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# Rademacher-Type Series

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# RADEMACHER-TYPE SERIES

by

**V. KIRIA–KAISERBERG**

(Under the Direction of Andrew Sills)

## ABSTRACT

In this thesis we use the classical circle method to find the coefficients $\tilde{p}(n)$ of $\sum_{n=0}^{\infty} \tilde{p}(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2}$. We introduce a Mathematica package which automates steps in the the circle method and computes the Rademacher-type series expressions for the Fourier coefficients of other Ramanujan-type series.

INDEX WORDS: Partitions, q-series, Circle method.

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# PARTITIONS AND GENERATING FUNCTIONS

## 1.1 Partitions

A *partition* of a positive integer $n$ is a representation of $n$ as a sum of positive integers where order of summands (parts) does not matter. For example, all the partitions of 4 are

4

3+1

2+2

2+1+1

1+1+1+1,

so, we have five partions of 4. The eleven partitions of 6 are

6

5+1

4+2

4+1+1

3+3

3+2+1

3+1+1+1

2+2+2

2+2+1+1

2+1+1+1+1

1+1+1+1+1+1.

Let $p(n)$ represent the number of partitions of $n$. Some values of $p(n)$ are given below:

$p(0) = 1$

$p(1) = 1$

$p(2) = 2$

$p(3) = 3$

$p(4) = 5$

$p(5) = 7$

$p(6) = 11$

$p(7) = 15$

$p(8) = 22$

$p(9) = 30$

$p(10) = 42$

$p(100) = 190,569,192$

$p(200) = 3,972,999,029,388$

As could be seen from the above list, $p(n)$ grows rapidly as $n$ increases. However, the pattern of growth is not obvious and a question was raised whether there is a closed formula for the partition function $p(n)$. G. H. Hardy and S. Ramanujan [9] in 1918 obtained the following asymptotic expression for the partition function

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}}, \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where $a(n) \sim b(n)$ as $n \to \infty$ means

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1.$$

Later, in 1937, H. Rademacher [10, 11] was able to express $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right), \tag{1.2}$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{\pi i(s(h,k)-2nh/k)},$$

is a Kloosterman sum, and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right)$$

is a Dedekind sum.

In 2011, Ken Ono and Jan Bruinier [4] announced a new formula that expresses $p(n)$ as a *finite* sum.

## 1.2 Generating functions

In the theory of integer partitions an important role is played by *generating functions*. The *generating function* of a sequence $b_n$ is a formal power series in one unknown, whose coefficients are the $b_n$, i.e.,

$$b_0 + b_1x + b_2x^2 + b_3x^3 + ... = \sum_{n=0}^{\infty} b_nx^n \tag{1.3}$$

is a generating function for the sequence $\{b_n\}_{n=0}^{\infty}$. L. Euler proved [6] that the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{1.4}$$

From elementary calculus we know that

$$\sum_{n=0}^{\infty} (x^k)^n = \frac{1}{1-x^k}, \quad |x| < 1,\ k \in \mathbb{N}. \tag{1.5}$$

Expanding right hand side of (1.4) we have that

$$\begin{aligned}
\prod_{k=1}^{\infty} \frac{1}{1-x^k} &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\cdots \\
&= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^{2n}\right)\left(\sum_{n=0}^{\infty} x^{3n}\right)\left(\sum_{n=0}^{\infty} x^{4n}\right)\cdots \\
&=(1+x+x^2+x^3+x^4+x^5+x^6+\cdots) \\
&\times(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+\cdots) \\
&\times(1+x^3+x^6+x^9+x^{12}+x^{15}+x^{18}+\cdots) \\
&\times(1+x^4+x^8+x^{12}+x^{16}+x^{20}+x^{24}+\cdots)\cdots \\
&=(1+x+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^{1+1+1+1+1}+\cdots) \\
&\times(1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+x^{2+2+2+2+2}+\cdots) \\
&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+x^{3+3+3+3+3}+\cdots) \\
&\times(1+x^4+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+x^{4+4+4+4+4}+\cdots)\cdots \\
&=1+x+(x^{1+1}+x^2)+(x^{1+1+1}+x^{1+2}+x^3) \\
&+(x^{1+1+1+1}+x^{1+1+2}+x^{2+2}+x^{1+3}+x^4)+\cdots \\
&=1+x+2x^2+3x^3+5x^4+7x^5+11x^6+\cdots \\
&=\sum_{n=0}^{\infty} p(n)x^n. \qquad (1.6)
\end{aligned}$$

This is exactly the generating function for $p(n)$ because coefficient of $x^n$ is the number of partions of $n$. As could be seen from this example, generating functions by themselves tell us nothing about how to find the coefficient of $x^n$ in the series expansion. We need some additional tools to extract coefficients for each $n$. The Circle Method, developed by G. Hardy and S. Ramanujan and improved by H. Rademacher, will allow us to do precisely this. In chapter 2 we use this method to find the formula for the coefficient of $x^n$ for one of the Ramanujan-type identities.

# CHAPTER 2

## Formula for $\tilde{p}(n)$

Let

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m} \tag{2.1}$$

be Euler's generating function for $p(n)$. H. Rademacher used the classical circle method to find the coefficients of $q^n$. There are many other infinite products to which this method could be applied. We introduce one of these infinite products here and derive the formula for the coefficients of $q^n$. Define

$$G(q) := \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \frac{(f(q^2))^2}{f(q)}. \tag{2.2}$$

Let $\tilde{p}(j)$ denote the coefficient of $q^j$ in the expansion of $G(q)$, i.e.

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j. \tag{2.3}$$

Our goal is to find a closed expression for $\tilde{p}(j)$. We are about to use the classical circle method to achieve this goal. It is a long, tedious calculation, many steps of which have been implemented in a Mathematica package written by the author and his advisor. See Appendix A for the Mathematica code for this package. At key points of this chapter, the Mathematica package is demonstrated with input and output enclosed in a box.

We begin with some preliminaries.

**Farey fractions.** The sequence $\mathcal{F}_N$ of *proper Farey fractions of order N* is the set of all $h/k$ with $(h,k)=1$ and $0 \le h/k < 1, k \le N$ arranged in increasing order, where $(h,k)$ denotes the gcd$(h,k)$. Thus, e.g., $\mathcal{F}_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$.

For a given $N$, let $h_p$, $h_s$, $k_p$ and $k_s$ be such that $\dfrac{h_p}{k_p}$ is the immediate predecessor of

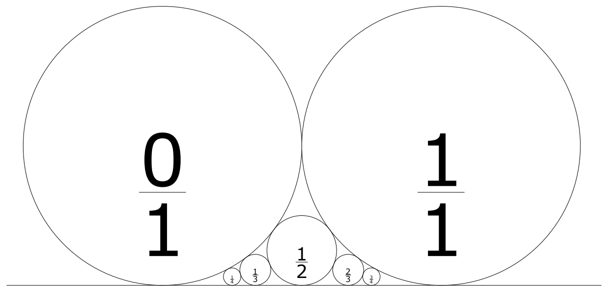


Figure 2.1: Ford Circles of order 4.

$\frac{h}{k}$ and $\frac{h_s}{k_s}$ is the immediate successor of $\frac{h}{k}$ in $\mathcal{F}_N$ and $\frac{h}{k} = \frac{h_p + h_s}{k_p + k_s}$.

**Ford circles and Rademacher path.** Let $h$ and $k$ be as above. The *Ford circle* $C(h, k)$ [7] is the circle in $\mathbb{C}$ of radius $\frac{1}{2k^2}$ centered at the point

$$\frac{h}{k} + \frac{1}{2k^2} i.$$

The *upper arc* $\gamma(h, k)$ *of the Ford circle* $C(h, k)$ is those points of $C(h, k)$ from the initial point

$$I_{h,k} := \frac{h}{k} - \frac{k_p}{k(k^2 + k_p^2)} + \frac{1}{k^2 + k_p^2} i \tag{2.4}$$

to the terminal point

$$T_{h,k} := \frac{h}{k} + \frac{k_s}{k(k^2 + k_s^2)} + \frac{1}{k^2 + k_s^2} i \tag{2.5}$$

traversed in the negative direction. We note that

$$I_{0,1} = T_{N-1,N}.$$

Every Ford circle is in the upper half plane. For $\frac{h_1}{k_1}, \frac{h_2}{k_2} \in \mathcal{F}_N$, $C(h_1, k_1)$ and $C(h_2, k_2)$ are either tangent or do not intersect [7].

