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On Some Length Biased Inequalities for Reliability Measures

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In this note, inequalities for length biased and the original residual life function and equilibrium distribution function with monotone hazard rate and mean residual life functions are derived. We also obtain estimates of the length biased probability density function and hazard function under random censoring. Finally, the Bayesian exponential reliability estimate under length biased sampling using a conjugate prior for the scale parameter is given.

Keywords: Inequalities; Life distribution; Reliability function;
Maximum likelihood estimator

AMS 1991 Subject Classification: 39C05

1 INTRODUCTION

In survival analysis, the usual estimators of cell kinetic parameters resulting from labeled mitosis experiments are biased, because cells with longer DNA synthesis periods have greater probability of being labeled. The effect of size-biased sampling in cell kinetic problems and the distributions associated with cell populations have been studied by several authors including Brockwell and Trucco [2], Schotz and Zelen [16] to mention a few. Length biased distributions have also been applied in other areas of scientific studies including but not limited to reliability studies, renewal theory, and wildlife populations (Patil and Rao [13]). The importance and implications of the property of monotone hazard rate and mean residual life function is well investigated. The purpose of this article is to establish bounds on the distance between length biased residual life distributions, equilibrum distributions in the class of distributions having increasing or decreasing hazard rate and the exponential distribution. The estimates of the length biased probability density function (pdf), hazard rate and exponential reliability function are also given.

Consider the family of distributions with pdf

$$g(t;\theta) = \begin{cases} tf(t;\theta)/\mu, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

where $0 < \mu = E(T) < \infty$, θ is an unknown positive parameter for a given parametric set $\Theta \subset [0, \infty)$. If the random variable Y is distributed according to the density (1.1) where f is the exponential pdf with parameter θ, then the survival or reliability function is given by

$$\begin{aligned} \bar{G}(y;\theta) &= P(Y > y;\theta), \\ &= \{(y+\theta)/\theta\}\exp\{-y/\theta\}, \quad y \geq 0. \end{aligned} \tag{1.2}$$

In general, from (1.1)

$$\bar{G}(y) = \bar{F}(y)\{y + \delta_F(y)\}/\mu_F,$$

where

$$\delta_F(y) = \int_y^\infty \bar{F}(u)\,\mathrm{d}u/\bar{F}(y) \quad \text{and} \quad 0 < \mu_F = \int_0^\infty \bar{F}(y)\,\mathrm{d}y < \infty. \tag{1.3}$$

Clearly, $\bar{G}(y) \geq \bar{F}(y)$, for all $y \geq 0$, so that $y + \int_y^\infty \bar{F}(u)\,\mathrm{d}(u)/\bar{F}(y) \geq \int_0^\infty \bar{F}(y)\,\mathrm{d}y$, for all $y \geq 0$, that is

$$E_f(Y \mid Y > y) \geq \int_0^\infty \bar{F}(y)\,\mathrm{d}y, \quad \forall y \geq 0, \tag{1.4}$$

where E_f denotes expectation with respect to the probability density function f. This note is organized as follows. In Section 2, some basic results, including several estimates of the parameters are given. In Sections 3 and 4 stability results and inequalities for length biased residual life function and equilibrium distribution function with monotone hazard functions and mean residual life functions are derived.

In Section 5, estimation of the density function and hazard rate under random censoring are presented. In Section 6, we consider the Bayesian estimates of the length biased exponential reliability function. This section is primarily concerned with various estimates of the reliability function.

2 BASIC RESULTS

Let $X_1, X_2, \ldots, X_n$ denote a random sample from the model in (1.2), where $f(x;\theta) = \theta^{-1}\exp\{-x/\theta\}$, $x > 0$, $\theta > 0$. We write $X \sim LB\exp(\theta)$ to denote that X has the length biased exponential density, with survival function given by (1.2). The maximum likelihood estimator of θ is given by

$$\hat{\theta} = S_n/2n, \tag{2.1}$$

where $S_n = \sum_{i=1}^{n} X_i$.

The statistic S_n which is sufficient for θ, has a gamma distribution with shape parameter $2n$ and scale parameter θ. That is $S_n \sim \Gamma(2n, \theta)$. Consider the estimator of $\theta = \lambda^{-1}$ of the form $\hat{\alpha}(c) = c/S_n$, where $c > 0$. Then

$$E(c/S_n) = c\lambda/(2n-1), \tag{2.2}$$

and

$$\begin{aligned}\mathrm{Var}(c/S_n) &= \lambda^2\{c^2 - 4c(n-1) + (2n-1)(2n-2)\}/\{(2n-1)(2n-2)\} \\ &= \lambda^2\left\{\frac{c^2 - 4c(n-1)}{(2n-1)(2n-2)} + 1\right\}. \end{aligned} \tag{2.3}$$

Note that the MLE of λ is obtained by setting $c_1 = 2n$, while the minimum mean squared error estimator (MMSE) in the class $\{c : \hat{\lambda}(c) = c/S_n, c > 0\}$ is obtained by setting $c_2 = 2(n-1)$.

