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Precovering Dgc-Complexes

Jonathan G. Crosby

PRECOVERING DG$\mathcal{C}$-COMPLEXES

by

JONATHAN G. CROSBY

(Under the Direction of Alina Iacob)

ABSTRACT

In this thesis we prove that with the hypotheses that dg$\widetilde{\mathcal{C}}$ is injectively resolving and that the class $\mathcal{C}$ is closed under arbitrary direct limits we have that the following are equivalent to dg$\mathcal{C}$ being covering: Every complex of $\mathcal{C}$-modules is in dg$\widetilde{\mathcal{C}}$ and every exact complex of $\mathcal{C}$-modules is in $\widetilde{\mathcal{C}}$.

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JONATHAN G. CROSBY

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JONATHAN G. CROSBY

Major Professor: Alina Iacob

Committee: Ionut Iacob
Goran Lesaja

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CHAPTER 1

INTRODUCTION

1.1 Background

The process of extending homological algebra from the category of modules to that of complexes began with the last chapter of Cartan's and Eilenberg's book "Homological Algebra", published in 1956.

For quite some time, a major problem was the fact that there was no sufficiently general result for the existence of resolutions of unbounded complexes. It is true that in the category of complexes one can construct injective resolutions and projective resolutions in a similar manner as for modules, but there are complexes of complexes, not complexes of modules. So, for instance, if we consider a module M regarded as a complex concentrated at the zeroth place, it has an injective resolution, but this injective resolution does not agree with the classical injective resolution of the module M. The goal was to introduce "injective" (and "projective") resolutions that are complexes of modules and such that in the particular case described above one recovers the classical injective (projective) resolution of the module M.

This was accomplished with Avramov's and Holperin's work ("Through the looking glass: a dictionary between rational homotopy theory and local algebra", 1986) and with Spaltenstein's work ("Resolutions of unbounded complexes", Compositio Mathematica, 1988). Avramov and Halperin introduced the dg-resolutions. Then Avramov and Foxby used the tools to define the injective (projective) dimension of complexes by means of dg-injective (dg-projective) resolutions.

We recall that a complex I is dg-injective if each component I_n is an injective module and every map of complexes $V : E \to I$ from an exact complex E to I is homotopic to zero.

Their definition is working with complexes of modules (instead of complexes of complexes as in the alternate definition) and for a module M regarded as a complex concentrated at zero, one recovers the usual injective dimension of the module M.

The dg-injective complexes also play a part in model categories theory. M. Hovey showed that there is a close connection between Quillen abelian model structures and complete cotorsion pairs. The dg-injective complexes are associated with the classical cotorsion pair $(\mathcal{M}od, \mathcal{I}nj)$. In a series of papers Hovey and Gillespie applied Hovey's approach to define new and interesting model structures, associated with complete cotorsion pairs $(\mathcal{F}, \mathcal{C})$. To accomplish this they introduce generalizations of dg-injective (dg-projective) complexes with respect to such a cotorsion pair. Thet called these classes of complexes dg$\mathcal{F}$- and dg$\mathcal{C}$-complexes. They also introduced the $\mathcal{F}$-complexes and (respectively) $\mathcal{C}$-complexes (that generalize the dg-projective and, respectively, dg-injective complexes). For convenience, we recall the definitions:

Let $(\mathcal{F}, \mathcal{C})$ be a fixed cotorsion pair of modules. Then

- An $\mathcal{F}$-complex is an exact complex X such that $Z_n(X) \in \mathcal{F}$ for all $n \in \mathbb{Z}$.
- An $\mathcal{C}$-complex is an exact complex Y such that $Z_n(Y) \in \mathcal{C}$ for all $n \in \mathbb{Z}$.
- A complex X is a dg$\mathcal{F}$-complex if each component X_n is a module in $\mathcal{F}$ and if every map $f : X \to Y$ is homotopic to zero for every $Y \in \mathcal{C}$.
- A complex Y is a dg$\mathcal{C}$-complex if each component Y_n is a module in $\mathcal{C}$ and if every map $g : Y \to X$ is homotopic to zero for every $X \in \mathcal{F}$.

An important (known) result is the following ([3], Theorem 3.1):

Proposition 0 - 1. *If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair cogenerated by a set, then the induced pairs*

$(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ and $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ are complete cotorsion pairs.

This means that dg$\widetilde{\mathcal{C}}$ is preenveloping. When $(\mathcal{F}, \mathcal{C}) = (\mathcal{M}od, \mathcal{I}nj)$ one obtains the well-known cotorsion pair $(\epsilon, \text{dg-}\mathcal{I}nj)$, with $\epsilon =$ the class of exact complexes. Dually, if $(\mathcal{F}, \mathcal{C}) = (\mathcal{P}roj, \mathcal{M}od)$, then we obtain $(\text{dg-}\mathcal{P}roj, \epsilon)$. So, by the above result, if $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set, then dg$\widetilde{\mathcal{C}}$ is preenveloping.

The question that we consider here is: if the class $\mathcal{C}$ is also (pre)covering (in $\mathcal{M}od$), when is the class dg$\widetilde{\mathcal{C}}$ (pre)covering in the category of complexes?

Our main results are the following:

1. We give a necessary condition in order for the class of dg$\widetilde{\mathcal{C}}$-complexes to be covering. We prove that if dg$\widetilde{\mathcal{C}}$ is covering, then every complex of $\mathcal{C}$-modules is a dg$\widetilde{\mathcal{C}}$-complex.

2. We provide sufficient conditions on the class $\mathcal{C}$ that make dg$\widetilde{\mathcal{C}}$ covering.

More precisely, we prove that with the additional hypotheses that dg$\widetilde{\mathcal{C}}$ is injectively resolving and that the class $\mathcal{C}$ is closed under arbitrary direct limits we have:

Proposition 0 - 2. *The following are equivalent:*

1. *$dg\mathcal{C}$ is covering.*

2. *Every complex of $\mathcal{C}$-modules is in $dg\widetilde{\mathcal{C}}$.*

3. *Every exact complex of $\mathcal{C}$-modules is in $\widetilde{\mathcal{C}}$.*

1.2 Preliminaries

Throughout this paper, we will assume the reader is at least at the graduate level in mathematics and is familiar with ring theory, modules, and basic properties of modules.