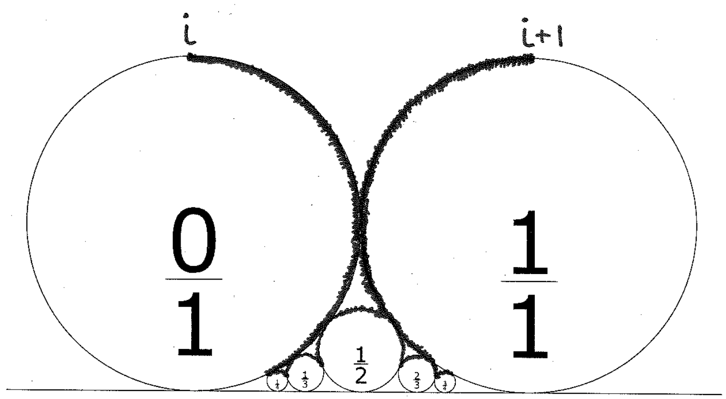


Figure 2.2: Rademacher path of order 4.

The *Rademacher path P(N) of order N* is the path in the upper half of the $\tau$-plane from $i$ to $i+1$ consisting of

$$\bigcup_{\frac{h}{k}\in\mathcal{F}_N} \gamma(h,k) \tag{2.6}$$

traversed left to right in negative direction.

We note that the left half of the Ford circle $C(0,1)$ and the corresponding upper arc $\gamma(0,1)$ are shifted to the right by 1 unit. This is justified since the function that is to be integrated over Rademacher path is periodic.

**Convergence and Cauchy Residue Theorem.**

Considering $q$ as a complex variable in

$$\prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \prod_{m=1}^{\infty} \frac{1}{(1+q^m)(1-q^{2m})} = \prod_{m=1}^{\infty} \frac{1}{(1-(-q^m))(1-q^{2m})} \tag{2.7}$$

we see from the right hand side that infinite product and thus also infinite series are convergent for $|q| < 1$ since

$$\sum_{n=0}^{\infty} (q^k)^n = \frac{1}{(1-q^k)} \tag{2.8}$$

is a geometric series which converges for $|q| < 1$ for any fixed $k \geq 1$.

Next, we note that from

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j) q^j, \tag{2.9}$$

we get that

$$\frac{G(q)}{q^{n+1}} = \sum_{j=0}^{\infty} \frac{\tilde{p}(j) q^j}{q^{n+1}} \quad \text{if} \quad 0 < |q| < 1. \tag{2.10}$$

The series on the right of (2.10) is a Laurent series of $\frac{G(q)}{q^{n+1}}$. It has a pole of order $n+1$ at $q = 0$ with residue $\tilde{p}(n)$. Applying Cauchy's Residue Theorem we get that

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_C \frac{G(q)}{q^{n+1}} \, dq = \frac{1}{2\pi i} \int_C \frac{(f(q^2))^2}{f(q) q^{n+1}} \, dq, \tag{2.11}$$

where $C$ is any positively oriented simple closed countour lying inside the unit circle. The change of the variable $q = e^{2\pi i \tau}$ maps the unit disk $|q| < 1$ into an infinite vertical strip of width 1 in the $\tau$-plane. To see this we note that from $q = e^{2\pi i \tau}$ we get $\log q = 2\pi i \tau$, so $\tau = \frac{\log q}{2\pi i}$. Choosing the branch cut to be [0,1], we get

$$\tau = \frac{\log |q|}{2\pi i} + \frac{\operatorname{Arg}(q)}{2\pi}. \tag{2.12}$$

As $q$ traverses a circle centered at $q = 0$ of radius $e^{-2\pi}$ in the positive direction, the point $\tau$ varies from $i$ to $i+1$ along a horizontal segment as could be easly deduced from (2.12).

Replacing the segment by the Rademacher path composed of upper arcs of the Ford circles formed by the Farey series $\mathcal{F}_N$, (2.11) becomes

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_i^{i+1} \frac{(f(e^{4\pi i \tau}))^2 2\pi i e^{2\pi i \tau}}{f(e^{2\pi i \tau}) e^{2\pi i \tau (n+1)}} \, d\tau, \tag{2.13}$$

which simplifies to

$$\tilde{p}(n) = \int_{i}^{i+1} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau, \tag{2.14}$$

$$= \int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau, \tag{2.15}$$

The above can be written as

$$\int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau = \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ (h,k)=1}} \int_{\gamma(h,k)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau \tag{2.16}$$

where $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$. Consider another change of variable

$$\tau = \frac{h}{k} + \frac{iz}{k}, \tag{2.17}$$

so that

$$z = -\, ik\left(\tau - \frac{h}{k}\right) \tag{2.18}$$

$$dz = -\, ik\, d\tau. \tag{2.19}$$

Under this transformation the Ford circle $C(h,k)$ in the $\tau$-plane with center at $\frac{h}{k} + i\frac{1}{2k^2}$ and radius $\frac{1}{2k^2}$ is mapped to a negatively oriented circle $C_k$ in the $z$-plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$. This follows from the fact that any point on the Ford circle $C(h,k)$ is given by

$$\tau = \left(\frac{h}{k} + i\frac{1}{2k^2}\right) + \frac{1}{2k^2}e^{i\theta}, \quad 0 \le \theta < 2\pi. \tag{2.20}$$

Substitution of (2.20) into (2.18) gives

$$z = \frac{1}{2k} + \frac{1}{2k}(-ie^{i\theta}) \tag{2.21}$$

which is a circle centered at $\frac{1}{2k}$ with radius $\frac{1}{2k}$. Now we make change of variable in (2.16). This gives

$$\tilde{p}(n) = i\sum_{k=1}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \frac{(f(e^{4\pi i h/k-4\pi i z/k}))^2}{f(e^{2\pi i h/k-2\pi z/k})} e^{2\pi n z/k}\, dz \tag{2.22}$$

where the initial point

$$s_{h,k} = \frac{k}{k^2+k_p^2} + \frac{k_p}{k^2+k_p^2} i \tag{2.23}$$

and the terminal point

$$t_{h,k} = \frac{k}{k^2+k_s^2} - \frac{k_s}{k^2+k_s^2} i \tag{2.24}$$

both obtained from (2.4), (2.5) and (2.18) through change of variable.

Next, we note that

$$f(q) = f(e^{2\pi i\tau}) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}, \tag{2.25}$$

where $\eta(\tau)$ is the Dedekind eta function. Rewriting modular functional equation [3, p. 96] for $\eta(\tau)$ in terms of $f(q) = f(e^{2\pi i\tau}) = f(e^{2\pi i h/k-2\pi z/k})$ we get

$$f(e^{2\pi i h/k-2\pi z/k}) = \omega(h,k)\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)\sqrt{z} f\left(\exp\left(2\pi i\frac{iz^{-1}+H}{k}\right)\right) \tag{2.26}$$

where

$$\omega(h,k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right)\right), \tag{2.27}$$

and $hH \equiv -1 \pmod{k}$, $(h,k)=1$.

To evaluate (2.22) we would like to express

$$G(q) = G(e^{2\pi i\tau}) = G(e^{2\pi i h/k-2\pi z/k}) = \frac{(f(e^{4\pi i h/k-4\pi i z/k}))^2}{f(e^{2\pi i h/k-2\pi z/k})} \tag{2.28}$$

in the same way we did for $f(q)$ above. Two cases have to be considered: $(k,2)=1$ and $(k,2)=2$. When $(k,2)=1$ we will replace $h$ by $2h$ and $z$ by $2z$, and when $(k,2)=2$, $k$ will be replaced by $k/2$ in order to obtain $f(q^2)$ from $f(q)$. Hence, we have

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}\dfrac{\omega^2(2h,k)e^{\frac{\pi((2z)^{-1}-2z)}{6k}}2zf^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{\omega(h,k)e^{\frac{\pi(z^{-1}-z)}{12k}}\sqrt{z}f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if} \quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)e^{\frac{\pi(z^{-1}-z)}{3k}}zf^2(e^{4\pi i(iz^{-1}+H_1)/k})}{\omega(h,k)e^{\frac{\pi(z^{-1}-z)}{12k}}\sqrt{z}f(e^{2\pi i(iz^{-1}+H_1)/k})}, & \text{if} \quad (k,2)=2,\end{cases} \tag{2.29}$$

which simplifies to

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}2\dfrac{\omega^2(2h,k)}{\omega(h,k)}e^{-\frac{\pi z}{4k}}\sqrt{z}\dfrac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if} \quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)}{\omega(h,k)}e^{\frac{\pi(z^{-1}-z)}{4k}}\sqrt{z}\dfrac{f^2(e^{4\pi i(iz^{-1}+H_1)/k})}{f(e^{2\pi i(iz^{-1}+H_1)/k})}, & \text{if} \quad (k,2)=2,\end{cases} \tag{2.30}$$

or, equivalently

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}2\dfrac{\omega^2(2h,k)}{\omega(h,k)}e^{-\frac{\pi z}{4k}}\sqrt{z}\dfrac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if} \quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)}{\omega(h,k)}e^{\frac{\pi(z^{-1}-z)}{4k}}\sqrt{z}G(e^{2\pi i(iz^{-1}+H_1)/k}), & \text{if} \quad (k,2)=2,\end{cases} \tag{2.31}$$

where $hH_j\equiv -1 \pmod{k}$ and $j|H_j$ for $j=1,2$.