The Pitman estimator of λ is

$$\hat{\lambda}_{\mathrm{P}} = \left\{\int_0^{\infty} \lambda U(\mathbf{X}\mid\lambda)\,\mathrm{d}\lambda\right\} \Big/ \left\{\int_0^{\infty} U(\mathbf{X}\mid\lambda)\,\mathrm{d}\lambda\right\}, \tag{2.4}$$

where $U(\mathbf{X} \mid \lambda)$ is the likelihood function, that is

$$\hat{\lambda}_{\mathrm{P}} = \left\{ \int_0^{\infty} \lambda \cdot \lambda^{2n} \mathrm{e}^{-S_n \lambda} \, \mathrm{d}\lambda \right\} \Big/ \left\{ \int_0^{\infty} \lambda^{2n} \mathrm{e}^{-S_n \lambda} \, \mathrm{d}\lambda \right\}$$
$$= (2n+1)/S_n. \tag{2.5}$$

We assume that the uncertainty about the parameter $\lambda = \theta^{-1}$ can be expressed as a gamma distribution, $\lambda \sim \Gamma(\alpha, \beta)$ with prior density

$$h(\lambda; \alpha, \beta) = \begin{cases} \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} \mathrm{e}^{-\beta\lambda}, & \lambda, \alpha, \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

For a single observation $X = x$, the joint distribution of (X, λ) has the density

$$f_{X,\lambda}(x, \lambda) = h(\lambda; \alpha, \beta) g(x; \lambda)$$
$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha+1} x \mathrm{e}^{-(\beta+x)\lambda}, \quad x, \lambda > 0. \tag{2.7}$$

The statistic $S_n = \sum_{i=1}^{n} X_i$ is sufficient for $\theta \in \Theta \subset [0, \infty)$ and its pdf is given by

$$g_{S_n}(s; \theta) = \begin{cases} \dfrac{\theta^{-2n}}{\Gamma(2n)} s^{2n-1} \mathrm{e}^{-s/\theta}, & s > 0, \ \theta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

With $\lambda = \theta^{-1}$ (2.8) can be written as

$$g_{S_n}(s; \lambda) = \begin{cases} \dfrac{\lambda^{2n}}{\Gamma(2n)} s^{2n-1} \mathrm{e}^{-\lambda s}, & s > 0, \ \lambda > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Let $T_n = S_n / n$, then

$$g_{T_n}(t; \lambda) = \begin{cases} \dfrac{\lambda^{2n}}{\Gamma(2n)} n^{2n} t^{2n-1} \mathrm{e}^{-\lambda n t}, & \lambda > 0, \ t > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

The joint distribution of $(S_n/n, \lambda)$ is

$$f_{S_n,\lambda}(s, \lambda; \alpha, \beta) = \begin{cases} \dfrac{\beta^\alpha \lambda^{2n+\alpha-1}}{\Gamma(\alpha)\Gamma(2n)} s^{2n-1} e^{-\lambda(s+\beta)}, & s > 0,\ \lambda > 0,\ \alpha > 0,\ \beta > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

and the joint density of (T_n, λ) is given by

$$f_{T_n,\lambda}(t, \lambda; \alpha, \beta) = \begin{cases} \dfrac{n^{2n}\beta^\alpha \lambda^{2n+\alpha-1}}{\Gamma(\alpha)\Gamma(2n)} t^{2n-1} e^{-\lambda(nt+\beta)}, & t > 0,\ \lambda > 0,\ \alpha > 0,\ \beta > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Suppose however, we consider observation from the original pdf $f(y \mid \lambda)$ rather than the length biased pdf ($X \sim LB\exp(\theta)$) then the predictive distribution has the density given by

$$\begin{aligned} f_{Y|X}(y \mid x) &= \int_0^\infty f(y; \lambda) f(\lambda; x)\, d\lambda \\ &= \int_0^\infty \lambda e^{-\lambda y} \frac{\lambda^{\alpha+1}\beta^\alpha}{\Gamma(\alpha)} x e^{-(\beta+x)\lambda} \\ &= \frac{\beta^\alpha x}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha+2} e^{-(\beta+x+y)\lambda}\, d\lambda \\ &= \frac{\beta^\alpha x}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{(\beta+x+y)^{\alpha+1}} \\ &= \frac{\alpha\beta^\alpha x}{(\beta+x+y)^{\alpha+1}}, \quad y > 0. \end{aligned} \tag{2.13}$$

In medical applications of survival analysis and various other settings, there is usually censoring. For the ith individual there is a survival time X_i and a censoring time C_i and we observe (T_i, δ_i), where $T_i = X_i \wedge C_i$ and $\delta_i = I(X_i \le C_i)$. Under length biased exponential sampling, if d is the number of uncensored observations, then the loglikelihood based on

$T_1, T_2, \ldots, T_n$ is given by

$$\ln L(\lambda) = K + \sum_u \ln \gamma_G(t_j) + \sum_u \ln \bar{G}(t_j), \tag{2.14}$$

where K is constant, and γ_G, $\bar{G}$ given by (1.3).