Let R denote an associative ring with 1. We start by recalling a few basic properties about:

- modules and their elementary properties
- R-homomorphisms
- R-submodules
- quotient modules
- direct products
- direct sums
- functors (specifically, covariant and contravariant functors)

CHAPTER 2

MODULES AND SEQUENCES OF MODULES

2.1 Modules

By an *R-module* M, we will mean a *unitary left* R*-module*, that is, an abelian group M with a map $R \times M \to M$ denoted $(r,\ x) \mapsto\ rx$ such that for every $x, y \in M,$ for every $r, s \in R$

$$\begin{aligned} r(x+y) &= rx + ry \\ (r+s)x &= rx + sx \\ (rs)x &= r(sx) \\ 1x &= x \text{ where } 1 \in R. \end{aligned}$$

A *unitary right* R*-module* is defined similarly.

Examples:

- Let $R = \mathbb{R}$ or $\mathbb{C}$, $x \in \mathbb{R}^n$ be a column vector. Then we have exactly the usual vector spaces. Multiplying by an element of R coincides with scalar multiplication and addition of two elements $x_1, x_2 \in \mathbb{R}^n$ coincides with addition of vectors. Thus, modules are an extension of vector spaces.

- Let $R = \mathbb{Z}$. Then any abelian group M is a $\mathbb{Z}$-module with the scalar multiplication defined by $nx = x + x + \cdots + x$ if $n > 0$.

Let M, N be R-modules, then by $Hom_R(M, N)$ we mean all the R-homomor-phisms from M to N. Note that $Hom_R(M, N)$ is an abelian group under addition of functions.

An R-module is said to be *free* if it is a direct sum of copies of R.

Remark: Classical homological algebra can be viewed as based on the classes of projective, injective, and flat modules. Since one characterization of projective modules is in terms of free modules (see section 2.3) we recall that a free R-module is a direct sum of copies of R. It is known that every module is a quotient of a free module. We include the result here:

Proposition 2 - 1. *Every R-module is a quotient of a free R-module.*

Proof. Let M be an R-module and $\{x_i : i \in I\}$ be a set of generators of M. Then $\oplus_{i\in I} M_i$ is a free R-module. Define a map $\varphi : \oplus_{i\in I} M_i \to M$ by $\varphi((r_i)_{i\in I}) = \sum_{i\in I} r_i x_i$. Then φ is onto and so $M \cong \oplus_{i\in I} M_i / Ker\varphi$.

□

If $f \in Hom_R(M, N)$ where M and N are R-modules, then the *kernel* of f, denoted Ker f, is defined as usual. The *cokernel* of f, denoted Coker f, is defined to be $N/\mathrm{Im}\ f$ where Im f is the *image* of f.

Since flat modules are defined in terms of the tensor product, we will include its definition (in terms of the universal balanced map).

Let M be a right R-module, N a left R-module, and G an abelian group. Then a map $\sigma : M \times N \to G$ is said to be *balanced* (or *bilinear*) if it is additive in both variables. That is, if

$$\begin{aligned} \sigma(x + x', y) &= \sigma(x, y) + \sigma(x', y) \\ \sigma(x, y + y') &= \sigma(x, y) + \sigma(x, y') \\ \sigma(xr, y) &= \sigma(x, ry) \end{aligned}$$

$\sigma : M \times N \to G$ is said to be a *universal balanced map* if for every abelian group G' and balanced map $\sigma' : M \times N \to G'$, there exists a unique map $h : G \to G'$ such that $\sigma' = h \circ \sigma$. In other words, the following diagram commutes for every G' and σ':

$$\begin{tikzcd}
M \times N \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & G \\
& G' \arrow[u, dashed, "h"']
\end{tikzcd}$$

A *tensor product* of a right R-module M and a left R-module N is an abelian group T together with a universal balanced map $\sigma : M \times N \to T$. The tensor product of two R-modules M, N is denoted $M \otimes N$.

Proposition 2 - 2. *The tensor product of M_R and ${}_RN$ exists.*

Proof. Let F be the free abelian group with base $M \times N$. That is,

$$F = \left\{ \sum_i m_i(x_i, y_i) : m_i \in \mathbb{Z}, (x_i, y_i) \in M \times N \right\} \cong \mathbb{Z}^{M \times N}$$

Let S be the subgroup of F generated by elements of F of the form

$$(x + x', y) - (x, y) - (x', y)$$
$$(x, y + y') - (x, y) - (x, y')$$
$$(rx, y) - (x, ry)$$

where $x, x' \in M,\ y, y' \in N,\ r \in R$. Define a map $\sigma : M \times N \to F/S$ by $\sigma(x, y) = (x, y) + S$. Then σ is clearly balanced. Now let $\sigma' : M \times N \to G'$ be a balanced map into an abelian group g'. But F is free on $M \times N$. So there is a unique homomorphism $h' : F \to G'$ that extends σ' (that is, $h'(x, y) = \sigma(x, y)$). But $S \subset Kerh'$ since σ is balanced. So we get a unique induced map $h : F/S \to G'$ such that $\sigma' = h\sigma$. Thus $F/S = M \otimes_R N$.

$\square$

Remark: One important property of the tensor product is that it commutes with arbitrary direct sums.

Proposition 2 - 3. *Let $(M_i)_I$ be a family of right R-modules and N a left R-module. Then*

$$(\oplus_I M_i) \otimes_R N \cong \oplus_I (M_i \otimes N).$$

Proof. The map $(\oplus_I M_i) \times N \to \oplus_I (M_i \otimes N)$ given by $((x_i)_I, y) \mapsto (x_i \otimes y)_I$ is balanced and so we have a unique homomorphism $h : (\oplus_I M_i) \otimes N \to \oplus_I (M_i \otimes N)$ such that $h((x_i)_I \otimes y) = (x_i \otimes y)_I$. Similarly ones gets a unique homomorphism $h' : \oplus_I (M_i \otimes N) \to (\oplus_I M_i) \otimes N$ given by $h'(x_i \otimes y_i)_I \ \sum_I e_i(x_i) \otimes y_i$. It is easy to see that $h' = h^{-1}$.

