MATLAB

$$\mathbf{W}_C(2) = \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

and

$$u_1(t) = -58.82\, e^{0.5\,t} \; + \; 27.96\, e^{t}\,.$$

From (2.20)

$$\mathbf{x}(t) = e^{\mathbf{A}t}\, [\, \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau} \mathbf{B}\mathbf{u}(\tau)\, d\,\tau\,]$$

we obtain

$$x_1 = 30.03\, e^{-.5\,t} - 29.41\, e^{.5\,t} + 9.38\, e^{t}\,,$$

$$x_2 = 24.2333\, e^{-t} - 39.2133\, e^{.5\,t} + 13.98\, e^{t}$$

The simulation results are shown in Figure (3.5), where $u_1(2) \approx 45$.

Case II: We want to find $u_2(t)$ such that $\mathbf{x}(4) = 0$.

For this case we have

$$\mathbf{W}_C(4) = \int_0^4 \left(\begin{bmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} [\,\frac{1}{2} \quad 1\,] \begin{bmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) \mathrm{d}\tau\,,$$

and

$$u(t) = -[\,\frac{1}{2} \;\; 1\,] \begin{bmatrix} e^{-\frac{1}{2}(4-t)} & 0 \\ 0 & e^{-(4-t)} \end{bmatrix} \mathbf{W}_C^{-1}(4) \begin{bmatrix} e^{-2} & 0 \\ 0 & e^{-4} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix}.$$

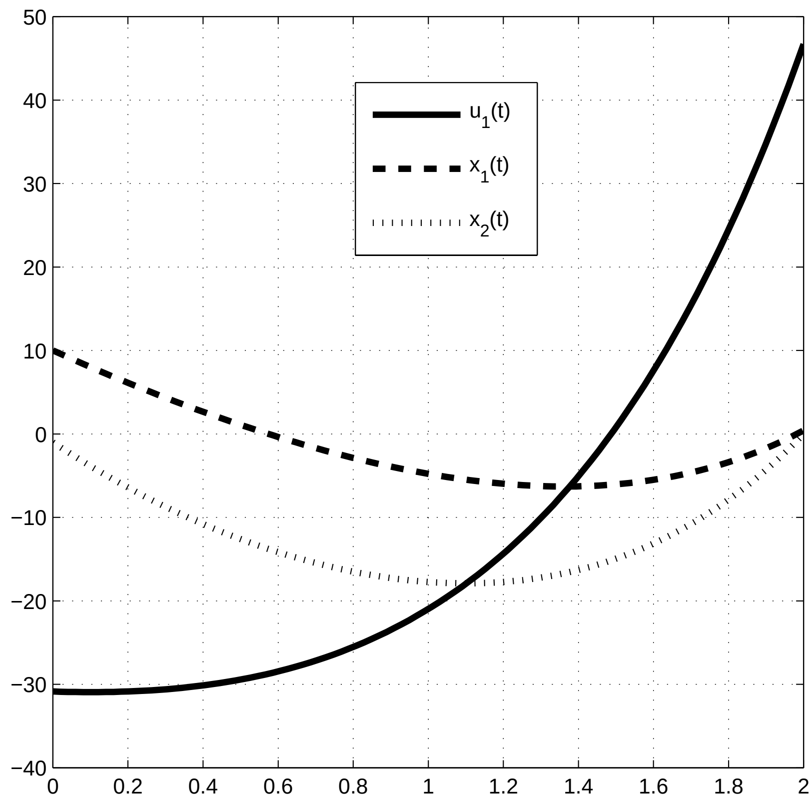

Figure 3.5: Transfer $\mathbf{x}(0) = [\,10 \;\; -1\,]^T$ to $[\,0 \;\; 0\,], 2$ seconds

From MATLAB we have $u_2(4) \approx 9$ for this case.

Case III Consider again platform system shown in Figure (3.4), where the viscous friction coefficient and the spring constants of both spring systems are assumed to be equal to 1. Then

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{u}\,.$$

Check Condition II of Theorem 3.3.1:

$$\text{rank}(\mathcal{C}) = \text{rank}[\,\mathbf{B}\;\;\mathbf{AB}\,] = \text{rank}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = 1 \;\;\Rightarrow\;\; \text{uncontrollable}\,.$$

Remarks:

(1) The smaller the time interval is, the larger the input magnitude will be.

(2) If no restriction is imposed on the input, we can transfer $\mathbf{x}(0) \neq 0$ to zero in an arbitrarily small time interval; however, the input magnitude may become very large.

(3) If $|u(t)| < 9$, then we cannot transfer $\mathbf{x}(0)$ to $\mathbf{0}$ for less than 4 seconds.

CHAPTER 4

OBSERVABILITY

In this chapter, we deal with a second concept of interest for studying linear systems. It involves the effect of the state vector on the output of the linear state equation. It is simplest to consider the case of zero-input, and this does not entail loss of generality since the concept is unchanged in the presence of a known input signal. Therefore, we may consider the system of equations (3.1) and the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4.1}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{4.2}$$

to be equivalent. We note that the term $\mathbf{D}(t)\mathbf{u}(t)$ as previously introduced in Definition 3.1.1 is not necessary in (4.2), because $\mathbf{D}(t)$ and $\mathbf{u}(t)$ are known and hence $\mathbf{D}(t)\mathbf{u}(t)$ may be omitted without any loss of generality. We discuss observability, which is the possibility of determining the initial state of a system by measuring only the output, we introduce a precise definition for observability, and we investigate criteria for a constant system. In particular, we prove sufficiency and necessity of stated observability conditions. Furthermore, we give some consequences regarding the duality and decomposition for a system that is not controllable. Several interesting examples of the models will be given, and we will test them for observability using criteria.

4.1 Definition of Observability.

The concept of observability is closely related to that of controllability. The basic idea of observability is to study the possibility of estimating state variables from only the output.

For example, when a political party wins the most votes in a national election, it usually claims that its policies are supported by a majority of the electorate; however, could the state of opinion of the voters on a particular point at issue be determined from the overall election result?

Definition 4.1.1. *Definition of Observability.*

The system, described by the linear state equations (4.1), (4.2)

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

is said to be **completely observable** *if for any unknown initial state* $\mathbf{x}(t_0) = \mathbf{x}_0$*, there exists a finite time* $t_1 > t_0$ *such that the input* $\mathbf{u}(t)$ *and output* $\mathbf{y}(t)$ *over* $t_0 \leq t \leq t_1$ *are sufficient information to uniquely determine the initial condition* $\mathbf{x}_0$*.*

The qualifying term 'completely' implies that the definition holds for *all* x_0 and t_0. The control $\mathbf{u}(t)$ is assumed *piecewise continuous* in the interval t_0 to t_1, that is, continuous except at a finite number of points in the interval.

In fact, there is no loss of generality if $\mathbf{u}(t)$ is assumed to be identically zero throughout the interval $[t_0, t_1]$. Since the output response for time invariant case is given by

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

We can substitude the solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

into the preceding expression for $\mathbf{y}(t)$ to obtain

$$\mathbf{y}(t) = \mathbf{C}\,[e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau\,] + \mathbf{D}\mathbf{u}(t)\,.$$

If $\mathbf{u}(t) \equiv 0$,then

$$\mathbf{y}_0(t) = \mathbf{y}(t) = \mathbf{C}\, e^{\mathbf{A}t}\mathbf{x}_0$$

is the zero-input response. Since all the terms in $\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$ are known, the system is observable if and only if $\mathbf{x}_0$ can be uniquely determined from its zero-input response over a finite time interval. Therefore, the term $\mathbf{D}\mathbf{u}(t)$ may be deleted.

For time variant case:

$$\mathbf{y}(t) = \mathbf{C}(t)\,\{\mathbf{\Phi}(t,t_0)\,[\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau\,]\} + \mathbf{D}(t)\mathbf{u}(t)$$

$$\Rightarrow \mathbf{C}(t)\mathbf{\Phi}(t,t_0)\mathbf{x}_0 = \mathbf{y}(t) - \mathbf{C}(t)\mathbf{\Phi}(t,t_0)\int_{t_0}^{t} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau - \mathbf{D}(t)\mathbf{u}(t) \equiv \mathbf{y}_0(t)\,.$$

If $\mathbf{u}(t) \equiv 0$, then

$$\mathbf{y}_0(t) = \mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t,t_0)\mathbf{x}_0$$

is the **zero-input response**. Since all the terms in

$$\mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t_0,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\,d\tau$$

are known, the system is observable if and only if $\mathbf{x}_0$ can be uniquely determined from its zero-input response over a finite time interval. Therefore there is no loss of generality in assuming that $\mathbf{u}(t) \equiv 0$ throughout any interval of consideration as $[t_0, t_1]$.

Conclusion: the term $\mathbf{D}(t)\mathbf{u}(t)$ may be deleted.

Example 4.1.2. *Unobservable system.*

Consider the system described by

$$\dot{x}_1(t) = a_1 x_1(t) + b_1\mathbf{u}(t)$$

$$\dot{x}_2(t) = a_2 x_2(t) + b_2\mathbf{u}(t)$$

and output signal

$$y(t) = x_1(t) \, .$$

The first equation shows that $x_1(t)$ is completely determined by $\mathbf{u}(t)$ and $x_1(t_0)$ because $x_1(t) = y(t)$. However it is impossible to determine $x_2(t_0)$ by measuring the output, so the system is not completely observable.

Note: Applying condition (II) of Theorem 3.3.1, we see that the system is completely controllable, provided $a_1 \neq a_2$, $b_1 \neq 0$, $b_2 \neq 0$. Since

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} .$$

Then the controllability matrix

$$\mathcal{C} = [\mathbf{B} \ \mathbf{AB}] = \begin{bmatrix} b_1 & a_1 b_1 \\ b_2 & a_2 b_2 \end{bmatrix} .$$

$$\Rightarrow \det \mathcal{C} = b_1 a_2 b_2 - b_2 a_1 b_1 = b_1 b_2 (a_2 - a_1) \neq 0 \, .$$

$$\Rightarrow a_1 \neq a_2, \;\; b_1 \neq 0, \;\; b_2 \neq 0 \, .$$

4.2 Conditions of Observability

The basic characterization of observability is similar in form to the controllability case. Since only $\mathbf{A}$ and $\mathbf{C}$ appear in the expression

$$\mathbf{y}_0(t) = \mathbf{C} e^{\mathbf{A}t} \mathbf{x}_0 \, ,$$

we may refer to the observability of the pair $[\mathbf{A}, \mathbf{C}]$ rather then the triad $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$. Now we state more convenient criteria for observability corresponding to the general criterion of Theorem (3.3.1), developed for controllability in Chapter 3.

Theorem 4.2.1. *The n-dimensional linear time invariant system* $(\mathbf{A}, \mathbf{C})$ *is observable if and only if:*

(I) The following $n \times n$ *matrix (observability Grammian matrix)*

$$\mathbf{W}_0(t) = \int_0^t e^{\mathbf{A}^T \tau} \mathbf{C}^T \mathbf{C} e^{\mathbf{A}\tau} d\,\tau$$

is non-singular for any $t > 0$.

(II) The observability matrix

$$\mathcal{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA^{n-1}} \end{bmatrix}$$

has full column rank, i.e.

$$\mathrm{rank}(\mathcal{O}) = n\,.$$

(III) The $(n+p) \times n$ *matrix*

$$\begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} \\ \mathbf{C} \end{bmatrix}$$

has full column rank at each eigenvalue λ *of* $\mathbf{A}$, *i.e.,*

$$\mathrm{rank} \begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} \\ \mathbf{C} \end{bmatrix} = n\,.$$

(IV) For the matrix $\mathbf{A}$ *whose eigenvalues are located in the open left-half of the complex plane, the Lyapunov equation:*

$$\mathbf{A}^T \mathbf{W}_0 + \mathbf{W}_0 \mathbf{A} = -\mathbf{C}^T \mathbf{C}$$

has the non-singular solution

$$\mathbf{W}_0 = \int_0^\infty e^{\mathbf{A}^T \tau} \mathbf{C}^T \mathbf{C} e^{\mathbf{A}\tau} d\,\tau\,.$$

4.3 Proof of Observability Conditions

(1) *Sufficiency* of **Condition (I)**:

For

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{y}_0(t),$$

we multiply both sides on the left by $e^{\mathbf{A}^T t}\mathbf{C}^T$ and integrate to obtain

$$\int_0^{t_1} (e^{\mathbf{A}^T\tau}\mathbf{C}^T)\mathbf{C}e^{\mathbf{A}\tau}d\,\tau\,\mathbf{x}_0 = \int_0^{t_1} (e^{\mathbf{A}^T\tau}\mathbf{C}^T)\mathbf{y_0}(\tau)d\,\tau\,,$$

which is the same as

$$\mathbf{W}_0(t_1)\mathbf{x}_0 = \int_0^{t_1} e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{y_0}(\tau)d\,\tau\,.$$

If $\mathbf{W}_0(t_1)$ is *non-singular*, then

$$\mathbf{x}_0 = \mathbf{W}_0^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{y_0}(\tau)d\,\tau\,,$$

and $\mathbf{x}_0$ is uniquely determined by $\mathbf{y}_0(t)$.

(2) *Necessity* of **Condition (I)**: (proof by contradiction).

Assume that $(\mathbf{A},\mathbf{C})$ is observable. If $\mathbf{W}_0(t_1)$ is singular for some t_1 (hence not positive definite), then there exists a nonzero vector $\mathbf{v}$ such that

$$0 = \mathbf{v}^T\mathbf{W}_0(t_1)\mathbf{v} = \int_0^{t_1} (\mathbf{v}^T e^{\mathbf{A}^T\tau}\mathbf{C}^T)(\mathbf{C}e^{\mathbf{A}\tau}\mathbf{v})d\,\tau = \int_0^{t_1} ||\mathbf{C}e^{\mathbf{A}\tau}\mathbf{v}||^2 d\,\tau\,,$$

which implies that $\mathbf{C}e^{\mathbf{A}t}\mathbf{v} = 0$ for all t in $[0, t_1]$.

Assume that the initial condition $\mathbf{x}_0^{(1)}$ satisfies

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0^{(1)} = \mathbf{y}_0(t),$$

then by choosing $\mathbf{x}_0^{(2)} = (\mathbf{x}_0^{(1)} - \mathbf{v})$, we also have

$$\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0^{(2)} = \mathbf{C}e^{\mathbf{A}t}(\mathbf{x}_0^{(1)} - \mathbf{v}) = \mathbf{y}_0(t)\,.$$

Thus the same $\mathbf{y}_0(t)$ yields two different initial states, i.e. we cannot uniquely determine $\mathbf{x}_0$, and hence the system is not observable. In other words, if the system is observable, then $\mathbf{W}_0(t)$ is nonsingular for any $t > 0$.

(3) *Sufficiency* of **Condition II**

By using the exponential matrix expansion

$$e^{\mathbf{A}\,t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}\,t^2 + \frac{\mathbf{A}^3}{3!}\,t^3 + \cdots + \frac{\mathbf{A}^{n-1}}{(n-1)!}\,t^{n-1} + \cdots\,,$$

we have that

$$\begin{aligned}\mathbf{y}_0(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0\\ &= \mathbf{C}\mathbf{x}_0 + \mathbf{C}\mathbf{A}t\mathbf{x}_0 + \mathbf{C}\frac{\mathbf{A}^2}{2!}\,t^2\mathbf{x}_0 + \mathbf{C}\frac{\mathbf{A}^3}{3!}\,t^3\,\mathbf{x}_0 + \cdots + \cdots\\ &+ \mathbf{C}\frac{\mathbf{A}^{n-1}}{(n-1)!}\,t^{n-1}\,\mathbf{x}_0 + \cdots\,.\end{aligned}$$

Now we let

$$t = 0$$

to obtain

$$\mathbf{y}_0(0) = \mathbf{C}\mathbf{x}_0\,.$$

Next we take the first derivative of $\mathbf{y}_0(t)$ with respect to t, and then let $t = 0$ to obtain

$$\mathbf{y}_0'(0) = \mathbf{C}\mathbf{A}\mathbf{x}_0\,.$$

Then we take the second derivative of $\mathbf{y}_0(t)$ with respect to t, and then let $t = 0$ to obtain

$$\mathbf{y}_0''(0) = \mathbf{C}\mathbf{A}^2\mathbf{x}_0\,.$$

We continue in this matter,

$$\vdots$$

so that after n-1 such stages, we take the $(n-1)^{th}$ derivative of $\mathbf{y}_0(t)$ with respect to t, and then let $t = 0$ to obtain

$$\mathbf{y}_0^{(n-1)}(0) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}_0 \,.$$

We can stop here because of the Cayley-Hamilton theorem. Thus, we have the following equation:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA^{n-1}} \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \mathbf{y}_0(0) \\ \mathbf{y}_0'(0) \\ \vdots \\ \mathbf{y}_0^{(n-1)}(0) \end{bmatrix},$$

Therefore we can solve for $\mathbf{x}_0$ uniquely if

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA^{n-1}} \end{bmatrix} = n\,.$$

(4) *Necessity* of **Condition II.**

Suppose that the system (4.1), (4.2) is completely observable and we wish to show that rank$(\mathcal{O}) = n$. Assume rank$(\mathcal{O}) < n$, then there exists a nonzero vector $\mathbf{p}$ in $\mathbb{R}^n$ such that

$$\mathcal{O}\mathbf{p} = 0 \;\Rightarrow \mathbf{Cp} = 0,\; \mathbf{CAp} = 0,\; \mathbf{CA}^2\mathbf{p} = 0, \cdots, \mathbf{CA}^{n-1}\mathbf{p} = 0$$

Let $\mathbf{x}_0 = \mathbf{p}$ in $\mathbf{y}(\mathbf{t}) = \mathbf{Ce^{At}x_0}$, then we have that

