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Rademacher-Type Series

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RADEMACHER-TYPE SERIES

by

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(Under the Direction of Andrew Sills)

ABSTRACT

In this thesis we use the classical circle method to find the coefficients $\tilde{p}(n)$ of $\sum_{n=0}^{\infty} \tilde{p}(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2}$. We introduce a Mathematica package which automates steps in the the circle method and computes the Rademacher-type series expressions for the Fourier coefficients of other Ramanujan-type series.

INDEX WORDS: Partitions, q-series, Circle method.

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CHAPTER 1

PARTITIONS AND GENERATING FUNCTIONS

1.1 Partitions

A *partition* of a positive integer n is a representation of n as a sum of positive integers where order of summands (parts) does not matter. For example, all the partitions of 4 are

4

3+1

2+2

2+1+1

1+1+1+1,

so, we have five partions of 4. The eleven partitions of 6 are

6

5+1

4+2

4+1+1

3+3

3+2+1

3+1+1+1

2+2+2

2+2+1+1

2+1+1+1+1

1+1+1+1+1+1.

Let $p(n)$ represent the number of partitions of n. Some values of $p(n)$ are given below:

$p(0) = 1$

$p(1) = 1$

$p(2) = 2$

$p(3) = 3$

$p(4) = 5$

$p(5) = 7$

$p(6) = 11$

$p(7) = 15$

$p(8) = 22$

$p(9) = 30$

$p(10) = 42$

$p(100) = 190,569,192$

$p(200) = 3,972,999,029,388$

As could be seen from the above list, $p(n)$ grows rapidly as n increases. However, the pattern of growth is not obvious and a question was raised whether there is a closed formula for the partition function $p(n)$. G. H. Hardy and S. Ramanujan [9] in 1918 obtained the following asymptotic expression for the partition function

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}}, \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where $a(n) \sim b(n)$ as $n \to \infty$ means

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1.$$

Later, in 1937, H. Rademacher [10, 11] was able to express $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right), \tag{1.2}$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} e^{\pi i(s(h,k)-2nh/k)},$$

is a Kloosterman sum, and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right)$$

is a Dedekind sum.

In 2011, Ken Ono and Jan Bruinier [4] announced a new formula that expresses $p(n)$ as a *finite* sum.

1.2 Generating functions

In the theory of integer partitions an important role is played by *generating functions*. The *generating function* of a sequence b_n is a formal power series in one unknown, whose coefficients are the b_n, i.e.,

$$b_0 + b_1x + b_2x^2 + b_3x^3 + ... = \sum_{n=0}^{\infty} b_nx^n \tag{1.3}$$

is a generating function for the sequence $\{b_n\}_{n=0}^{\infty}$. L. Euler proved [6] that the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{1.4}$$

From elementary calculus we know that

$$\sum_{n=0}^{\infty} (x^k)^n = \frac{1}{1-x^k}, \quad |x| < 1,\ k \in \mathbb{N}. \tag{1.5}$$

Expanding right hand side of (1.4) we have that

$$\begin{aligned}
\prod_{k=1}^{\infty}\frac{1}{1-x^k} &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\cdots \\
&= \left(\sum_{n=0}^{\infty}x^n\right)\left(\sum_{n=0}^{\infty}x^{2n}\right)\left(\sum_{n=0}^{\infty}x^{3n}\right)\left(\sum_{n=0}^{\infty}x^{4n}\right)\cdots \\
&=(1+x+x^2+x^3+x^4+x^5+x^6+\cdots) \\
&\times(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+\cdots) \\
&\times(1+x^3+x^6+x^9+x^{12}+x^{15}+x^{18}+\cdots) \\
&\times(1+x^4+x^8+x^{12}+x^{16}+x^{20}+x^{24}+\cdots)\cdots \\
&=(1+x+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^{1+1+1+1+1}+\cdots) \\
&\times(1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+x^{2+2+2+2+2}+\cdots) \\
&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+x^{3+3+3+3+3}+\cdots) \\
&\times(1+x^4+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+x^{4+4+4+4+4}+\cdots)\cdots \\
&=1+x+(x^{1+1}+x^2)+(x^{1+1+1}+x^{1+2}+x^3) \\
&+(x^{1+1+1+1}+x^{1+1+2}+x^{2+2}+x^{1+3}+x^4)+\cdots \\
&=1+x+2x^2+3x^3+5x^4+7x^5+11x^6+\cdots \\
&=\sum_{n=0}^{\infty}p(n)x^n. \qquad (1.6)
\end{aligned}$$

This is exactly the generating function for $p(n)$ because coefficient of x^n is the number of partions of n. As could be seen from this example, generating functions by themselves tell us nothing about how to find the coefficient of x^n in the series expansion. We need some additional tools to extract coefficients for each n. The Circle Method, developed by G. Hardy and S. Ramanujan and improved by H. Rademacher, will allow us to do precisely this. In chapter 2 we use this method to find the formula for the coefficient of x^n for one of the Ramanujan-type identities.

CHAPTER 2

Formula for $\tilde{p}(n)$

Let

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m} \tag{2.1}$$

be Euler's generating function for $p(n)$. H. Rademacher used the classical circle method to find the coefficients of q^n. There are many other infinite products to which this method could be applied. We introduce one of these infinite products here and derive the formula for the coefficients of q^n. Define

$$G(q) := \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \frac{(f(q^2))^2}{f(q)}. \tag{2.2}$$

Let $\tilde{p}(j)$ denote the coefficient of q^j in the expansion of $G(q)$, i.e.

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j. \tag{2.3}$$

Our goal is to find a closed expression for $\tilde{p}(j)$. We are about to use the classical circle method to achieve this goal. It is a long, tedious calculation, many steps of which have been implemented in a Mathematica package written by the author and his advisor. See Appendix A for the Mathematica code for this package. At key points of this chapter, the Mathematica package is demonstrated with input and output enclosed in a box.

We begin with some preliminaries.

Farey fractions. The sequence $\mathcal{F}_N$ of *proper Farey fractions of order N* is the set of all h/k with $(h,k)=1$ and $0 \le h/k < 1, k \le N$ arranged in increasing order, where (h,k) denotes the $\gcd(h,k)$. Thus, e.g., $\mathcal{F}_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$.