□

Note that with the appropriate hypotheses there is an isomorphism

$$M \otimes_R (\oplus N_i) \cong \oplus_I (M \otimes N_i).$$

2.2 Sequences and Complexes

Our main results concern certain classes of complexes of modules: the $\mathcal{C}$-complexes and dg$\mathcal{C}$-complexes. In order to define these complexes, we must go through an overview of complexes, exact complexes, and (later) cotorsion pairs.

By a (*chain*) *complex* $\mathbf{C}$ of R-modules we mean a sequence

$$\mathbf{C} : \cdots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2}$$

of R-modules and R-homomorphisms such that $\delta_{n-1} \circ \delta_n = 0$, for every $n \in \mathbb{Z}$. $\mathbf{C}$ is denoted by $((C_n), (\delta_n))$.

Let $\mathbf{C}$ and $\mathbf{C}'$ be complexes of R-modules. Then by a *map* (or *chain map*) $\mathbf{f} : \mathbf{C} \to \mathbf{C}'$ we mean a sequence of maps $f_n : C_n \to C'_n$ such that the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\delta_n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C'_n & \xrightarrow{\delta'_n} & C'_{n-1} \end{array}$$

is commutative for each $n \in \mathbb{Z}$. $\mathbf{f}$ is denoted by (f_n).

Let $\mathbf{C} = ((C_n), (f_n))$ be a complex. Then the nth *homology module* of $\mathbf{C}$ is defined to be Ker δ_n/Im δ_{n+1} and is denoted by $H_n(\mathbf{C})$. Ker δ_n and Im δ_{n+1} will be denoted $Z_n(\mathbf{C}), B_n(\mathbf{C})$ and their elements are called *n-cycles* and *n-boundaries*, respectively.

A chain complex of the form

$$\mathbf{C} : \cdots \to C^{-2} \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \to \cdots$$

such that $\delta^n \circ \delta^{n-1} = 0$ for every $n \in \mathbb{Z}$ is called a *cochain complex*. The duals to homology modules, n-cycles, and n-boundaries are defined similarly. Since the cochain complex can be treated as a special type of chain complex (with a change in indexes), then we will only consider the chain complex (or complex for short).

A complex of R-modules and R-homomorphisms

$$\cdots \to M_2 \to M_1 \xrightarrow{\delta_1} M_0 \xrightarrow{\delta_0} M_{-1} \to \delta_{-1} M_{-2} \to \cdots$$

is said to be *exact at M_i* if Im $\delta_{i+1} =$ Ker δ_i. The sequence is said to be *exact* if it is exact at each M_i.

An exact sequence of the form

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is called a *short exact sequence.*

Example: Let $R = \mathbb{Z}$ and consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_2 \to 0$$

where $f(x_1) = 2x_1$ and $g(x_2) = x_2 \bmod 2$. A quick calculation will show that this is a short exact sequence. Sometimes, we wish to have an alternative way of checking whether a sequence is a short exact sequence. We give this equivalent definition here:

Proposition 2 - 4. *A sequence $0 \to A \xrightarrow{f} B$ of R-modules is exact if and only if f is one-to-one, and a sequence $B \xrightarrow{g} C \to 0$ is exact if and only if g is onto. Altogether the sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a short exact sequence if and only if f is one-to-one, g is onto, and Im f = Ker g.

Proof. Let $0 \xrightarrow{\varphi_4} M' \xrightarrow{\varphi_3} M \xrightarrow{\varphi_2} M'' \xrightarrow{\varphi_1} 0$ be a short exact sequence. Then Im φ_4 = Ker $\varphi_3 \Leftrightarrow$ Ker $\varphi_3 = 0 \Leftrightarrow \varphi_3$ is one-to-one. Similarly Im φ_2 = Ker $\varphi_1 \Leftrightarrow$ Im $\varphi_3 = M'' \Leftrightarrow \varphi_3$ is onto. □

Looking at the previous example, we notice that the sequence satisfies the three conditions. A particular case of short exact sequences is the split exact sequence. Since we work with projective and injective modules and both classes can be characterized in terms of split exact sequences, we give the following definition:

A short exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is called *split exact* if $\mathrm{Im}\, f$ is a direct summand of M.

Example: Let $R = \mathbb{R}$ and consider the sequence

$$0 \to \mathbb{R} \xrightarrow{\iota_1} \mathbb{C} \xrightarrow{\pi_2} \mathbb{R} \to 0$$

where $\iota_1(x) = x \in \mathbb{C}$ and $\pi_2(a + bi) = b \in \mathbb{R}$. A quick check will show that this is a split exact sequence. In the previous example, the sequence is a short exact sequence that is not split exact.

Proposition 2 - 5. *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence. The following are equivalent:*

- *The sequence is split exact.*
- *There is an R-homomorphism $f' : M \to M'$ such that $f' \circ f = id_{M'}$.*
- *There is an R-homomorphism $g'' : M'' \to M$ such that $g \circ g'' = id_{M''}$.*

Proof. Proof can be found in [1] (Proposition 1.2.15). □

2.3 Projective, Injective, and Flat Modules

As we already mentioned, there are three classes of modules that play a crucial part in classical homological algebra: projective, injective, and flat modules. We give the definition and some properties of these classes of modules. Since we also include some characterizations of these classes in terms of $Hom(A, \_)$, $Hom(\_, B)$, we start by recalling their definitions.

Let N be an R-module. Then by $Hom(A, \_)$ we mean a mapping from N into the set of morphisms $Hom(A, N)$.

Let M be an R-module. Then by $Hom(\_, B)$ we mean a mapping from M into the set of morphisms $Hom(M, B)$.

$Hom(A, \_)$ will associate the module N with the abelian group $Hom(A, N)$ while associating a homomorphism $f : N \to N'$ with $Hom(A, f)$ defined by $Hom(A, f)(h) = f \circ h,$ for every $h \in Hom(A, N)$. Similarly, $Hom(\_, B)$ associates a module M with the abelian group $Hom(M, B)$ and associates a homomorphism $g : M' \to M$ with $Hom(g, B)$ defined by $Hom(g, B)(h) = h \circ g,$ for every $h \in Hom(M, B)$.