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{C}\,[\,\mathbf{r_0 I} + \mathbf{r_1 A} + \mathbf{r_2 A^2} + \cdots + \mathbf{r_{n-1} A^{n-1}}\,]\,\mathbf{p} \\ &= r_0\mathbf{Cp} + r_1\mathbf{CAP} + r_2\mathbf{CA}^2\mathbf{p} + \cdots + r_{n-1}\mathbf{CA}^{n-1}\mathbf{p} = 0\,. \end{aligned}$$

where $r_0\mathbf{I}+r_1\mathbf{A}+r_2\mathbf{A}^2+\cdots+r_{n-1}\mathbf{A}^{n-1}$ is polynomial in $\mathbf{A}$ of degree $\leq n-1$. Thus $\mathbf{y}(t)=0$ for $0\leq t\leq t_1$, when $\mathbf{x}_0=p\neq 0$. However, we also have $\mathbf{y}(t)=0$ when $\mathbf{x}_0=0$, which contradicts the assumption that the system is completely observable. Therefore, we must have that the rank$(\mathcal{O})=n$

4.4 Duality and Dual Problems

For a system in the form of (3.1) represented by matrices $[\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}]$, the *adjoint system* is represented by the matrices $[-\mathbf{A^T},\mathbf{B^T},\mathbf{C^T},\mathbf{D^T}]$. That is,

$$\begin{aligned}\dot{\mathbf{x}} &= -\mathbf{A}^T\mathbf{x}+\mathbf{C}^T\mathbf{u}\,,\\ \mathbf{y} &= \mathbf{B}^T\mathbf{x}+\mathbf{D}^T\mathbf{u}.\end{aligned} \tag{4.3}$$

Controllability and observability are dual properties, because the controllability of $(\mathbf{A},\mathbf{B})$ is the same as the observability of $(-\mathbf{A^T},\mathbf{C^T})$. Using the fact that if the state transition matrix defined by (2.26)

$$\mathbf{\Phi}(\mathbf{t},\mathbf{t_0})=\mathbf{X}(t)\mathbf{X}^{-1}(t_0)\,,$$

then $[\mathbf{\Phi}^{-1}(t,t_0)]^T$ is the transition matrix for the system $\dot{\mathbf{z}}=-\mathbf{A}^T(t)$ (the adjoint system) [1]. By comparing equation (3.2)

$$\mathbf{W}_C(t_0,t_1)=\int_{t_0}^{t_1}\mathbf{\Phi}(t_o,\tau)\mathbf{B}(\tau)\mathbf{B}^T(\tau)\mathbf{\Phi}^T(t_0,\tau)\,d\,\tau$$

and equation

$$\mathbf{W}_0(t)=\int_0^t\mathbf{\Phi}^T(\tau,t_0)\mathbf{C}^T(\tau)\mathbf{C}(\tau)\mathbf{\Phi}(\tau,t_0)d\,\tau\,,$$

we can see that the controllability Grammian matrix $\mathbf{W_C}$ is identical to the observability Grammian matrix $\mathbf{W_O}$ associated with the pair $[-\mathbf{A}^T(t)\mathbf{B}^T(t)]$. Conversely, the observability Grammian matrix $\mathbf{W_O}$ is identical to the matrix $\mathbf{W_C}$ associated with the pair $[-\mathbf{A}^T(t),\mathbf{C}^T(t)]$. We have thus established:

Theorem 4.4.1. *(Theorem of Duality). The system defined in (4.1), (4.2) is completely controllable if and only if the dual system*

$$\begin{aligned} \dot{\mathbf{x}} &= -\mathbf{A}^T\mathbf{x} + \mathbf{C}^T\mathbf{u}\,, \\ \mathbf{y} &= \mathbf{B}^T\mathbf{x} \end{aligned} \tag{4.4}$$

is completely observable, and conversely.

In other words, the pair $(\mathbf{A}, \mathbf{B})$ is controllable is equivalent to the pair $(-\mathbf{A}^T, \mathbf{C}^T)$ is observable.

Note: If the matrices have complex elements, transpose in preceding equation is replaced by conjugate transpose.

Remarks:

(1) Since the system observability is a dual property of the system controllability, therefore all discussions for controllability can be applied to observability in a similar way.

(2) Designing an observer (state estimator) is thus a dual problem of designing a state-feedback, and same design procedures can apply to both problems.

(3) The duality theorem is very useful, since it enables us to deduce immediately from a controllability result the corresponding one on observability (and conversely).

    For example: to obtain the observability criterion for the time invariant case, one may apply the Condition II of Theorem 3.3.1 (check rank$(\mathcal{C}) = \mathbf{n}$) to (4.4), since transposition does not affect its rank.

4.5 Canonical Forms. Kalman Canonical Decomposition

Given a transfer function, it can be written in either the controllable canonical form or observable canonical form.

For a system that is *not controllable* (usually by this, we mean not all models of the system are controllable) one can perform a decomposition to separate the controllable and uncontrollable models. Consider the system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned} \tag{4.5}$$

Let

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

then

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \tilde{\mathbf{D}}\mathbf{u}\,,\end{aligned} \tag{4.6}$$

where

$$\tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \quad \tilde{\mathbf{B}} = \mathbf{P}\mathbf{B} \quad \tilde{\mathbf{D}} = \mathbf{D}$$

All properties of (4.5), including controllability and observability, are preserved in (4.6). We also have

$$\tilde{\mathcal{C}} = [\,\tilde{\mathbf{B}}\ \tilde{\mathbf{A}}\mathbf{B}\ \cdots\ \tilde{\mathbf{A}}^{\mathbf{n-1}}\tilde{\mathbf{B}}\,] = [\,\mathbf{PB}\ \mathbf{PAB}\ \cdots\ \mathbf{PA}^{\mathbf{n-1}}\mathbf{B}\,] = \mathbf{P}\mathcal{C}$$

and

$$\tilde{\mathcal{O}} = \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{C}}\mathbf{A} \\ \vdots \\ \tilde{\mathbf{C}}\mathbf{A}^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C}\mathbf{P}^{-1} \\ \mathbf{C}\mathbf{A}\mathbf{P}^{-1} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1}\mathbf{P}^{-1} \end{bmatrix} = \mathcal{O}\mathbf{P}^{-1}\,.$$

Lemma 4.5.1. *For the system described by (4.5) if*

$$\mathrm{rank}[\,\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}\,] = n_1 < n\,,$$

one can build a matrix

$$\mathbf{Q} := \mathbf{P}^{-1} = [\,\mathbf{q}_1\ \mathbf{q}_2\ \cdots\ \mathbf{q}_{n_1}\ \ \mathbf{q}_{n_1+1}\ \mathbf{q}_{n_1+2}\ \cdots\ \mathbf{q}_n\,]\,,$$

where $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n_1}$ *are* n_1 *linearly independent columns chosen directly from*

$$[\,\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}\,]\,,$$

and

$$\mathbf{q}_{n_1+1}\ \mathbf{q}_{n_1+2}\ \cdots\ \mathbf{q}_n$$

are $(n - n_1)$ *vectors arbitrarily chosen to make* $\mathbf{Q}$ *invertible.*

By taking the matrix $\mathbf{P}$ as the transformation matrix, the system can be transformed into the following decomposed system model:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_{11} & \tilde{\mathbf{A}}_{12} \\ 0 & \tilde{\mathbf{A}}_{22} \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = [\,\tilde{\mathbf{C}}_1\ \tilde{\mathbf{C}}_2\,], \quad \tilde{\mathbf{D}} = \tilde{\mathbf{D}}\,. \tag{4.7}$$

Now we want to show that the transfer function matrices of (4.7) and (4.5) are the same. We have the following theorem.

Theorem 4.5.2. *The sub-system* $(\tilde{\mathbf{A}}_{11}, \tilde{\mathbf{B}}_1, \tilde{\mathbf{C}}_1, \mathbf{D}$*, defined by (4.7) is controllable (i.e. contains all the controllable models) and yields the same transfer function as the original system* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$*.*

Proof. Since

$$\mathrm{rank}[\tilde{\mathbf{B}}\ \tilde{\mathbf{A}}\mathbf{B}\ \cdots\ \tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{B}} = \mathrm{rank}\begin{bmatrix} \tilde{\mathbf{B}}_1 & \tilde{\mathbf{A}}_{11}\tilde{\mathbf{B}}_1 & \cdots & \tilde{\mathbf{A}}_{11}^{n-1}\tilde{\mathbf{B}}_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} = n_1 < n\,,$$

therefore rank$[\,\tilde{\mathbf{B}}_1\ \tilde{\mathbf{A}}_{11}\tilde{\mathbf{B}}_1\ \cdots\ \tilde{\mathbf{A}}_{11}^{n-1}\tilde{\mathbf{B}}_1\,] = n_1$, which means $(\tilde{\mathbf{A}}_{11}, \tilde{\mathbf{B}}_1)$ is controllable. The transfer function is given by

$$\mathbf{G}(s) = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \mathbf{D} = [\,\tilde{\mathbf{C}}_1\ \ \tilde{\mathbf{C}}_2\,] \begin{bmatrix} s\mathbf{I}_1 - \tilde{\mathbf{A}}_{11} & -\tilde{\mathbf{A}}_{12} \\ 0 & s\mathbf{I}_2 - \tilde{\mathbf{A}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} + \mathbf{D}$$

$$= \tilde{\mathbf{C}}_1(s\mathbf{I} - \tilde{\mathbf{A}}_{11})^{-1}\tilde{\mathbf{B}}_1 + \mathbf{D} = \mathbf{G}_1(s)\,.$$

□

Note: The inverse of the block triangular matrix is calculated as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ 0 & \mathbf{C} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{-1} \\ 0 & \mathbf{C}^{-1} \end{bmatrix}$$

The above decomposition (4.7) can also be written using the following notations:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_c & \tilde{\mathbf{A}}_{12} \\ 0 & \tilde{\mathbf{A}}_n \end{bmatrix}, \ \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_c \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{C}} = [\,\tilde{\mathbf{C}}_c\ \tilde{\mathbf{C}}_n\,], \quad \tilde{\mathbf{D}} = \tilde{\mathbf{D}}\,,$$

or

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{x}}_c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\mathbf{x}}_n \end{bmatrix},$$

where the subscript c means controllable models and n means uncontrollable models.

Note: The input-output description (transfer function) does not show the uncontrollable models. However, these models exist as the internal dynamics of the system.

Consider an unobservable system which means not all states are observable. Similar decomposition can be performed on an unobservable system with the following transformation matrix:

$$\mathbf{P} := \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \cdots \\ \mathbf{q}_{n_1} \\ \mathbf{q}_{n_1+1} \\ \mathbf{q}_{n_1+2} \\ \cdots \\ \mathbf{q}_n \end{bmatrix},$$

where $\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_{n_1}$ are n_1 linearly independent columns chosen directly from the observability matrix $\mathcal{O}$, and $\mathbf{q}_{n_1+1} \ \mathbf{q}_{n_1+2} \ \cdots \ \mathbf{q}_n$ are $(n-n_1)$ vectors arbitrarily chosen to make $\mathbf{P}$ invertible. Then the system can be separated into observable models and unobservable models.

$$\begin{bmatrix} \dot{\tilde{\mathbf{x}}}_o \\ \dot{\tilde{\mathbf{x}}}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_o & 0 \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_n \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_o \\ \tilde{\mathbf{x}}_n \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}}_o \\ \tilde{\mathbf{B}}_n \end{bmatrix} \mathbf{u}\,, \quad \mathbf{y} = [\tilde{\mathbf{C}}_o \ \tilde{\mathbf{C}}_n] \begin{bmatrix} \mathbf{x}_o \\ \mathbf{x}_n \end{bmatrix} + \mathbf{D}\mathbf{u}\,.$$

Similarly, the observable sub-system $(\tilde{\mathbf{A}}_o, \tilde{\mathbf{B}}_o, \tilde{\mathbf{C}}_o, \tilde{\mathbf{D}})$ also yields the same transfer function as the original system $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$.

Finally, by combining the above two decomposition methods and apply them to a system that is not completely controllable and observable, we can perform the so-called Kalman decomposition, and the system is then divided into four sub-systems, which correspond to four models respectively:

(1) controllable and observable ($\mathbf{CO}$)

(2) controllable but unobservable ($\mathbf{C\bar{O}}$)

(3) observable but uncontrollable ($\mathbf{O\bar{C}}$)

(4) uncontrollable and unobservable ($\bar{\mathbf{O}}\bar{\mathbf{C}}$).

Theorem 4.5.3. *Every state-space equation can be transformed, by an equivalence transformation, into the following canonical form*

$$\begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{co} \\ \dot{\tilde{\mathbf{x}}}_{c\bar{o}} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}o} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_{co} & 0 & \tilde{\mathbf{A}}_{13} & 0 \\ \tilde{\mathbf{A}}_{21} & \tilde{\mathbf{A}}_{c\bar{o}} & \tilde{\mathbf{A}}_{23} & \tilde{\mathbf{A}}_{24} \\ 0 & 0 & \tilde{\mathbf{A}}_{\bar{c}o} & 0 \\ 0 & 0 & \tilde{\mathbf{A}}_{43} & \tilde{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{co} \\ \dot{\tilde{\mathbf{x}}}_{c\bar{o}} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}o} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}}_{co} \\ \tilde{\mathbf{B}}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [\, \tilde{\mathbf{C}}_{co} \; 0 \; \tilde{\mathbf{C}}_{\bar{c}o} \; 0 \;] \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_{co} \\ \dot{\tilde{\mathbf{x}}}_{c\bar{o}} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}o} \\ \dot{\tilde{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u} \;,$$

where the vector $\tilde{\mathbf{x}}_{co}$ *is controllable and observable,* $\tilde{\mathbf{x}}_{c\bar{o}}$ *is controllable but not observable,* $\tilde{\mathbf{x}}_{\bar{c}o}$ *is observable but not controllable, and* $\tilde{\mathbf{x}}_{\bar{c}\bar{o}}$ *is neither controllable not observable.*

Furthermore, the state equation is zero state equivalent to the controllable and observable state equation

$$\dot{\tilde{\mathbf{x}}}_{co} = \tilde{\mathbf{A}}_{co}\tilde{\mathbf{x}}_{co} + \tilde{\mathbf{B}}_{co}\mathbf{u} \tag{4.8}$$

$$\mathbf{y} = \tilde{\mathbf{C}}_{co}\tilde{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u}\,.$$

Lemma 4.5.4. *The transfer function of the system equals the transfer function of the controllable and observable sub-system:*

$$\mathbf{G}(s) = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}} + \mathbf{D} = \tilde{\mathbf{C}}_{co}(s\mathbf{I} - \tilde{\mathbf{A}}_{co})^{-1}\tilde{\mathbf{B}} + \mathbf{D} = \mathbf{G}_{co}(s) \tag{4.9}$$

Note: By (4.9), the uncontrollable or unobservable modes will disappear in calculating the transfer function via pole zero cancellation.

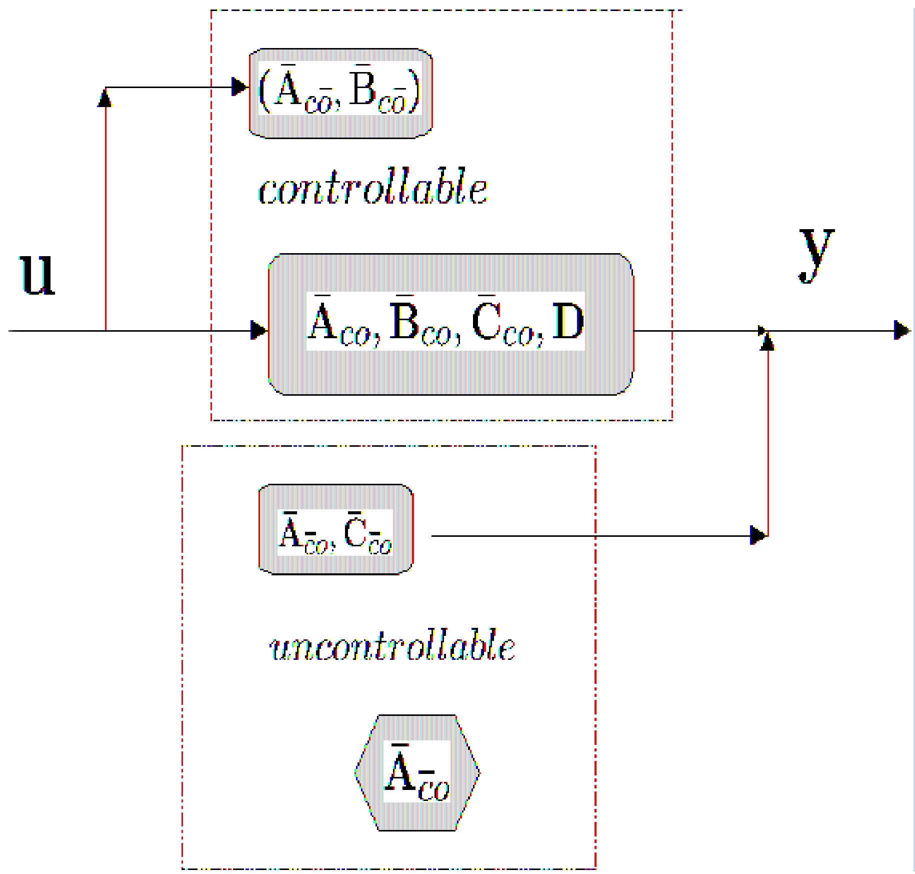

Figure 4.1: Diagram for Kalman decomposition

Last theorem can be illustrated symbolically as shown in Figure 4.1 Generally speaking, given a system, it is desirable for the uncontrollable modes to to be stable (stabilizability), and the unobservable modes to be stable (detectability), i.e. $\tilde{\mathbf{A}}_{co}$, $\tilde{\mathbf{A}}_{c\tilde{o}}$, and $\tilde{\mathbf{A}}_{\tilde{c}o}$ all have negative real eigenvalues solely, hence $\tilde{\mathbf{x}}_{\tilde{c}o} \to 0$, $\tilde{\mathbf{x}}_{c\tilde{o}} \to 0$, and $\tilde{\mathbf{x}}_{\tilde{c}\tilde{o}} \to 0$, as $t \to 0$.