For a given N, let h_p, h_s, k_p and k_s be such that $\dfrac{h_p}{k_p}$ is the immediate predecessor of

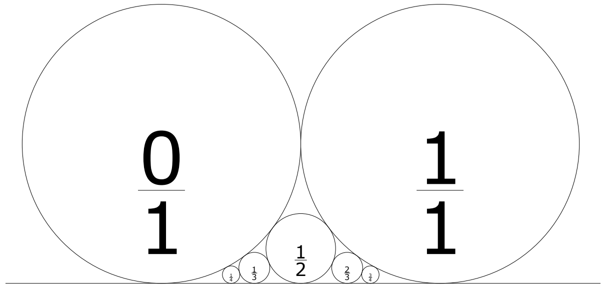

Figure 2.1: Ford Circles of order 4.

$\dfrac{h}{k}$ and $\dfrac{h_s}{k_s}$ is the immediate successor of $\dfrac{h}{k}$ in $\mathcal{F}_N$ and $\dfrac{h}{k} = \dfrac{h_p + h_s}{k_p + k_s}$.

Ford circles and Rademacher path. Let h and k be as above. The *Ford circle* $C(h, k)$ [7] is the circle in $\mathbb{C}$ of radius $\dfrac{1}{2k^2}$ centered at the point

$$\frac{h}{k} + \frac{1}{2k^2} i.$$

The *upper arc* $\gamma(h, k)$ *of the Ford circle* $C(h, k)$ is those points of $C(h, k)$ from the initial point

$$I_{h,k} := \frac{h}{k} - \frac{k_p}{k(k^2 + k_p^2)} + \frac{1}{k^2 + k_p^2} i \tag{2.4}$$

to the terminal point

$$T_{h,k} := \frac{h}{k} + \frac{k_s}{k(k^2 + k_s^2)} + \frac{1}{k^2 + k_s^2} i \tag{2.5}$$

traversed in the negative direction. We note that

$$I_{0,1} = T_{N-1,N}.$$

Every Ford circle is in the upper half plane. For $\dfrac{h_1}{k_1}, \dfrac{h_2}{k_2} \in \mathcal{F}_N$, $C(h_1, k_1)$ and $C(h_2, k_2)$ are either tangent or do not intersect [7].

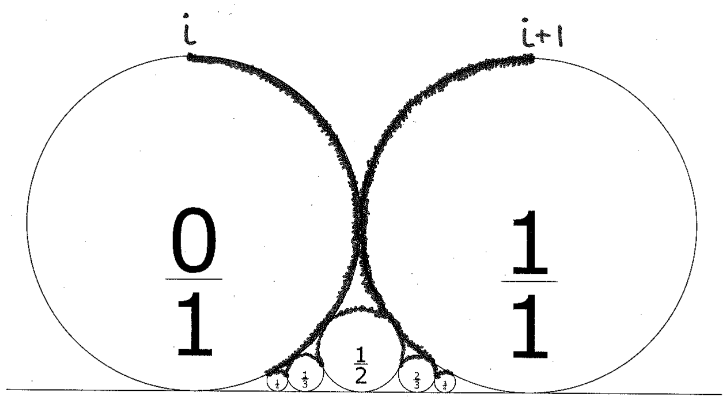

Figure 2.2: Rademacher path of order 4.

The *Rademacher path P(N) of order N* is the path in the upper half of the τ-plane from i to $i+1$ consisting of

$$\bigcup_{\frac{h}{k}\in\mathcal{F}_N} \gamma(h,k) \tag{2.6}$$

traversed left to right in negative direction.

We note that the left half of the Ford circle $C(0,1)$ and the corresponding upper arc $\gamma(0,1)$ are shifted to the right by 1 unit. This is justified since the function that is to be integrated over Rademacher path is periodic.

Convergence and Cauchy Residue Theorem.

Considering q as a complex variable in

$$\prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \prod_{m=1}^{\infty} \frac{1}{(1+q^m)(1-q^{2m})} = \prod_{m=1}^{\infty} \frac{1}{(1-(-q^m))(1-q^{2m})} \tag{2.7}$$

we see from the right hand side that infinite product and thus also infinite series are convergent for $|q| < 1$ since

$$\sum_{n=0}^{\infty} (q^k)^n = \frac{1}{(1-q^k)} \tag{2.8}$$

is a geometric series which converges for $|q| < 1$ for any fixed $k \geq 1$.

Next, we note that from

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j) q^j, \tag{2.9}$$

we get that

$$\frac{G(q)}{q^{n+1}} = \sum_{j=0}^{\infty} \frac{\tilde{p}(j) q^j}{q^{n+1}} \quad \text{if} \quad 0 < |q| < 1. \tag{2.10}$$

The series on the right of (2.10) is a Laurent series of $\dfrac{G(q)}{q^{n+1}}$. It has a pole of order $n+1$ at $q = 0$ with residue $\tilde{p}(n)$. Applying Cauchy's Residue Theorem we get that

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_C \frac{G(q)}{q^{n+1}}\, dq = \frac{1}{2\pi i} \int_C \frac{(f(q^2))^2}{f(q) q^{n+1}}\, dq, \tag{2.11}$$

where C is any positively oriented simple closed countour lying inside the unit circle. The change of the variable $q = e^{2\pi i \tau}$ maps the unit disk $|q| < 1$ into an infinite vertical strip of width 1 in the τ-plane. To see this we note that from $q = e^{2\pi i \tau}$ we get $\log q = 2\pi i \tau$, so $\tau = \dfrac{\log q}{2\pi i}$. Choosing the branch cut to be [0,1], we get

$$\tau = \frac{\log |q|}{2\pi i} + \frac{\operatorname{Arg}(q)}{2\pi}. \tag{2.12}$$

As q traverses a circle centered at $q = 0$ of radius $e^{-2\pi}$ in the positive direction, the point τ varies from i to $i+1$ along a horizontal segment as could be easly deduced from (2.12).