We will extensively use the fact that $Hom_R(\_, N)$ is a covariant functor on the class of right R-modules and $Hom_R(M, \_)$ is a contravariant functor on the class of left R-modules. Even so, a proper justification of these facts will require an introduction to functors, which is outside the scope of this paper. Therefore, we will present only the facts without proof. The proofs can be found easily in most texts that discuss functors or in Relative Homological Algebra ([1]).

Proposition 2 - 6. *If* $0 \to N' \xrightarrow{f} N \xrightarrow{g} N''$ *is an exact sequence of* R*-modules, then for each* R*-module* M *the sequence*

$$0 \to Hom_R(M, N') \to Hom_R(M, N) \to Hom_R(M, N'')$$

is also exact.

Proposition 2 - 7. *If* $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *is an exact sequence of* R*-modules, then for* *each R-module N the sequence*

$$0 \to Hom_R(M'', N) \to Hom_R(M, N) \to Hom_R(M', N)$$

is also exact.

An R-module is said to be *projective* if given an exact sequence $A \xrightarrow{\varphi} B \to 0$ of R-modules and an R-homomorphism $f : P \to B$, there exists an R-homomorphism $g : P \to A$ such that $f = \varphi \circ g$. That is, the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ \mathbb{A} \arrow[r, "\varphi"'] & \mathbb{B} \arrow[r] & 0 \end{tikzcd}$$

Proposition 2 - 8. *The following are equivalent:*

1. *P is projective.*
2. *$Hom(P, \_)$ is right exact.*
3. *Every exact sequence $0 \to A \to B \to P \to 0$ is split exact.*
4. *P is a direct summand of a free R-module.*

Proof. $(1 \Rightarrow 2)$

This is trivial.

$(2 \Rightarrow 3)$

If $B \xrightarrow{\varphi} P \to 0$ is exact, then $Hom(P, B) \to Hom(P, P) \to 0$ is exact, which implies the sequence $B \to P \to 0$ splits by Proposition 2 - 5.

($3 \Rightarrow 4$)

This follows from Proposition 2 - 1 easily.

($4 \Rightarrow 1$)

Let P be a direct summand of a free R-module F. There is a map $s : F \to P$ such that $s \circ \iota = \mathrm{id}_P$ where $\iota : P \to F$ is the inclusion. Now let $A \xrightarrow{\varphi} B \to 0$ be exact and $f : P \to B$ be an R-homomorphism. Then there is a map $g : F \to A$ such that $\varphi \circ g = f \circ s$. But then $\varphi \circ g \circ \iota = f \circ s \circ \iota = f$. Thus P is projective. □

Let $R = \mathbb{Z}$. Then $\mathbb{Z}^n$ is free for $n \geq 1$. Therefore, $\mathbb{Z}^n$ is projective by Proposition 2 - 9.

Remark: By Proposition 2 - 1 every module has a *free* (and hence *projective*) *resolution*. Let us start with the sequence $M \to 0$. Then we use proposition 2.1 to construct the projective module P_0 and map $K_0 : P_0 \to M$ such that $P_0 \xrightarrow{\varphi} M \to 0$ is an exact sequence. Then we repeat the process for Ker K_0 to extend the sequence to the exact sequence $P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$. We continue this process to generate the projective resolution $\cdots P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0$.

Let $\cdots P_1 \to P_0 \to M \to 0$ be a projective resolution of an R-module M and consider the deleted projective resolution $\cdots P_1 \to P_0 \to 0$. Then the ith cohomology module of the complex $0 \to Hom(P_0, N) \to Hom(P_1, N) \to \cdots$ is denoted $\mathrm{Ext}^i_R(M, N)$. Note that $\mathrm{Ext}^0_R(M, N) = Hom(M, N)$ since $0 \to Hom(M, N) \to Hom(P_0, N) \to Hom(P_1, N) \to \cdots$ is exact.

The dual notion of a projective module is the injective module:

An R-module is said to be *injective* if given R-modules $A \subset B$ and a homomorphism $f : A \to E$, there exists a homomorphism $g : B \to E$ such that $g|_A = f$. Equivalently, the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

for the exact sequence $0 \to A \to B$.

Over a principle ideal domain there is a nicer description of injectiv emodules; they are precisely the divisible modules:

Proposition 2 - 9. *(Theorem 3.1.4 in Relative Homological Algebra)*
Let R be a principle ideal domain (PID). Then an R-module M is injective if and only if it is divisible.

We recall that M is a divisible R-module if for every $y \in M$ and for every $s \in R$ $\exists$ $x \in M$ such that $y = sx$.

In particular, $\mathbb{Z}$ is a PID and $\mathbb{Q}$ is divisible. Therefore, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Also, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module because it is not divisible.

It is known (Theorem 3.1.7 in [1]) that every module can be embedded in an injective module. As a consequence every module N has a so-called *injective resolution*, i.e. and exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots$$

with each E^i injective. It is known that for any R-modules M and N, the groups $Ext^i(M, N)$ can be computed either by using a projective resolution (as described above) or by using an injective resolution.

Let $0 \to N \to E^0 \to E^1 \to \cdots$ be an injective resolution and consider the deleted injective resolution $0 \to E^0 \to E^1 \to \cdots$. Then the ith homology module of the complex $0 \to Hom(M, E^0) \to Hom(M, E^1) \to \cdots$ is exactly $\mathrm{Ext}^i_R(M, N)$.

Proposition 2 - 10. *The following are equivalent:*

1. *E is injective.*

2. *$Hom(\_, E)$ is right exact.*

3. *E is a direct summand of every E-module containing E.*

4. *$Ext^i(M, E) = 0$ for every R-modules M and for every $i \geq 1$.*

5. *$Ext^1(M, E) = 0$ for every R-modules M.*

Proof. Proof can be found in [1] (Theorems 3.1.2 and 3.1.9). □

An R-module is said to be *flat* if given any exact sequence $0 \to A \to B$ of right R-modules, the tensored sequence $0 \to A \otimes_R F \to B \otimes_R F$ is exact. The corresponding homology modules are denoted $Tor^R_i(M_i, F)$.