For $LB\exp(\lambda)$, $\gamma_G(y) = \lambda^2 y/(1+\lambda y)$ and $\bar{G}(y) = (1+\lambda y)\mathrm{e}^{-\lambda y}$, $y > 0$, so that

$$\begin{aligned} \ln L(\lambda) &= K + \sum_u \ln\left(\frac{\lambda^2 t_j}{1+\lambda t_j}\right) + \sum_u \ln[(1+\lambda t_j)\mathrm{e}^{-\lambda t_j}] \\ &= K + 2\sum_u \ln\lambda + \sum_j t_j - \sum_j \lambda t_j. \end{aligned}$$

The maximum likelihood estimate of λ is given by $\hat{\lambda} = \sum_{j=1}^{n} t_j/2d$ and the MLE estimate of the reliability function is

$$\bar{\bar{G}}_l(y; \hat{\lambda}) = \left\{1 + \left(\frac{\sum_{j=1}^{n} t_j}{2d}\right) y\right\} \exp\left\{-y \sum_{j=1}^{n} t_j/2d\right\}. \tag{2.15}$$

3 INEQUALITIES FOR RELIABILITY MEASURES

In this section, inequalities for the length biased residual life function and equilibrum distribution function with monotone hazard and mean residual life functions are established.

Consider a renewal process with life distribution $F(x)$ and length biased distribution $G(x)$. Let X_t denote the residual lifetime of the unit functioning at time t. Then as $t \to \infty$, X_t has the limiting pdf (Ross [14])

$$f_{\mathrm{e}}(x) = \bar{F}(x)/\mu_F, \quad x \geq 0. \tag{3.1}$$

The survival function or equilibrium survival function is given by

$$\bar{F}_{\mathrm{e}}(x) = \frac{1}{\mu_F}\int_x^{\infty} \bar{F}(u)\,\mathrm{d}u, \quad x \geq 0, \tag{3.2}$$

and the length biased equilibrium survival function is

$$\begin{aligned}\bar{G}_{\mathrm{e}}(x) &= \frac{1}{\mu_G}\int_x^{\infty}\bar{G}(u)\,\mathrm{d}u \\ &= \frac{1}{\mu_F^2+\sigma_F^2}\int_x^{\infty}\bar{F}(u)\{u+\delta_F(u)\}\,\mathrm{d}u, \qquad (3.3)\end{aligned}$$

where $\delta_F(u) = \int_u^{\infty}\bar{F}(t)\,\mathrm{d}t/\bar{F}(u)$, $u\geq 0$, and σ_F^2 is the variance of F.

Let $\{S_j(x)\}$, $j=0,1,2,\ldots,J$, be a sequence of decreasing functions given by

$$S_j(x) = \begin{cases} \bar{F}(x), & j=0, \\ \int_0^{\infty}\bar{F}(x+t)\dfrac{t^{j-1}}{(j-1)!}\,\mathrm{d}t, & j=1,2,\ldots,J. \end{cases} \qquad (3.4)$$

We let $S_{-1}(x)=f(x)$ be the pdf of F if it exists. Then $S_j(0)=\mu_j/j!$, $S_j'(x) = -S_{j-1}(x)$, $j=0,1,2,\ldots,J$. The ratio $S_{j-1}(x)/S_j(x)$ is a hazard function of a distribution function with survival function $S_j(x)/S_j(0)$. The following is a modified version of the lemma given by Barlow *et al.* [1]:

LEMMA 3.1 (Barlow *et al.* [1]) *If F has decreasing mean residual life (DMRL), then*

$$S_k(x) \leq S_k(0)\mathrm{e}^{-x/\mu}, \quad k=1,2,\ldots,$$

and

$$S_k(x) \geq \mu S_{k-1}(0)\mathrm{e}^{-x/\mu} - \mu S_{k-1}(0) + S_k(0), \quad k=2,3,\ldots \qquad (3.5)$$

The inequalities are reversed if F has increasing mean residual life (IMRL).

Let $t>0$ be fixed. The distribution function

$$G_{l_t}(x) = \frac{G(t+x)-G(t)}{1-G(t)}, \quad x\geq 0, \qquad (3.6)$$

is called the residual life distribution corresponding to the length biased distribution G. The corresponding survival function is given by

$$\bar{G}_{l_t}(x) = \frac{\bar{G}(t+x)}{\bar{G}(t)}, \quad x\geq 0. \qquad (3.7)$$

THEOREM 3.1 *Let $\bar{G}_e$ be the length biased equilibrium survival function with decreasing hazard rate (DHR). Then*

$$\int_0^\infty \left|\bar{G}_e(x) - x\mathrm{e}^{-x/\mu}\right| \mathrm{d}x \geq 2\mathrm{e}^{-\epsilon/\mu}(\mu - \mu\epsilon - 1), \quad \textit{for } \epsilon > \mu.$$