Example 4.5.5. *Decomposition.*

For a given uncontrollable system, perform the transformation into decomposed form. Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [\,1\ 1\ 1\,]\,\mathbf{x}$$

$$\text{rank}[\,\mathbf{B}\ \mathbf{AB}\,] = \text{rank} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 2 < 3 \Rightarrow \textit{uncontrollable.}$$

We form the 3×3 transformation matrix

$$\mathbf{P}^{-1} \equiv \mathbf{Q} \equiv \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where first two columns are linearly independent columns of $[\,\mathbf{B}\ \mathbf{AB}\,]$, and third column is chosen as long as $\mathbf{P}$ is nonsingular. Then the equivalence transformation $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$ or $\mathbf{x} = \mathbf{P}^{-1}\,\tilde{\mathbf{x}}$ will transfer given system into decomposed system model (4.7):

$$\tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{P}\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} = [\,1\ 1\ 1\,] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [\,1\ 2\ 1\,]\,.$$

Therefore decomposed equation is controllable and has the same transfer function matrix as given system.

CHAPTER 5

REALIZABILITY

In this chapter we address questions related to the input-output (zero-state) behavior of the standard linear state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

with zero initial state assumed. From the knowledge of the fundamental matrix solution of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ we can determine the input-output behavior and the weighting pattern $\mathbf{G}$. Our interest here is the reversal of the computation, and in particular, we need to establish conditions on a specified $\mathbf{G}$, that guarantee existence of a corresponding linear state equation. At the beginning of this chapter we present the main ideas and historical facts of realizability. Then we consider questions related to realization of time-invariant systems, discrete-time systems and time varying systems. Furthermore, we discuss minimum realization and introduce the realizability criteria. Some interesting examples illustrate the material and complete the chapter.

5.1 Introduction

In previous chapters we have analyzed questions about three theoretic properties that arise naturally when studying a basic system. *Observability* which means that all information about the state x should in principle, be recoverable from knowledge of the measurements $\mathbf{y}$ that result. *Controllability* which has to do with finding controls $\mathbf{u}$ that allow one to attain a desired state $\mathbf{x}$. In the case of linear systems, this analogy can be made precise through the idea of *duality* and it permits obtaining most results for observability as immediate corollaries of those for controllability. This duality extends to a precise correspondence between optimal control problems (in which one studies the trade-off between cost of control and speed of control), and filtering problems (which are based on the trade-off between magnitude of noise and speed of estimation).

Since outputs (measurements) are incorporated into the basic definition of a system, one can pose questions that involve the relationship between controls $\mathbf{u}$ and outputs (observations) $\mathbf{y}$. It is then of interest to characterize the class of *input/output behaviors* that can be obtained and conversely. This type of analysis is called the *realization* problem.

The main question that arises immediately is 'Given an observed input/output behavior, what possible systems could give rise to it?' In other words, if we start with a 'black box' model that provides only information about how $\mathbf{u}$ affects $\mathbf{y}$, how does one deduce the differential (or, in discrete-time, difference) equation that is responsible for this behavior?

The answers to these questions lead to the concept of realizability which is, roughly speaking, a mathematical representation of a system.

Besides obvious philosophical interest, such questions are closely related to *identification* problems, where one desires to estimate the input/output behavior itself from partial or possible noisy data, whereas state-space descriptions serve to parameterize such possible input/output behavior. Conversely, from a synthesis viewpoint, realization techniques allow us to compute a state representation, and, if desired, also construct a physical system, that satisfies given input/output specifications.

The main results on the realization problem presented in this chapter show that realizations essentially are unique provided that they satisfy certain minimality or redundancy requirements.

Later in this chapter we will provide the main theorems of realization theory for linear systems. The underlying properties turn out to be closely related to other system theoretic notions such as controllability and observability.

5.2 History

Over the past 60 years, realizability has developed into a subject of such dimensions that a comprehensive overview would require a heavy book. Nowadays such book does not exist but a list of publications I found exceeded hundred of articles. For realizability has many faces, each of them turned towards different areas of Logic, Mathematics, Number Theory, Computer Science and Control Theory. Jaap van Oosten ([17], p.239-263) developed a fairly good sketch of a few basic topics in the history of realizability. Oosten divided the history of realizability into two periods, 1940-1980 and 1980-2000. Realizability was introduced in Stephen Cole Kleene's original 1945 paper, ([18], p.109-124). The definition specifies, in an inductive way, what it means that a natural number n *realizes* a sentence φ of the language of arithmetic. However, concept of realizability has been used widely in different areas.

In Mathematical Control Theory, many researchers have worked in area of realizability: Arbib, Falb, Kalman in late 1960's [21], Rubio in early 1970's [20], Barnett, Cameron [2], Silverman [16], and Wimmer [19] in 1970's. Later Davidson and Wang, Porter and Crossley. More recently M. di Bernardo, E.D.Sontag [14] (in 1990's), T.-H.S.Li and D.W.Fausett, L.V.Fausett, K.N.Murty [11], [12], [13].

5.3 Realization of a Time Invariant System

In order to give a formal definition for realization, consider a time-invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\,, \tag{5.1}$$

where $\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $n \times m$ constant matrices, $\mathbf{x}$(t) is the $n-$vector of state variables and $\mathbf{u}(t)$ is the $m-$vector of input or control variables.

Suppose that we have r output variables, each a linear combination of the x_i, so that

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{5.2}$$

where $\mathbf{C}$ is an $r \times n$ constant matrix. Taking Laplace transforms of (5.1) and assuming the initial condition:

$$\mathbf{x}(t_0) = 0\,,$$

we have

$$s\bar{\mathbf{x}}(s) = \mathbf{A}\bar{\mathbf{x}}(s) + \mathbf{B}\bar{\mathbf{u}}(s)\,,$$

or after rearrangement

$$\bar{\mathbf{x}}(s)(s\mathbf{I}_n - \mathbf{A}) = \mathbf{B}\bar{\mathbf{u}}(s)\,.$$

Therefore

$$\bar{\mathbf{x}}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\bar{\mathbf{u}}(s)\,.$$

Notice from (5.2) the Laplace transform of the output is

$$\bar{\mathbf{y}}(s) = \mathbf{C}\bar{\mathbf{x}}(s)\,.$$

Substituting $\bar{\mathbf{x}}$ into this equation gives

$$\bar{\mathbf{y}} = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\bar{\mathbf{u}}(s)\,.$$

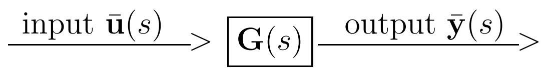

Figure 5.1: Transfer Function Matrix Diagram

Now we define the $r \times m$ matrix $\mathbf{G}$ such that:

$$\mathbf{G} := \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} \,. \tag{5.3}$$

Thus we have

$$\bar{\mathbf{y}} = \mathbf{G}\bar{\mathbf{u}}(s) \,, \tag{5.4}$$

where the matrix $\mathbf{G}$ is called the **transfer matrix**.

Note: By analogy with the scalar case, described in Chapter 2 (section 2.3), the transfer matrix $\mathbf{G}$ relates the Laplace transform of the output vector to that of the input vector. A block diagram representation of the form shown in Figure (2.1) for $g(s)$ can still be drawn. Figure (5.3) illustrates the equation (5.4) with input and output vectors, and the operator now is the matrix $\mathbf{G}$. In case when $r = m = 1$, the matrix $\mathbf{G}(s)$ reduces to $g(s)$.

The transfer matrix $\mathbf{G}(s)$ defined in (5.3) is associated with a time-invariant system. In practice it is often happens that the mathematical description of a system in terms of a differential equation is not known, but $\mathbf{G}(s)$ can be determined from experimental measurements or other considerations. Our interest is to find a system in our usual linear state space form to which $\mathbf{G}(s)$ corresponds. **For example,** an analogue simulator can then be constructed, this is essentially a device (usually electronic) which duplicates the behavior of the physical system and thus can be conveniently used to study its properties.

In formal terms, given an $r \times m$ matrix $\mathbf{G}(s)$ whose elements are rational functions of s, we wish to find constant matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ having dimensions $n \times n$, $n \times m$ and

$r \times n$ respectively such that

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,,$$

and the system equations will then be

$$\dot{\mathbf{x}} = \mathbf{A}x(t) + \mathbf{B}\mathbf{u}(t)$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x}(t)\,.$$

Definition 5.3.1. *The triple* $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ *is termed a realization of* $G(s)$ *of order* n*, and is not, of course, unique. Among all such realizations, some will include matrices* $\mathbf{A}$ *having least dimensions; these are called minimal realizations, since the corresponding systems involve the smallest possible number of state variables.*

Note: Since each element in

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{adj(s\mathbf{I} - \mathbf{A})}{det(s\mathbf{I} - \mathbf{A})}$$

has the degree of the numerator less that of the denominator it follows that

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \longrightarrow 0$$

as $s \longrightarrow \infty$, and we shall assume that any given $\mathbf{G}(s)$ also has this property. In such a case $\mathbf{G}(s)$ is termed **strictly proper**.

The transfer function for the scalar output case defined in (2.3), after substitutions for $\beta(s)$ and $k(s)$ has a form such as

$$g(s) = \frac{\beta_0 s^m + \beta_1 s^{m-1} + \ldots\ldots + \beta_{m-1}s + \beta_m}{s^n + k_1 s^{n-1} + \ldots\ldots + k_{n-1}s + k_n}\,, \qquad (5.5)$$

with $m < n$. Our interest here is to determine (if it is possible) the transfer function.

Theorem 5.3.2. *The transfer function (5.5) can be expressed as*

$$g(s) = \mathbf{r}(s\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}\,,$$

with the 'companion form' matrix

$$\mathbf{C} := \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ -k_n & -k_{n-1} & -k_{n-2} & \cdot & \cdot & \cdot & -k_1 \end{bmatrix},$$

and

$$\mathbf{d} := [\,0,\ 0,\ \cdots,\ 0,\ 1]^T\ ,$$

$$\mathbf{r} := [\,\beta_m, \cdots, \beta_1, \beta_0, 0, \cdots, 0\,]$$

Proof. Consider the product $\mathbf{r}(s\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ and show it does have the form (5.5).

$$\mathbf{r}(s\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} =$$

$$= [\,\beta_m, \cdots, \beta_1,\ \beta_0,\ 0, \cdots, 0\,]\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{C})}{\det(s\mathbf{I} - \mathbf{C})}\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

Since $\mathbf{C}$ is the companion form matrix, then its characteristic polynomial is

$$\det(s\mathbf{I} - \mathbf{C}) = s^n + k_1 s^{n-1} + \cdots + k_n = k(s)\,,$$

thus we have

$$\mathbf{r}(s\mathbf{I}-\mathbf{C})^{-1}\mathbf{d}$$

$$= [\,\beta_m, \cdots, \beta_1,\ \beta_0,\ 0, \cdots, 0\,]\frac{\text{adj}(s\mathbf{I}-\mathbf{C})}{k(s)}\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

Notice that matrix $s\mathbf{I}-\mathbf{C}$ has the form

$$s\mathbf{I}-\mathbf{C} = \begin{bmatrix} s & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & s & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & s & -1 \\ k_n & k_{n-1} & k_{n-2} & \cdot & \cdot & k_2 & s+k_1 \end{bmatrix}.$$

From matrix multiplication it follows that because column vector $\mathbf{d}$ has zero elements except for the last one, the product

$$\text{adj}(s\mathbf{I}-\mathbf{C})\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix},$$

which is the last column of $adj(s\mathbf{I}-\mathbf{C})$.

Now we have

$$\mathbf{r}(s\mathbf{I}-\mathbf{C})^{-1}\mathbf{d} = \frac{[\,\beta_m, \cdots, \beta_1,\ \beta_0,\ 0, \cdots, 0\,]\begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix}}{k(s)}$$

$$= \frac{\beta_m + \beta_{m-1}s + \cdots + \beta_1 s^{m-1} + \beta_0 s^m}{k(s)}$$

$$= \frac{\beta_m + \beta_{m-1}s + \cdots + \beta_1 s^{m-1} + \beta_0 s^m}{s^n + k_1 s^{n-1} + \cdots + k_{n-1}s + k_n},$$

which has the same form as the transfer function in equation (5.5). □

Furthermore, there is a need to generalize the result of Theorem 5.3.2. This leads us to find a simple realization for a transfer function matrix, although it will not in general be minimal.

Theorem 5.3.3. *Let*

$$g(s) = s^q + g_1 s^{q-1} + \cdots + g_q$$

be the monic least common denominator of all the elements $g_{ij}(s)$ *of* $\mathbf{G}(s)$. *Let*

$$g(s)\mathbf{G}(s) = s^{q-1}\mathbf{G}_0 + s^{q-2}\mathbf{G}_1 + \cdots + \mathbf{G}_{q-1},$$

where the $\mathbf{G}_i$ *are constant* $r \times m$ *matrices. Then a realization of* $\mathbf{G}(s)$ *is given by*

$\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ *with*

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I}_m & 0 & \cdot & 0 \\ 0 & 0 & \mathbf{I}_m & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{I}_m \\ -g_q\mathbf{I}_m & -g_{q-1}\mathbf{I}_m & \cdot & \cdot & -g_1\mathbf{I}_m \end{bmatrix},$$

$$\mathbf{B} = \left.\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \mathbf{I}_m \end{bmatrix}\right\} q \text{ blocks}, \qquad \textit{and}$$

$$\mathbf{C} = \left[\mathbf{G}_{q-1}\ \mathbf{G}_{q-2}\ \cdots\ \mathbf{G}_0\right].$$

Furthermore, the pair $[\mathbf{A}, \mathbf{B}]$ *is completely controllable.*

Proof involves the Kronecker product, and is due to H.K.Winner [19] (pg.201-206) and S.Barnett and R.G.Cameron [1] (pg.139-141).

Now it is appropriate to discuss the idea of algebraic equivalence and its applications for the realization problems.

Definition 5.3.4. *If* $\mathbf{P}(t)$ *is an* $n \times n$ *matrix that is continuous and nonsingular for all* $t \geq t_0$, *then the system obtained by the transformation* $\tilde{\mathbf{x}}(t) = \mathbf{P}(t)\mathbf{x}(t)$ *is said to be algebraically equivalent to the system defined in equations (5.1), (5.2).*

Note: If the original system is completely controllable, then so is the algebraically equivalent system.

Observing equations (4.7), (4.6) and denote by $\mathbf{P}$ the transformation matrix which decomposes $\mathbf{A}$ and $\mathbf{B}$ in (5.1) into the form

$$\frac{d}{dt}\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\mathbf{u} \tag{5.6}$$

and $\mathbf{C}$ in (5.2) into the form

$$\mathbf{y} = [C_1\ C_2]\mathbf{x}\,,$$

where $x^{(1)}$ and $x^{(2)}$ have orders n_1 and $n - n_1$ respectively and the pair $[\,A_1, B_1\,]$ is completely controllable.

Note: Comparing last equation (5.6) with Example 3 from Chapter 3 (section 3.2), it is clear that in (5.6) the vector $\mathbf{x}^{(2)}$ is completely unaffected by $\mathbf{u}$.
Therefore the state space has been divided into two parts, one being *completely controllable* and other *uncontrollable.*

Using information presented in Chapter 4 (section 4.5) it follows that since the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant, then the transformation matrix $\mathbf{P}$ is also constant, and the transformation

$$\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

produces a system with matrices

$$\tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1},\ \ \tilde{\mathbf{B}} = \mathbf{P}\mathbf{B},\ \ \tilde{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1}$$

Theorem 5.3.5. *If the triple* $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ *represents a completely controllable (completely observable) system, then so does* $\{\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}\}$.

The proof follows by checking the rank of the controllability matrix and the observability matrix [1].

Lemma 5.3.6. *Two systems are algebraically equivalent if their transfer function matrices are identical:*

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}\,.$$

Now we prove the important result of this section, which links together the three basic concepts of controllability, observability, and realization.

Theorem 5.3.7. *A realization* $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ *of a given transfer function matrix* $\mathbf{G}(s)$ *is minimal if and only if the pair* $[\mathbf{A},\mathbf{B}]$ *is completely controllable, and the pair* $[\mathbf{A},\mathbf{C}]$ *is completely observable.*

Proof. Let $\mathcal{C}$ be the controllability matrix such that:

$$\mathcal{C} := [\,\mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B}\ \cdots \mathbf{A}^{n-1}\mathbf{B}\,]\,,$$

and $\mathcal{O}$ be the observability matrix defined by

$$\mathcal{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

We need to prove the sufficiency and necessity conditions.

Sufficiency.

In Barnett and Cameron's book [1] it is shown that if these matrices both have rank n then $\mathbf{G}$ has least order n. Suppose that there exists a realization $\{\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}}\}$ of $\mathbf{G}(s)$ with $\tilde{\mathbf{A}}$ having order $\tilde{n}$.

From

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \tilde{\mathbf{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}$$

it follows that

$$\mathbf{C}e^{\mathbf{A}\mathbf{t}}\mathbf{B} = \tilde{\mathbf{C}}e^{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}\,,$$

and recalling the series for the exponential matrix, we have that

$$\mathbf{C}\mathbf{A}^{i}\mathbf{B} = \tilde{\mathbf{C}}\tilde{\mathbf{A}}^{i}\tilde{\mathbf{B}}\,. \tag{5.7}$$

Consider the product

$$\mathcal{OC} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{B}\ \mathbf{AB}\ \mathbf{A^2B} \cdots \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Matrix multiplication gives

$$= \begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \cdot & \cdot & \mathbf{CA}^{n-1}\mathbf{B} \\ \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \cdot & \cdot & \mathbf{CA}^{n}\mathbf{B} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{CA}^{n-1}\mathbf{B} & \mathbf{CA}^{n}\mathbf{B} & \cdot & \cdot & \mathbf{CA}^{2n-2}\mathbf{B} \end{bmatrix},$$

and using result of (5.7), we can express this matrix as

$$= \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^2 \\ \vdots \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^{n-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{B}}\ \tilde{\mathbf{A}}\tilde{\mathbf{B}}\ \tilde{\mathbf{A}}^2\tilde{\mathbf{B}} \cdots \tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{B}} \end{bmatrix};$$

or

$$\mathcal{OC} = \tilde{\mathcal{O}}\tilde{\mathcal{C}}\,.$$

The matrix $\mathcal{OC}$ has rank n, so the matrix $\tilde{\mathcal{O}}\tilde{\mathcal{C}}$ has rank n too. Notice, the dimensions of $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{C}}$ are respectively $qn \times \tilde{n}$ and $\tilde{n} \times pn$, where q and p are positive integers, so that the rank of $\tilde{\mathcal{O}}\tilde{\mathcal{C}}$ cannot be greater than $\tilde{n}$. Therefore, $n \leq \tilde{n}$, so there is no realization of $\mathbf{G}$ having order less than n.