Proposition 2 - 11. *The following are equivalent:*

1. *F is flat.*

2. *$\_ \otimes_R F$ is left exact.*

3. *$Tor^R_i(M, F) = 0$ for every rightR-modules M and for every $i \geq 1$.*

4. *$Tor^R_1(M, F) = 0$ for every rightR-modules M.*

5. $Tor_1^R(M, F) = 0$ *for every finitely-generated right R-modules M.*

Proof. Proof can be found in [1] (Theorem 2.1.8). $\square$

CHAPTER 3

COVERS AND ENVELOPES

3.1 Covers

Let R be a ring and let $\mathcal{F}$ be a class of R-modules. Then for an R-module M, a morphism $\varphi : C \to M$ where $C \in \mathcal{F}$ is called an $\mathcal{F}$-*cover* of M if

1. any diagram with $C' \in \mathcal{F}$

$$\begin{tikzcd} C' \arrow[d, dashed] \arrow[dr] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

   can be completed to a commutative diagram and

2. the diagram

$$\begin{tikzcd} C \arrow[d, dashed] \arrow[dr, "\varphi"] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

   can be completed only by automorphisms of C.

If $\varphi : C \to M$ satisfies (1) but maybe not (2), then it is called an $\mathcal{F}$-*precover* of M.

Proposition 3 - 1. *Let $\mathcal{F}$ be a class of R-modules that is closed under direct sums. Then an R-module M has an $\mathcal{F}$-precover if and only if there exists a set I and a family $(C_i)_{i \in I}$ of* *elements of $\mathcal{F}$ and morphisms $\varphi : C_I \to M$ for every $i \in I$ such that any morphism $D \to M$ with $D \in \mathcal{F}$ has a factorization $D \to C_j \xrightarrow{\varphi} M$ for some $j \in I$.*

Proof. The only if part is trivial. For the if part, we simply note that $\oplus_{i\in I} C_i \xrightarrow{\oplus\varphi_j} M$ is an $\mathcal{F}$-precover. $\square$

3.2 Envelopes

Let R be a ring and let $\mathcal{F}$ be a class of R-modules. Then for an R-module M, a morphism $\varphi : M \to F$ where $M \in \mathcal{F}$ is called an $\mathcal{F}$-*envelope* of M if

1. any diagram with $F' \in \mathcal{F}$

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr] & F \arrow[d, dashed] \\ & F' \end{tikzcd}$$

   can be completed to a commutative diagram and

2. the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & F \arrow[d, dashed] \\ & F \end{tikzcd}$$

   can be completed only by automorphisms of F.

If $\varphi : M \to F$ satisfies (1) but maybe not (2), then it is called an $\mathcal{F}$-*preenvelope* of M.

A class $\mathcal{F}$ of R-modules is *(pre)enveloping* if every R-module has an $\mathcal{F}$-(pre)envelope.

CHAPTER 4

DIRECT AND INVERSE LIMITS

4.1 Direct limits

A set I is called a *directed set* if I is partially ordered and for every $i, j \in I$, $\exists\ k \in I$ such that $i, j, \leq k$.

Let I be a directed set and $\{M_i\}_{i \in I}$ be a family of R-modules. Suppose that for every $i, j \in I$ with $i \leq j$ $\exists$ an R-homomorphism $f_{ji} : M_i \to M_j$ such that

1. $f_{ii} = \mathrm{id}_{M_i}$ for every $i \in I$

2. if $i \leq j \leq k$, then $f_{kj} \circ f_{ji} = f_{ki}$.

Then we say that the R modules M_i together with the homomorphisms f_{ji} form a *direct* (or *injective*) *system* which is denoted $((M_i), (f_{ji}))$.

The *direct (inductive) limit* of a direct system $((M_i), (f_{ji}))$ of R-modules is an R-module M with R-homomorphisms $g_i : M_i \to M$ for $i \in I$ with $g_i = g_j \circ f_{ji}$ whenever $i \leq j$ and such that if $(N, \{h_i\})$ is another such family, then there is a unique R-homomorphism $f : M \to N$ such that $f \circ g_i = h$ for every $i \in I$.

The direct limit will be denoted $\lim\limits_{\rightarrow} M_i$. Note that $\lim\limits_{\rightarrow} M_i$ is unique up to isomorphism.

Proposition 4 - 1. *The direct limit of a direct system of R-modules always exists.*

Proof. Let $((M_i), (f_{ij}))$ be a direct system of R-modules and U be the disjoint union of the M_i. Define a relation on U by $x_i \sim x_j$ if there is a $k \geq i, j$ such that $f_{kj}(x_i) = f_{kj}(x_j)$ where $x_i \in M_i, x_j \in M_j$. Then $\sim$ is an equivalence relation. Now let M be the set of equivalence classes under this relation and let $[x]$ denote the equivalence class of x. Define operations on M by $r[x_i] = [rx_i]$ if $r \in R$ and $[x_i] + [x_j] = [y_k + y'_k]$ where $k \geq i, j$ and $y_k = f_{ki}(x_i), y'_k = f_{kj}(x_j)$. Then M is an R-module. Now define maps $g_i : M_i \to M$ by $g_i(x_i) = [x_i]$. Then it is easy to see that $(M, \{g_i\})$ is the direct limit. □

4.2 Inverse limit

Let I be a directed set and $\{M_i\}_{i \in I}$ be a family of R-modules. Suppose that for every $i, j \in I$ with $i \leq j$, there exists an R-homomorphism $f_{ij} : M_j \to M_i$ such that

1. $f_{ii} = \mathrm{id}_{M_i}$ for every $i \in I$
2. if $i \leq j \leq k$, then $f_{ij} \circ f_{jk} = f_{ik}$.

Then we say that the R modules M_i together with the homomorphisms f_{ij} form an *inverse* (or *projective*) *system* which is denoted $((M_i), (f_{ij}))$.

The *inverse (projective) limit* of an inverse system $((M_i), (f_{ij}))$ of R-modules is an R-module M with R-homomorphisms $g_i : M \to M_i$ for $i \in I$ with $g_i = f_{ij} \circ g_i$ whenever $i \leq j$ and such that if $(N, \{h_i\})$ is another such family, then there is a unique R-homomorphism $f : N \to M$ such that $h_i : g_i \circ f,$ for every $i \in I$.