The implementation of the above in Mathematica is:

```
In[1]:= ModularRHS[h, k, z, ω, f, qprime,{{1,-1},{2,2}}]
```

$$\texttt{Out[1]=} \begin{pmatrix} 1, & \left\{\frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{z}, -\frac{\pi z}{4k}, \frac{f(\sqrt{\texttt{qprime}})^2}{f(\texttt{qprime})}\right\} \\ 2, & \left\{\frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \sqrt{z}, \frac{\pi}{4kz} - \frac{\pi z}{4k}, \frac{f\left(\texttt{qprime}^2\right)^2}{f(\texttt{qprime})}\right\} \end{pmatrix}$$

We return to evaluation of (2.22). To proceed we note that

$$G(e^{2\pi i(iz^{-1}+H_1)/k}) = 1 + \{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\}. \tag{2.32}$$

Rewriting (2.22) in terms of (2.31) and (2.32) we obtain

$$\begin{aligned}
\tilde{p}(n) =& i \sum_{\substack{k=1 \\ (k,2)=1}}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} 2\frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})}\, dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{1 + (G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1)\} \\
&\times e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})}\, dz \\
=& 2i \sum_{\substack{k=1 \\ (k,2)=1}}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})}\, dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} (J_1(h,k) + J_2(h,k)),
\end{aligned} \tag{2.33}$$

where

$$J_1(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})}\, dz \tag{2.34}$$

and

$$J_2(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})}\, dz. \tag{2.35}$$

We will estimate the first term in (2.33) and will show that it is small for large $N$. To do this we change variable again by letting $\xi = zk$. Then the first term in (2.33) becomes

$$2i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\, d\xi \tag{2.36}$$

where

$$s^*_{h,k} = \frac{k^2}{k^2+k_p^2} + \frac{kk_p}{k^2+k_p^2} i \tag{2.37}$$

and

$$t^*_{h,k} = \frac{k^2}{k^2+k_s^2} - \frac{kk_s}{k^2+k_s^2} i \tag{2.38}$$

are initial and terminal points obtained from (2.23) and (2.24) respectively. Under this change of variable circle $C_k$ in $z$-plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$ is mapped to a circle $C^*_k$ in $\xi$-plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$. Note also that the mapping $w = \frac{1}{\xi}$ maps the circle $C^*_k$ and its interior onto a half-plane $\Re(w) \ge 1$ (where $\Re(w)$ denotes the real part of complex variable $w$ and $\Im(w)$ is the imaginary part). For, from elementary complex analysis we have that $\Re(w) = \frac{x}{x^2+y^2}$ and $\Im(w) = \frac{-y}{x^2+y^2}$, where $x + iy = \xi$. It is readily seen that the segment $0 < x \le 1$ in the $\xi$-plane is mapped to an infinite strip $[1,\infty)$ in the $w$-plane. So, it follows that inside and on the circle $C^*_k$ we have that $0 < \Re(\xi) \le 1$ and $\Re\left(\frac{1}{\xi}\right) \ge 1$. We now show that $\Re\left(\frac{1}{\xi}\right) = 1$

on the circle $C_k^*$ . To see this note that in the polar form $\xi = \frac{1}{2} + \frac{1}{2}e^{i\theta}$ on $C_k^*$, $0 \le \theta \le 2\pi$. From this we get that

$$\begin{aligned}
\frac{1}{\xi} =& \frac{2}{1+e^{i\theta}} = \frac{2}{(1+\cos\theta)+i\sin\theta} \\
=& \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta} \\
=& \frac{2(1+\cos\theta)}{2+2\cos\theta} - i\frac{2\sin\theta}{2+2\cos\theta} \\
=& 1 - i\frac{\sin\theta}{1+\cos\theta}.
\end{aligned} \tag{2.39}$$

so, $\Re\left(\frac{1}{\xi}\right) = 1$.

Furthermore, we may move path of integration from the arc joining $s_{h,k}^*$ and $t_{h,k}^*$ to a segment connecting these two points on the circle $C_k^*$. By [3, p. 104, Theorem 5.9] the length of the path of integration is bounded by $2\sqrt{2}k/N$, and on the segment connecting $s_{h,k}^*$ and $t_{h,k}^*$, $|\xi| < \sqrt{2}k/N$.

Next, let us define $\tilde{p}^*(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^*(m)q^m = \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})}. \tag{2.40}$$

which is a part of the integrand in (2.36). Then, estimating the integrand in (2.36) we get

$$
\begin{aligned}
&\left|\sqrt{\xi}e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|-1+\frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})}\right| \\
&= |\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(\frac{2\pi im(i(\frac{\xi}{k})^{-1}+H_2)}{k}\right)\right| \\
&= |\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(-\frac{2\pi m}{\xi}\right)\exp\left(\frac{2\pi imH_2}{k}\right)\right| \\
&\leq |\xi|^{1/2}\, e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}\left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(-\frac{2\pi m}{\xi}\right)\exp\left(\frac{2\pi imH_2}{k}\right)\right| \\
&\leq |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\exp\left(-2\pi m\,\Re\left(\frac{1}{\xi}\right)\right) \\
&\leq |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, e^{-2\pi m} \\
&= |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, y^m, \quad (\text{where} \quad y = e^{-2\pi}) \\
&= c|\xi|^{1/2}, \qquad (2.41)
\end{aligned}
$$

where

$$
c = e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, y^m. \qquad (2.42)
$$

Note that $c$ does not depend on $\xi$ or $N$. It depends on $n$, but $n$ remains fixed in the above analysis. So,

$$
\begin{aligned}
\left|\int_{s^*(h,k)}^{t^*(h,k)}\sqrt{\xi}\frac{f^2\big(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k}\big)}{f\big(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k}\big)}e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\,d\xi\right| &\leq c|\xi|^{1/2} \leq c\left(\frac{\sqrt{2}k}{N}\right)^{1/2}\frac{2\sqrt{2}N}{N} \\
&< \alpha k^{3/2}N^{-3/2} \qquad (2.43)
\end{aligned}
$$

for some constant $\alpha$ and we have that

$$\left| 2i \sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} \, d\xi \right|$$
$$\le 2 \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ (h,k)=1}} \alpha k^{-1} N^{-3/2}$$
$$\le 2\alpha N^{-3/2} \sum_{k=1}^{N} 1 = 2\alpha N^{-1/2}. \tag{2.44}$$

This completes the estimation of the first term in (2.33). We proceed to the second term. First, we will show that

$$J_2(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})} \, dz$$

is small for large $N$. Making change of variable $\xi = zk$ as before, we get that

$$J_2(h,k) = k^{-3/2} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\{G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})} \, d\xi$$

where $s^*_{h,k}$ and $t^*_{h,k}$ are as in (2.37) and (2.38) respectively. As before, we define $\tilde{p}^{**}(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^{**}(m) q^m = G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}) - 1. \tag{2.45}$$

Then, estimating the integrand, we see that

$$
\begin{aligned}
&\left|\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\right| \times \left|G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k})-1\right| \\
=&|\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}e^{\frac{\pi}{4\xi}}\right| \times \left|\sum_{m=0}^{\infty}\tilde{p}^{**}(m)\exp\left(\frac{2\pi i m(\frac{ik}{\xi}+H_1)}{k}\right)-1\right| \\
\leq&|\xi|^{1/2}e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}e^{\frac{\pi}{4}\Re(\frac{1}{\xi})}\left|\sum_{m=1}^{\infty}\tilde{p}^{**}(m)\exp\left(\frac{-2\pi m}{\xi}\right)\exp\left(\frac{2\pi i m H_1}{k}\right)\right| \\
\leq&|\xi|^{1/2}e^{2\pi n}e^{\frac{\pi}{4}\Re(\frac{1}{\xi})}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(-2\pi m\Re\left(\frac{1}{\xi}\right)\right) \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(\left(-2\pi m+\frac{\pi}{4}\right)\Re\left(\frac{1}{\xi}\right)\right) \\
\leq&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(\left(-2\pi m+\frac{\pi}{4}\right)\right) \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\,e^{-\frac{\pi}{4}(8m-1)} \\
\leq&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,e^{-\frac{\pi}{4}(8m-1)} \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,x^{8m-1}, \quad (\text{where} \quad x=e^{-\frac{\pi}{4}}) \\
=&b|\xi|^{1/2}, \qquad (2.46)
\end{aligned}
$$

where

$$b = e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,x^{8m-1}.$$

Note that $b$ does not depend on $\xi$ or $N$. It depends on $n$, but $n$ is fixed. It follows, therefore, that

$$|J_2(h,k)| \leq b\left(\frac{\sqrt{2}k}{N}\right)^{1/2}\frac{2\sqrt{2}N}{N} < \beta k^{3/2}N^{-3/2} \qquad (2.47)$$

for some constant $\beta$. Then we have that

$$\left| i \sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_2(h,k) \right| < \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \beta k^{-1} N^{-3/2}$$
$$\le \beta N^{-3/2} \sum_{k=1}^{N} 1 = \beta N^{-1/2}. \qquad (2.48)$$