Necessity. We use the method of contradiction to prove that if the pair $[\mathbf{A}, \mathbf{B}]$ is not completely controllable, then there exists a realization of $\mathbf{G}(s)$ having order less than n. We use duality to derive the corresponding part of the proof involving observability.

Let the rank of $\mathcal{C}$ be $n_1 < n$ and let $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n_1}$ be any set of n_1 linearly independent columns of $\mathcal{C}$.

Define the $n \times n$ matrix $\mathbf{P}$ from its inverse (Lemma 4.5.1)

$$\mathbf{P}^{-1} := [\, \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n_1}, \mathbf{q}_{n_1+1}, \cdots, \mathbf{q}_n \,] \,.$$

So that the corresponding transformation is

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x} \,,$$

where the columns $\mathbf{q}_{n_1+1}, \cdots, \mathbf{q}_n$ are any vectors which make the matrix $\mathbf{P}$ in (5.6) nonsingular. Since rank of $\mathcal{C}$ is n_1, it follows that each of its columns is a linear combination of the basis $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{n_1}$. Notice the matrix

$$\mathbf{A}\mathcal{C} = [\, \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\; \cdots]$$

contains all but the first m columns of $\mathcal{C}$, so in particular it follows that the vectors $\mathbf{Au}_i$, $\; i = 1, 2, \ldots, n_1$, can be expressed in terms of the same basis. If we multiply the equation (5.6) on the left by $\mathbf{P}$, we see that $\mathbf{Pq}_i$ is equal to the $i - th$ column of $\mathbf{I}_n$.

Combining these facts together, we obtain

$$\begin{aligned}\tilde{\mathbf{A}} &= \mathbf{PAP}^{-1}\\ &= \mathbf{P}[\mathbf{Aq}_1, \cdots, \mathbf{Aq}_{n_1}, \cdots, \mathbf{Au}_n]\\ &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_3 \end{bmatrix},\end{aligned}$$

where the matrix $\mathbf{A}_1$ is $n_1 \times n_1$.

Similarly, since $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$ also form the basis for the columns of $\mathbf{B}$, we have

$$\tilde{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix},$$

where the matrix $\mathbf{B}_1$ is $n_1 \times m$. For the transformation

$$\tilde{\mathbf{C}} = \mathbf{CP}^{-1} = [\,\mathbf{C}_1\,\mathbf{C}_2\,],$$

we have

$$\begin{aligned}\mathbf{G}(s) &= \tilde{\mathcal{C}}(s\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{B}\\ &= [\,\mathbf{C}_1\ \mathbf{C}_2\,]\begin{bmatrix} s\mathbf{I} - \mathbf{A}^{-1} & -\mathbf{A}_2 \\ 0 & s\mathbf{I} - \mathbf{A}_3 \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}\\ &= [\,\mathbf{C}_1\ \mathbf{C}_2\,]\begin{bmatrix} (s\mathbf{I} - \mathbf{A}_1)^{-1} & (s\mathbf{I} - \mathbf{A}^{-1})^{-1}\mathbf{A}_2(s\mathbf{I} - \mathbf{A}_3)^{-1} \\ 0 & (s\mathbf{I} - \mathbf{A}_3)^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix}.\end{aligned}$$

This means that $\{\mathbf{A_1}, \mathbf{B_1}, \mathbf{C_1}\}$ is a realization of $\mathbf{G}(s)$ of order $n_1 < n$. This contradicts the assumption that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is minimal, hence $[\mathbf{A}, \mathbf{B}]$ must be completely controllable. □

5.4 Examples

Example 5.4.1. *Realization For Scalar Transfer Function.*

Computing a realization to illustrate Theorem 5.3.2 for scalar case. Consider the scalar transfer function

$$g(s) = \frac{2s+7}{s^2-5s+6}\,.$$

Using Theorem 5.3.2,

$$g(s) = \mathbf{r}(s\mathbf{I}-\mathbf{C})^{-1}\mathbf{d}\,,$$

in particular, in this case $m = 1, n = 2$,

$$g(s) = \frac{\beta_0 s+\beta_1}{s^2+k_1 s+k_2} = \frac{2s+7}{s^2-5s+6}\,.$$

From last equation:

$$\beta_0 = 2 \qquad \beta_1 = 7 \qquad k_1 = -5 \qquad k_2 = 6\,.$$

Therefore

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix},$$

$$\mathbf{d} = [0\ 1]^T\,,$$

and

$$\mathbf{r} = [\beta_1\ \beta_0] = [7\ 2]\,.$$

We observe that

$$s^2-5s+6 = det(s\mathbf{I}-\mathbf{A})\,,$$

which is possible if we have

$$s\mathbf{I}-\mathbf{A} = \begin{bmatrix} s & 1 \\ -6 & s-5 \end{bmatrix}.$$

or

$$s\mathbf{I}-\mathbf{A} = \begin{bmatrix} s-2 & 0 \\ 0 & s-3 \end{bmatrix}.$$

$\Rightarrow$ Thus one realization of $g(s)$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = [7\ 2]\,.$$

Conclusion : From this example it is clear that when $m = 1$ the matrices $\mathbf{A}$ and $\mathbf{B}$ reduce to $\mathbf{C}$ and $\mathbf{d}$ in the controllable canonical form.

Note: There is one more realization of $g(s)$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = [-11\ 13]\,.$$

Both these realizations are minimal, since numerator and denominator of a scalar transfer function do not have any common factors.

Example 5.4.2. *Realization For The Matrix transfer Function.*

Computing a realization for the case of matrix transfer function. Consider the proper rational matrix

$$\mathbf{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}.$$

First column is

$$G_{C1} = \begin{bmatrix} \frac{4s-10}{2s+1} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{(4s-10)(s+2)}{(2s+1)(s+2} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{4s^2-2s-20}{2s^2+5s+2} \\ \frac{1}{2s^2+5s+2} \end{bmatrix}.$$

The built-in MATLAB function

$$[a, b, c, d] = tf2ss(num, den)$$

generates the controllable canonical form realization shown in Theorem 5.3.3. Typing

$$n1 = [4\ \ -2\ \ -20; 0\ 0\ 1];$$

$$d1 = [2\ 5\ 2];$$

$$[a, b, c, d] = tf2ss(n1, d1)\,.$$

From MATLAB we have the following realization for the first column of $\mathbf{G}(s)$

$$\dot{\mathbf{x}}_1 = \mathbf{A_1 x_1} + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1$$

$$\mathbf{y}_{C1} = \mathbf{C_1 x_1} + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & .5 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1 .$$

Similarly, find the realization for the second column of $\mathbf{G}(s)$

$$G_{C2} = \begin{bmatrix} \frac{3}{s+2} \\ \frac{s+1}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} \frac{3s+2}{s^2+4s+4} \\ \frac{s+1}{s^2+4s+4} \end{bmatrix} .$$

Typing in MATLAB Command window:

$$n2 = [0\ 3\ 2; 0\ 1\ 1];$$

$$d2 = [1\ 4\ 4];$$

$$[a, b, c, d] = tf2ss(n2, d2) ,$$

we have

$$\dot{\mathbf{x}}_2 = \mathbf{A_2 x_2} + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$\mathbf{y}_{C2} = \mathbf{C_2 x_2} + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_2 .$$

These two realizations can be combined as

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 & 0 \\ 0 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \mathbf{y}_{C1} + \mathbf{y}_{C2} = [\mathbf{C}_1\ \mathbf{C}_2]\mathbf{x} + [\mathbf{d}_1\ \mathbf{d}_2]\mathbf{u}$$
$$= \begin{bmatrix} -6 & -12 & 3 & 2 \\ 0 & .5 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

This realization of the transfer matrix function $\mathbf{G}(s)$ has dimension 4.

5.5 Realization of Discrete-Time Systems

Earlier in this chapter we have discussed only systems described by linear differential equations (linear continuous time systems). Previously in Chapter 2 (section 2.2) we have mentioned that the definitions of controllability and observability can be carried over to systems described by linear difference equation as given in equation (2.1),

$$X(k+n) + k_1 X(k+n-1) + \cdots + k_{n-1}X(k+1) + k_n X(k)$$
$$= \beta_0 u(k+m) + \beta_1 u(k+m-1) + \cdots + \beta_m u(k)\,,$$

(where k_i and β_i are constants), with only minor modifications.

When $\mathbf{A}$ and $\mathbf{B}$ are time invariant the controllability and observability criteria are the same as those for the continuous time case given in Chapters 2 and 3 with one important exception ([20], pg. 234). In the situation when the discrete-time system matrix $\mathbf{A}$ is singular, it is necessary to define a new concept of reachability.

Definition 5.5.1. *The linear system*

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{5.8}$$

is **completely reachable** *(from the origin) if given any state* $\mathbf{x}_i$ *there exists an integer* $N > 0$ *and a control sequence* $\mathbf{u}(0), \mathbf{u}(1), \cdots, \mathbf{u}(N-1)$ *such that if* $\mathbf{x}(0) = 0$*, then* $\mathbf{x}(N) = \mathbf{x}_f$*.*

When the matrix $\mathbf{A}$ is singular, this property is not exactly the same as complete controllability. In such a case the discrete analogue of the the contollability theorem, Theorem 3.3.1, is as presented in next Lemma [21].

Lemma 5.5.2. *The linear system described by (5.8) is completely reachable if and only if*

$$\operatorname{rank}([\,\mathbf{B}\ \mathbf{AB}\ \cdots .\mathbf{A}^{n-1}\mathbf{B}\,]) = n$$

and the set of reachable states is a subset of the set of controllable states.

Note: For the continuous time systems, the concepts of reachability and controllability are completely equivalent.

As was discussed in Chapter 2 (section 2.2), for discrete-time system, we apply the method of $z-$transforms. To deal with the realization problem, using $z-$transform techniques for the system (5.8), and assuming $\mathbf{x}(0) = 0$, we obtain

$$z\tilde{\mathbf{x}}(z) = \mathbf{A}\tilde{\mathbf{x}}(z) + \mathbf{B}\tilde{\mathbf{u}}(z)\,.$$

After rearranging

$$\tilde{\mathbf{x}}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\tilde{\mathbf{u}}(z)\,,$$

therefore the output $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$ has $z-$transform

$$\tilde{\mathbf{y}}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\tilde{\mathbf{u}}(z)\,. \tag{5.9}$$

We define the transfer function matrix driving from a given initial state to any state for the discrete-time system by

$$\mathbf{G}_d = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,.$$

Note: The transfer function matrix $\mathbf{G}_d$, relating $\tilde{\mathbf{y}}(z)$ and $\tilde{\mathbf{u}}(z)$, has the same form as that for the continuous-time case $\mathbf{G}$. Therefore, the theory developed in section 5.3 carries over directly for (5.8).

Example 5.5.3. *Discrete time system.*

We want to investigate the situation when the continuous-time constant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

is sampled at times kT, where $k = 0, 1, 2, 3, \cdots$ Assume the input is constant throughout each sampling interval, i.e.

$$\mathbf{u}(t) = \mathbf{u}(k), \quad kT \leq t \leq (k+1)T\,.$$

Recall the solution for the $n-$order linear constant system from Chapter 2 (2.19),

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\,[\,\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t_0, \tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau\,]\,.$$

Let

$$t = (k+1)T\,, \quad t_0 = kT, \quad \mathbf{x}(kT) \equiv \mathbf{x}(k)\,,$$

and using the previous result, we find the solution for (5.9) as

$$\mathbf{x}(k+1) = e^{\mathbf{A}T}\mathbf{x}(k) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\,d\tau\,\mathbf{B}\mathbf{u}(k)\,. \tag{5.10}$$

Using the change of variable $\theta = (k+t)T - \tau$, in the integral in (5.10), we obtain

$$\mathbf{x}(k+1) = e^{\mathbf{A}T}\mathbf{x}(k) + \int_{0}^{T} e^{\mathbf{A}\theta}\,d\theta\,\mathbf{B}\mathbf{u}(k)\,.$$

Let

$$\mathbf{A}_d = e^{\mathbf{A}T}$$

and

$$\mathbf{B}_d = \int_{0}^{T} e^{\mathbf{A}\theta}\,d\theta\,\mathbf{B}$$

Now we have the equation in the standard discrete-time form (5.8) with

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k)\,.$$

5.6 Realization of Time Varying Systems

Consider again linear time varying systems described by the equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

and

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)\,,$$

where $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{B}$ is an $n \times m$ matrix, $\mathbf{C}$ is an $r \times n$ matrix.

As we discussed in Chapter 2 (section 2.7) for the case of the transition matrix for time varying systems, we find that many of the properties established for time-invariant systems in previous section (5.3) still hold. However, as is to be expected, we cannot give analytical methods for calculation of realizations.

Similar to the constant case, assume that $\mathbf{x}(t_0) = 0$. Recall the solution for the time-varying system (2.29),

$$\mathbf{x}(t) = \Phi(t, t_0)\left[\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\right] .$$

The output is

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) \\
&= \mathbf{C}(t) \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \\
&= \int_{t_0}^{t} \mathbf{K}(t, \tau)\mathbf{u}(\tau)\, d\tau\,,
\end{aligned}$$

where the matrix $\mathbf{\Phi}$ is defined in (2.25).

Definition 5.6.1. *The matrix* $\mathbf{K}(t, \tau)$ *defined by*

$$\mathbf{K}(t, \tau) := \mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau) \tag{5.11}$$

is called the **weighting pattern matrix**.

Now the realization problem is to find a triple $\{\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)\}$ for a given $\mathbf{K}(t,\tau)$ such that (5.11) is satisfied. The meaning of minimality of a realization is the same as before, that $\mathbf{A}$ should have the least possible dimension. In the case when $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant matrices the transition matrix is

$$\mathbf{\Phi}(t,\tau) = e^{[\mathbf{A}(t-\tau)]} \tag{5.12}$$

and the Laplace transform of $\mathbf{K}(t,\tau)$ is

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\,.$$

Substituting (5.12) into (5.11) produces

$$\mathbf{K}(t,\tau) = \mathbf{C}(t)e^{\mathbf{A}t}e^{-\mathbf{A}\tau}\mathbf{B}(\tau)\,.$$

Note: This last equation shows that $\mathbf{K}(t,\tau)$ can be written as a product of functions t and τ.

Now we establish that this representation holds even when $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are time-varying.

Theorem 5.6.2. *For the matrix* $\mathbf{K}(t,\tau)$ *a realization exists if and only if it can be expressed in the form*

$$\mathbf{K}(t,\tau) = \mathbf{L}(t)\mathbf{M}(\tau)\,, \tag{5.13}$$

where $\mathbf{L}$ *and* $\mathbf{M}$ *are matrices having finite dimensions.*

Proof. From the expression for the state transition matrix, defined in section 2.7:

$$\tilde{\mathbf{\Phi}}(t,t_0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$$

and (5.11) if $\mathbf{K}$ possess a realization then

$$\mathbf{K}(t,\tau) = \mathbf{C}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{B}(\tau)\,,$$

so a necessary condition holds for (5.13).

Conversely, if (5.13) holds, then a realization of $\mathbf{K}(t,\tau)$ is $\{\mathbf{0}_n, \mathbf{M}(t), \mathbf{L}(t)\}$, where $\mathbf{0}_n$ is an $n \times n$ zero matrix, since then $\mathbf{\Phi}(t,\tau) = \mathbf{I}$. □

It is interesting that the fundamental result on controllability and observability established in Theorem 5.3.7 still holds for the time varying systems.

Theorem 5.6.3. *A realization*

$$R = \{\, \mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)\,\}$$

of $\mathbf{K}(t,\tau)$ *is minimal if and only if it is completely controllable and completely observable.*

The proof is similar to that for the time-invariant case, and is given in Barnett's book.

An interesting question is under what conditions a given weighting matrix has a time invariant realization:

Theorem 5.6.4. *A given matrix* $\mathbf{K}(t,\tau)$ *has a realization* $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ *in which* $\mathbf{A},\mathbf{B},\mathbf{C}$ *are constant matrices if and only if (5.13) holds and if in addition*

$$\mathbf{K}(t,\tau) = \mathbf{K}(t+s,\tau+s), \quad t_0 \le t,\ \tau s \le t_1\,.$$

Unfortunately the proof given in [16] does not provide a practical method of constructing such a realization, so it will be omitted.