The inverse limit will be denoted $\lim\limits_{\leftarrow} M_i$.

Proposition 4 - 2. *The inverse limit of an inverse system of R-modules always exists.*

Proof. Let $((M_i), f_{ij}))$ be an inverse system. Then for each $i \in I$, let $\pi_i : \prod M_i \to M_i$ by the ith projection map. We set $M = \{(x_i)_I \in \prod M_i : x_i = f_{ij}(x_j)$ whenever $i \leq j\}$ and define $g_i : M \to M_i$ by $g_i = \pi_i|_M$. Then $(M, \{g_i\})$ is an inverse limit. □

Proposition 4 - 3. *Let $\mathcal{F}' = ((M_i'), (f_{ij}'))$, $\mathcal{F} = ((M_i), (f_{ij}))$, and $\mathcal{F}'' = ((M_i''), (f_{ij}''))$ be inverse systems over the same directed set and suppose there exist maps $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ such that*

$$0 \to M_i' \xrightarrow{\sigma_i} M_i \xrightarrow{\tau_i} M_i''$$

is exact for every i. Then the induced sequence

$$0 \to \lim_{\leftarrow} M_i' \xrightarrow{\lim\limits_{\leftarrow} \sigma_i} \lim_{\leftarrow} M_i \xrightarrow{\lim\limits_{\leftarrow} \tau_i'} \lim_{\leftarrow} M_i'' \to 0$$

is exact. If, furthermore, the set of indices is $\mathbb{N}$ and if the maps f_{ij}' are surjective, then when

$$0 \to M_i' \to M_i \to M_i'' \to 0$$

is exact for every i, the corresponding induced sequence is exact.

Proof. Proof can be found in [1] (Theorem 1.5.13). □

Proposition 4 - 4. *If N is an R-module, then*

1. $Hom(N, \lim\limits_{\leftarrow} M_i) \cong \lim\limits_{\leftarrow} Hom(N, M_i)$
2. $Hom(\lim\limits_{\rightarrow} M_i, N) \cong \lim\limits_{\leftarrow} Hom(M_i, N)$

Proof. Proof can be found in [1] (Theorem 1.5.14). □

CHAPTER 5

COTORSION PAIRS

Let $\mathcal{C}$ be a class of R-modules.

1. The class ${}^{\perp}\mathcal{C}$ of R-modules F such that $\mathrm{Ext}^1_R(F, C) = 0,$ for every $C \in \mathcal{C}$ and

2. the class $\mathcal{C}^{\perp}$ of R-modules G such that $\mathrm{Ext}^1_R(C, G) = 0,$ for every $C \in \mathcal{C}$

are called *orthogonal classes* of $\mathcal{C}$.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cortorsion pair* (for the category of R-modules) if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$.

When $(\mathcal{F}, \mathcal{C}) = (\mathcal{M}od, \mathcal{I}nj)$ one obtains the well-known cotorsion pairs $(\epsilon, \text{dg-}\mathcal{I}nj)$, with $\epsilon =$ the class of exact complexes. Dually, if $(\mathcal{F}, \mathcal{C}) = (\mathcal{P}roj, \mathcal{M}od)$, then we obtain $(\text{dg-}\mathcal{P}roj, \epsilon)$. So, by the above result, if $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set, then $\mathrm{dg}\widetilde{\mathcal{C}}$ is preenveloping.

A class $\mathcal{D}$ is is said to *generate* the cotorsion pair if ${}^{\perp}\mathcal{D} = \mathcal{F}$ and a class $\mathcal{G}$ is said to *cogenerate* the cotorsion pair if $\mathcal{G}^{\perp} = \mathcal{C}$.

Example: Let $\mathcal{M}$ denote the class of left R-modules. Let $\mathcal{I}nj$ and $\mathcal{P}roj$ denote the classes of injective and projective modules, respectively. Then $(\mathcal{M}, \mathcal{I}nj)$ and $(\mathcal{P}roj, \mathcal{M})$ are cotorsion pairs.

Remark: Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair. Then

- $\mathcal{F}$ and $\mathcal{C}$ are both closed under extensions and summands.

- $\mathcal{F}$ contains all the projective modules while $\mathcal{C}$ contains all the injective modules.

- $\mathcal{F}$ is closed under arbitrary direct sums while $\mathcal{C}$ is closed under arbitrary direct products.

- If $(\mathcal{F}, \mathcal{C})$ is generated (cogenerated) by a set X (so not just a class), then $(\mathcal{F}, \mathcal{C})$ is generated by the single module $\prod_{M \in X} M$ ($\bigoplus_{M \in X} M$).

Important types of cotorsion pairs are those that are complete, hereditary, and closed under extensions. In our main result, we will require all three. Therefore, we include them all together here.

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair. Then $(\mathcal{F}, \mathcal{C})$ is called *complete* if for every R-module M, there exist exact sequences

$$0 \to C \to F \to M \to 0 \quad \text{and} \quad 0 \to M \to C' \to F' \to 0$$

such that $C, C' \in \mathcal{C}$ and $F, F' \in \mathcal{F}$. We note that this is also related to the idea of *having enough projectives (injectives).*

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called *hereditary* if $\mathrm{Ext}^i_R(F, C) = 0,$ for all $F \in \mathcal{F},$ for all $C \in \mathcal{C},$ for all $i \geq 1$.

Equivalently, $(\mathcal{F}, \mathcal{C})$ is called *hereditary* if when

$$0 \to F' \to F \to F'' \to 0 \quad \text{and} \quad 0 \to C' \to C \to C'' \to 0$$

are exact with $F, F'' \in \mathcal{F}$ and $C', C \in \mathcal{C}$, then $F' \in \mathcal{F}$ and $C'' \in \mathcal{C}$ as well.

If $(\mathcal{F}, \mathcal{C})$ satisfies the first condition, then $\mathcal{F}$ is called *projectively resolving.* If $(\mathcal{F}, \mathcal{C})$ satisfies the second condition, then $\mathcal{C}$ is called *injectively resolving.*

A class $\mathcal{C}$ is *closed under extensions* if for any exact sequence $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$ with $\mathbf{A}, \mathbf{C} \in \mathcal{C}$, we have that $\mathbf{B} \in \mathcal{C}$.