Proof Let $\bar{G}_e$ have DHR, then there exist $\epsilon \geq \mu$ such that $\bar{G}_e(x) \leq x\exp(-x/\mu)$ or $\bar{G}_e(x) \geq x\exp(-x/\mu)$ as $x \leq \epsilon$ or $x \geq \epsilon$. It follows, therefore, that

$$\begin{aligned}
&\int_0^\infty \left|\bar{G}_e(x) - x\exp(-x/\mu)\right| \mathrm{d}x \\
&= 2\int_\epsilon^\infty (\bar{G}_e(x) - x\exp(-x/\mu))\,\mathrm{d}x \\
&\geq 2\int_\epsilon^\infty (\bar{F}_e(x) - x\exp(-x/\mu))\,\mathrm{d}x \\
&= 2\int_\epsilon^\infty \left(\frac{1}{\mu}\int_x^\infty \bar{F}(y)\,\mathrm{d}y - x\exp(-x/\mu)\right)\mathrm{d}x \\
&= \frac{2}{\mu}\int_\epsilon^\infty (S_1(x) - \mu x\exp(-x/\mu))\,\mathrm{d}x \\
&= \frac{2}{\mu}(S_2(\epsilon) - \{\mu^2\epsilon\exp(-\epsilon/\mu) + \mu\exp(-\epsilon/\mu)\}) \\
&\geq 2S_1(\epsilon) - 2\exp(-\epsilon/\mu)\{\mu\epsilon + 1\} \\
&\geq 2\mu S_0(\epsilon) - 2\exp(-\epsilon/\mu)\{\mu\epsilon + 1\} \\
&= 2\exp(-\epsilon/\mu)\{\mu - \mu\epsilon - 1\}.
\end{aligned}$$

The first inequality follows from the fact that $\bar{G}_e(x) \geq \bar{F}_e(x)$, for all $x \geq 0$. The last two inequalities follows from the fact that

$$S_k(x) \geq \mu S_{k-1}(x), \quad \forall x \geq 0,\ k \geq 1,$$

where

$$S_k(x) = \int_x^\infty S_{k-1}(u)\,\mathrm{d}u.$$

THEOREM 3.2 *If $\bar{G}_e$ has increasing hazard rate (IHR), then*

$$\int_0^\infty \left|\bar{G}_e(x) - x e^{-x/\mu}\right| dx \le 2\delta,$$

where $\delta = \mu^2 - \mu_2/2\mu$.

Proof Let $D = \{x \mid \bar{G}_e(x) \le x\exp\{-x/\mu\}\}$. Then, for $x > 0$, we have

$$\begin{aligned}
&\int_0^\infty \left|\bar{G}_e(x) - x\exp\{-x/\mu\}\right| dx \\
&= \int_D \{x\exp(-x/\mu) - \bar{G}_e(x)\}\, dx - \int_{D^c} \{x\exp(-x/\mu) - \bar{G}_e(x)\}\, dx \\
&\le 2\int_D \{x\exp(-x/\mu) - \bar{G}_e(x)\}\, dx \\
&\le 2\int_0^\infty \{x\exp(-x/\mu) - \bar{F}_e(x)\}\, dx \\
&= 2\int_0^\infty \left\{x\exp(-x/\mu) - \left(\frac{1}{\mu}\int_x^\infty \bar{F}(y)\, dy\right)\right\} dx \\
&= 2\int_0^\infty \{x\exp(-x/\mu) - S_1(x)/\mu\}\, dx \\
&= 2(\mu^2 - S_2(0)/\mu) \\
&= 2\left(\mu^2 - \frac{\mu_2}{2\mu}\right) \\
&= 2\delta.
\end{aligned}$$

THEOREM 3.3 *Let $\bar{G}_{l_t}$ have DHR, then*

$$\int_0^\infty \left|\bar{G}_{l_t}(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\}\right| dx \ge 2e^{-\epsilon/\mu}\{\alpha + \beta\} \quad \textit{for } \epsilon \ge \mu,$$

where $\alpha = \mu(e^{-t/\mu} - (\epsilon/(\mu + t)) - 1)$ and $\beta = -(\mu + t)^{-1}$.

Proof Let $t > 0$ be fixed and $\bar{G}_{l_t}$ have DHR, then there exist $\epsilon \geq \mu$ such $\bar{G}_{l_t}(x) \leq (\geq)\{1 + x/(\mu + t)\}\exp\{-x/\mu\}$ as $x \leq (\geq)\epsilon$, so that

$$\begin{aligned}
&\int_0^\infty \left|\bar{G}_{l_t}(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\}\right| \mathrm{d}x \\
&= 2\int_\epsilon^\infty (\bar{G}_{l_t}(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\})\,\mathrm{d}x \\
&\geq 2\int_\epsilon^\infty (\bar{F}_t(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\})\,\mathrm{d}x \\
&\geq 2\int_\epsilon^\infty (\bar{F}(x + t) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\})\,\mathrm{d}x \\
&= 2S_1(\epsilon + t) - 2\mu\exp\{-\epsilon/\mu\} \\
&\quad - \frac{2}{(\mu + t)}(\mu\epsilon\exp\{-\epsilon/\mu\} + \exp\{-\epsilon/\mu\}) \\
&\geq 2\mu S_0(\epsilon + t) - 2\mu \mathrm{e}^{-\epsilon/\mu} - \frac{2}{(\mu + t)}(\mu\epsilon \mathrm{e}^{-\epsilon/\mu} + \mathrm{e}^{-\epsilon/\mu}) \\
&= 2\mu \mathrm{e}^{-(\epsilon+t)/\mu} - 2\mu \mathrm{e}^{-\epsilon/\mu} - \frac{2}{(\mu + t)}(\mu\epsilon \mathrm{e}^{-\epsilon/\mu} + \mathrm{e}^{-\epsilon/\mu}) \\
&= 2\mu \mathrm{e}^{-\epsilon/\mu}\left(\mathrm{e}^{-t/\mu} - \frac{\epsilon}{\mu + t} - 1\right) - \frac{2\mathrm{e}^{-\epsilon/\mu}}{\mu + t} \\
&= 2\mathrm{e}^{-\epsilon/\mu}\left\{\mu\left(\mathrm{e}^{-t/\mu} - \frac{\epsilon}{\mu + t} - 1\right) - \frac{1}{\mu + t}\right\} \\
&= 2\mathrm{e}^{-\epsilon/\mu}\{\alpha + \beta\}.
\end{aligned}$$