Example 5.6.5. *Time Variant System*

Calculating the realization for the time variant case. Consider

$$g(t) = te^{\lambda t}$$

or

$$g(t,\tau) = g(t-\tau) = (t-\tau)e^{\lambda(t-\tau)}\,.$$

Then

$$g(t-\tau) = te^{\lambda(t-\tau)} - \tau e^{\lambda(t-\tau)} = [\,e^{\lambda(t-\tau)}\ \ te^{\lambda t}\,]\begin{bmatrix} -\tau e^{-\lambda\tau} \\ e^{-\lambda\tau} \end{bmatrix}.$$

The linear time variant realization is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} -te^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix}\mathbf{u}(t)$$

$$\mathbf{y}(t) = [\,e^{\lambda t}\ \ te^{\lambda t}\,]\,\mathbf{x}(t)\,.$$

For linear time invariant realization

$$g(s) = \mathcal{L}[te^{\lambda t}] = \frac{1}{(s-\lambda)^2} = \frac{1}{s^2 - 2\lambda s + \lambda^2} = \frac{1}{s^2 + \alpha_1 s + \alpha_2}\,.$$

Then the controllable canonical realization is

$$\dot{\mathbf{x}} = \begin{bmatrix} 2\lambda & -\lambda^2 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbf{u}(t)$$

$$\mathbf{y} = [\,0\ 1\,]\,\mathbf{x}(t)\,.$$

CHAPTER 6

FIRST-ORDER MATRIX SYLVESTER SYSTEMS

This chapter presents several fundamental results concerning the controllability, observability and realizability criteria for a first-order matrix Sylvester system

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) + \mathbf{C}(t)\mathbf{U}(t)\,,$$

with output signal $\mathbf{Y}(t) = \mathbf{K}(t)\mathbf{X}(t)$ and control $\mathbf{U}(t)$. The case of $\mathbf{A}(t) = \mathbf{B}(t)$ is known as a Lyapunov system. We start with some historical information about Sylvester, Lyapunov, and Lyapunov-like equations, and introduce the names of some researchers who made a significant impact in active studies of applications for Sylvester and Lyapunov matrix equations. Then, the general solution of the system is presented in terms of the fundamental matrix solutions of

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \quad \text{and} \quad \dot{\mathbf{X}}(\mathrm{t}) = \mathbf{B}^*(\mathrm{t})\mathbf{X}(\mathrm{t})\,.$$

Next, a set of necessary and sufficient conditions is presented for the complete controllability and complete observability of the Sylvester system. Conditions for realizability and minimal realizability are given for the linear system with zero-initial state, for the linear system with periodic coefficient matrices, and for the time-invariant linear system. Finally, the results for linear systems are extended to give conditions for the controllability, observability and realizability of non-linear control systems.

6.1 History

The Sylvester, Lyapunov and Lyapunov-like matrix equations appear in control theory as well as in many different engineering and mathematical perspectives such as system theory, optimization, power systems, signal processing, linear algebra, differential equations, boundary value problems, and communications. The Lyapunov equation is named after the Russian mathematician Alexander Michailovitch Lyapunov, who in his doctoral dissertation, in 1892, introduced his famous stability theory of linear and nonlinear systems. A complete English translation of Lyapunov's 1892 doctoral dissertation was published in *International Journal of Control* in March of 1992. According to his definition of stability, one can check the stability of a system by finding certain functions, called the Laypunov functions. Although there is no general procedure for finding a Lyapunov function for nonlinear systems, for the linear time invariant systems, the procedure comes down to the problem of solving the matrix Lyapunov equation. Since linear systems are mathematically very convenient and often give fairly good approximations for nonlinear systems, mathematicians and engineers frequently base their analyses on the linearized models. Therefore, the solutions of the Lyapunov matrix equations give much insight into the behavior of dynamical systems.

The most famous of the Lyapunov-like equations, known as the Sylvester equation (Sylvester, 1884), is fully present on the list of applications of the Lyapunov and Lyapunov-like equations in science and engineering. The Sylvester equation represents a generalization of the Lyapunov equation.

The Lyapunov and Sylvester equations have been the subject of research since the beginning of the last century (Weddeburn; 1904). For example, the quadratic performance measure of a linear feedback system is given in terms of the solution of the Lyapunov equation. Many problems of control theory are based on the Lyapunov and Sylvester equation such as: concepts of controllability and observability Grammians (Chen, 1984), balancing transformation (Moore, 1981), stability robustness to parameter variations (Patel and Toda, 1980; Yedavalli, 1985), reduced-order modeling and control Hyland and Bernstein, 1985, 1986; Safonov and Chaing, 1989), filtering with singular mesurement noise (Haddad and Bernstein, 1987; Haveli, 1989), and power systems (Ilic, 1989). The Lyapunov and/or Lyapunov-like equations also appear in differential games (Petrovic and Gajic,1988), singular systems (Lewis and Ozcaldiran, 1989), signal processing (Anderson, 1986), differential equations (Dou, 1966), boundary value problems in partial differential equations (Kreisselmeir, 1972), and interpolation problems for rational matrix functions (Lerer and Rodman, 1993). The Sylvester equation in the Jordan form was studied by (Rutherford, 1932) using an expansion method into a set of linear algebraic equations, and in (Ma, 1966) by using a finite series method. Numerical solution of the algebraic Sylvester equation can be obtained by using the Bartels-Stewart algorithm (1972). Nowadays the very popular Krylov subspace method is implemented in (Hu and Reichel, 1992) for solving the Sylvester algebraic equation. The matrix sign function method for numerical solution of the continuous-time algebraic Sylvester equation is discussed in (L.Jodar, 1987).

The Sylvester (or Lyapunov) equation is a simple linear equation, but extremely reaching in its applications. Due to its broad applications, the Sylvester matrix equation has been the subject of very active research for the past forty years. Although Lyapunov theory was introduced at the end of the nineteenth century, it was not recognized for its vast applications until the 1960's. Since then it has had a major part in control theory. Around 1965, several researchers such as MacFarlane, Barnett and Storey, Chen and Shieh, Bingulac, and Lancaster presented solutions to the Lyapunov matrix equation. In the 1970's, when growing use of digital computers became part of almost every scientific field, the need for efficient numerical solution was felt. This resulted in celebrated algorithms for numerical solutions of the continuous-time algebraic Lyapunov equation (Bartels and Stewart, 1972), and for discrete systems, which is slightly different. Active research is still ongoing.

6.2 Definitions

The first order matrix Sylvester system is defined by

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) + \mathbf{C}(t)\mathbf{U}(t), \tag{6.1}$$

with the initial conditions

$$\mathbf{X}(t_0) = \mathbf{X}_0\,, \tag{6.2}$$

and the output signal

$$\mathbf{y}(t) = \mathbf{K}(t)\mathbf{X}(t)\,, \tag{6.3}$$

where $\mathbf{A}, \mathbf{B}$ are $(n \times n)$ continuous matrices on $I = [\,t_0, t_1\,]$, $\mathbf{C}$ is an $(n \times m)$ continuous matrix, $\mathbf{u}$ is an $(m \times n)$ matrix called the control, and the matrix $\mathbf{K}(t)$ is $(p \times n)$.

The case when $\mathbf{A} = \mathbf{B}$ is called a Lyapunov system.

The controllability and observability criteria for (6.1), (6.2), (6.3) were recently discussed by many authors [9], [11]. More convenient criteria for controllability and observability are available in [13].

More specifically, this chapter is organized as follows. First, we state a sufficient condition for controllability and observability under strengthened smoothness hypotheses on the linear state equation coefficients. Second, we present necessary and sufficient conditions for the time-invariant state equations (6.1), (6.2), (6.3) to be completely controllable and completely observable. Finally, we address questions related to input-output (zero state) behavior of the Sylvester system (6.1), (6.2), (6.3). Section 6.3 presents the general solutions of (6.1) in terms of the fundamental matrix solutions of

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \quad \text{and} \quad \dot{\mathbf{x}} = \mathbf{B}^*(\mathrm{t})\mathbf{x}(\mathrm{t})\,.$$

In next section, Section 6.4, we address controllability and observability criteria under a strengthened smoothness hypothesis. In Section 6.5, we address the realizability and minimal realizability criteria under more strengthened forms of controllability and observability. Finally, in Section 6.6, we consider non-linear control systems of the form

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t, \mathbf{X}(t), \mathbf{U}(t))\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t, \mathbf{X}(t), \mathbf{U}(t)) + \mathbf{C}(t, \mathbf{X}(t), \mathbf{U}(t))\mathbf{U}(t) \,.$$

and present certain sufficient conditions for controllability, observability and realizability by assuming that $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{U}$ and $\mathbf{X}$ are all continuous matrices with respect to their arguments.

6.3 General Solution Of The Sylvester Systems

In this section, we establish the general solution of the first order matrix Sylvester system (6.1), (6.2) in terms of the fundamental matrix solution of

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{6.4}$$

and

$$\dot{\mathbf{X}}(t) = \mathbf{B}^*(t)\mathbf{X}(t) \,, \tag{6.5}$$

where $\mathbf{B}^*$ is Hermitian transpose (conjugate transpose) matrix.

Throughout the remainder of this paper, $\mathbf{Y}(t)$ stands for a fundamental matrix solution of (6.4) and $\mathbf{Z}(t)$ stands for a fundamental matrix solution of (6.5).

Theorem 6.3.1. *Any solution of the homogeneous equation*

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) \tag{6.6}$$

is of the form $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t)$ *where* $\mathbf{\Lambda}$ *is a constant square matrix of order* n.

Proof. It can be directly verified that $\mathbf{X}$ defined by

$$\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t)$$

is a solution of (6.6). To prove that every solution of (6.6) is of this form, let $\mathbf{X}$ be a solution and $\mathbf{J}$ be a square matrix defined by

$$\mathbf{J}(t) = \mathbf{Y}^{-1}(t)\mathbf{X}(t)\,.$$

Then

$$\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{J}(t)$$

and

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t)$$

if and only if

$$\dot{\mathbf{Y}}(t)\mathbf{J}(t) + \mathbf{Y}(t)\dot{\mathbf{J}}(t) = \mathbf{A}(t)\mathbf{Y}(t)\mathbf{J}(t) + \mathbf{J}(t)\mathbf{X}(t)\mathbf{B}^*(t)$$

if and only if $\dot{\mathbf{J}}(t) = \mathbf{J}(t)\mathbf{B}(t)$ and if and only if $\dot{\mathbf{J}}^*(t) = \mathbf{B}^*(t)\mathbf{J}^*(t)\,.$ Since $\mathbf{Z}$ is a fundamental matrix solution of $\dot{\mathbf{X}} = \mathbf{B}^*\mathbf{X}$, it follows that there exists constant nonsingular matrix $\mathbf{\Lambda}^*$ such that $\dot{\mathbf{J}}^*(t) = \mathbf{Z}(t)\mathbf{\Lambda}^*$. Hence $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}^*\mathbf{Z}^*(t)$ $\square$

Theorem 6.3.2. *Any solution of (6.1), (6.2) is of the form*

$$\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t) + \tilde{\mathbf{X}}(t) \tag{6.7}$$

where $\tilde{\mathbf{X}}(t)$ *is a particular solution of (6.1), (6.2).*

Proof. It can be directly verified that $\mathbf{X}(t)$ defined by (6.7) is a solution of (6.1), (6.2).

Now, to prove that every solution of (6.1), (6.2) is of this form, let $\mathbf{X}(t)$ be any solution of (6.1), (6.2) and $\tilde{\mathbf{X}}(t)$ be a particular solution of (6.1), (6.2). Then

$$\mathbf{X}(t) - \tilde{\mathbf{X}}(t)$$

is a solution of (6.1), (6.2) and hence by Theorem 6.3.1,

$$\mathbf{X}(t) - \tilde{\mathbf{X}}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t)\,,$$

or

$$\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t) + \tilde{\mathbf{X}}(t).$$

□

Theorem 6.3.3. *A particular solution of (6.1), (6.2) is given by*

$$\tilde{\mathbf{X}}(t) = \mathbf{Y}(t)\,[\int_a^t \mathbf{Y}^{-1}(s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{Z}^{*-1}(s)\,ds\ \ ]\ \ \mathbf{Z}^*(t).$$

Proof. It can be directly verified that $\tilde{\mathbf{X}}(t)$ is a solution of (6.1).

Using variation of parameters we can write

$$\tilde{\mathbf{X}} = \mathbf{Y}(t)\tilde{\mathbf{C}}(t)\mathbf{Z}^*(t)\,.$$

Substituting in (6.1), (6.2) and solving for $\tilde{\mathbf{C}}(t)$, we see that

$$\tilde{\mathbf{C}}(t) = \int_a^t \mathbf{Y}^{-1}(s)\mathbf{C}(s)\,\mathbf{U}(s)\mathbf{Z}^{*-1}(s)\,ds.$$

□

Theorem 6.3.4. *Any solution $\mathbf{X}(t)$ of the initial value problem (6.1), satisfying $\mathbf{X}(t_0) = \mathbf{X}_0$ is given by*

$$\begin{aligned}\mathbf{X}(t) &= \mathbf{\Phi}(t,t_0)\,\mathbf{X}_0\mathbf{\Psi}^*(t_0,t)\\ &\quad + \mathbf{\Phi}(t,t_0)\,[\int_{t_0}^t \mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s,t_0)\,ds\ \ ]\ \ \mathbf{\Psi}^*(t_0,t)\,,\end{aligned}$$

where

$$\mathbf{\Phi}(t,t_0) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)$$

and

$$\mathbf{\Psi}^*(t_0,t) = \mathbf{Z}^{*-1}(t_0)\mathbf{Z}^*(t)\,.$$

Proof. Any solution $\mathbf{X}(t)$ of (6.1), (6.2) is of the form

$$\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{\Lambda}\mathbf{Z}^*(t) + \mathbf{Y}(t)\int_{t_0}^{t} \mathbf{Y}^{-1}(s)\mathbf{C}(s)\mathbf{u}(s)\mathbf{Z}^{*-1}(s)\,\mathrm{d}s\,\mathbf{Z}^*(t).$$

Notice that

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

implies that

$$\mathbf{X}(t_0) = \mathbf{Y}(t_0)\mathbf{\Lambda}\mathbf{Z}^*(t_0)$$

or

$$\mathbf{\Lambda} = \mathbf{Y}^{-1}(t_0)\mathbf{X}(t_0)\mathbf{Z}^{*-1}(t_0).$$

Hence

$$\begin{aligned}\mathbf{X}(t) &= \mathbf{\Phi}(t,t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0,t) + \int_{t_0}^{t} \mathbf{\Phi}(t,t_0)\mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(t_0,t)\,\mathrm{d}s \\ &= \mathbf{\Phi}(t,t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0,t) \\ &+ \mathbf{\Phi}(t,t_0)\,[\int_{t_0}^{t} \mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s,t_0)\,\mathrm{d}s\,]\,\mathbf{\Psi}^*(t_0,t)\,.\end{aligned}$$

□

6.4 Criteria For Controllability and Observability Of The Linear Sylvester System

In this section,we address the fundamental concepts of controllability and observability of the Sylvester system (6.1), (6.2), (6.3). In fact, for a time varying linear state equation (6.1), (6.2), (6.3) the connection of the input signal to the state variables can change with time. Therefore, the concept of controllability is tied to a specific finite time interval defined by $[\,t_0, t_1\,]$.

Definition 6.4.1. *The linear time varying continuous system (6.1), (6.2) is completely controllable on* $[\,t_0, t_1\,]$ *if for any initial time* t_0*, any initial state* $\mathbf{X}(t_0) = \mathbf{X}_0$*, there exists a continuous input signal* $\mathbf{U}(t)$ *such that the corresponding solution of (6.1) satisfies* $\mathbf{X}(t_1) = \mathbf{X}_1$*.*

Theorem 6.4.2. *The linear state equation (6.1), (6.2), (6.3) is completely controllable on* $[\,t_0, t_1\,]$ *if and only if the* $(n \times n)$ *symmetric matrix (the controllability Grammian)*

$$\mathbf{W}(t_0, t_1) = \int_{t_0}^{t_1} \mathbf{\Phi}(t_0, s)\mathbf{C}(s)\mathbf{C}^*(s)\mathbf{\Phi}^*(t_0, s)\,\mathrm{d}s$$

is non-singular.

Proof. First suppose that $\mathbf{W}(t_0, t_1)$ is non-singular. Then given an $(n \times n)$ matrix $\mathbf{X}_0$, choose

$$\mathbf{U}(t) = -\mathbf{C}^*(t)\mathbf{\Phi}^*(t_0, t)\mathbf{W}^{-1}(t_0, t_1)\,[\,\mathbf{X}_0 - \Phi(t_0, t_1)\mathbf{X}_1\mathbf{\Psi}^*(t_1, t_0)\,]\,\mathbf{\Psi}^*(t_0, t). \tag{6.8}$$

Recall that the input-signal $\mathbf{U}$ is continuous on the interval and that the corresponding solution of (6.1), satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ by Theorem c6t4 is given by

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\,\mathbf{X}_0\mathbf{\Psi}^*(t_0, t) + \mathbf{\Phi}(t, t_0)\,[\int_{t_0}^{t} \mathbf{\Phi}(t_0, s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s, t_0)\,\mathrm{d}s\ \ ]\ \ \Psi^*(t_0, t).$$

Now

$$\mathbf{X}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0, t_1) + \mathbf{\Phi}(t_1, t_0)\,[\int_{t_0}^{t_1} \mathbf{\Phi}(t_0, s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s, t_0)\,\mathrm{d}s\ \ ]\ \ \mathbf{\Psi}^*(t_0, t_1).$$

Substituting $\mathbf{U}(s)$ from (6.8), we get

$$\mathbf{X}(t_1) = \mathbf{\Phi}(t_1, t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0, t_1) - \mathbf{\Phi}(t_1, t_0)\,[\int_{t_0}^{t_1} \mathbf{\Phi}(t_0, s)\mathbf{C}(s)\,[\,\mathbf{C}^*(s)\mathbf{\Phi}^*(t_0, s)\,\mathrm{d}s$$
$$\times\ \mathbf{W}^{-1}(t_0, t_1)\,[\ \mathbf{X}_0 - \mathbf{\Phi}(t_0, t_1)\mathbf{X}_1\mathbf{\Psi}^*(t_1, t_0)\,]\ \ \mathbf{\Psi}^*(t_0, t_1).$$

Simplifying, we get

$$\begin{aligned}
\mathbf{X}(t_1) &= \mathbf{\Phi}(t_1,t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0,t_1) - \mathbf{\Phi}(t_1,t_0)\,[\int_{t_0}^{t_1}\mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{C}^*(s)\mathbf{\Phi}^*(t_0,s)\,\mathrm{d}s\,] \\
&\quad \times \mathbf{W}^{-1}(t_0,t_1)\,[\,\mathbf{X}_0 - \mathbf{\Phi}(t_0,t_1)\mathbf{X}_1\mathbf{\Psi}^*(t_1,t_0)\,]\,\mathbf{\Psi}^*(t_0,t_1) \\
&= \mathbf{\Phi}(t_1,t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0,t_1) - \mathbf{\Phi}(t_1,t_0)\,[\,\mathbf{X}_0 - \mathbf{\Phi}(t_0,t_1)\mathbf{X}_1\mathbf{\Psi}^*(t_1,t_0)\,]\,\mathbf{\Psi}^*(t_0,t_1) \\
&= \mathbf{X}_1.
\end{aligned}$$

Thus, the state equation is controllable and this is true for all $t_1 > t_0$, so it follows that the state equation is completely controllable.