Replacing the segment by the Rademacher path composed of upper arcs of the Ford circles formed by the Farey series $\mathcal{F}_N$, (2.11) becomes

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_i^{i+1} \frac{(f(e^{4\pi i \tau}))^2 2\pi i e^{2\pi i \tau}}{f(e^{2\pi i \tau}) e^{2\pi i \tau (n+1)}}\, d\tau, \tag{2.13}$$

which simplifies to

$$\tilde{p}(n) = \int_{i}^{i+1} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau, \tag{2.14}$$

$$= \int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau, \tag{2.15}$$

The above can be written as

$$\int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau = \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ (h,k)=1}} \int_{\gamma(h,k)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n}\, d\tau \tag{2.16}$$

where $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$. Consider another change of variable

$$\tau = \frac{h}{k} + \frac{iz}{k}, \tag{2.17}$$

so that

$$z = -\, ik\left(\tau - \frac{h}{k}\right) \tag{2.18}$$

$$dz = -\, ik\, d\tau. \tag{2.19}$$

Under this transformation the Ford circle $C(h,k)$ in the τ-plane with center at $\frac{h}{k} + i\frac{1}{2k^2}$ and radius $\frac{1}{2k^2}$ is mapped to a negatively oriented circle C_k in the z-plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$. This follows from the fact that any point on the Ford circle $C(h,k)$ is given by

$$\tau = \left(\frac{h}{k} + i\frac{1}{2k^2}\right) + \frac{1}{2k^2}e^{i\theta}, \quad 0 \le \theta < 2\pi. \tag{2.20}$$

Substitution of (2.20) into (2.18) gives

$$z = \frac{1}{2k} + \frac{1}{2k}(-ie^{i\theta}) \tag{2.21}$$

which is a circle centered at $\frac{1}{2k}$ with radius $\frac{1}{2k}$. Now we make change of variable in (2.16). This gives

$$\tilde{p}(n) = i\sum_{k=1}^{N} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \frac{(f(e^{4\pi i h/k-4\pi i z/k}))^2}{f(e^{2\pi i h/k-2\pi z/k})} e^{2\pi n z/k}\, dz \tag{2.22}$$

where the initial point

$$s_{h,k} = \frac{k}{k^2+k_p^2} + \frac{k_p}{k^2+k_p^2} i \tag{2.23}$$

and the terminal point

$$t_{h,k} = \frac{k}{k^2+k_s^2} - \frac{k_s}{k^2+k_s^2} i \tag{2.24}$$

both obtained from (2.4), (2.5) and (2.18) through change of variable.

Next, we note that

$$f(q) = f(e^{2\pi i \tau}) = \frac{e^{\pi i \tau/12}}{\eta(\tau)}, \tag{2.25}$$

where $\eta(\tau)$ is the Dedekind eta function. Rewriting modular functional equation [3, p. 96] for $\eta(\tau)$ in terms of $f(q) = f(e^{2\pi i \tau}) = f(e^{2\pi i h/k-2\pi z/k})$ we get

$$f(e^{2\pi i h/k-2\pi z/k}) = \omega(h,k)\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)\sqrt{z} f\left(\exp\left(2\pi i \frac{iz^{-1}+H}{k}\right)\right) \tag{2.26}$$

where

$$\omega(h,k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right)\right), \tag{2.27}$$

and $hH \equiv -1 \pmod{k}$, $(h,k)=1$.

To evaluate (2.22) we would like to express

$$G(q) = G(e^{2\pi i \tau}) = G(e^{2\pi i h/k-2\pi z/k}) = \frac{(f(e^{4\pi i h/k-4\pi i z/k}))^2}{f(e^{2\pi i h/k-2\pi z/k})} \tag{2.28}$$

in the same way we did for $f(q)$ above. Two cases have to be considered: $(k,2)=1$ and $(k,2)=2$. When $(k,2)=1$ we will replace h by $2h$ and z by $2z$, and when $(k,2)=2$, k will be replaced by $k/2$ in order to obtain $f(q^2)$ from $f(q)$. Hence, we have

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}\dfrac{\omega^2(2h,k)e^{\frac{\pi((2z)^{-1}-2z)}{6k}}2zf^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{\omega(h,k)e^{\frac{\pi(z^{-1}-z)}{12k}}\sqrt{z}f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if}\quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)e^{\frac{\pi(z^{-1}-z)}{3k}}zf^2(e^{4\pi i(iz^{-1}+H_1)/k})}{\omega(h,k)e^{\frac{\pi(z^{-1}-z)}{12k}}\sqrt{z}f(e^{2\pi i(iz^{-1}+H_1)/k})}, & \text{if}\quad (k,2)=2,\end{cases} \tag{2.29}$$

which simplifies to

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}2\dfrac{\omega^2(2h,k)}{\omega(h,k)}e^{-\frac{\pi z}{4k}}\sqrt{z}\dfrac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if}\quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)}{\omega(h,k)}e^{\frac{\pi(z^{-1}-z)}{4k}}\sqrt{z}\dfrac{f^2(e^{4\pi i(iz^{-1}+H_1)/k})}{f(e^{2\pi i(iz^{-1}+H_1)/k})}, & \text{if}\quad (k,2)=2,\end{cases} \tag{2.30}$$

or, equivalently

$$G(e^{2\pi ih/k-2\pi z/k})=\begin{cases}2\dfrac{\omega^2(2h,k)}{\omega(h,k)}e^{-\frac{\pi z}{4k}}\sqrt{z}\dfrac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})}, & \text{if}\quad (k,2)=1,\\ \dfrac{\omega^2(h,k/2)}{\omega(h,k)}e^{\frac{\pi(z^{-1}-z)}{4k}}\sqrt{z}G(e^{2\pi i(iz^{-1}+H_1)/k}), & \text{if}\quad (k,2)=2,\end{cases} \tag{2.31}$$

where $hH_j\equiv -1 \pmod k$ and $j|H_j$ for $j=1,2$.