Combining the results from (2.44) and (2.48) we have that

$$\begin{aligned} \tilde{p}(n) = & i \sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_1(h,k) + O(\beta N^{-1/2} + 2\alpha N^{-1/2}) \\ = & i \sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_1(h,k) + O(N^{-1/2}). \end{aligned} \qquad (2.49)$$

Finally, we turn our attention to

$$J_1(h,k) = k^{-3/2} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}(2n - \frac{1}{4}) + \frac{\pi}{4\xi}\right)} \, d\xi. \qquad (2.50)$$

We note that

$$J_1(h,k) = \int_{C^*_k} - \int_{s^*_{h,k}}^{0} - \int_{0}^{t^*_{h,k}} = \int_{C^*_k} -S_1 - S_2, \qquad (2.51)$$

where $C^*_k$ is a circle in the $\xi$-plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, as before. It is easily seen that the length of the arc connecting 0 and $s^*_{h,k}$ is less then

$$2\pi \frac{|s^*_{h,k}|}{2} \le \pi |s^*_{h,k}| \le \pi \sqrt{2} \frac{k}{N} \qquad (2.52)$$

From the disscussion above we know that $\Re\left(\frac{1}{\xi}\right) = 1$ and $0 < \Re(\xi) \le 1$ on $C^*_k$. So, the integrand in $S_1$ could be estimated as

$$\begin{aligned}\left|\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\right| =&|\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right|\left|e^{\frac{\pi}{4\xi}}\right| \\ =&|\xi|^{1/2}e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}e^{\frac{\pi}{4}\Re\left(\frac{1}{\xi}\right)} \\ \leq&2^{1/4}\frac{k^{1/2}}{N^{1/2}}e^{2\pi n}e^{\frac{\pi}{4}}. \end{aligned} \tag{2.53}$$

Combining the rsults in (2.52) and (2.53) we get

$$|S_1| < \gamma k^{3/2}N^{-3/2}, \tag{2.54}$$

where $\gamma$ is a constant. We can obtain similar estimate for $S_2$ and, as before, we get an error term $O(N^{-1/2})$ in the formula for $\tilde{p}(n)$. Therefore, we can write

$$\tilde{p}(n) = i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\leq h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{C_k^*} \sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\,d\xi + O(N^{-1/2}). \tag{2.55}$$

Letting $N\to\infty$ we have that

$$\tilde{p}(n) = i\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\leq h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{C_k^*} \sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\,d\xi. \tag{2.56}$$

The above could be implemented in Mathematica as:

```
In[2]:=ZToZetaIntegrandComponents[n, h, k, z, ξ, ω,{{1,-1},{2,2}}]
```

$$\text{Out[2]=} \begin{pmatrix} 1, & \left\{\frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2}\right\} \\ 2, & \left\{\frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2} + \frac{\pi}{4\zeta}\right\} \end{pmatrix}$$

We introduce another change of varaible

$$\xi = \frac{1}{w}, \quad d\xi = -\frac{1}{w^2}.$$

Then (2.56) becomes

$$\tilde{p}(n) = \frac{1}{i}\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{1-\infty i}^{1+\infty i} w^{-5/2} e^{\left(\frac{\pi}{k^2}\left(2n-\frac{1}{4}\right)\frac{1}{w}+\frac{\pi w}{4}\right)}\, dw. \tag{2.57}$$

Let $t = \dfrac{\pi w}{4}$ in (2.57), then the above becomes

$$\tilde{p}(n) = 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \frac{1}{2\pi i}\int_{\pi/4-\infty i}^{\pi/4+\infty i} t^{-5/2} e^{\left(t+\frac{\pi^2}{4k^2}\left(2n-\frac{1}{4}\right)\frac{1}{t}\right)}\, dt. \tag{2.58}$$

The implementation of the above in Mathematica is:

```
In[3]:= IntegrandInT[n, h, k, ξ, dξ, ω, t, dt,{{1,-1},{2,2}}]
```

$$\text{Out[3]=}\left( 2, \quad \left\{-\frac{1}{8}ie^{-\frac{2ihn\pi}{k}}\left(\frac{1}{k}\right)^{5/2}\pi^{3/2}, \frac{8n\pi^2-\pi^2}{16k^2t}+t, -\frac{5}{2}, \frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \mathtt{dt}\right\} \right)$$

In Watson's Treatise on Bessel functions [13, p. 181] we find a formula equivalent to the following:

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-\nu-1}e^{t+(z^2/4t)}\,dt \quad ,(\text{if} \quad c>0, \quad \Re(\nu)>0). \tag{2.59}$$

Let

$$\frac{z}{2} = \left\{\frac{\pi^2}{4k^2}\left(2n-\frac{1}{4}\right)\right\}^{1/2} \tag{2.60}$$

and $\nu = 3/2$. Then we have

$$\begin{aligned}\tilde{p}(n) =& 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1}^{\infty}k^{-5/2}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}\frac{\pi^{-3/2}\left(2n-\frac{1}{4}\right)^{-3/4}}{4^{-3/4}k^{-3/2}}I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right)\\ =&\frac{2\pi\left(2n-\frac{1}{4}\right)^{-\frac{3}{4}}}{\sqrt{8}}\sum_{k=1}^{\infty}k^{-1}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right).\end{aligned} \tag{2.61}$$

The implementation of the above in Mathematica is:

```
In[4]:= RawFormula[n, h, k, ω, {{1,-1},{2,2}}]
```

$$\texttt{Out[4]=}\left(2, \quad \frac{2e^{-\frac{2ihn\pi}{k}}\pi\texttt{BESSELI}\left(\frac{3}{2},\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{2}\right)^2}{k(8n-1)^{3/4}\omega(h,k)}\right)$$

Note that Bessel functions of this order can be expressed as

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left( \frac{\sinh z}{z} \right). \tag{2.62}$$

Expanding (2.62) we have that

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \left( \frac{\cosh z}{z} - \frac{\sinh z}{z^2} \right) \tag{2.63}$$

Substituting (2.60) into (2.63), we get

$$\begin{aligned}
I_{\frac{3}{2}}(z) =& I_{\frac{3}{2}} \left( \frac{\pi}{k} \left( 2n - \frac{1}{4} \right)^{\frac{1}{2}} \right) = I_{\frac{3}{2}} \left( \frac{\pi\sqrt{8n-1}}{2k} \right) \\
=& \sqrt{\frac{2\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi}} \left( \frac{\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\left(\frac{\pi\sqrt{8n-1}}{2k}\right)} - \frac{\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\left(\frac{\pi\sqrt{8n-1}}{2k}\right)^2} \right) \\
=& \frac{(8n-1)^{1/4}}{\sqrt{k}} \left( \frac{2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\frac{\pi}{k}\sqrt{8n-1}} - \frac{\frac{4k}{\pi}\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\frac{\pi}{k}(8n-1)} \right) \\
=& \frac{1}{\pi\sqrt{\frac{\sqrt{8n-1}}{k}}} \left( 2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}} \right)
\end{aligned} \tag{2.64}$$

Multiplying (2.64) by

$$\frac{2\pi\left(2n - \frac{1}{4}\right)^{-\frac{3}{4}}}{k\sqrt{8}} = \frac{2\pi}{k(8n-1)^{\frac{3}{4}}} \tag{2.65}$$

we get

$$\frac{2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}}. \tag{2.66}$$

Finally, we rewrite (2.61) in terms of (2.66) to get

$$\tilde{p}(n) = \sum_{k=1}^{\infty} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \frac{2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}} \tag{2.67}$$

which is in agreement with the formula obtained using our Mathematica package.

In[5]:=RTF[ $\frac{\texttt{QPochhammer[q,q]}}{\texttt{QPochhammer}[\texttt{q}^2,\texttt{q}^2]^2}$ , p, 3.14]

If $\prod_{n=1}^{\infty} \frac{1-q^n}{(1-q^{2n})^2} = \sum_{n=0}^{\infty} p_{3.14}(n) q^n$

, then . . .