THEOREM 3.4 *If $\bar{G}_{l_t}$ has IHR, then*

$$\int_0^\infty \left|\bar{G}_{l_t}(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\}\right| \mathrm{d}x \leq 2\mu(1 + \gamma),$$

where $\gamma = \mu/(\mu + t) - (\mu_2/2\mu^2)$.

Proof Let $B = \{x \mid \bar{G}_{l_t}(x) \leq (1 + x/(\mu + t))\exp(-x/\mu)\}$. Then for fixed $t > 0$ and $x > 0$, we have

$$
\begin{aligned}
&\int_0^\infty \left|\bar{G}_{l_t}(x) - \{1 + x/(\mu + t)\}\exp\{-x/\mu\}\right| \mathrm{d}x \\
&= \int_B ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{G}_{l_t}(x))\,\mathrm{d}x \\
&\quad - \int_{B^c} ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{G}_{l_t}(x))\,\mathrm{d}x \\
&\leq 2\int_B ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{G}_{l_t}(x))\,\mathrm{d}x \\
&\leq 2\int_0^\infty ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{G}_{l_t}(x))\,\mathrm{d}x \\
&\leq 2\int_0^\infty ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{F}_t(x))\,\mathrm{d}x \\
&\leq 2\int_0^\infty ((1 + x/(\mu + t))\exp(-x/\mu) - \bar{F}(x + t))\,\mathrm{d}x \\
&\leq 2\int_0^\infty ((1 + x/(\mu + t))\exp(-x/\mu) - S_1(x + t)/\mu)\,\mathrm{d}x \\
&= 2\left(\mu + \frac{\mu^2}{\mu + t} - \frac{\mu_2}{2\mu}\right) \\
&= 2\mu\left(1 + \frac{\mu}{\mu + t} - \frac{\mu_2}{2\mu^2}\right) \\
&= 2\mu(1 + \gamma).
\end{aligned}
$$

4 INEQUALITIES AND STABILITY RESULTS FOR REPAIRABLE SYSTEMS

In this section, we present results and inequalities for repairable systems. Let $\{X_i\}_{i=1}^\infty$ be a sequence of operating times from a repairable system that starts functioning at time $t = 0$. The sequence of times $\{X_i\}_{i=1}^\infty$ forms a renewal-type stochastic point process. Following Kijima [7], if a system has virtual age $T_{m-1} = t$ immediately after the $(m-1)$th repair, then the length of the mth cycle X_m has the distribution function

$$G_t(x) = P(X_m \leq x \mid T_{m-1} = t) = \frac{F(x+t) - F(t)}{\bar{F}(t)}, \quad x \geq 0, \tag{4.1}$$

where $\bar{F}(x) = 1 - F(x)$, is the reliability function of a new system. When $t = \sum_{i=1}^{j} X_i, j = 1, 2, \ldots, m-1$, minimal repair is performed, keeping the virtual age intact and when $t = 0$ we have perfect repair. The virtual age of a system is equal to its operating time for the case of minimal repair.

The reliability function corresponding to the distribution function (4.1) is (see also Brown [3]),

$$\bar{G}_t(x) = \bar{F}(x+t)/\bar{F}(t), \quad x \geq 0. \tag{4.2}$$

THEOREM 4.1 *If $\bar{G}_t$ has DHR, then*

$$\int_0^\infty \left|\bar{G}_t(x) - e^{-x/\mu}\right| dx \geq 2\mu e^{-\epsilon/\mu}(e^{-t/\mu} - 1).$$

Proof Let $\bar{G}_t$ have DHR, then there exist $\epsilon \geq \mu$ such that $\bar{G}_t(x) \geq e^{-x/\mu}$ or $\bar{G}_t(x) \leq e^{-x/\mu}$ as $x \geq \epsilon$ or $x \leq \epsilon$. We have

$$\begin{aligned}\int_0^\infty \left|\bar{G}_t(x) - e^{-x/\mu}\right| dx &= 2\int_\epsilon^\infty (\bar{G}_t(x) - e^{-x/\mu})\, dx \\ &\geq 2\int_\epsilon^\infty (\bar{F}(x+t) - e^{-x/\mu})\, dx \\ &= 2(S_1(\epsilon + t) - \mu e^{-\epsilon/\mu}) \\ &\geq 2\mu S_0(\epsilon + t) - 2\mu e^{-\epsilon/\mu} \\ &= 2\mu e^{-(\epsilon+t)/\mu} - 2\mu e^{-\epsilon/\mu} \\ &= 2\mu e^{-\epsilon/\mu}(e^{-t/\mu} - 1).\end{aligned}$$