The implementation of the above in Mathematica is:

```
In[1]:= ModularRHS[h, k, z, ω, f, qprime,{{1,-1},{2,2}}]
```

$$\text{Out[1]=} \begin{pmatrix} 1, & \left\{\frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{z}, -\frac{\pi z}{4k}, \frac{f(\sqrt{\mathtt{qprime}})^2}{f(\mathtt{qprime})}\right\} \\ 2, & \left\{\frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \sqrt{z}, \frac{\pi}{4kz} - \frac{\pi z}{4k}, \frac{f\left(\mathtt{qprime}^2\right)^2}{f(\mathtt{qprime})}\right\} \end{pmatrix}$$

We return to evaluation of (2.22). To proceed we note that

$$G(e^{2\pi i(iz^{-1}+H_1)/k}) = 1 + \{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\}. \tag{2.32}$$

Rewriting (2.22) in terms of (2.31) and (2.32) we obtain

$$\begin{aligned}
\tilde{p}(n) =& i \sum_{\substack{k=1 \\ (k,2)=1}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} 2\frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})}\, dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{1 + (G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1)\} \\
&\times e^{(\frac{\pi z}{k}(2n-\frac{1}{4}) + \frac{\pi}{4zk})}\, dz \\
=& 2i \sum_{\substack{k=1 \\ (k,2)=1}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})}\, dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^{N} k^{-1} \sum_{\substack{0 \le h < k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} (J_1(h,k) + J_2(h,k)),
\end{aligned} \tag{2.33}$$

where

$$J_1(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} e^{(\frac{\pi z}{k}(2n-\frac{1}{4}) + \frac{\pi}{4zk})}\, dz \tag{2.34}$$

and

$$J_2(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\}e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})}\,dz. \tag{2.35}$$

We will estimate the first term in (2.33) and will show that it is small for large N. To do this we change variable again by letting $\xi = zk$. Then the first term in (2.33) becomes

$$2i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\,d\xi \tag{2.36}$$

where

$$s^*_{h,k} = \frac{k^2}{k^2+k_p^2} + \frac{kk_p}{k^2+k_p^2}i \tag{2.37}$$

and

$$t^*_{h,k} = \frac{k^2}{k^2+k_s^2} - \frac{kk_s}{k^2+k_s^2}i \tag{2.38}$$

are initial and terminal points obtained from (2.23) and (2.24) respectively. Under this change of variable circle C_k in z-plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$ is mapped to a circle C^*_k in ξ-plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$. Note also that the mapping $w = \frac{1}{\xi}$ maps the circle C^*_k and its interior onto a half-plane $\Re(w) \ge 1$ (where $\Re(w)$ denotes the real part of complex variable w and $\Im(w)$ is the imaginary part). For, from elementary complex analysis we have that $\Re(w) = \frac{x}{x^2+y^2}$ and $\Im(w) = \frac{-y}{x^2+y^2}$, where $x+iy = \xi$. It is readily seen that the segment $0 < x \le 1$ in the ξ-plane is mapped to an infinite strip $[1,\infty)$ in the w-plane. So, it follows that inside and on the circle C^*_k we have that $0 < \Re(\xi) \le 1$ and $\Re\left(\frac{1}{\xi}\right) \ge 1$. We now show that $\Re\left(\frac{1}{\xi}\right) = 1$

on the circle C_k^* . To see this note that in the polar form $\xi = \frac{1}{2} + \frac{1}{2}e^{i\theta}$ on C_k^*, $0 \leq \theta \leq 2\pi$. From this we get that

$$\begin{aligned}\frac{1}{\xi} =& \frac{2}{1+e^{i\theta}} = \frac{2}{(1+\cos\theta)+i\sin\theta}\\ =& \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta}\\ =& \frac{2(1+\cos\theta)}{2+2\cos\theta} - i\frac{2\sin\theta}{2+2\cos\theta}\\ =& 1 - i\frac{\sin\theta}{1+\cos\theta}. \end{aligned} \tag{2.39}$$

so, $\Re\left(\frac{1}{\xi}\right) = 1$.

Furthermore, we may move path of integration from the arc joining $s_{h,k}^*$ and $t_{h,k}^*$ to a segment connecting these two points on the circle C_k^*. By [3, p. 104, Theorem 5.9] the length of the path of integration is bounded by $2\sqrt{2}k/N$, and on the segment connecting $s_{h,k}^*$ and $t_{h,k}^*$, $|\xi| < \sqrt{2}k/N$.

Next, let us define $\tilde{p}^*(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^*(m)q^m = \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})}. \tag{2.40}$$

which is a part of the integrand in (2.36). Then, estimating the integrand in (2.36) we get

$$
\begin{aligned}
&\left|\sqrt{\xi}e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|-1+\frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})}\right| \\
&= |\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(\frac{2\pi i m(i(\frac{\xi}{k})^{-1}+H_2)}{k}\right)\right| \\
&= |\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right| \times \left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(-\frac{2\pi m}{\xi}\right)\exp\left(\frac{2\pi i m H_2}{k}\right)\right| \\
&\le |\xi|^{1/2}\, e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}\left|\sum_{m=1}^{\infty}\tilde{p}^*(m)\exp\left(-\frac{2\pi m}{\xi}\right)\exp\left(\frac{2\pi i m H_2}{k}\right)\right| \\
&\le |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\exp\left(-2\pi m\,\Re\left(\frac{1}{\xi}\right)\right) \\
&\le |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, e^{-2\pi m} \\
&= |\xi|^{1/2}\, e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, y^m, \quad (\text{where} \quad y=e^{-2\pi}) \\
&= c|\xi|^{1/2},
\end{aligned} \tag{2.41}
$$

where

$$
c = e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^*(m)|\, y^m. \tag{2.42}
$$

Note that c does not depend on ξ or N. It depends on n, but n remains fixed in the above analysis. So,

$$
\begin{aligned}
\left|\int_{s^*(h,k)}^{t^*(h,k)}\sqrt{\xi}\frac{f^2\big(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k}\big)}{f\big(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k}\big)}e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\,d\xi\right| &\le c|\xi|^{1/2} \le c\left(\frac{\sqrt{2}k}{N}\right)^{1/2}\frac{2\sqrt{2}N}{N} \\
&< \alpha k^{3/2}N^{-3/2}
\end{aligned} \tag{2.43}
$$

for some constant α and we have that

$$\left| 2i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi}\, \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\, d\xi \right|$$

$$\le 2\sum_{k=1}^{N} \sum_{\substack{0\le h<k\\(h,k)=1}} \alpha k^{-1} N^{-3/2}$$

$$\le 2\alpha N^{-3/2} \sum_{k=1}^{N} 1 = 2\alpha N^{-1/2}. \tag{2.44}$$

This completes the estimation of the first term in (2.33). We proceed to the second term. First, we will show that

$$J_2(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})}\, dz$$

is small for large N. Making change of variable $\xi = zk$ as before, we get that

$$J_2(h,k) = k^{-3/2} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\{G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\, d\xi$$

where $s^*_{h,k}$ and $t^*_{h,k}$ are as in (2.37) and (2.38) respectively. As before, we define $\tilde{p}^{**}(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^{**}(m) q^m = G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}) - 1. \tag{2.45}$$