Out[5] $= p_{3.14}(n)$

$$= \sum_{k=1}^{\infty}\sum_{h=0}^{k-1} \frac{2e^{-\frac{2ihn\pi}{k}} \left\lfloor \frac{1}{\gcd(h,k)} \right\rfloor \left(2\cosh\left(\frac{\sqrt{8n-1}\pi}{2k}\right) - \frac{4k\sinh\left(\frac{\sqrt{8n-1}\pi}{2k}\right)}{\sqrt{8n-1}\pi}\right)}{\sqrt{k}(8n-1)\omega(h,k)}$$

$$\times \chi(\gcd(2,k)=2)\omega\left(h,\frac{k}{2}\right)^2$$

# CHAPTER 3

# RADEMACHER-TYPE FORMULAE FOR FOURIER COEFFICIENTS OF RAMANUJAN-TYPE SERIES

## 3.1 Ramanujan-type identities

There are numerous Ramanujan-type infinite products. Each product could be represented as an infinite series in veraible $q$. We use our Mathematica package to extract the coefficients of $q^n$ for the given infinite product.

In his lost notebook [1, 2], Ramanujan recorded these identities:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^n)(1-q^{12n})} \qquad \text{[1, Entry 11.3.1, p. 254]} \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})(1-q^{12n})}{(1-q^n)(1-q^{6n})} \qquad \text{[1, Entry 11.3.2, p. 254]} \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{12n})^2}{(1-q^n)^2(1-q^{6n})(1-q^{4n})} \qquad \text{[1, Entry 11.3.3, p. 254]} \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q;q)_n(-q;q)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^n)(1-q^{3n})} \qquad \text{[1, Entry 11.3.4, p. 255]} \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})^5}{(1-q^n)(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^2} \qquad \text{[1, Entry 11.3.5, p. 255]} \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{(-q)^n(q;q^2)_n}{(-q;-q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2(1-q^n)(1-q^{4n})^2}{(1-q^{2n})^4(1-q^{6n})} \qquad \text{[1, Entry 11.3.5, p. 255]} \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{2n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.3, p. 102]} \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{6n})^2}{(1-q^{2n})(1-q^{3n})(1-q^{18n})} \qquad \text{[2, Entry 5.3.4, p. 103]} \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{3n})^2}{(1-q^{n})(1-q^{4n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{4n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^{3n})(1-q^{4n})} \qquad \text{[2, Entry 5.3.7, p. 105]} \tag{3.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{8n})(1-q^{2n})^2(1-q^{12n})^2}{(1-q^{24n})(1-q^{4n})^2(1-q^{n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.8, p. 105]} \tag{3.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q^3;q^6)_n}{(q^4;q^4)_n(q;q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})}{(1-q^{n})(1-q^{12n})} \qquad \text{[2, Entry 5.3.9, p. 106]} \tag{3.13}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{2n})^2} \qquad \text{[2, Entry 4.2.6, p. 84]} \tag{3.14}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{4n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.15}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^{2n^2}}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{2n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.3, p. 102]} \tag{3.16}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})(1-q^{6n})^5}{(1-q^{3n})^2(1-q^{2n})^2(1-q^{12n})^2} \qquad \text{[2, Entry 4.2.7, p. 85]} \tag{3.17}$$

The following identities are due to Slater:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})} \qquad [8, \text{ p. } 152, \text{ Eq. } (3, 23)] \tag{3.18}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} \qquad [8, \text{ p. } 152, \text{ Eq. } (2)] \tag{3.19}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{4n})}{(1-q^{2n})^2} \qquad [8, \text{ p. } 152, \text{ Eq. } (5)] \tag{3.20}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})}{(1-q^{2n})} \qquad [8, \text{ p. } 152, \text{ Eq. } (7)] \tag{3.21}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n(n+1)}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})^5}{(1-q^{2n})^2(1-q^{3n})^2(1-q^{12n})^2} \qquad [8, \text{ p. } 152, \text{ Eq. } (48)] \tag{3.22}$$

The following identities are due to Bowman, McLaughlin and Sills:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^5}{(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^2} \qquad [5, \text{ p. } 314, \text{ Eq. } (2.19)] \tag{3.23}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n^2}}{(-q;q)_{2n}(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{6n})(1-q^{2n})} \qquad [5, \text{ p. } 314, \text{ Eq. } (2.19)] \tag{3.24}$$

The following identity is due to Starcher

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)^2} \qquad [12, \text{ p. } 805, \text{ Eq. } (3.6)] \tag{3.25}$$

## 3.2 Formulae for the Fourier Coefficients of the series presented in 3.2

Let $\Pi_j(q)$ represent the right hand side of Eq.$(j)$, where $j$ runs from (3.1) to (3.25) and let $\tilde{p}_j(n)$ be defined by

$$\sum_{n=0}^{\infty} \tilde{p}_j(n) q^n = \Pi_j(q).$$

Our Mathematica package conjectures the following Rademacher-type formulae. These formulae appear to be new:

$$\tilde{p}_{3.1}(n) = \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{4\sqrt{3}k}\right)\omega(12h,k)\omega(h,k)}{k\sqrt{24n-1}\quad \omega(6h,k)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=12}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right)\omega\left(h,\frac{k}{12}\right)\omega(h,k)}{k\sqrt{24n-1}\quad \omega\left(h,\frac{k}{6}\right)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right)\omega\left(3h,\frac{k}{4}\right)\omega(h,k)}{\sqrt{3}k\sqrt{24n-1}\quad \omega\left(3h,\frac{k}{2}\right)^2}. \tag{3.26}$$

$$\tilde{p}_{3.2}(n) = \sum_{\substack{k=1\\(12,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right)\omega\left(h,\frac{k}{6}\right)\omega(h,k)}{2k\sqrt{3n+1}\quad \omega\left(h,\frac{k}{3}\right)\omega\left(2h,\frac{k}{6}\right)}$$
$$+ \sum_{\substack{k=1\\(12,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right)\omega\left(3h,\frac{k}{2}\right)\omega(h,k)}{2\sqrt{3}k\sqrt{3n+1}\quad \omega(3h,k)\omega\left(6h,\frac{k}{2}\right)}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{3n+1}\pi}{\sqrt{6}k}\right)\omega(6h,k)\omega(h,k)}{4k\sqrt{3n+1}\quad \omega(3h,k)\omega(12h,k)}. \tag{3.27}$$

$$\tilde{p}_{3.3}(n) = \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right)\omega\left(2h,\frac{k}{3}\right)\omega(4h,k)\omega(h,k)^2}{8k\sqrt{3n+2}\quad \omega(2h,k)^2\omega\left(4h,\frac{k}{3}\right)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\sqrt{\frac{5}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right)\omega(4h,k)\omega(6h,k)\omega(h,k)^2}{8k\sqrt{3n+2}\quad \omega(2h,k)^2\omega(12h,k)^2}. \tag{3.28}$$

$$\tilde{p}_{3.4}(n) = \sum_{\substack{k=1 \\ (6,k)=3}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(h, \frac{k}{3}\right) \omega(h,k)}{2\sqrt{2}k\sqrt{3n+1} \quad \omega\left(2h, \frac{k}{3}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (6,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega(3h,k)\omega(h,k)}{2\sqrt{6}k\sqrt{3n+1} \quad \omega(6h,k)^2}. \tag{3.29}$$

$$\tilde{p}_{3.5}(n) = \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right) \omega\left(h, \frac{k}{3}\right)^2 \omega(h,k)\omega(2h,k)\omega\left(4h, \frac{k}{3}\right)^2}{2k\sqrt{120n+5} \quad \omega\left(2h, \frac{k}{3}\right)^5 \omega(4h,k)}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right) \omega(h,k)\omega(2h,k)\omega(3h,k)^2\omega(12h,k)^2}{2\sqrt{3}k\sqrt{120n+5} \quad \omega(4h,k)\omega(6h,k)^5}. \tag{3.30}$$

$$\tilde{p}_{3.6}(n) = \sum_{\substack{k=1 \\ (12,k)=6}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)^4}{k\sqrt{120n+5} \quad \omega\left(h, \frac{k}{3}\right)^2 \omega(h,k)\omega\left(2h, \frac{k}{2}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)^4}{\sqrt{3}k\sqrt{120n+5} \quad \omega(h,k)\omega\left(2h, \frac{k}{2}\right)^2 \omega(3h,k)^2}. \tag{3.31}$$

$$\tilde{p}_{3.7}(n) = \sum_{\substack{k=1 \\ (6,k)=6}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \quad \omega\left(h, \frac{k}{3}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (6,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1} \quad \omega(3h,k)^2}. \tag{3.32}$$