CHAPTER 6

MAIN RESULT

We recall the following types of complexes as they will be used in the main result:

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair. Then

- An $\mathcal{F}$-*complex* is an exact complex X such that $Z_n(X) \in \mathcal{F}$ for all $n \in \mathbb{Z}$.
- A complex X is a *dg*$\mathcal{F}$-*complex* if each component X_n is a module in $\mathcal{F}$ and if every map $f : X \to Y$ is homotopic to zero for every $Y \in \mathcal{C}$.
- An $\mathcal{C}$-*complex* is an exact complex Y such that $Z_n(Y) \in \mathcal{C}$ for all $n \in \mathbb{Z}$.
- A complex Y is a *dg*$\mathcal{C}$-*complex* if each component Y_n is a module in $\mathcal{C}$ and if every map $g : Y \to X$ is homotopic to zero for every $X \in \mathcal{F}$.

We recall the following results as they will be used in the main result:

Proposition 6 - 1. *Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair cogenerated by a set. Then the induced pairs $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ and $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ are complete cotorsion pairs.*

Proof. Proof can be found in Cotorsion pairs, model structures, and homotopy categories. □

Proposition 6 - 2. *If $\mathcal{C}$ is covering (in $\mathcal{M}$) and closed under direct limits, products, and extensions, then the class of complexes of modules in $\mathcal{C}$ is covering in Ch(R).*

Proof. Proof can be found in [5]. □

Proposition 6 - 3. *(**Wakamatsu's Lemma:**) If $\mathcal{F}$ is a class of modules closed under extensions and if $\varphi : F \to M$ is an $\mathcal{F}$-cover, then Ker $\varphi \in \mathcal{F}^{\perp}$.*

Proof. Let $G \in \mathcal{F}$. We want to argue that $\mathrm{Ext}^i(G, \mathrm{Ker}\ \varphi) = 0$. Let $0 \to S \to P \to G \to 0$ be exact with P projective. Then we need that any $f : S \to \mathrm{Ker}\varphi$ can be extended to a linear $P \to \mathrm{Ker}\varphi$. But if we consider the commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & P \arrow[d, "0"] \arrow[dl, dashed, "g"'] \\ F \arrow[r] & M \end{tikzcd}$$

where $S \to F$ agrees with f we see that any $g : P \to F$ that makes the diagram commutative has its image in Ker φ and so gives the desired extension. $\square$

Proposition 6 - 4. *(Proposition 4 - 3)*
Let $\mathcal{F}' = ((M_i'), (f_{ij}'))$, $\mathcal{F} = ((M_i), (f_{ij}))$, $\mathcal{F}'' = ((M_i''), (f_{ij}''))$ be inverse systems over the same directed set and suppose $\exists$ maps $\mathcal{F}' \xrightarrow{\{\sigma_i\}} \mathcal{F} \xrightarrow{\{\tau_i\}} \mathcal{F}''$ such that $0 \to M_i' \xrightarrow{\sigma_i} M_i \xrightarrow{\tau_i} M_i''$ is exact for every i. Then the induced sequence

$$0 \to \varprojlim M_i' \xrightarrow{\varprojlim \sigma_i} \varprojlim M_i \xrightarrow{\varprojlim \tau_i} \varprojlim M_i'' \to 0$$

is exact.

If, furthermore, the set of indices is $\mathbb{N}$ and if the maps f_{ij}' are surjective, then when $0 \to M_i' \to M_i \to M_i'' \to 0$ is exact for every i, the induced sequence

$$0 \to \varprojlim M_i' \xrightarrow{\varprojlim \sigma_i} \varprojlim M_i \xrightarrow{\varprojlim \tau_i} \varprojlim M_i'' \to 0$$

is exact.

The question that we consider here is: if the class $\mathcal{C}$ is also covering (in $\mathcal{M}od$), when is the class dg$\widetilde{\mathcal{C}}$ covering in the category of complexes?

Lemma 1. *If dg$\widetilde{\mathcal{C}}$ is covering, then every $\mathcal{C}$-complex is a dg$\mathcal{C}$-complex.*

Proof. Let $\mathbf{X} = \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$ be a $\mathcal{C}$-complex. Then for each $n \geq 1$, we let

$$\mathbf{X}(n) = 0 \to X_n \to X_{n-1} \to \cdots$$

Then $\mathbf{X}(n)$ is a dg$\mathcal{C}$-complex by definition and $\mathbf{X} = \varinjlim \mathbf{X}(n)$ by construction.
Let $\mathbf{D} \to \mathbf{X}$ be a dg$\mathcal{C}$-cover and $K = \mathrm{Ker}(\mathbf{D} \to \mathbf{X})$. Then by Wakamatsu's Lemma, $\mathrm{Ext}^1_R(C, K) = 0$ for every $C \in \mathrm{dg}\widetilde{\mathcal{C}} \Rightarrow Ext^1_R(\mathbf{X}(n), K) = 0$ for every $n \geq 1$. Thus, we have the exact sequence

$$0 \to \mathbf{X}(n) \to \mathbf{X}(n+1) \to \frac{\mathbf{X}(n+1)}{\mathbf{X}(n)} \to 0$$

of dg$\mathcal{C}$-complexes for every $n \geq 1$ which gives an exact sequence

$$0 \to Hom\left(\frac{\mathbf{X}(n{+}1)}{\mathbf{X}(n)}, \mathbf{K}\right) \to Hom(\mathbf{X}(n{+}1), \mathbf{K}) \to Hom(\mathbf{X}(n), \mathbf{K}) \to 0$$

This imples that the maps $Hom(\mathbf{X}(n{+}1), \mathbf{K}) \to Hom(\mathbf{X}(n), \mathbf{K})$ are onto for every $n \geq 1$. Therefore, we have the following exact complex:

$$0 \to \varprojlim Hom(\mathbf{X}(n), \mathbf{K}) \to \varprojlim Hom(\mathbf{X}(n), \mathbf{D}) \to \varprojlim Hom(\mathbf{X}(n), \mathbf{X}) \to 0$$