Then, estimating the integrand, we see that

$$\begin{aligned}
&\left|\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\right| \times \left|G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k})-1\right| \\
=&|\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}e^{\frac{\pi}{4\xi}}\right| \times \left|\sum_{m=0}^{\infty}\tilde{p}^{**}(m)\exp\left(\frac{2\pi im(\frac{ik}{\xi}+H_1)}{k}\right)-1\right| \\
\leq&|\xi|^{1/2}e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}e^{\frac{\pi}{4}\Re(\frac{1}{\xi})}\left|\sum_{m=1}^{\infty}\tilde{p}^{**}(m)\exp\left(\frac{-2\pi m}{\xi}\right)\exp\left(\frac{2\pi imH_1}{k}\right)\right| \\
\leq&|\xi|^{1/2}e^{2\pi n}e^{\frac{\pi}{4}\Re(\frac{1}{\xi})}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(-2\pi m\Re\left(\frac{1}{\xi}\right)\right) \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(\left(-2\pi m+\frac{\pi}{4}\right)\Re\left(\frac{1}{\xi}\right)\right) \\
\leq&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\exp\left(\left(-2\pi m+\frac{\pi}{4}\right)\right) \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(m)|\,e^{-\frac{\pi}{4}(8m-1)} \\
\leq&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,e^{-\frac{\pi}{4}(8m-1)} \\
=&|\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,x^{8m-1}, \quad (\text{where} \quad x=e^{-\frac{\pi}{4}}) \\
=&b|\xi|^{1/2}, \qquad (2.46)
\end{aligned}$$

where

$$b=e^{2\pi n}\sum_{m=1}^{\infty}|\tilde{p}^{**}(8m-1)|\,x^{8m-1}.$$

Note that b does not depend on ξ or N. It depends on n, but n is fixed. It follows, therefore, that

$$|J_2(h,k)| \leq b\left(\frac{\sqrt{2}k}{N}\right)^{1/2}\frac{2\sqrt{2}N}{N} < \beta k^{3/2}N^{-3/2} \qquad (2.47)$$

for some constant β. Then we have that

$$\left| i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_2(h,k) \right| < \sum_{k=1}^{N} \sum_{\substack{0\le h<k\\(h,k)=1}} \beta k^{-1} N^{-3/2}$$
$$\le \beta N^{-3/2} \sum_{k=1}^{N} 1 = \beta N^{-1/2}. \tag{2.48}$$

Combining the results from (2.44) and (2.48) we have that

$$\begin{aligned}\tilde{p}(n) =& i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_1(h,k) + O(\beta N^{-1/2} + 2\alpha N^{-1/2})\\ =& i\sum_{k=1}^{N} k^{-5/2} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} J_1(h,k) + O(N^{-1/2}).\end{aligned} \tag{2.49}$$

Finally, we turn our attention to

$$J_1(h,k) = k^{-3/2} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi}\right)} \, d\xi. \tag{2.50}$$

We note that

$$J_1(h,k) = \int_{C^*_k} - \int_{s^*_{h,k}}^{0} - \int_{0}^{t^*_{h,k}} = \int_{C^*_k} - S_1 - S_2, \tag{2.51}$$

where C^*_k is a circle in the ξ-plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, as before. It is easily seen that the length of the arc connecting 0 and $s^*_{h,k}$ is less then

$$2\pi \frac{|s^*_{h,k}|}{2} \le \pi |s^*_{h,k}| \le \pi\sqrt{2}\frac{k}{N} \tag{2.52}$$

From the disscussion above we know that $\Re\left(\frac{1}{\xi}\right) = 1$ and $0 < \Re(\xi) \le 1$ on C^*_k. So, the integrand in S_1 could be estimated as

$$\begin{aligned}\left|\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\right| =&|\xi|^{1/2}\left|e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})}\right|\left|e^{\frac{\pi}{4\xi}}\right| \\ =&|\xi|^{1/2}e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)}e^{\frac{\pi}{4}\Re\left(\frac{1}{\xi}\right)} \\ \leq&2^{1/4}\frac{k^{1/2}}{N^{1/2}}e^{2\pi n}e^{\frac{\pi}{4}}.\end{aligned} \tag{2.53}$$

Combining the rsults in (2.52) and (2.53) we get

$$|S_1| < \gamma k^{3/2}N^{-3/2}, \tag{2.54}$$

where γ is a constant. We can obtain similar estimate for S_2 and, as before, we get an error term $O(N^{-1/2})$ in the formula for $\tilde{p}(n)$. Therefore, we can write

$$\tilde{p}(n) = i\sum_{k=1}^{N} k^{-5/2}\sum_{\substack{0\leq h<k \\ (h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}\int_{C_k^*}\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\,d\xi + O(N^{-1/2}). \tag{2.55}$$

Letting $N\to\infty$ we have that

$$\tilde{p}(n) = i\sum_{k=1}^{\infty} k^{-5/2}\sum_{\substack{0\leq h<k \\ (h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}\int_{C_k^*}\sqrt{\xi}e^{(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi})}\,d\xi. \tag{2.56}$$

The above could be implemented in Mathematica as:

In[2]:=ZToZetaIntegrandComponents[n, h, k, z, ξ, ω,{{1,-1},{2,2}}]

Out[2]= $\begin{pmatrix} 1, & \left\{\frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2}\right\} \\ 2, & \left\{\frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2} + \frac{\pi}{4\zeta}\right\} \end{pmatrix}$

We introduce another change of varaible

$$\xi = \frac{1}{w}, \quad d\xi = -\frac{1}{w^2}.$$

Then (2.56) becomes

$$\tilde{p}(n) = \frac{1}{i}\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{1-\infty i}^{1+\infty i} w^{-5/2} e^{\left(\frac{\pi}{k^2}\left(2n-\frac{1}{4}\right)\frac{1}{w}+\frac{\pi w}{4}\right)}\, dw. \tag{2.57}$$

Let $t = \dfrac{\pi w}{4}$ in (2.57), then the above becomes

$$\tilde{p}(n) = 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \frac{1}{2\pi i} \int_{\pi/4-\infty i}^{\pi/4+\infty i} t^{-5/2} e^{\left(t+\frac{\pi^2}{4k^2}\left(2n-\frac{1}{4}\right)\frac{1}{t}\right)}\, dt. \tag{2.58}$$