THEOREM 4.2 *If $\bar{G}_t$ has IHR, then*

$$\int_0^\infty \left|\bar{G}_t(x) - e^{-x/\mu}\right| dx \leq 2\mu(1 - \mu_2/2\mu^2).$$

Proof Let $B = \{x \mid \bar{G}_t(x) \leq e^{-x/\mu}\}$. Then for $x > 0$,

$$\begin{aligned}
\int_0^\infty |\bar{G}_t(x) - e^{-x/\mu}|\,dx &\leq 2\int_B (e^{-x/\mu} - \bar{G}_t(x))\,dx \\
&\leq 2\int_0^\infty (e^{-x/\mu} - \bar{F}(x+t))\,dx \\
&\leq 2\int_0^\infty (e^{-x/\mu} - \bar{F}(x+t))\,dx \\
&\leq 2\int_0^\infty \left(e^{-x/\mu} - \frac{S_1(x+t)}{\mu}\right)dx \\
&= 2(e^{x/\mu} - S_2(0)/\mu) \\
&= 2(\mu - \mu_2/2\mu) \\
&= 2\mu(1 - \mu_2/2\mu^2).
\end{aligned}$$

The results given in previous sections concerning the length biased equilibrium survival functions also applies to the original equilibrium survival function. The results are stated below. The proofs are similar to the proof of Theorems 3.1 and 3.2 respectively.

THEOREM 4.3 *Let $\bar{F}_e$ have DHR, then*

$$\int_0^\infty |\bar{F}_e(x) - e^{-x/\mu}|\,dx \geq 2\mu e^{-\epsilon/\mu}(1-\mu).$$

Proof We have by virtue of $\bar{F}_l$ having DHR, that

$$\begin{aligned}
\int_0^\infty |\bar{F}_e(x) - e^{-x/\mu}|\,dx &= 2\int_\epsilon^\infty (\bar{F}_e(x) - e^{-x/\mu})\,dx, \quad \text{for } \epsilon \geq \mu, \\
&= \frac{2}{\mu}\int_\epsilon^\infty (S_1(x) - \mu e^{-x/\mu})\,dx \\
&= \frac{2}{\mu}(S_2(\epsilon) - \mu(\mu e^{-\epsilon/\mu})) \\
&\geq 2S_1(\epsilon) - 2\mu^2 e^{-\epsilon/\mu} \\
&\geq 2\mu S_0(\epsilon) - 2\mu^2 e^{-\epsilon/\mu} \\
&= 2\mu e^{-\epsilon/\mu}(1-\mu).
\end{aligned}$$

The inequalities follows from the fact that

$$S_k(x) \geq \mu S_{k-1}(x), \quad \forall x \geq 0, \ k \geq 1.$$

THEOREM 4.4 *If $\bar{F}_e$ has IHR, then*

$$\int_0^\infty |\bar{F}_e(x) - e^{-x/\mu}| \, dx \leq 2\mu - \frac{\mu_2}{\mu}.$$

Proof Let $D = \{x \mid \bar{F}_e(x) \leq e^{-x/\mu}\}$. Then for $x > 0$,

$$\begin{aligned}
\int_0^\infty |\bar{F}_e(x) - e^{-x/\mu}| \, dx &\leq 2 \int_D (e^{-x/\mu} - \bar{F}_e(x)) \, dx \\
&\leq 2 \int_0^\infty \left(e^{-x/\mu} - \left(\frac{1}{\mu} \int_x^\infty \bar{F}(y) \, dy \right) \right) dx \\
&= 2\mu - 2S_2(x)/\mu \\
&= 2\mu - \frac{\mu_2}{\mu}.
\end{aligned}$$

5 ESTIMATION FROM CENSORED DATA

In this section we consider estimation of the length biased density function and related functions with and without censoring. Vardi [17] derived the nonparametric maximum likelihood estimate (NPMLE), $\hat{F}$ of F on the basis two independent samples of sizes m and n from $F(y)$ and $G(y) = 1/\mu_F \int_0^y x \, dF(x)$, respectively. As in Section 2, let the random variable X_i be censored on the right by the random variable C_i, leading to the observation of only $T_i = X_i \wedge C_i$ and $\delta_i = I(X_i \leq C_i)$, where $\wedge$ denotes minimum and $I(\cdot)$ is the indicator random variable of the event in parenthesis. The censoring times C_i, $i = 1, 2, \ldots, n$ are assumed independent and identically distributed (iid) and independent of X_i, $i = 1, 2, \ldots, n$.

Consider the well known uncensored kernel density estimator of f (Padgett and McNichols [11])

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n} \right), \tag{5.1}$$

where $\{h_n; n \geq 1\}$ is a sequence of positive constants tending to zero, K is a known pdf (kernel). A natural estimator of g, the length biased pdf based on the complete set of observations is

$$g_n(x) = \frac{xf_n(x)}{\hat{\mu}_n}, \tag{5.2}$$

where $\hat{\mu}$ is an estimator of μ, and a possible estimator is given by $\hat{\mu}_n = n/\sum_{i=1}^{n} X_1^{-1}$. Another estimator is $\bar{X}_n$ since $\hat{\mu}_n \leq \bar{X}_n$.