$$\tilde{p}_{3.8}(n) = \sum_{\substack{k=1 \\ (18,k)=18}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{18}\right) \omega\left(h, \frac{k}{3}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \quad \omega\left(h, \frac{k}{9}\right) \omega\left(h, \frac{k}{6}\right)^2}$$

$$+ \sum_{\substack{k=1 \\ (18,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{10\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{60n-5}\pi}{9k}\right) \omega(3h,k) \omega\left(9h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{3\sqrt{3}k\sqrt{60n-5} \quad \omega\left(3h, \frac{k}{2}\right)^2 \omega(9h,k)}$$

$$+ \sum_{\substack{k=1 \\ (18,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\sqrt{\frac{2}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{12n-1}\pi}{9k}\right) \omega(2h,k)\omega(3h,k)\omega(18h,k)}{3k\sqrt{12n-1} \quad \omega(6h,k)^2 \omega(9h,k)}. \tag{3.33}$$

$$\tilde{p}_{3.9}(n) = \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{4\sqrt{3}k}\right) \omega(4h,k)\omega(6h,k)\omega(h,k)}{3k\sqrt{8n-1} \quad \omega(2h,k)\omega(3h,k)^2}$$

$$+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{4}\right) \omega(h,k)}{k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{3}\right)^2 \omega\left(h, \frac{k}{2}\right)}$$

$$+ \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right) \omega\left(3h, \frac{k}{2}\right) \omega(h,k)}{\sqrt{3}k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{2}\right) \omega(3h,k)^2}. \tag{3.34}$$

$$\tilde{p}_{3.10}(n) = \sum_{\substack{k=1 \\ (4,k)=4}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right)}{k\sqrt{8n-1}\omega(h,k)}. \tag{3.35}$$

$$\tilde{p}_{3.11}(n) = \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{2\sqrt{3}k}\right) \omega\left(h, \frac{k}{4}\right) \omega(3h,k)}{k\sqrt{24n+5} \quad \omega\left(3h, \frac{k}{2}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{12k}\right) \omega(4h,k)\omega(3h,k)}{\sqrt{3}k\sqrt{24n+5} \quad \omega(6h,k)^2}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{12k}\right) \omega\left(h, \frac{k}{3}\right) \omega(4h,k)}{k\sqrt{24n+5} \quad \omega\left(2h, \frac{k}{3}\right)^2}. \tag{3.36}$$

$$\tilde{p}_{3.12}(n) = \sum_{\substack{k=1 \\ (24,k)=24}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{24}\right) \omega\left(h, \frac{k}{6}\right) \omega(h,k) \omega\left(h, \frac{k}{4}\right)^2}{k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{12}\right)^2 \omega\left(h, \frac{k}{8}\right) \omega\left(h, \frac{k}{2}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (24,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5}\sqrt{8n-1}\pi}{6k}\right) \omega(h,k)\omega\left(3h, \frac{k}{2}\right) \omega\left(6h, \frac{k}{4}\right) \omega\left(h, \frac{k}{4}\right)^2}{3k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{2}\right)^2 \omega\left(2h, \frac{k}{4}\right) \omega\left(3h, \frac{k}{4}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (24,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\sqrt{\frac{5}{2}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5}\sqrt{8n-1}\pi}{12k}\right) \omega(h,k)\omega(4h,k)^2\omega(6h,k)\omega(24h,k)}{3k\sqrt{8n-1} \quad \omega(2h,k)^2\omega(8h,k)\omega(12h,k)^2}$$
$$+ \sum_{\substack{k=1 \\ (24,k)=3}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{4k}\right) \omega(h,k)\omega\left(2h, \frac{k}{3}\right) \omega(4h,k)^2\omega\left(8h, \frac{k}{3}\right)}{\sqrt{2}k\sqrt{8n-1} \quad \omega(2h,k)^2\omega\left(4h, \frac{k}{3}\right)^2 \omega(8h,k)}. \tag{3.37}$$

$$\tilde{p}_{3.13}(n) = \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{3\sqrt{2}k}\right) \omega(12h,k)\omega(h,k)}{3k\sqrt{8n-1} \quad \omega(4h,k)\omega(6h,k)}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{12}\right) \omega(h,k)}{k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{4}\right)}. \tag{3.38}$$

$$\tilde{p}_{3.14}(n)=\sum_{\substack{k=1\\(2,k)=2}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2e^{-\frac{2i\pi hn}{k}}\omega\left(h,\frac{k}{2}\right)^2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)-\frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}\quad\omega(h,k)}. \quad (3.39)$$

$$\tilde{p}_{3.15}(n)=\sum_{\substack{k=1\\(4,k)=4}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2\pi e^{-\frac{2i\pi hn}{k}}I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{4}\right)}{k\sqrt{8n-1}\quad\omega(h,k)}. \quad (3.40)$$

$$\tilde{p}_{3.16}(n)=\sum_{\substack{k=1\\(6,k)=6}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2\pi e^{-\frac{2i\pi hn}{k}}I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n-1}\quad\omega\left(h,\frac{k}{3}\right)^2}$$
$$+\sum_{\substack{k=1\\(6,k)=2}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2\pi e^{-\frac{2i\pi hn}{k}}I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(3h,\frac{k}{2}\right)\omega\left(h,\frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1}\quad\omega(3h,k)^2}. \quad (3.41)$$

$$\tilde{p}_{3.17}(n)=\sum_{\substack{k=1\\(12,k)=4}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(3h,\frac{k}{4}\right)^2\omega(3h,k)^2\omega\left(h,\frac{k}{2}\right)^2}{\sqrt{3}k\sqrt{8n-1}\omega(h,k)\omega\left(3h,\frac{k}{2}\right)^5}$$
$$+\sum_{\substack{k=1\\(12,k)=2}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{4e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2\sqrt{3}k}\right)\omega(3h,k)^2\omega\left(6h,\frac{k}{2}\right)^2\omega\left(h,\frac{k}{2}\right)^2}{3k\sqrt{8n-1}\omega(h,k)\omega\left(3h,\frac{k}{2}\right)^5}$$
$$+\sum_{\substack{k=1\\(12,k)=12}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{12}\right)^2\omega\left(h,\frac{k}{3}\right)^2\omega\left(h,\frac{k}{2}\right)^2}{k\sqrt{8n-1}\omega\left(h,\frac{k}{6}\right)^5\omega(h,k)} \quad (3.42)$$

$$\tilde{p}_{3.18}(n)=\sum_{\substack{k=1\\(2,k)=2}}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{2\pi e^{-\frac{2i\pi hn}{k}}I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{24n-1}\quad\omega(h,k)}. \quad (3.43)$$

$$\tilde{p}_{3.19}(n) = \sum_{\substack{k=1 \\ (2,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{6\sqrt{2}k}\right) \omega(h,k)}{k\sqrt{24n+1} \quad \omega(2h,k)}. \tag{3.44}$$

$$\tilde{p}_{3.20}(n) = \sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{3\sqrt{2}k}\right) \omega\left(h,\frac{k}{2}\right)^2}{k\sqrt{24n+1} \quad \omega(h,k)\omega\left(2h,\frac{k}{2}\right)}. \tag{3.45}$$

$$\begin{aligned}\tilde{p}_{3.21}(n) = &\sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{3\sqrt{2}k}\right) \omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n+1} \quad \omega\left(2h,\frac{k}{2}\right)} \\ &+ \sum_{\substack{k=1 \\ (4,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{6\sqrt{2}k}\right) \omega(2h,k)}{2k\sqrt{12n+1} \quad \omega(4h,k)}.\end{aligned} \tag{3.46}$$

$$\begin{aligned}\tilde{p}_{3.22}(n) = &\sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2}k}\right) \omega\left(h,\frac{k}{3}\right)^2 \omega(2h,k)^2 \omega\left(4h,\frac{k}{3}\right)^2}{4k\sqrt{n} \quad \omega\left(2h,\frac{k}{3}\right)^5 \omega(4h,k)} \\ &+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{2}{3}}\sqrt{n}\pi}{k}\right) \omega\left(h,\frac{k}{2}\right)^2 \omega(3h,k)^2 \omega\left(6h,\frac{k}{2}\right)^2}{3k\sqrt{n} \quad \omega\left(2h,\frac{k}{2}\right) \omega\left(3h,\frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2}k}\right) \omega(2h,k)^2 \omega(3h,k)^2 \omega(12h,k)^2}{4\sqrt{3}k\sqrt{n} \quad \omega(4h,k)\omega(6h,k)^5}.\end{aligned} \tag{3.47}$$

$$\tilde{p}_{3.23}(n) = \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right)\omega(2h,k)\omega\left(4h,\frac{k}{3}\right)^2\omega\left(h,\frac{k}{3}\right)^2}{k\sqrt{12n-1}\quad \omega\left(2h,\frac{k}{3}\right)^5}$$

$$+ \sum_{\substack{k=1\\(12,k)=12}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{12}\right)^2\omega\left(h,\frac{k}{2}\right)\omega\left(h,\frac{k}{3}\right)^2}{k\sqrt{12n-1}\quad \omega\left(h,\frac{k}{6}\right)^5}$$

$$+ \sum_{\substack{k=1\\(12,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{2}\right)\omega\left(3h,\frac{k}{4}\right)^2\omega(3h,k)^2}{\sqrt{3}k\sqrt{12n-1}\quad \omega\left(3h,\frac{k}{2}\right)^5}$$