The implementation of the above in Mathematica is:

In[3]:= IntegrandInT[n, h, k, ξ, $d\xi$, ω, t, dt,{{1,-1},{2,2}}]

Out[3]= $\begin{pmatrix} 2, & \left\{-\frac{1}{8}ie^{-\frac{2ihn\pi}{k}}\left(\frac{1}{k}\right)^{5/2}\pi^{3/2}, \frac{8n\pi^2-\pi^2}{16k^2t}+t, -\frac{5}{2}, \frac{\omega\left(h,\frac{k}{2}\right)^2}{\omega(h,k)}, \mathtt{dt}\right\} \end{pmatrix}$

In Watson's Treatise on Bessel functions [13, p. 181] we find a formula equivalent to the following:

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-\nu-1}e^{t+(z^2/4t)}\,dt \quad ,(\text{if} \quad c>0, \quad \Re(\nu)>0). \tag{2.59}$$

Let

$$\frac{z}{2} = \left\{\frac{\pi^2}{4k^2}\left(2n-\frac{1}{4}\right)\right\}^{1/2} \tag{2.60}$$

and $\nu = 3/2$. Then we have

$$\begin{aligned}\tilde{p}(n) =& 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1}^{\infty}k^{-5/2}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}\frac{\pi^{-3/2}\left(2n-\frac{1}{4}\right)^{-3/4}}{4^{-3/4}k^{-3/2}}I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right)\\ =& \frac{2\pi\left(2n-\frac{1}{4}\right)^{-\frac{3}{4}}}{\sqrt{8}}\sum_{k=1}^{\infty}k^{-1}\sum_{\substack{0\le h<k\\(h,k)=1}}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k}I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right).\end{aligned} \tag{2.61}$$

The implementation of the above in Mathematica is:

```
In[4]:= RawFormula[n, h, k, ω, {{1,-1},{2,2}}]
```

$$\texttt{Out[4]=}\left(2, \quad \frac{2e^{-\frac{2ihn\pi}{k}}\pi\texttt{BESSELI}\left(\frac{3}{2},\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{2}\right)^2}{k(8n-1)^{3/4}\omega(h,k)}\right)$$

Note that Bessel functions of this order can be expressed as

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right). \tag{2.62}$$

Expanding (2.62) we have that

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \left(\frac{\cosh z}{z} - \frac{\sinh z}{z^2} \right) \tag{2.63}$$

Substituting (2.60) into (2.63), we get

$$\begin{aligned}
I_{\frac{3}{2}}(z) =& I_{\frac{3}{2}} \left(\frac{\pi}{k} \left(2n - \frac{1}{4} \right)^{\frac{1}{2}} \right) = I_{\frac{3}{2}} \left(\frac{\pi\sqrt{8n-1}}{2k} \right) \\
=& \sqrt{\frac{2\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi}} \left(\frac{\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\left(\frac{\pi\sqrt{8n-1}}{2k}\right)} - \frac{\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\left(\frac{\pi\sqrt{8n-1}}{2k}\right)^2} \right) \\
=& \frac{(8n-1)^{1/4}}{\sqrt{k}} \left(\frac{2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\frac{\pi}{k}\sqrt{8n-1}} - \frac{\frac{4k}{\pi}\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\frac{\pi}{k}(8n-1)} \right) \\
=& \frac{1}{\pi\sqrt{\frac{\sqrt{8n-1}}{k}}} \left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}} \right)
\end{aligned} \tag{2.64}$$

Multiplying (2.64) by

$$\frac{2\pi\left(2n - \frac{1}{4}\right)^{-\frac{3}{4}}}{k\sqrt{8}} = \frac{2\pi}{k(8n-1)^{\frac{3}{4}}} \tag{2.65}$$

we get

$$\frac{2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}}. \tag{2.66}$$

Finally, we rewrite (2.61) in terms of (2.66) to get

$$\tilde{p}(n) = \sum_{k=1}^{\infty} k^{-1} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \frac{2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}} \tag{2.67}$$

which is in agreement with the formula obtained using our Mathematica package.

$$\texttt{In[5]:=RTF}\left[\frac{\texttt{QPochhammer[q,q]}}{\texttt{QPochhammer}[\texttt{q}^2,\texttt{q}^2]^2}, \texttt{p}, \texttt{3.14}\right]$$

$$\text{If } \prod_{n=1}^{\infty} \frac{1-q^n}{(1-q^{2n})^2} = \sum_{n=0}^{\infty} p_{3.14}(n) q^n$$

, then . . .

$$\texttt{Out[5]} = p_{3.14}(n)$$

$$= \sum_{k=1}^{\infty} \sum_{h=0}^{k-1} \frac{2e^{-\frac{2ihn\pi}{k}} \left\lfloor \frac{1}{\gcd(h,k)} \right\rfloor \left(2\cosh\left(\frac{\sqrt{8n-1}\pi}{2k}\right) - \frac{4k\sinh\left(\frac{\sqrt{8n-1}\pi}{2k}\right)}{\sqrt{8n-1}\pi}\right)}{\sqrt{k}(8n-1)\omega(h,k)}$$

$$\times \chi(\gcd(2,k)=2)\omega\left(h, \frac{k}{2}\right)^2$$

CHAPTER 3

RADEMACHER-TYPE FORMULAE FOR FOURIER COEFFICIENTS OF RAMANUJAN-TYPE SERIES

3.1 Ramanujan-type identities

There are numerous Ramanujan-type infinite products. Each product could be represented as an infinite series in veraible q. We use our Mathematica package to extract the coefficients of q^n for the given infinite product.