Let $W_n(x) = \sum_{i=1}^{n} I(T_i \geq x)$ and $M_n(x) = \sum_{i=1}^{n} I(T_i \leq x \wedge C_i)$, then $W_n(x)$ and $M_n(x)$ are the number of observations censored or uncensored greater than or equal to x and the number of uncensored observations less than or equal to x respectively.

An estimator of g based on the censored data $\{(T_i, \delta_i)\}_{i=1}^{n}$ is given by

$$g_n(x) = \frac{x\tilde{f}_n(x)}{\tilde{\mu}_n}, \tag{5.3}$$

where

$$\tilde{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x - T_i}{h_n}\right)p_i, \quad p_i = \hat{F}_n^*(T_{(i)}) - \hat{F}_n^*(T_{(i-1)}),$$

$$p_1 = \hat{F}_n(T_{(1)}), \quad i = 2, \ldots, n,$$

$$\hat{\bar{F}}_n^*(x) = 1 - \hat{F}_n^*(x) = \prod_{i:T_{(i)} \leq x < T_{(n)}} \left(\frac{n-i}{n-i+1}\right)^{\delta_i},$$

$\tilde{\mu}_n = n/\sum_{i=1}^{n} T_i^{-1}$ and $T_{(i)}$ is the order statistic of T_i. Note that $\hat{\bar{F}}^*(x)$ is the Kaplan–Meier (K–M) estimator of $\hat{F}(x)$. Note that an empirical estimator of the length biased cummulative hazard function is the well known Nelson estimator and is given by

$$\hat{\Lambda}_n(x) = \int_{-\infty}^{x} \frac{\mathrm{d}M_n(y)}{W_n(y)}. \tag{5.4}$$

In estimating the hazard rate $\gamma_G(x)$ by $\hat{\gamma}_G(x)$ we may use (4.4) or use the fact that

$$\tilde{G}_n(x) = \int_0^x g_n(y)\,\mathrm{d}y, \tag{5.5}$$

where $g_n(x)$ is given by (5.3) and estimate $\gamma_G(x)$ as

$$\tilde{\gamma}_G(x) = g_n(x)/\bar{\tilde{G}}_n(x), \tag{5.6}$$

by consider values of x for which $\bar{\tilde{G}}(x) > 0$.

THEOREM 5.1 *Under appropriate assumptions* [9,12,15], *and if* $\sqrt{hn_n}(\tilde{\mu}_n - \mu) \xrightarrow{P} 0$, *then*

$$(nh)^{1/2}(g_n(x) - E(g_n(x))) \xrightarrow{D} N(0, \sigma^2(x)),$$

for $x \in [0, a]$, $a < \infty$, *where*

$$\sigma^2(x) = g(x)\int_{\Re} K^2(u)\,\mathrm{d}y. \tag{5.7}$$

Proof Follows from Parzen [12], p. 1073 and Roussas [15], Theorem 3.1. The asymptotic variance of $g_n(x)$ is

$$\frac{g(x)\int_{\Re} K^2(y)\,\mathrm{d}y}{nh_n}.$$

If however, (T_i, δ_i), $i = 1, 2, \ldots, n$ is available from

$$G(x) = \frac{1}{\mu_F}\int_0^x y\,\mathrm{d}F(y), \tag{5.8}$$

$0 < \mu_F < \infty$, then an estimate of f is given by

$$\tilde{f}_n^*(x) = \tilde{\mu}_n^* \tilde{g}_n^*(x)/x, \tag{5.9}$$

where

$$\tilde{g}_n^*(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x - T_i}{h_n}\right) q_i,$$

$$q_i = \hat{G}_n^*(T_{(i)}) - \hat{G}_n^*(T_{(i-1)}), \quad i = 2, 3, \ldots, n,$$

$$q_1 = \hat{G}_n^*(T_{(1)}), \quad \tilde{\mu}_n^* = n \Big/ \sum_{i=1}^{n} T_i^{-1}$$

or any consistent estimator of μ and

$$\hat{\bar{G}}_n^*(x) = \prod_{i:T_{(i)} \leq x} \left(\frac{n - i}{n - i + 1}\right)^{\delta_i}, \quad x \leq T_{(n)}. \tag{5.10}$$

THEOREM 5.2 *If $\tilde{\mu}_n^*$ is a consistent estimator of μ, then $\tilde{f}_n^*$ is a consistent estimator of $f(x)$, provided $\tilde{g}_n^*(x) \xrightarrow{P} g(x)$, where $\xrightarrow{P}$ denote convergence in probability.*