Conversely, suppose the state equation (6.1), (6.2) is completely controllable on $[\,t_0,t_1\,]$. Then it is claimed that $\mathbf{W}(t_0,t_1)$ is non-singular. To the contrary, suppose that (t_0,t_1) is non-invertible. Then there exists an $(n \times 1)$ vector α such that

$$\alpha^T\mathbf{W}(t_0,t_1)\alpha = \int_{t_0}^{t_1}\alpha^T\mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{C}^*(s)\mathbf{\Phi}^*(t_0,s)\,\alpha\,\mathrm{d}s = 0.$$

Because of the fact that the integrand in the expression is the non-negative continuous function

$$||\alpha^T\mathbf{\Phi}(t_0,s)\mathbf{C}(s)||^2\,,$$

it follows that

$$\alpha^T\mathbf{\Phi}(t_0,s)\mathbf{C}(s) = 0, \quad s \in [t_0,t_1].$$

Since the equation is completely controllable on $[\,t_0,t_1\,]$, there exists a control $\mathbf{U}$ such that $\mathbf{x}(t_1) = 0$ if $\mathbf{X}(t_0) = \alpha_0\,\mathbf{g}$, where $\mathbf{g}$ is any non-zero $(1 \times n)$ matrix. Hence

$$0 = \mathbf{\Phi}(t_1,t_0)\,[\,\alpha_0\,\mathbf{g} + \int_{t_0}^{t_1}\mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s,t_0)\,\mathrm{d}s\,]\,\mathbf{\Psi}^*(t_0,t_1).$$

Therefore,

$$\alpha\mathbf{g} = -\int_{t_0}^{t_1}\mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s,t_0)\,ds$$

and consequently

$$||\alpha \mathbf{g}||^2 = (\alpha \mathbf{g})^T(\alpha \mathbf{g}) = -\int_{t_0}^{t_1} (\alpha \mathbf{g})^* \mathbf{\Phi}(t_0, s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s, t_0)\, \mathrm{d}s = 0,$$

and this contradicts the fact that $\alpha \mathbf{g} \neq 0$. Hence the proof of the theorem is complete.

□

Note: The controllability Grammian $\mathbf{W}(t_0, t_1)$ has the property that for every $t_1 > t_0$ it is symmetric and positive definite. Thus the state equation (6.1), (6.2), (6.3) is completely controllable on $[\,t_0, t_1\,]$ if and only if $\mathbf{W}(t_0, t_1)$ is positive definite. If the state equation is not controllable on $[t_0, t_1]$, it might become so if t_1 were increased. And controllability might be lost if t_1 is lowered. Analogous observations can be made in regard to changing t_0.

We now consider more convenient criteria for controllability under strengthened smoothness hypotheses.

Definition 6.4.3. *Corresponding to the linear state equation (6.1), (6.2), (6.3), we define a sequence of* $(n \times m)$ *matrix functions* $\mathbf{K}_j(t)$ *by*

$$\mathbf{K}_0(t) = \mathbf{C}(t)$$

$$\mathbf{K}_j(t) = -\mathbf{A}(t)\mathbf{K}_{j-1}(t) - \mathbf{K}_{j-1}(t)\mathbf{B}(t) + \dot{\mathbf{K}}_{j-1}(t), \quad j = 1, 2, 3, \ldots$$

and let

$$\mathbf{W}_i(t) = [\,\mathbf{K}_0(t), \mathbf{K}_1(t), \ldots\ldots \mathbf{K}_{i-1}(t)\,]\,, \;\; i = 1, 2, \ldots$$

We have the following useful Lemma.

Lemma 6.4.4. *For all* t, s

$$\frac{\partial^j}{\partial s^j}\,[\,\mathbf{\Phi}(t, s)\mathbf{C}(s)\mathbf{\Psi}^*(t, s)\,] = \mathbf{\Phi}(t, s)\mathbf{K}_j(s)\mathbf{\Psi}^*(t, s), \quad j = 0, 1, 2, 3, \ldots \tag{6.9}$$

Proof. For $j=0$, we have

$$\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s) = \mathbf{\Phi}(t,s)\mathbf{K}_0(s)\mathbf{\Psi}^*(t,s),$$

and for $j=1$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\left[\,\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s)\,\right] &= \frac{\partial}{\partial s}\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s) + \mathbf{\Phi}(t,s)\dot{\mathbf{C}}(s)\mathbf{\Psi}^*(t,s) \\
&\quad + \mathbf{\Phi}(t,s)\mathbf{C}(s)\frac{\partial}{\partial s}\mathbf{\Psi}^*(t,s) \\
&= -\mathbf{\Phi}(t,s)\mathbf{A}(s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s) \\
&\quad + \mathbf{\Phi}(t,s)[\mathbf{K}_1(s) + \mathbf{A}(s)\mathbf{K}_0(s) + \mathbf{K}_0(s)\mathbf{B}(s)]\mathbf{\Psi}^*(t,s) \\
&\quad - \mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{B}(s)\mathbf{\Psi}^*(t,s) \\
&= \mathbf{\Phi}(t,s)\mathbf{K}_1(s)\mathbf{\Psi}^*(t,s).
\end{aligned}$$

Therefore, the result is true for $j=1$, and the result follows by induction. For, assume that the result is true for $j=n$, i.e.

$$\frac{\partial^n}{\partial s^n}\left[\,\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s)\,\right] = \mathbf{\Phi}(t,s)\mathbf{K}_n(s)\mathbf{\Psi}^*(t,s).$$

Then

$$\frac{\partial^{n+1}}{\partial s^{n+1}}\left[\,\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s)\right] = \frac{\partial}{\partial s}[\mathbf{\Phi}(t,s)\mathbf{K}_n(s)\mathbf{\Psi}^*(t,s)]$$

From [12] the result is

$$= \mathbf{\Phi}(t,s)\mathbf{K}_{n+1}(s)\mathbf{\Psi}^*(t,s).$$

□

At $t=s$, equation (6.9) gives a simple interpretation of the matrices in Definition 6.4.3. Since at $t=s$,

$$\mathbf{\Phi}(t,t) = \mathbf{\Psi}^*(t,t) = \mathbf{I}\,,$$

we have

$$\mathbf{K}_j(t) = \frac{\partial^j}{\partial s^j}\left[\,\mathbf{\Phi}(t,s)\mathbf{C}(s)\mathbf{\Psi}^*(t,s)\,\right], \quad j=0,1,2,3,\ldots$$

Theorem 6.4.5. *Suppose q is a positive integer such that for $t \in [t_0, t_1]$, $\mathbf{C}(t)$ is q times continuously differentiable, and $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $(q-1)$ times continuously differentiable on $[t_0, t_1]$. Then the linear state equation is completely controllable on $[t_0, t_1]$ if for some $t_c \in [t_0, t_1]$,* $\operatorname{rank}[\mathbf{K}_0(t_c), \mathbf{K}_1(t_c), \ldots, \mathbf{K}_q(t_c)] = n$.

Proof. Suppose for some $t_c \in [t_0, t_1]$ the rank condition holds. Then it is claimed that the state equation (6.1), (6.2),(6.3) is completely controllable. To the contrary, suppose that $\mathbf{W}(t_0, t_1)$ is non-invertible. Then there exists an $(n \times 1)$ vector α such that

$$\alpha^T \mathbf{\Phi}(t_0, t)\mathbf{C}(t)\mathbf{\Psi}^*(t_0, t) = 0 \qquad \text{for} \quad t \in [t_0, t_1]. \tag{6.10}$$

Let β be a non-zero vector defined by

$$\beta = \mathbf{\Phi}^T(t_0, t_c)\, \alpha\,.$$

Then

$$\beta^T \mathbf{\Phi}(t_c, t)\, \mathbf{C}(t)\mathbf{\Psi}^*(t_0, t) = 0 \qquad \text{for} \quad t \in [t_0, t_1]\,.$$

At $t = t_c$, we have

$$\beta^T \mathbf{\Phi}(t_c, t_c)\mathbf{C}(t_c)\mathbf{\Psi}^*(t_0, t_c) = 0\,,$$

i.e.

$$\beta^T \mathbf{K}_0(t_c)\mathbf{\Psi}^*(t_0, t_c) = 0\,.$$

Next, differentiating (6.10) with respect to t gives at $t = t_c$,

$$\beta^T \mathbf{K}_1(t_c)\mathbf{\Psi}^*(t_0, t_c) = 0\,.$$

Continuing in this way, we get

$$\beta^T (\mathbf{K}_0(t_c), \mathbf{K}_1(t_c), \ldots, \mathbf{K}_q(t_c))\mathbf{\Psi}^*(t_0, t_c) = 0\,.$$

Since $\mathbf{\Psi}$ is non-singular, it follows that

$$\mathrm{rank}\,[\,\mathbf{K}_0(t_c), \mathbf{K}_1(t_c), \ldots, \mathbf{K}_q(t_c)\,] \; < n\,,$$

which is a contradiction. □

Theorem 6.4.6. *The time-invariant linear state equation (6.1), (6.2), (6.3) is completely controllable on* $[\,t_0, t_1\,]$ *if and only if the* $(n\times nm)$ *controllability matrices satisfy* $\mathrm{rank}\,[\,\mathbf{C}, \mathbf{AC}, ...\mathbf{A}^{n-1}\mathbf{C}\,] = n$ *or* $\mathrm{rank}\,[\,\mathbf{C}, \mathbf{CB}, ...\mathbf{CB}^{n-1}\,] = n$*.*

This theorem is also given by K.N.Murty and L.V.Fausett in [12].

We now consider the observability of the linear state equation (6.1), (6.2), (6.3).

Definition 6.4.7. *The linear state equation (6.1), (6.2), (6.3) is said to be completely observable on* $I = [t_0, t_1]$ *if any initial state* $\mathbf{x}(t_0) = \mathbf{x}_0$ *is uniquely determined by the corresponding response* $\mathbf{y}(t)$ *for* $t \in [\,t_0, t_1\,]$*.*

The basic characterization of observability is similar in form to the controllability case, though the proof is a bit simpler [12].

Theorem 6.4.8. *The linear state equation (6.1), (6.2), (6.3) is completely observable if and only if the* $(n \times n)$ *matrix (the observability Grammian)*

$$\mathbf{M}(t_0, t_1) = \int_{t_0}^{t_1} \mathbf{\Phi}^*(s, t_0)\mathbf{K}^*(s)\mathbf{K}(s)\mathbf{\Phi}(s, t_0)\,\mathrm{d}s$$

is non-singular.

Proof. Suppose $\mathbf{M}(t_0, t_1)$ is non-singular. Without loss of generality suppose that $\mathbf{U}(t) = 0$ on I. Then

$$\mathbf{X}(t) = \mathbf{\Phi}(t, t_0)\mathbf{X}_0\mathbf{\Psi}^*(t_0, t)\,.$$

Therefore

$$\mathbf{y}(t) = \mathbf{K}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0\mathbf{\Psi}^*(t_0, t)\,.$$

Hence,

$$\mathbf{\Phi}^*(t,t_0)\mathbf{K}^*(t)\mathbf{y}(t)\mathbf{\Psi}^*(t,t_0) = \mathbf{\Phi}^*(t,t_0)\mathbf{K}^*(t)\mathbf{K}(t)\mathbf{\Phi}(t,t_0)\mathbf{X}_0\,.$$

Integrating from t_0 to t_1 yields

$$\mathbf{M}(t_0,t_1)\mathbf{x}_0 = \int_{t_0}^{t_1} \mathbf{\Phi}^*(s,t_0)\mathbf{K}^*(s)\mathbf{y}(s)\mathbf{\Psi}^*(s,t_0)\,\mathrm{d}s\,,$$

or

$$\mathbf{X}_0 = \mathbf{M}^{-1}(t_0,t_1)\int_{t_0}^{t_1} \mathbf{\Phi}^*(s,t_0)\mathbf{K}^*(s)\mathbf{y}(s)\mathbf{\Psi}^*(s,t_0)\,\mathrm{d}s.$$

Thus, $\mathbf{X}_0$ is uniquely determined.

Conversely, suppose the state linear equation (6.1), (6.2), (6.3) is completely observable. Since $\mathbf{M}(t_0,t_1)$ is symmetric, we construct a quadratic form

$$\alpha^*\mathbf{M}(t_0,t_1)\alpha = \int_{t_0}^{t_1} \alpha^*\mathbf{\Phi}^*(s,t_0)\mathbf{K}^*(s)\mathbf{K}(s)\mathbf{\Phi}(s,t_0)\,\alpha\,\mathrm{d}s = \int_{t_0}^{t_1} ||\theta(s,t_0)||^2\,\mathrm{d}s$$

where

$$\theta(s,t_0) = \mathbf{K}(s)\mathbf{\Phi}(s,t_0)\,\alpha \geq 0\,.$$

Therefore $\mathbf{M}(t_0,t_1)$ is positive definite.

Now, suppose that some column matrix exists such that

$$\mathbf{r}_1 \neq 0, \quad \text{but } \ \mathbf{r}_1^*\,\mathbf{M}(t_0,t_1)\,\mathbf{r}_1 = 0\,.$$

Then

$$\int_{t_0}^{t_1} ||\,\mathbf{K}(s)\mathbf{\Phi}(s,t_0)\,\mathbf{r}_1\,||^2\,\mathrm{d}s = 0\,,$$

i.e.

$$||\,\mathbf{K}(s)\mathbf{\Phi}(s,t_0)\mathbf{r}_1\,|| = 0 \qquad \textit{on}\ \ [\,t_0,t_1\,]\,.$$

If $\mathbf{X}_0 = \mathbf{r}_1\mathbf{s}$ ($\mathbf{s}$ is any row vector) then the output is

$$\mathbf{y}(t) = \mathbf{K}(t)\mathbf{\Phi}(t,t_0)\mathbf{r}_1\mathbf{s}\mathbf{\Psi}^*(t_0,t) = 0.$$

Therefore $\mathbf{X}_0$ cannot be determined with the knowledge of $\mathbf{y}(t)$ in this case. This contradicts the assumption that the linear state-equation is completely observable. Therefore $\mathbf{M}(t_0, t_1)$ is non-singular. □

We now state more convenient criteria for observability similar to the criteria developed for controllability.

Definition 6.4.9. *Corresponding to the linear state equation (6.1), (6.2), (6.3), we define the $(p \times m)$ matrix functions*

$$\mathbf{L}_0(t) = \mathbf{K}(t)\,,$$

$$\mathbf{L}_j(t) = -\mathbf{A}(t)\mathbf{L}_{j-1}(t) - \mathbf{L}_{j-1}(t)\mathbf{B}(t) + \dot{\mathbf{L}}_{j-1}(t).$$

Then we have the following lemma.

Lemma 6.4.10. *For all $t, s \in [\,t_0, t_1\,]$*

$$\frac{\partial^j}{\partial s^j}\,[\,\mathbf{\Phi}(t,s)\mathbf{K}(s)\mathbf{\Psi}^*(t,s)\,] = \mathbf{\Phi}(t,s)\mathbf{L}_j(s)\mathbf{\Psi}^*(t,s), \quad j = 0, 1, 2, 3, \ldots$$

For $s = t$, we have

$$\frac{\partial^j}{\partial s^j}\,[\,\mathbf{\Phi}(t,s)\mathbf{K}(s)\mathbf{\Psi}^*(t,s)\,] = \mathbf{L}_j(s), \quad j = 0, 1, 2, 3, \ldots ...$$

Theorem 6.4.11. *Suppose q is a positive integer such that for $t \in [t_0, t_1]$, $K(t)$ is q times continuously differentiable and $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $(q-1)$ times continuously differentiable. Then the linear state equation (6.1), (6.2), (6.3), is completely observable on $[\,t_0, t_1\,]$ if for some $t_c \in [\,t_0, t_1\,]$, $\operatorname{rank}[\,\mathbf{L}_0(t_c), \mathbf{L}_1(t_c), \ldots, \mathbf{L}_q(t_c)\,] = n$.*

6.5 Realizability Criteria For The Linear and Periodic Sylvester System

In this section, we address questions related to the input-output (zero state) behavior of the standard linear state equation (6.1), (6.2), (6.3) with zero initial state assumed.

The output signal $\mathbf{y}(t)$ is given by

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}_1(t,s)\mathbf{U}(s)\mathbf{G}_2^*(s,t)\,\mathrm{d}s \tag{6.11}$$

where

$$\mathbf{G}_1(t,s) = \mathbf{K}(t)\mathbf{\Phi}(t,s)\mathbf{C}(s)$$

and

$$\mathbf{G}_2^*(s,t) = \mathbf{\Psi}^*(s,t)\,.$$

This expression for $\mathbf{y}(t)$ follows directly from equation (6.3) and Theorem 6.3.4, since Theorem 6.3.4, with $\mathbf{X}_0 = 0$ gives

$$\mathbf{X}(t) = \mathbf{\Phi}(t,t_0)\,[\int_{t_0}^{t} \mathbf{\Phi}(t_0,s)\mathbf{C}(s)\mathbf{U}(s)\mathbf{\Psi}^*(s,t_0)\,\mathrm{d}s\ \ ]\ \ \mathbf{\Psi}^*(t_0,t).$$

Of course, given the state equation (6.1), (6.2), (6.3), in principle $\mathbf{G}_1$ and $\mathbf{G}_2$ can be computed so that the input-output behavior is known according to (6.11). Our interest here is the reversal of the computation, and we wish to establish conditions on a specified $\mathbf{G}_1(t,s)$ and $\mathbf{G}_2^*(s,t)$ that guarantee existence of a corresponding linear state equation.