But we know that

$$\lim_{\leftarrow} Hom(\mathbf{X}(n), \mathbf{K}) \cong Hom(X, K)$$
$$\lim_{\leftarrow} Hom(\mathbf{X}(n), \mathbf{D}) \cong Hom(X, D)$$
$$\lim_{\leftarrow} Hom(\mathbf{X}(n), \mathbf{X}) \cong Hom(X, X)$$

So we have an exact complex

$$0 \to Hom(\mathbf{X}, \mathbf{K}) \to Hom(\mathbf{X}, \mathbf{D}) \to Hom(\mathbf{X}, \mathbf{X}) \to 0$$

Then $\exists$ a morphism $\mathbf{X} \to \mathbf{D}$ such that $\mathbf{X} \to \mathbf{D} \to \mathbf{X}$ is the identity on $\mathbf{X}$. So $\mathbf{D} \to \mathbf{X}$ is onto and

$$0 \to \mathbf{K} \to \mathbf{D} \to \mathbf{X} \to 0$$

is split exact $\Rightarrow \mathbf{X}$ is isomorphic to a direct summand of $\mathbf{D}$. Since $\mathbf{D} \in \mathrm{dg}\widetilde{\mathcal{C}}$, it follows that $\mathbf{X} \in \mathrm{dg}\widetilde{\mathcal{C}}$ as well. $\square$

Our main results are:

1. A necessary condition in order for the class of $\mathrm{dg}\widetilde{\mathcal{C}}$-complexes to be covering. We prove that if $\mathrm{dg}\widetilde{\mathcal{C}}$ is precovering, then every complex of $\mathcal{C}$-modules is a $\mathrm{dg}\widetilde{\mathcal{C}}$-complex.

2. Sufficient conditions on the class $\mathcal{C}$ that make $\mathrm{dg}\widetilde{\mathcal{C}}$ covering.

More precisely, we prove that with the additional hypotheses that $\mathrm{dg}\widetilde{\mathcal{C}}$ is injectively resolving and that the class $\mathcal{C}$ is closed under arbitrary direct limits we have the following:

Proposition 6 - 5. *Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair in $\mathcal{M}od$ cogenerated by a set. If $\mathcal{C}$ is covering and closed under direct limits, and $dg\widetilde{\mathcal{C}}$ is injectively resolving, then the following are equivalent:*

1. *dg-$\mathcal{C}$ is covering.*

2. *Every complex of $\mathcal{C}$-modules is in $dg\widetilde{\mathcal{C}}$.*

3. *Every exact complex of $\mathcal{C}$-modules is in $\widetilde{\mathcal{C}}$.*

Proof. $(1 \Rightarrow 2)$

This is Lemma 1.

$(2 \Rightarrow 1)$

By Proposition 6 - 2, the class of complexes of $\mathcal{C}$ modules is covering in Ch(R). By (2), this coincides with dg$\widetilde{\mathcal{C}}$.

$(1 \Rightarrow 3)$

Let $\mathbf{X}$ be an exact complex of $\mathcal{C}$-modules. By (1), $\mathbf{X}$ is an exact complex in dg$\widetilde{\mathcal{C}}$. By definition, these are precisely the complexes in $\widetilde{\mathcal{C}}$.

$(3 \Rightarrow 1)$

Let $\mathbf{Y}$ be a complex of $\mathcal{C}$-modules. Since $(\widetilde{\mathcal{F}}, \text{dg}\widetilde{\mathcal{C}})$ is a complete cotorsion pair in Ch(R), $\exists$ an exact sequence

$$0 \to \mathbf{C} \to \mathbf{F} \to \mathbf{Y} \to 0$$

with $\mathbf{F} \in \widetilde{\mathcal{F}}$ and $\mathbf{C} \in \text{dg}\widetilde{\mathcal{C}}$. For each $n \in \mathbb{Z}$ we have an exact sequence

$$0 \to C_n \to F_n \to Y_n \to 0$$

with $C_n, Y_n \in \mathcal{C}$ for every $n \in \mathbb{Z}$. Since $\mathcal{C}$ is closed under extensions, $F_n \in \mathcal{C}$ for every $n \in \mathbb{Z}$. Thus $\mathbf{F}$ is an exact complex of $\mathcal{C}$-modules. By (3), $\mathbf{F} \in \widetilde{\mathcal{C}} \subseteq \mathrm{dg}\widetilde{\mathcal{C}}$. Now we have that $\mathbf{C}, \mathbf{F} \in \mathrm{dg}\widetilde{\mathcal{C}}$. Since $\mathrm{dg}\widetilde{\mathcal{C}}$ is injectively resolving, we have $\mathbf{Y} \in \mathrm{dg}\widetilde{\mathcal{C}}$. $\square$

CHAPTER 7

CLOSING REMARKS

We recall that if $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair cogenerated by a set, then $\mathrm{dg}\widetilde{\mathcal{C}}$ is preenveloping:

Proposition 7 - 1. *If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair cogenerated by a set, then the induced pairs $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ and $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ are complete cotorsion pairs.*

When $(\mathcal{F}, \mathcal{C}) = (\mathcal{M}od, \mathcal{I}nj)$ one obtains the well-known cotorsion pair $(\epsilon, \text{dg-}\mathcal{I}nj)$, with $\epsilon =$ the class of exact complexes. Dually, if $(\mathcal{F}, \mathcal{C}) = (\mathcal{P}roj, \mathcal{M}od)$, then we obtain $(\text{dg-}\mathcal{P}roj, \epsilon)$.

The question that we considered was the following: if the class $\mathcal{C}$ is also covering (in $\mathcal{M}od$), then when is the class $\mathrm{dg}\widetilde{\mathcal{C}}$ covering in the category of complexes? Our answer was the following:

Proposition 7 - 2. *Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair in $\mathcal{M}od$ cogenerated by a set. If $\mathcal{C}$ is covering and closed under direct limits, and $dg\widetilde{\mathcal{C}}$ is injectively resolving, then the following are equivalent:*

1. *dg-$\mathcal{C}$ is covering.*

2. *Every complex of $\mathcal{C}$-modules is in $dg\widetilde{\mathcal{C}}$.*

3. *Every exact complex of $\mathcal{C}$-modules is in $\widetilde{\mathcal{C}}$.*

CHAPTER 8

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