In his lost notebook [1, 2], Ramanujan recorded these identities:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^n)(1-q^{12n})} \qquad [1, \text{Entry 11.3.1, p. 254}] \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})(1-q^{12n})}{(1-q^n)(1-q^{6n})} \qquad [1, \text{Entry 11.3.2, p. 254}] \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{12n})^2}{(1-q^n)^2(1-q^{6n})(1-q^{4n})} \qquad [1, \text{Entry 11.3.3, p. 254}] \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q;q)_n(-q;q)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^n)(1-q^{3n})} \qquad [1, \text{Entry 11.3.4, p. 255}] \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})^5}{(1-q^n)(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^2} \qquad [1, \text{Entry 11.3.5, p. 255}] \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{(-q)^n(q;q^2)_n}{(-q;-q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2(1-q^n)(1-q^{4n})^2}{(1-q^{2n})^4(1-q^{6n})} \qquad [1, \text{Entry 11.3.5, p. 255}] \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{2n})(1-q^{6n})} \qquad [2, \text{Entry 5.3.3, p. 102}] \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{6n})^2}{(1-q^{2n})(1-q^{3n})(1-q^{18n})} \qquad \text{[2, Entry 5.3.4, p. 103]} \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{3n})^2}{(1-q^{n})(1-q^{4n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{4n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^2}{(1-q^{3n})(1-q^{4n})} \qquad \text{[2, Entry 5.3.7, p. 105]} \tag{3.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{8n})(1-q^{2n})^2(1-q^{12n})^2}{(1-q^{24n})(1-q^{4n})^2(1-q^{n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.8, p. 105]} \tag{3.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q^3;q^6)_n}{(q^4;q^4)_n(q;q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})}{(1-q^{n})(1-q^{12n})} \qquad \text{[2, Entry 5.3.9, p. 106]} \tag{3.13}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{2n})^2} \qquad \text{[2, Entry 4.2.6, p. 84]} \tag{3.14}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{4n})} \qquad \text{[2, Entry 5.3.6, p. 104]} \tag{3.15}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2 q^{2n^2}}{(q^2;q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{2n})(1-q^{6n})} \qquad \text{[2, Entry 5.3.3, p. 102]} \tag{3.16}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{n})(1-q^{6n})^5}{(1-q^{3n})^2(1-q^{2n})^2(1-q^{12n})^2} \qquad \text{[2, Entry 4.2.7, p. 85]} \tag{3.17}$$

The following identities are due to Slater:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})} \qquad [8, p. 152, Eq. (3, 23)] \tag{3.18}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} \qquad [8, p. 152, Eq. (2)] \tag{3.19}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{(q^2;q^2)_n(-q;q^2)_{n+1}} = \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{4n})}{(1-q^{2n})^2} \qquad [8, p. 152, Eq. (5)] \tag{3.20}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})}{(1-q^{2n})} \qquad [8, p. 152, Eq. (7)] \tag{3.21}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n(n+1)}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})^5}{(1-q^{2n})^2(1-q^{3n})^2(1-q^{12n})^2} \qquad [8, p. 152, Eq. (48)] \tag{3.22}$$

The following identities are due to Bowman, McLaughlin and Sills:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n^2}}{(q;q^2)_n(q^4;q^4)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{6n})^5}{(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^2} \qquad [5, p. 314, Eq. (2.19)] \tag{3.23}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n^2}}{(-q;q)_{2n}(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{(1-q^{3n})^2}{(1-q^{6n})(1-q^{2n})} \qquad [5, p. 314, Eq. (2.19)] \tag{3.24}$$

The following identity is due to Starcher

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)^2} \qquad [12, p. 805, Eq. (3.6)] \tag{3.25}$$

3.2 Formulae for the Fourier Coefficients of the series presented in 3.2

Let $\Pi_j(q)$ represent the right hand side of Eq.(j), where j runs from (3.1) to (3.25) and let $\tilde{p}_j(n)$ be defined by

$$\sum_{n=0}^{\infty} \tilde{p}_j(n) q^n = \Pi_j(q).$$

Our Mathematica package conjectures the following Rademacher-type formulae. These formulae appear to be new:

$$\tilde{p}_{3.1}(n) = \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{4\sqrt{3}k}\right) \omega(12h,k)\omega(h,k)}{k\sqrt{24n-1}\quad \omega(6h,k)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=12}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right) \omega\left(h,\frac{k}{12}\right)\omega(h,k)}{k\sqrt{24n-1}\quad \omega\left(h,\frac{k}{6}\right)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right) \omega\left(3h,\frac{k}{4}\right)\omega(h,k)}{\sqrt{3}k\sqrt{24n-1}\quad \omega\left(3h,\frac{k}{2}\right)^2}. \tag{3.26}$$

$$\tilde{p}_{3.2}(n) = \sum_{\substack{k=1\\(12,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(h,\frac{k}{6}\right)\omega(h,k)}{2k\sqrt{3n+1}\quad \omega\left(h,\frac{k}{3}\right)\omega\left(2h,\frac{k}{6}\right)}$$
$$+ \sum_{\substack{k=1\\(12,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(3h,\frac{k}{2}\right)\omega(h,k)}{2\sqrt{3}k\sqrt{3n+1}\quad \omega(3h,k)\omega\left(6h,\frac{k}{2}\right)}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{3n+1}\pi}{\sqrt{6}k}\right) \omega(6h,k)\omega(h,k)}{4k\sqrt{3n+1}\quad \omega(3h,k)\omega(12h,k)}. \tag{3.27}$$

$$\tilde{p}_{3.3}(n) = \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right) \omega\left(2h,\frac{k}{3}\right)\omega(4h,k)\omega(h,k)^2}{8k\sqrt{3n+2}\quad \omega(2h,k)^2\omega\left(4h,\frac{k}{3}\right)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\sqrt{\frac{5}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right) \omega(4h,k)\omega(6h,k)\omega(h,k)^2}{8k\sqrt{3n+2}\quad \omega(2h,k)^2\omega(12h,k)^2}. \tag{3.28}$$

$$\tilde{p}_{3.4}(n) = \sum_{\substack{k=1\\(6,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right)\omega\left(h,\frac{k}{3}\right)\omega(h,k)}{2\sqrt{2}k\sqrt{3n+1}\quad\omega\left(2h,\frac{k}{3}\right)^2}$$
$$+\sum_{\substack{k=1\\(6,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right)\omega(3h,k)\omega(h,k)}{2\sqrt{6}k\sqrt{3n+1}\quad\omega(6h,k)^2}. \tag{3.29}$$

$$\tilde{p}_{3.5}(n) = \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right)\omega\left(h,\frac{k}{3}\right)^2\omega(h,k)\omega(2h,k)\omega\left(4h,\frac{k}{3}\right)^2}{2k\sqrt{120n+5}\quad\omega\left(2h,\frac{k}{3}\right)^5\omega(4h,k)}$$
$$+\sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right)\omega(h,k)\omega(2h,k)\omega(3h,k)^2\omega(12h,k)^2}{2\sqrt{3}k\sqrt{120n+5}\quad\omega(4h,k)\omega(6h,k)^5}. \tag{3.30}$$