Definition 6.5.1. *A linear state equation of dimension n*

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) + \mathbf{C}(t)\mathbf{U}(t) \tag{6.12}$$

$$\mathbf{y}(t) = \mathbf{K}(t)\mathbf{X}(t)\,,$$

is called a realization of the weighting patterns $\mathbf{G}_1(t,s)$, $\mathbf{G}_2^*(s,t)$ if, for all t and s,

$$\mathbf{G}_1(t,s) = \mathbf{K}(t)\mathbf{\Phi}(t,s)\mathbf{C}(s) \tag{6.13}$$

and

$$\mathbf{G}_2^*(s,t) = \mathbf{\Psi}^*(s,t).$$

If a realization (6.12) exists, then the weighting pattern is called realizable; if the realization is of dimension n *then (6.12) is called a minimal realization.*

Theorem 6.5.2. *The Sylvester system (6.1), (6.2), (6.3) is completely realizable iff it is completely controllable and completely observable.*

Proof. Suppose the system is completely controllable and completely observable. Then the Grammians $\mathbf{W}(t_0, t_1)$ and $\mathbf{M}(t_0, t_1)$ are invertible and hence the system is completely realizable. If the system is completely realizable, then it is claimed that the system is completely controllable and completely observable. To the contrary, if the system is either not completely controllable, or not completely observable, then the Grammian matrices $\mathbf{W}(t_0, t_1)$ or $\mathbf{M}(t_0, t_1)$ are non-invertible. Then proceeding as in Theorem 6.4.2 or Theorem 6.4.8, we arrive at a contradiction. □

Theorem 6.5.3. *If the weighting patterns* $\mathbf{G}_1(t,s)$ *and* $\mathbf{G}_2^*(s,t)$ *are realizable then* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are separable; that is, there exist continuous matrices* $\mathbf{H}_1$ *of order* $(p \times n)$ *and* $\mathbf{F}_1$ *of order* $(n \times m)$ *such that for all* t *and* s*,*

$$\mathbf{G}_1(t,s) = \mathbf{H}_1(t)\mathbf{F}_1(s)\,,$$

and $(n \times n)$ *continuous matrices* F_2 *and* H_2 *such that*

$$\mathbf{G}_2^*(s,t) = \mathbf{F}_2^*(s)\mathbf{H}_2^*(t).$$

Proof. Suppose that $\mathbf{G}_1(t,s)$ and $\mathbf{G}_2^*(s,t)$ are realizable. We can assume that the linear state equation (6.12) is one realization. Then using the composition property of $\mathbf{\Phi}$ we get

$$\begin{aligned}\mathbf{G}_1(t,s) &= \mathbf{K}(t)\mathbf{\Phi}(t,0)\mathbf{\Phi}(0,s)\mathbf{C}(s)\\ &= \mathbf{H}_1(t)\mathbf{F}_1(s)\end{aligned}$$

where $\mathbf{H}_1(t) = \mathbf{K}(t)\mathbf{\Phi}(t,0)$, and $\mathbf{F}_1(s) = \mathbf{\Phi}(0,s)\mathbf{C}(s)$.

Similarly, using the composition property of Ψ we get

$$\begin{aligned}\mathbf{G}_2^*(s,t) &= \mathbf{\Psi}^*(s,0)\mathbf{\Phi}^*(0,t)\\ &= \mathbf{F}_2^*(s)\mathbf{H}_2^*(t)\end{aligned}$$

where $\mathbf{H}_2^*(t) = \mathbf{\Psi}^*(0,t)$ and $\mathbf{F}_2^*(s) = \mathbf{\Psi}^*(s,0)$. □

Theorem 6.5.4. *If there exist continuous matrices $\mathbf{H}_1$ of order $(p \times n)$ and $\mathbf{F}_1$ of order $(n \times m)$ such that for all t and s,*

$$\mathbf{G}_1(t,s) = \mathbf{H}_1(t)\mathbf{F}_1(s)$$

and

$$\mathbf{G}_2^*(s,t) = \mathbf{I}\,,$$

then the weighting patterns $\mathbf{G}_1(t,s)$ and $\mathbf{G}_2^*(s,t)$ are realizable. Furthermore, there is a realization of the form

$$\dot{\mathbf{X}}(t) = \mathbf{F}_1(t)\mathbf{U}(t); \qquad \mathbf{y}(t) = \mathbf{H}_1(t)\mathbf{X}(t)\,.$$

Proof. Suppose there exist continuous matrix functions $\mathbf{H}_1(t)$ and $\mathbf{F}_1(s)$ so that

$$\mathbf{G}_1(t,s) = \mathbf{H}_1(t)\mathbf{F}_1(s)$$

and that

$$\mathbf{G}_2^*(s,t) = \mathbf{I}\,.$$

Since

$$\mathbf{G}_1(t,s) = \mathbf{K}(t)\mathbf{\Phi}(t,s)\mathbf{C}(s) = \mathbf{H}_1(t)\mathbf{F}_1(s)\,,$$

we can find a realization with $\mathbf{A} = 0$ by taking

$$\mathbf{K}t) = \mathbf{H}_1(t), \qquad \mathbf{C}(s) = \mathbf{F}_1(s) \quad \text{and} \quad \mathbf{\Phi}(\mathrm{t},\mathrm{s}) = \mathbf{I}\,.$$

Furthermore, we have $\mathbf{B} = 0$, since $\mathbf{G}_2^*(s,t) = \mathbf{\Psi}^*(s,t) = \mathbf{I}$.

Thus, the linear state equation

$$\dot{\mathbf{X}}(t) = \mathbf{F}_1(t)\mathbf{U}(t) \quad \mathbf{y}(t) = \mathbf{H}_1(t)\mathbf{X}(t)$$

is a realization of $\mathbf{G}1$ and $\mathbf{G}_2$ and the proof of the theorem is complete. □

We now address the case of periodic linear state equations.

Definition 6.5.5. *A linear state equation of dimension n*

$$\begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{A}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t) + \mathbf{C}(t)\mathbf{U}(t) \\ \mathbf{y}(t) &= \mathbf{K}(t)\mathbf{X}(t) \end{aligned} \tag{6.14}$$

is periodic if there exists a finite positive constant τ such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{K}$ are all periodic with the same period τ.

The weighting patterns $\mathbf{G}_1(t,s)$, $\mathbf{G}_2^*(s,t)$ are periodic if there exists a finite positive constant τ such that

$$\mathbf{G}_1(t+\tau, s+\tau) = \mathbf{G}_1(t,s) \tag{6.15}$$

and

$$\mathbf{G}_2^*(s+\tau, t+\tau) = \mathbf{G}_2^*(s,t)$$

for all t and s.

Theorem 6.5.6. *The weighting patterns $\mathbf{G}_1(t,s)$, $\mathbf{G}_2^*(s,t)$ are realizable by a periodic linear state equation if and only if they are realizable and they are periodic. If these conditions hold, then there exists a minimal realization of $\mathbf{G}_1(t,s), \mathbf{G}_2^*(s,t)$ that is periodic.*

Proof. If $\mathbf{G}_1(t,s)$, $\mathbf{G}_2^*(s,t)$ has a periodic realization with period τ, then obviously $\mathbf{G}_1(t,s), \mathbf{G}_2^*(s,t)$ is realizable. Furthermore in terms of the realization, we can write

$$\mathbf{G}_1(t,s) = \mathbf{K}(t)\mathbf{\Phi}(t,s)\mathbf{C}(s) \ \ \textit{and} \ \ \mathbf{G}_2^*(s,t) = \mathbf{\Psi}^*(s,t).$$

Since $\mathbf{\Phi}(t,t_0)$ is a fundamental matrix solution of

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}$$

with $\mathbf{A}(t+\tau) = \mathbf{A}(t)$, we have $\mathbf{\Phi}(t+\tau, s+\tau) = \mathbf{\Phi}(t,s)$, and similarly

$$\mathbf{\Psi}^*(t+\tau, s+\tau) = \mathbf{\Psi}^*(t,s)\,,$$

so we have

$$\mathbf{G}_1(t+\tau, s+\tau) = \mathbf{G}_1(t,s)\,.$$

A similar argument holds for $\mathbf{G}_2^*(s,t)$.

Conversely, suppose that

$$\mathbf{G}_1(t,s), \quad \mathbf{G}_2^*(s,t)$$

is realizable and (6.15) holds. Assume further that

$$\mathbf{G}_1(t,s) = \mathbf{H}_1(t)\mathbf{F}_1(s)$$

and that $\mathbf{G}_2^*(s,t) = \mathbf{I}$, so that there is a realization of the form

$$\dot{\mathbf{X}}(t) = \mathbf{C}(t)\mathbf{U}(t)\,,$$

$$\mathbf{y}(t) = \mathbf{K}(t)\mathbf{X}(t)\,.$$

We first show that there is a minimal realization.

Since

$$\mathbf{H}_1(t) = \mathbf{K}(t) \quad \text{and} \qquad \mathbf{F}_1(s) = \mathbf{C}(s),$$

we write

$$\mathbf{G}_1(t,s) = \mathbf{K}(t)\mathbf{C}(s)\,. \tag{6.16}$$

Using the periodicity of $\mathbf{G}_1$, we have

$$\mathbf{K}(t+\tau)\mathbf{C}(s+\tau) = \mathbf{K}(t)\mathbf{C}(s)\,.$$

Replacing s by $s-\tau$, we have

$$\mathbf{K}(t+\tau)\mathbf{C}(s) = \mathbf{K}(t)\mathbf{C}(s-\tau)\,. \tag{6.17}$$

There exist finite times t_0 and t_1 with $t_0 < t_1$, so that we can define the non-singular, constant matrices

$$\mathbf{W}(t_0,t_1) = \int_{t_0}^{t_1} \mathbf{C}(s)\mathbf{C}^*(s)\,\mathrm{d}s \quad \text{and} \quad \mathbf{M}(t_0,t_1) = \int_{t_0}^{t_1} \mathbf{K}^*(t)\mathbf{K}(t)\,\mathrm{d}t$$

Similarly, we define

$$\hat{\mathbf{W}}(t_0,t_1) = \int_{t_0}^{t_1} \mathbf{C}(s-\tau)\mathbf{C}^*(s)\,\mathrm{d}s\,. \quad \text{and} \quad \hat{\mathbf{M}}(t_0,t_1) = \int_{t_0}^{t_1} \mathbf{K}^*(s)\mathbf{K}(s+\tau)\,\mathrm{d}t\,.$$

Post-multiplying (6.17) by $\mathbf{C}^*(s)$ and integrating with respect to s from t_0 to t_1, yields (for all t)

$$\mathbf{K}(t+\tau) = \mathbf{K}(t)\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1)\,. \tag{6.18}$$

Similarly, premultiplying (6.17) by $\mathbf{K}^*(t)$ and integrating with respect to t yields (for all s)

$$\mathbf{C}(s-\tau) = \mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\mathbf{C}s). \tag{6.19}$$

Substituting (6.18) and (6.19) into (6.17) gives

$$\mathbf{K}(t)\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1)\mathbf{C}(s) = \mathbf{K}(t)\mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\mathbf{C}(s)\,.$$

Pre-multiplying by $\mathbf{K}^*(t)$ and post-multiplying by $\mathbf{C}^*(s)$ gives

$$\mathbf{K}^*(t)\mathbf{K}(t)\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1)\mathbf{C}(s)\mathbf{C}^*(s) = \mathbf{K}^*(t)\mathbf{K}(t)\mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\mathbf{C}(s)\mathbf{C}^*(s)\,.$$

Then, integrating with respect to s and t gives

$$\mathbf{M}(t_0,t_1)\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1)\mathbf{W}(t_0,t_1) = \mathbf{M}(t_0,t_1)\mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\mathbf{W}(t_0,t_1)\,.$$

This implies that

$$\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1) = \mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\,.$$

We now show that $\hat{\mathbf{W}}$ and $\hat{\mathbf{M}}$ are invertible, which ensures that the realization is minimal. Let

$$\mathbf{P} = \hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1) = \mathbf{M}^{-1}(t_0,t_1)\hat{\mathbf{M}}(t_0,t_1)\,.$$

If $\mathbf{P}$ is not invertible, then there exists an $(n \times 1)$ vector α such that $\alpha^T\mathbf{P} = 0$. Then (6.19) gives

$$\alpha^T\mathbf{C}s-\tau) = 0$$

for all s. This implies that

$$\alpha^T\,[\int_{t_0}^{t_1+\tau}\mathbf{C}s-\tau)\mathbf{C}^*(s-\tau)\,\mathrm{d}s\,]\,\alpha = 0\,,$$

and a change of integration variable shows that

$$\alpha^T\mathbf{W}(t_0-\tau,t_1)\alpha = 0\,.$$

But then

$$\alpha^T\mathbf{W}(t_0,t_1)\alpha = 0$$

which contradicts the invertibility of $\mathbf{W}(t_0,t_1)$.

To show that the realization is periodic, we must show that $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are periodic. ($\mathbf{A}(t) = \mathbf{B}(t) = 0$, so they are clearly periodic). We use the simple mathematical fact that (since $\mathbf{P}$ is a constant matrix) there exists a real $(n \times n)$ matrix ζ such that

$$\mathbf{P}^2 = e^{\zeta 2\tau} \quad or \quad \mathbf{P} = e^{\zeta\tau}\,.$$

If we let

$$\mathbf{H}_1(t) = \mathbf{K}(t)e^{-\zeta t} \qquad \mathbf{F}_1(t) = e^{\zeta t}C(t)\,,$$

then from (6.16), the state equation

$$\begin{aligned}\dot{R}(t) &= \mathbf{A}(t)\mathbf{R}(t) + \mathbf{F}_1(t)\mathbf{U}(t)\,,\\ \mathbf{y}(t) &= \mathbf{H}_1(t)\mathbf{R}(t)\end{aligned} \tag{6.20}$$

is a realization of $\mathbf{G}_1(t,s)$. Furthermore, from (6.18) and the definition of $\mathbf{P}$, we have

$$\mathbf{K}(t+\tau) = \mathbf{K}(t)\hat{\mathbf{W}}(t_0,t_1)\mathbf{W}^{-1}(t_0,t_1) = \mathbf{K}(t)\mathbf{P}.$$

Then

$$\begin{aligned}\mathbf{H}_1(t+\tau) &= \mathbf{K}(t+\tau)e^{-\zeta(t+\tau)}\\ &= \mathbf{K}(t)\mathbf{P}e^{-\zeta(t+\tau)}\\ &= \mathbf{K}(t)e^{-\zeta t}\\ &= \mathbf{H}_1(t)\end{aligned}$$

so $\mathbf{H}_1$ is τ periodic. Since $\mathbf{K}(t) = \mathbf{H}_1(t)e^{\zeta t}$ we have that $\mathbf{K}(t)$ is also τ periodic. A similar reasoning holds for $\mathbf{G}_2^*(s,t)$.

□

6.6 Controllability, Observability and Realizability of Non-linear Sylvester Systems

In this section, we discuss non-linear control systems associated with the Sylvester system

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t,\mathbf{X}(t),\mathbf{U}(t))\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}(t,\mathbf{X}(t),\mathbf{U}(t)) + \mathbf{C}(t,\mathbf{X}(t),\mathbf{U}(t))\mathbf{U}(t)\,, \tag{6.21}$$

$$\mathbf{y}(t) = \mathbf{K}(t, \mathbf{X}(t), \mathbf{U}(t))\mathbf{X}(t),$$

and present certain sufficient conditions for the controllability, observability and realizability criteria. Most of the results presented in this section are extensions of the results presented in the previous sections and hence the proofs are not given [12].

Theorem 6.6.1. *The system (6.21) is completely controllable on* $I = [t_0, t_1]$ *if and only if the* $(n \times n)$ *symmetric matrix*

$$\mathbf{W}(t_0, t_1, \tilde{\mathbf{X}}(t_1), \tilde{\mathbf{U}}(t_1)) = \int_{t_0}^{t_1} \mathbf{\Phi}(s, t_0, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))\mathbf{C}(s, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))C^*(s, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))\mathbf{\Phi}^*(s, t_0, \tilde{\mathbf{X}}(s)\tilde{\mathbf{U}}(s))\,\mathrm{d}s$$

is non-singular.

Theorem 6.6.2. *The Sylvester non-linear system (6.21) is completely observable if and only if the* $(n \times n)$ *symmetric matrix*

$$\mathbf{M}(t_0, t_1, \tilde{\mathbf{X}}(t_1)\tilde{\mathbf{U}}(t_1)) = \int_{t_0}^{t_1} \mathbf{\Phi}^*(s, t_0, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))K^*(s, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))\mathbf{K}(s, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))\mathbf{\Phi}(s, t_0, \tilde{\mathbf{X}}(s), \tilde{\mathbf{U}}(s))\,\mathrm{d}s$$

is non-singular.

Theorem 6.6.3. *The non-linear Sylvester system (6.21) is completely realizable if and only if it is completely controllable and completely observable.*

CHAPTER 7

A CONVENIENT METHOD FOR COMPUTING MATRIX EXPONENTIALS

In this chapter we develop a convenient method of computing the matrix exponential function which was introduced in Chapter 2. We start with the linear first order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.1}$$

and show that the state transition matrix $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is the unique solution of the $n-th$ order matrix differential equation

$$\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}^{(1)} + c_0\mathbf{\Phi} = 0$$

with specified initial conditions. Furthermore, we develop an elegant method to compute $e^{\mathbf{A}t}$. This method requires calculation of linearly independent solutions and power of matrix $\mathbf{A}$. We illustrate the theory with several examples.