Remark 5.1 (i) Note that $\tilde{\mu}_n^* = n/\sum_{i=1}^{n} T_i^{-1} \leq \hat{\mu}_n = \sum_{i=1}^{n} T_i/n$, is a consistent estimator of μ. If T_i^{-1} has finite second moment, then

$$\sqrt{nh_n}(\tilde{\mu}_n^* - \mu) \xrightarrow{P} 0. \tag{5.11}$$

(ii) If $E(T_i^{2S}) < \infty$, where S is a natural number, then

$$E(\tilde{f}_n^*(x) - f(x))^2 \to 0 \quad \text{as } n \to \infty,$$

where

$$\tilde{f}_n^*(x) = n\tilde{g}_n^*(x) \Big/ \left(x \sum_{i=1}^{n} T_i^{-1}\right). \tag{5.12}$$

6 BAYESIAN LENGTH BIASED EXPONENTIAL RELIABILITY ESTIMATION

In this section, estimates of the length biased exponential reliability function are presented. The maximum likelihood estimate, Bayes estimator, Pitman estimate are presented. For the Bayes estimate, direct calculation of the risk of estimators of reliability function is often impossible and numerical calculation is very difficult. Asymptotic approximation or numerical computation might be possible but is not objective of this manuscript and is not given here. Some results on asymptotic approximation for risk of the exponential reliability function calculated with respect to the quadratic loss function are given by Hurt *et al.* [6], Chao [4], among others. See also (Martz and Waller [8], Padgett [10]). The difficulty in the calculation of the risk of the exponential reliability function is compounded by the fact that computations depends on sampling distribution, prior distribution, type of estimates and choice of loss function.

It is well known that in the case of exponential distribution, the asymptotic properties of the Bayes risk for the typical estimators of the reliability function indicates that the best estimator is the Bayes estimator.

The MLE of the reliability function in (1.2) is given by

$$\hat{\bar{G}}(y;\hat{\theta}) = (1 + y/\hat{\theta})\exp(-y/\hat{\theta}), \\ = (1 + \hat{\lambda}y)\exp(-\hat{\lambda}y), \tag{6.1}$$

where $\hat{\theta}$ and hence $\hat{\lambda}$ is given by (2.1).

Substituting the Pitman estimate $\hat{\lambda}_{\mathrm{P}}$ of λ in (1.2), we get the estimate

$$\hat{G}_{\mathrm{P}}(y;\hat{\lambda}_{\mathrm{P}}) = (1 + \hat{\lambda}_{\mathrm{P}}y)\exp(-\hat{\lambda}_{\mathrm{P}}y), \tag{6.2}$$

where $\hat{\lambda}_{\mathrm{P}}$ is given by (2.4). The Bayes estimator of $\bar{G}(y;\lambda)$ is obtained from

$$\begin{aligned} &E[\bar{G}(y;\lambda)\,|\,\mathbf{X}] \\ &= \frac{\int_0^\infty \mathrm{e}^{-\lambda y}\{1+\lambda y\}\lambda^{2n}\left(\prod_{i=1}^n x_i\right)\mathrm{e}^{-\lambda S_n}(\lambda^{2n}/\Gamma(2n))S_n^{2n-1}\mathrm{e}^{-\lambda S_n}\,\mathrm{d}\lambda}{\int_0^\infty \lambda^{2n}\left(\prod_{i=1}^n x_i\right)\mathrm{e}^{-\lambda S_n}(\lambda^{2n}/\Gamma(2n))S_n^{2n-1}\mathrm{e}^{-\lambda S_n}\,\mathrm{d}\lambda} \\ &= \frac{\int_0^\infty \lambda^{4n}\mathrm{e}^{-\lambda(y+2S_n)}\,\mathrm{d}\lambda + y\int_0^\infty \lambda^{4n+1}\mathrm{e}^{-\lambda(y+2S_n)}\,\mathrm{d}\lambda}{\int_0^\infty \lambda^{4n}\mathrm{e}^{2\lambda S_n}\,\mathrm{d}\lambda} \\ &= \left(\frac{2S_n}{y+2S_n}\right)^{4n+1}\left\{\frac{4ny+2y+2S_n}{y+2S_n}\right\}. \end{aligned}$$

Consequently, the Bayes estimator of $\bar{G}(y; \lambda)$ is given by

$$\hat{\bar{G}}_{\mathrm{B}}(y; \lambda) = \left(\frac{2\sum_{i=1}^{n} X_i}{y + 2\sum_{i=1}^{n} X_i}\right)^{4n+1} \left\{\frac{4ny + 2y + 2\sum_{i=1}^{n} X_i}{y + 2\sum_{i=1}^{n} X_i}\right\}$$

$$= \left(1 + \frac{4ny + y}{2n\sum_{i=1}^{n} X_i}\right)\left\{\frac{1}{1 + y/2\sum_{i=1}^{n} X_i}\right\}^{4n+1}. \tag{6.3}$$

From (6.1), we get a conservative estimate of $\bar{G}(y; \lambda)$ given by

$$\hat{\bar{G}}_c(y; \hat{\lambda}) = \exp\{-\hat{\lambda} y\}. \tag{6.4}$$

This follows from the fact that $F \overset{st}{\leq} G$, that is $\bar{G}(y) \geq \bar{F}(y)$ (Gupta and Keating [5]), for all $y \geq 0$. The estimate of the length biased exponential reliability function under random censoring is given by (2.15).

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