$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right)\omega(2h,k)\omega(3h,k)^2\omega(12h,k)^2}{\sqrt{3}k\sqrt{12n-1}\quad \omega(6h,k)^5}. \tag{3.48}$$

$$\tilde{p}_{3.24}(n) = \sum_{\substack{k=1\\(6,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n-1}\quad \omega\left(h,\frac{k}{3}\right)^2}$$

$$+ \sum_{\substack{k=1\\(6,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(3h,\frac{k}{2}\right)\omega\left(h,\frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1}\quad \omega(3h,k)^2}. \tag{3.49}$$

$$\tilde{p}_{3.25}(n) = \sum_{\substack{k=1\\(2,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{e^{-\frac{2ihn\pi}{k}}\left(2\cosh\left(\frac{\sqrt{n}\pi}{k}\right) - \frac{2k\sinh\left(\frac{\sqrt{n}\pi}{k}\right)}{\sqrt{n}\pi}\right)\omega(h,k)^2}{8\sqrt{k}n\omega(2h,k)}. \tag{3.50}$$

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# Appendix A

## RADEMACHER.M MATHEMATICA PACKAGE

```
Print["The Rademacher Package"]
Print["Version of January 19, 2011"]
Print["V. Kiria and A. Sills"]

FindL[ simplifiedinput_List]:=
Apply[LCM,PowersOfQ[simplifiedinput]]

PowersOfQ[simplifiedinput_List]:=
Table[ simplifiedinput[[i,1]] ,{i,1,Length[simplifiedinput]}]

PowersOfP[simplifiedinput_List]:=
Table[ simplifiedinput[[i,2]] ,{i,1,Length[simplifiedinput]}]

BigOmega[d_Integer,h_Symbol,k_Symbol,
omega_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[ (omega[h,k ]/.{
k-> k/GCD[d,qpowers[[i]]],
h->  qpowers[[i]]*h/GCD[d,qpowers[[i]]]
 }
```

```
)^ppowers[[i]]    ,{i,1,numfactors}]
]

BigZ[ d_Integer, z_Symbol, simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[ (Sqrt[z]/.{
z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]]
 }
)^ppowers[[i]]    ,{i,1,numfactors}]
]

BigExp[d_Integer,z_Symbol,k_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Collect[Simplify[
Sum[
ppowers[[i]]*(Pi (1/z - z)/12/k)/.
{k-> k/GCD[d,qpowers[[i]]],
 z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]]}
,{i,1,numfactors}
```

```
]
]
,{z,z^(-1)}
]
]

BigP[d_Integer,P_Symbol,qprime_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[  P[qprime^(GCD[d,qpowers[[i]]]^2/qpowers[[i]])]^ppowers[[i]]
  ,{i,1,numfactors}]
]

OnePieceModularRHS[d_Integer,h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
{BigOmega[d,h,k,omega,simplifiedinput],BigZ[ d, z, simplifiedinput],
BigExp[d,z,k,simplifiedinput],BigP[d,P,qprime,simplifiedinput]}

ModularRHS[h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
Module[{d,L},
L=FindL[ simplifiedinput];
d=Divisors[L];
```

```
Table[{d[[i]],OnePieceModularRHS[ d[[i]],  h,k,z,omega,
P,qprime,simplifiedinput]},{i,1, Length[d]}]
]

ZToZetaIntegrandComponents[n_,h_,k_,z_,
zeta_,omega_,simplifiedinput_]:=
Module[{extractchange,transstuff,P,Q},
transstuff=ModularRHS[h,k,z,omega,P,Q,simplifiedinput];
extractchange =
Table[
{transstuff[[i,1]],
{transstuff[[i,2,1]],
transstuff[[i,2,2]],2 Pi n z/k + transstuff[[i,2,3]]
}/.{z-> zeta/k}
} ,{i,Length[transstuff]}]
]

ExtractNonTrivialCases[n_,h_,k_,z_,zeta_,
omega_,simplifiedinput_]:=Module[{tmp},
tmp=ZToZetaIntegrandComponents[n,h,k,z,
zeta,omega,simplifiedinput];
Complement[
Table[
If[ Coefficient[tmp[[i,2,3]], zeta,-1]>0,
tmp[[i]]
```

```
]
, {i,1,Length[tmp]}],
{Null}
]
]

ZetaToTTransformation[ expo_, zeta_, t_]:=
zeta-> Coefficient[ expo, zeta,-1]/t

DZetaToDt[expo_,zeta_,dzeta_,t_,dt_]:=
dzeta-> D[Extract[ZetaToTTransformation[
expo, zeta, t],2],t]dt

IntegrandInT[n_,h_,k_,zeta_,dzeta_,omega_,
t_,dt_, simplifiedinput_]:= Module[{z, zetacases,tmp},
zetacases=ExtractNonTrivialCases[
n,h,k,z,zeta,omega,simplifiedinput];
(*Print[zetacases];*)
tmp=
Table[
{zetacases[[i,1]],
{I/k^2 *Exp[-2 Pi I n h/k ],
zetacases[[i,2,3]]/.ZetaToTTransformation[
 zetacases[[i,2,3]], zeta, t] ,
zetacases[[i,2,1]],
```

```
zetacases[[i,2,2]]/.ZetaToTTransformation[
 zetacases[[i,2,3]], zeta, t],
dzeta/.DZetaToDt[zetacases[[i,2,3]],zeta,dzeta,t,dt]
}
},{i,1,Length[zetacases]}
];
(*Print[tmp];*)
Table[
{tmp[[i,1]] ,
{tmp[[i,2,1]]*(tmp[[i,2,5]]*tmp[[i,2,4]])/.{t->1,dt-> 1},
Collect[Together[tmp[[i,2,2]]],{t,t^(-1)}],
Exponent[ tmp[[i,2,5]] ,t]+
Exponent[PowerExpand[tmp[[i,2,4]] ],t],
tmp[[i,2,3]],
dt
}
}
,{i,1,Length[tmp]}]
]

RawFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,temp,zeta,dzeta,t,dt},temp=IntegrandInT[
n,h,k,zeta,dzeta,omega,t,dt, simplifiedinput];
(*Print[temp];*)
Table[{temp[[i,1]],
```

```
PowerExpand[
temp[[i,2,1]]*2*Pi*I/
PowerExpand[Simplify[(Sqrt[Coefficient[temp[[i,2,2]],
t^(-1)]])^(-(1+temp[[i,2,3]]))]]
*temp[[i,2,4]]*
BESSELI[(-(1+temp[[i,2,3]])),(2PowerExpand[Simplify[
Sqrt[Coefficient[temp[[i,2,2]],t^(-1)]]]])]
]
}
,{i,1,Length[temp]}]
]

FormattedSummand[ n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,tmp,L},
L=FindL[simplifiedinput];
tmp=RawFormula[n,h,k,omega,simplifiedinput];
Sum[
tmp[[i,2]]*Floor[1/GCD[h,k]]*\[Chi][GCD[k,L]==tmp[[i,1]] ]
,{i,1,Length[tmp]}]//PowerExpand
]

DedekindSum[h_,k_]:=
Sum[(\[Mu]/k-Floor[\[Mu]/k]-1/2)*(h*\[Mu]/k-Floor[
h*\[Mu]/k]-1/2),{\[Mu],1,k-1}]
```

```
Omega[h_,k_]:=
Exp[Pi*I*DedekindSum[h,k]]


NearlyHeadlessNick[simplifiedinput_,q_,MaxN_]:=
Series[Product[(1/QPochhammer[q^(simplifiedinput[
[i,1]])])^simplifiedinput[[i,2]],
{i,1,Length[simplifiedinput]}],{q,0,MaxN}]


TestFormula[ n_,h_, k_,omega_,
simplifiedinput_, MaxK_,MaxN_]:=
Module[{calcvals,i,tmp,series,
summand,q,seriesvals},
summand=FormattedSummand[n,h,k,omega,
simplifiedinput];
tmp=Sum[  summand  ,{k,1,MaxK},{h,0,k-1}];
tmp=tmp/.{BESSELI->
BesselI,omega->Omega, \[Chi][True]-> 1,
 \[Chi][False]-> 0 };
(*Print[tmp];*)
calcvals=Table[ N[tmp/.n-> i],{i,1,MaxN}];
series= NearlyHeadlessNick[simplifiedinput,q,MaxN];
seriesvals=Table[ Coefficient[series,q,i] ,{i,1,MaxN}];