7.1 Introduction

Some idea of scalar functions can be extend to functions of a matrix. Similar to the scalar exponential function e^{zt} can be represented as a power series

$$e^{zt} = 1 + zt + \frac{(zt)^2}{2!} + \cdots + \frac{(zt)^n}{n!} + \cdots .$$

Given a constant $n \times n$ matrix $\mathbf{A}$, the corresponding power series

$$\mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \cdots + \frac{(\mathbf{A}t)^n}{n!} + \cdots$$

converges to an $(n \times n)$ matrix, the matrix exponential function denoted by $e^{\mathbf{A}t}$ (2.10). The topic of matrix exponential function has been discussed in Chapter 2 (section 2.4) and proof for convergence of infinite matrix series is given in [1](p.54-58); it follows that since e^{zt} converges for all finite scalars z and t then the corresponding matrix series converges for all finite t and all $n \times n$ matrices $\mathbf{A}$ having finite elements.

Using the matrix exponential function, the general solution of the linear first-order system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,,$$

satisfying the initial conditions

$$\mathbf{x}(0) = \mathbf{x_0}$$

can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0\,.$$

7.2 Solution For Linear First-Order Systems

In this section we develop a convenient method of computing $e^{\mathbf{A}t}$, and hence the solution of the initial value problem (7.1). Before deriving the solution of (7.1), we present the following two results.

Theorem 7.2.1. *Let* $\mathbf{A}$ *be an* $(n \times n)$ *constant matrix with the characteristic polynomial*

$$c(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^{\mathbf{n}} + \mathbf{c_{n-1}}\lambda^{\mathbf{n-1}} + \cdots + \mathbf{c_1}\lambda + \mathbf{c_0}\,. \tag{7.2}$$

Then $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ *is the unique solution of the* $n-th$ *order matrix differential equation*

$$\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}' + c_0\mathbf{\Phi} = 0$$

satisfying the initial conditions

$$\begin{cases} \mathbf{\Phi}(0) = \mathbf{I}, \\ \mathbf{\Phi}'(0) = \mathbf{A}, \\ \mathbf{\Phi}''(0) = \mathbf{A}^2, \\ \vdots \\ \mathbf{\Phi}^{(n-1)}(0) = \mathbf{A}^{(n-1)}. \end{cases} \tag{7.3}$$

Proof. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be solutions to the $n - th$ order matrix differential equation (7.1), satisfying the initial conditions (7.3), and let $\mathbf{\Phi} = \mathbf{\Phi}_1 - \mathbf{\Phi}_2$. Then $\mathbf{\Phi}$ is a solution of (7.1), and

$$\mathbf{\Phi}(0) = \mathbf{\Phi}'(0) = \mathbf{\Phi}^{(n-1)}(0) = 0\,.$$

Therefore, $\mathbf{\Phi}$ satisfies the initial value problem

$$\mathbf{x}^{(n)} + c_{n-1}\mathbf{x}^{(n-1)} + \cdots + c_1\mathbf{x}' + c_0\mathbf{x} = 0$$
$$\mathbf{x}(0) = \mathbf{x}'(0) = \cdots = \mathbf{x}^{(n-1)}(0) = 0$$

with solution $\mathbf{x}(t) \equiv 0$, and so $\mathbf{\Phi} \equiv 0$ for all t in $\mathbb{R}$, and hence $\mathbf{\Phi}_1(t) \equiv \mathbf{\Phi}_2(t)$ for all t in $\mathbb{R}$. Now, $\mathbf{A}$ is the constant $n \times n$ matrix with the same characteristic polynomial

$$c(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^{\mathbf{n}} + \mathbf{c_{n-1}}\lambda^{\mathbf{n-1}} + \cdots + \mathbf{c_1}\lambda + \mathbf{c_0}\,,$$

If

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}\,,$$

then

$$\begin{cases} \mathbf{\Phi}'(t) = \mathbf{A}e^{\mathbf{A}t}, \\ \mathbf{\Phi}''(t) = \mathbf{A}^2, e^{\mathbf{A}t}, \\ \vdots \\ \mathbf{\Phi}^{(n)}(t) = \mathbf{A}^n e^{\mathbf{A}t}. \end{cases}$$

So that

$$\begin{aligned}
&\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}'(t) + c_0\mathbf{\Phi}(t) \\
&= (\mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I})e^{\mathbf{A}t} \\
&= \mathbf{P}(\mathbf{A})e^{\mathbf{A}t} \\
&= 0\,,
\end{aligned}$$

by the Caley-Hamilton Theorem. Also

$$\begin{cases}
\mathbf{\Phi}(0) = \mathbf{I}, \\
\mathbf{\Phi}'(0) = \mathbf{A}, \\
\mathbf{\Phi}''(0) = \mathbf{A}^2, \\
\vdots \\
\mathbf{\Phi}^{(n-1)}(0) = \mathbf{A}^{(n-1)}\,,
\end{cases}$$

and therefore $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is the fundamental matrix solution of the initial value problem

$$\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}' + c_0\mathbf{\Phi} = 0\,,$$

$$\begin{cases}
\mathbf{\Phi}(0) = \mathbf{I}, \\
\mathbf{\Phi}'(0) = \mathbf{A}, \\
\mathbf{\Phi}''(0) = \mathbf{A}^2, \\
\vdots \\
\mathbf{\Phi}^{(n-1)}(0) = \mathbf{A}^{(n-1)}
\end{cases}$$

Thus the proof is completed. □

Theorem 7.2.2. *Let* $\mathbf{A}$ *be a constant* $n \times n$ *matrix with the characteristic polynomial*

$$c(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^{\mathbf{n}} + \mathbf{c_{n-1}}\lambda^{\mathbf{n-1}} + \cdots + \mathbf{c_1}\lambda + \mathbf{c_0}\,,$$

then

$$e^{\mathbf{A}t} = \mathbf{x_1}(t)\mathbf{I} + \mathbf{x_2}(t)\mathbf{A} + \cdots + \mathbf{x}_n(t)\mathbf{A}^{n-1}\,,$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are the* n *linearly independent solutions of the* n − th *order scalar differential equation*

$$\mathbf{x}^{(n)} + c_{n-1}\mathbf{x}^{(n-1)} + \cdots + c_1\mathbf{x}' + c_0\mathbf{x} = 0\,, \tag{7.4}$$

satisfying the initial conditions

$$\begin{cases} \mathbf{x}_1(0) = 1 \\ \mathbf{x}_1'(0) = 0 \\ \mathbf{x}_1''(0) = 0 \\ \vdots \\ \mathbf{x}_1^{(n-1)}(0) = 0 \end{cases} \quad \begin{cases} \mathbf{x}_2(0) = 0 \\ \mathbf{x}_2'(0) = 1 \\ \mathbf{x}_2''(0) = 0 \\ \vdots \\ \mathbf{x}_2^{(n-1)}(0) = 0 \end{cases} \quad \cdots\cdots\cdots \begin{cases} \mathbf{x}_n(0) = 0 \\ \mathbf{x}_n'(0) = 0 \\ \mathbf{x}_n''(0) = 0 \\ \vdots \\ \mathbf{x}_n^{(n-1)}(1) = 0. \end{cases} \tag{7.5}$$

Proof. Since $\mathbf{x}_1$, $\mathbf{x}_2$, $\cdots$, $\mathbf{x}_n$ are the n linearly independent solutions of (7.4), satisfying (7.5), we have

$$\begin{aligned} &\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}' + c_0\mathbf{\Phi} \\ &= (\mathbf{x}_1^{(n)} + c_{n-1}\mathbf{x}_1^{(n-1)} + \cdots + c_1\mathbf{x}_1' + c_0\mathbf{x}_1)\,\mathbf{I} \\ &+ (\mathbf{x}_2^{(n)} + c_{n-1}\mathbf{x}_2^{(n-1)} + \cdots + c_1\mathbf{x}_2' + c_0\mathbf{x}_2)\,\mathbf{A} \\ &+ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ &+ (\mathbf{x}_n^{(n)} + c_{n-1}\mathbf{x}_n^{(n-1)} + \cdots + c_1\mathbf{x}_n + c_0\mathbf{x}_n)\,\mathbf{A}^{(n-1)} \\ &= 0\cdot\mathbf{I} + 0\cdot\mathbf{A} + \cdots + 0\cdot\mathbf{A}^{n-1} \\ &= 0\,. \end{aligned}$$

Thus $\mathbf{\Phi}$ is a solution of (7.1) satisfying

$$\begin{cases} \mathbf{\Phi}(0) = \mathbf{x}_1(0)\mathbf{I} + \mathbf{x}_2(0)\mathbf{A} + \cdots + \mathbf{x}_n(0)\mathbf{A}^{n-1} = \mathbf{I} \\ \mathbf{\Phi}'(0) = \mathbf{x}_1'(0)\mathbf{I} + \mathbf{x}_2'(0)\mathbf{A} + \cdots + \mathbf{x}_n'(0)\mathbf{A}^{n-1} = \mathbf{A} \\ \vdots \\ \mathbf{\Phi}^{(n-1)}(0) = \mathbf{x}_1(0)^{(n-1)}\mathbf{I} + \mathbf{x}_2^{(n-1)}(0)\mathbf{A} + \cdots + \mathbf{x}_n^{(n-1)}(0)\mathbf{A}^{n-1} = \mathbf{A}^{n-1}\,. \end{cases}$$

Therefore,

$$\mathbf{\Phi}(t) = \mathbf{x}_1(t)\mathbf{I} + \mathbf{x}_2(t)\mathbf{A} + \cdots + \mathbf{x}_n(t)\mathbf{A}^{n-1}$$

satisfies the initial value problem

$$\mathbf{\Phi}^{(n)} + c_{n-1}\mathbf{\Phi}^{(n-1)} + \cdots + c_1\mathbf{\Phi}' + c_0\mathbf{\Phi} = 0$$

$$\begin{cases} \mathbf{\Phi}(0) = \mathbf{I}, \\ \mathbf{\Phi}'(0) = \mathbf{A}, \\ \mathbf{\Phi}''(0) = \mathbf{A}^2, \\ \vdots \\ \mathbf{\Phi}^{(n-1)}(0) = \mathbf{A}^{(n-1)}. \end{cases}$$

Therefore from the uniqueness of initial value problems,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{x}_1(t)\mathbf{I} + \mathbf{x}_2(t)\mathbf{A} + \cdots + \mathbf{x}_n(t)\mathbf{A}^{n-1}$$

for all t in $\mathbb{R}$.

□

7.3 Examples

The method described above is elegant and efficient. It allows one can to avoid the heavy calculation we described earlier to find a fundamental matrix as a given linear system with constant coefficients. We illustrate the theory of the previous section with the following examples.

Example 7.3.1. *Complex Eigenvalues.*

Given the system of equations by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

The characteristic polynomial is

$$c(\lambda) = \lambda^2 + 1 = 0\,,$$

and hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -i$ and $\lambda_2 = i$, and the corresponding scalar differential equation is

$$\mathbf{x}'' + \mathbf{x} = 0\,.$$

Therefore the general solution is given by

$$\mathbf{x}(t) = c_1 \cos(t) + c_2 \sin(t)\,.$$

The solution satisfying the initial conditions

$$\begin{cases} \mathbf{x}_1(0) = 1 \\ \mathbf{x}_1'(0) = 0 \end{cases}$$

is

$$\mathbf{x}_1(t) = \cos(t)\,;$$

and the solution satisfying the initial condition

$$\begin{cases} \mathbf{x}_2(0) = 0 \\ \mathbf{x}_2'(0) = 1 \end{cases}$$

is

$$\mathbf{x}_2(t) = \sin t\,.$$

Therefore

$$\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{x_1}(t)\mathbf{I} + \mathbf{x_2}(t)\mathbf{A} \\
&= \cos(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \\
&= \mathbf{\Phi}(t)\,.
\end{aligned}$$

Example 7.3.2. *Real Eigenvalues.*

Consider the lineal system

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}\,.$$

The characteristic polynomial $c(\lambda)$ is given by

$$c(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

and the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The corresponding scalar differential equation is

$$\mathbf{x}''' - 7\mathbf{x}'' + 16\mathbf{x}' - 12\mathbf{x} = 0$$

and the general solution is given by

$$\mathbf{x}(t) = c_1 t e^{2t} + c_2 e^{2t} + c_3 e^{3t}\,.$$

The solution $\mathbf{x}_1$ satisfying the initial conditions

$$\begin{cases} \mathbf{x}_1(0) = 1, \\ \mathbf{x}_1'(0) = \mathbf{x}_1''(0) = 0 \end{cases}$$

is

$$\mathbf{x}_1(t) = -6te^{2t} - 3e^{2t} + 4e^{3t}\,;$$

and the solution $\mathbf{x}_2$ satisfying the initial conditions

$$\begin{cases} \mathbf{x}_2(0) = 0, \\ \mathbf{x}_2'(0) = 1, \\ \mathbf{x}_2''(0) = 0 \end{cases}$$

is

$$\mathbf{x}_2(t) = 5te^{2t} + 4e^{2t} - 4e^{3t}\,;$$

and finally the solution $\mathbf{x}_3$ satisfying the initial conditions

$$\begin{cases} \mathbf{x}_3(0) = 0, \\ \mathbf{x}_3'(0) = 0, \\ \mathbf{x}_3''(0) = 1 \end{cases}$$

is

$$\mathbf{x}_3(t) = -te^{2t} - e^{2t} + 3e^{3t}\,.$$

From

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

we have

$$\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Therefore

$$
\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{x}_1(t)\mathbf{I} + \mathbf{x}_2(t)\mathbf{A} + \mathbf{x}_3(t)\mathbf{A}^2 \\
&= (-6te^{2t} - 3e^{2t} + 4e^{3t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&+ (5te^{2t} + 4e^{2t} - 4e^{3t}) \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&+ (-te^{2t} - e^{2t} + 3e^{3t}) \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & 0 & e^{3t} - e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \\
&= \mathbf{\Phi}(t)\,.
\end{aligned}
$$

Note:

The method can be applied to real distinct eigenvalues, repeated eigenvalues, and complex eigenvalues. The advantage is that one can avoid reduction to Jordan form or the necessity of finding generalized eigenvectors.

CHAPTER 8

SUMMARY AND CONCLUSIONS

Control Theory is a discipline whose development spans more than two thousand years but became really relevant in the twentieth century. The main focus of Control Theory is on one of the most ancient dreams of mankind: how to make a given system behave in a desired manner. Control Theory addresses the automation of physical processes, from Archimedes to Shuttle, spacecrafts, automatic probes, robots, home automation. Control systems are fundamental in many aspects of our daily life (airplanes, cars, boilers, nuclear reactors, CD players etc).

In this research project, we analyzed three important concepts of the control theory: controllability, observability and realizability. We introduced the Classical Control Theory (1920–1957), which makes mostly use of Laplace transforms (transfer functions) and Modern Control Theory (1965–) which mostly deals directly with the system equations. As new technology was introduced, new control strategies needed to be found.

We started with Classical Control Theory and described the Laplace transform for continuous time systems and $z-$transform for discrete time systems. We defined the state transfer matrix $\mathbf{\Phi}(t, t_0)$ and verified its four properties. We introduced the techniques of matrix operation and illustrated the theory with interesting examples of computing the spectral form solution, exponential matrix solution and fundamental matrix solution for the constant linear systems and time varying systems.

We considered the system governed by the state equation in standard form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\,,$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\,.$$

We discussed definitions for controllability, introduced the controllability Grammian matrix $\mathbf{W}_C(t)$ and the controllability matrix $\mathcal{C}$ and gave the proof of sufficiency of the condition of controllability. Eight real life examples of controllable and uncontrollable linearized models were given: *rotating wheel, two connected water tanks and three parallel water tanks systems, and the platform system (bus suspension).* We tested those systems for controllability and then we constructed the control function $\mathbf{u}(t)$ that allow for the platform system to attain a desired state $\mathbf{x}$ and depicted the simulation result for two different time intervals in the graphs.

We found that is it reasonable to restrict ourselves to consider only the case of zero-input and omit the term $\mathbf{D}(t)\mathbf{u}(t)$ without loss of generality. We introduced definition and conditions for observability using the observability Grammian matrix $\mathbf{W_O}$ and observability matrix $\mathcal{O}$.

Based on these results, we found that the system observability is a dual property of the system controllability, therefore all discussion for controllability can be applied to observability in a similar way. For a system that is not controllable we can perform a decomposition to separate the controllable and uncontrollable models. That material was illustrated with the Diagram for Kalman decomposition and with example for decomposed system model.

Furthermore, we focus our interest on the realization problem, to compute a state representation. We developed the method of computing $g(s)$ for a scalar output with the companion form matrix (Theorem 5.3.2). We generalized our result for a matrix case and supported our conclusions with examples for the cases of scalar and matrix transfer functions (using MATLAB). We expanded our discussion about finding a triple $\mathbf{A}, \mathbf{B}, \mathbf{C}$ for discrete time systems and time varying systems and provided example for calculating $g(t)$.

We discussed some major topics of Lyapunov and Sylvester systems as a part of recent research in Modern Control Theory. We established conditions on a specified weighting patterns $\mathbf{G}_1(t, s)$ and $\mathbf{G}_2^*(s, t)$ that guarantee existence of a corresponding linear state equation.

Every chapter consists conclusions and remarks for all important topics. Throughout the research project we presented about twenty examples to illustrate the teoretical part. Based on examples we concluded that the most complicated part is to compute the matrix exponential function. Therefore, we developed a convenient method of calculating $e^{\mathbf{A}t}$ which requires only calculation of linearly independent solutions and power of matrix $\mathbf{A}$. This method is elegant and efficient and can be applied to real distinct, repeated and complex eigenvalues. Examples highlight the convenience of this method.

It is our recommendation to use described method for computing matrix exponential.

For future work it is reasonable to analyze questions relating to positive controllability and bounded control, stability and state feedback control: BIBO (bounded input-bounded output), nonlinear systems (Lyapunov indirect method), and optimal control laws (Pontryagin method).

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