{Re[calcvals]-seriesvals, Im[calcvals]}
]
```

```
PrettyFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{fs,rf,L},
L=FindL[simplifiedinput ];
rf=RawFormula[n,h,k,omega,simplifiedinput]/.BESSELI-> BesselI;
(*Sum[
Sum[ HoldForm [TraditionalForm[Evaluate[rf[[i,2]] ] *
\[Chi][ GCD[h,k]==1 && GCD[k,L]==rf[[i,1]] ] ]]
,{k,1,Infinity},{h,0,k-1}]
,{i,Length[rf]} ]*)
(*fs=ToString[FormattedSummand[
n,h,k,omega,simplifiedinput]/.BESSELI->BesselI,
TraditionalForm]*)

fs = PowerExpand[ FormattedSummand[n,h,k,omega,
simplifiedinput]/.BESSELI->BesselI]//TraditionalForm;
(*SUM[ fs, {k,1,Infinity},{h,0,k-1}]*)
fs
]

ProductToList[ expr_, q_:q,f_:f,
 psi_:\[Psi], phi_:\[Phi] ]:=
Module[{tmp,tmplist,a,b,c,p,r,numer,
denom,ex,subs2,subs3,subs4,subs6},
tmp=expr;
tmp=tmp/. f[a_,b_]->  QPochhammer[-a, a*b]
```

```
 QPochhammer[-b, a*b] QPochhammer[a*b];
tmp= tmp/. psi[-q^a_.]->
QPochhammer[q^(2a)]/QPochhammer[-q^a,q^(2a)];
tmp=tmp/. psi[q^a_.]->
 QPochhammer[q^(2a)]/QPochhammer[q^a,q^(2a)];
tmp=tmp/. phi[-q^a_.]->
QPochhammer[q^a]/QPochhammer[-q^a];
tmp=tmp/. phi[q^a_.]->
QPochhammer[-q,q^(2a)]^2 QPochhammer[q^(2a)];
tmp=tmp/. QPochhammer[ a_,-q^p_]->
QPochhammer[ a,q^(2p)]*QPochhammer[-a*q^p,q^(2p)];
tmp = tmp/. QPochhammer[-q^r_.,q^p_.]->
QPochhammer[q^(2r),q^(2p)]/QPochhammer[q^r,q^p];
numer=Numerator[tmp]/. QPochhammer[a_,b_]^c_->
 QPochhammer[a,b];
If[numer[[0]]=== Times,
numer=numer/.Times-> List,
numer={numer}];
If[verbose,Print["numer=",numer]];
denom=Denominator[tmp]/. QPochhammer[a_,b_]^c_->
QPochhammer[a,b];
If[denom[[0]]===Times,
denom=denom/.Times-> List,
denom={denom}];
If[verbose,Print["denom=",denom]];
```

```
ex=Union[Table[Exponent[numer[[i,2]] ,q],{i,1,Length[numer]}],
Table[Exponent[denom[[i,2]],q],{i,1,Length[denom]}]
];
If[verbose,Print[ex]];
ex=Union[Flatten[Table[ Divisors[ex[[i]]],{i,1,Length[ex]}]]];
If[verbose,Print[ex]];
subs2= Table[
QPochhammer[q^(ex[[i]]/2),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/2) ]/QPochhammer[q^ex[[i]] ]
,{i,1,Length[ex]}];
If[verbose,Print["subs2 = ",subs2]];
tmp=tmp/.subs2;
If[verbose,Print["after subs 2, tmp = ",tmp ] ];
subs3= Table[
QPochhammer[q^(ex[[i]]/3),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/3) ]/QPochhammer[q^ex[[i]] ]/
QPochhammer[q^(2ex[[i]]/3),q^ex[[i]]]
,{i,1,Length[ex]}];
If[verbose,Print["subs3 = ",subs3]];
tmp=tmp/.subs3;
If[verbose,Print["after subs3, tmp = ",tmp ] ];
subs4= Table[
QPochhammer[q^(ex[[i]]/4),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/4) ,q^(ex[[i]]/2)]
QPochhammer[q^(3ex[[i]]/4),q^ex[[i]]]
```

```
,{i,1,Length[ex]}];
If[verbose,Print["subs4 = ",subs4]];
tmp=(tmp/.subs4)/.subs2;
If[verbose,Print["after subs4 & subs2, tmp = ",tmp ] ];
subs6= Table[
QPochhammer[q^(ex[[i]]/6),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/6) ,q^(ex[[i]]/3)]/
QPochhammer[q^(5ex[[i]]/6),q^ex[[i]]]/
QPochhammer[ q^(ex[[i]]/2) ,q^(ex[[i]])]
,{i,1,Length[ex]}];
If[verbose,Print["subs6 = ",subs6]];
tmp=(tmp/.subs6)/.subs2;
If[verbose,Print["after subs6 & subs2, tmp = ",tmp ] ];
(*tmp=(((tmp/.subs2)/.subs3)/.subs4)/.subs6;
If[verbose,Print["after repeated subs, tmp = ",tmp ] ];*)
numer=Numerator[tmp];
denom=Denominator[tmp];

If[numer[[0]]===Times,numer= numer/.Times->List,numer={numer}];
If[numer==={1},numer={}];
If[verbose,Print["numer = ",numer ] ];
If[denom[[0]]===Times,denom=denom/.Times->List,denom={denom}];
If[denom==={1},denom={}];
If[verbose,Print["denom = ",denom ] ];
Join[(numer/. QPochhammer[a_]^b_. ->
```

```
{Log[a]/Log[q],-b})//PowerExpand,
(denom/. QPochhammer[a_]^b_. ->
{Log[a]/Log[q],b})//PowerExpand
]
]

RademacherTypeFormula[
prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=PrettyFormula[
n,h,k,omega,ProductToList[ prodexpr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodexpr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]

LaTeXSummand[prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,
omega_:\[Omega]]:=Module[{pf,simplifiedinput},
simplifiedinput=ProductToList[prodexpr];
TeXForm[Subscript[a,eqcode][n]==
FormattedSummand[ n,h,k,omega,
  simplifiedinput]/.{BESSELI->BesselI,Floor[GCD[h, k]^(-1)]->1}
]
]
```

```
(*
RTF[prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=
 Style[\[CapitalSigma]\[CapitalSigma],Large]
 PrettyFormula[n,h,k,omega,ProductToList[
  prodexpr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodexpr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]
*)

ListToProduct[simplifiedinput_List,q_:q,n_:n]:=
Product[(1-q^(
simplifiedinput[[i,1]]n))^(-simplifiedinput[[i,2]]),
{i,1,Length[simplifiedinput]} ]

MakeBoxes[MySum[s_,{k_, k0_, k1_},
{n_, n0_, n1_}],TraditionalForm]:=
RowBox[{UnderoverscriptBox["\[Sum]",RowBox[
{MakeBoxes[k,TraditionalForm],"=",
MakeBoxes[k0,TraditionalForm]}],
MakeBoxes[k1,TraditionalForm]],
```

```
RowBox[{UnderoverscriptBox["\[Sum]",RowBox[
{MakeBoxes[n,TraditionalForm],"=",
MakeBoxes[n0,TraditionalForm]}],
MakeBoxes[n1,TraditionalForm]],
MakeBoxes[s,TraditionalForm]}]}]

MakeBoxes[MyProduct[s_,{k_, k0_, k1_}],TraditionalForm]:=
RowBox[{UnderoverscriptBox["\[
Product]",RowBox[{MakeBoxes[k,TraditionalForm],"=",
MakeBoxes[k0,TraditionalForm]}],
MakeBoxes[k1,TraditionalForm]],
MakeBoxes[s,TraditionalForm]}]

The Rademacher Package

Version of January 19, 2011

V. Kiria and A. Sills

RTF[prodexpr_,a_:a, eqcode_:{},
 n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=
 TraditionalForm[
 MySum[
```

```
   PrettyFormula[n,h,k,omega,ProductToList[
   prodexpr, q,f, \[Psi], \[Phi] ]],
  {k,1,Infinity},{h,0,k-1}
  ]
]
;
Print["If ", TraditionalForm[
 MyProduct[ ListToProduct[ProductToList[prodexpr] ],
 {n,1,Infinity}] ==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]

verbose=False;
```