$$\tilde{p}_{3.6}(n) = \sum_{\substack{k=1\\(12,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right)\omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{2}\right)^4}{k\sqrt{120n+5}\quad\omega\left(h,\frac{k}{3}\right)^2\omega(h,k)\omega\left(2h,\frac{k}{2}\right)^2}$$
$$+\sum_{\substack{k=1\\(12,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right)\omega\left(3h,\frac{k}{2}\right)\omega\left(h,\frac{k}{2}\right)^4}{\sqrt{3}k\sqrt{120n+5}\quad\omega(h,k)\omega\left(2h,\frac{k}{2}\right)^2\omega(3h,k)^2}. \tag{3.31}$$

$$\tilde{p}_{3.7}(n) = \sum_{\substack{k=1\\(6,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n-1}\quad\omega\left(h,\frac{k}{3}\right)^2}$$
$$+\sum_{\substack{k=1\\(6,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(3h,\frac{k}{2}\right)\omega\left(h,\frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1}\quad\omega(3h,k)^2}. \tag{3.32}$$

$$\tilde{p}_{3.8}(n) = \sum_{\substack{k=1 \\ (18,k)=18}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{18}\right) \omega\left(h, \frac{k}{3}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \quad \omega\left(h, \frac{k}{9}\right) \omega\left(h, \frac{k}{6}\right)^2}$$
$$+ \sum_{\substack{k=1 \\ (18,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{10\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{60n-5}\pi}{9k}\right) \omega(3h,k)\omega\left(9h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{3\sqrt{3}k\sqrt{60n-5} \quad \omega\left(3h, \frac{k}{2}\right)^2 \omega(9h,k)}$$
$$+ \sum_{\substack{k=1 \\ (18,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\sqrt{\frac{2}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{12n-1}\pi}{9k}\right) \omega(2h,k)\omega(3h,k)\omega(18h,k)}{3k\sqrt{12n-1} \quad \omega(6h,k)^2\omega(9h,k)}. \tag{3.33}$$

$$\tilde{p}_{3.9}(n) = \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{4\sqrt{3}k}\right) \omega(4h,k)\omega(6h,k)\omega(h,k)}{3k\sqrt{8n-1} \quad \omega(2h,k)\omega(3h,k)^2}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{4}\right) \omega(h,k)}{k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{3}\right)^2 \omega\left(h, \frac{k}{2}\right)}$$
$$+ \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right) \omega\left(3h, \frac{k}{2}\right) \omega(h,k)}{\sqrt{3}k\sqrt{8n-1} \quad \omega\left(h, \frac{k}{2}\right) \omega(3h,k)^2}. \tag{3.34}$$

$$\tilde{p}_{3.10}(n) = \sum_{\substack{k=1 \\ (4,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right)}{k\sqrt{8n-1}\omega(h,k)}. \tag{3.35}$$

$$\tilde{p}_{3.11}(n) = \sum_{\substack{k=1\\(12,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{2\sqrt{3}k}\right)\omega\left(h,\frac{k}{4}\right)\omega(3h,k)}{k\sqrt{24n+5}\quad \omega\left(3h,\frac{k}{2}\right)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{12k}\right)\omega(4h,k)\omega(3h,k)}{\sqrt{3}k\sqrt{24n+5}\quad \omega(6h,k)^2}$$
$$+ \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5}\pi}{12k}\right)\omega\left(h,\frac{k}{3}\right)\omega(4h,k)}{k\sqrt{24n+5}\quad \omega\left(2h,\frac{k}{3}\right)^2}. \tag{3.36}$$

$$\tilde{p}_{3.12}(n) = \sum_{\substack{k=1\\(24,k)=24}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{24}\right)\omega\left(h,\frac{k}{6}\right)\omega(h,k)\omega\left(h,\frac{k}{4}\right)^2}{k\sqrt{8n-1}\quad \omega\left(h,\frac{k}{12}\right)^2\omega\left(h,\frac{k}{8}\right)\omega\left(h,\frac{k}{2}\right)^2}$$
$$+ \sum_{\substack{k=1\\(24,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5}\sqrt{8n-1}\pi}{6k}\right)\omega(h,k)\omega\left(3h,\frac{k}{2}\right)\omega\left(6h,\frac{k}{4}\right)\omega\left(h,\frac{k}{4}\right)^2}{3k\sqrt{8n-1}\quad \omega\left(h,\frac{k}{2}\right)^2\omega\left(2h,\frac{k}{4}\right)\omega\left(3h,\frac{k}{4}\right)^2}$$
$$+ \sum_{\substack{k=1\\(24,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\sqrt{\frac{5}{2}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5}\sqrt{8n-1}\pi}{12k}\right)\omega(h,k)\omega(4h,k)^2\omega(6h,k)\omega(24h,k)}{3k\sqrt{8n-1}\quad \omega(2h,k)^2\omega(8h,k)\omega(12h,k)^2}$$
$$+ \sum_{\substack{k=1\\(24,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{4k}\right)\omega(h,k)\omega\left(2h,\frac{k}{3}\right)\omega(4h,k)^2\omega\left(8h,\frac{k}{3}\right)}{\sqrt{2}k\sqrt{8n-1}\quad \omega(2h,k)^2\omega\left(4h,\frac{k}{3}\right)^2\omega(8h,k)}. \tag{3.37}$$

$$\tilde{p}_{3.13}(n) = \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{3\sqrt{2}k}\right)\omega(12h,k)\omega(h,k)}{3k\sqrt{8n-1}\quad \omega(4h,k)\omega(6h,k)}$$
$$+ \sum_{\substack{k=1\\(12,k)=12}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{12}\right)\omega(h,k)}{k\sqrt{8n-1}\quad \omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{4}\right)}. \tag{3.38}$$

$$\tilde{p}_{3.14}(n) = \sum_{\substack{k=1 \\ (2,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2e^{-\frac{2i\pi hn}{k}}\omega\left(h,\frac{k}{2}\right)^2\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)-\frac{4k\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4}\sqrt{\frac{\sqrt{8n-1}}{k}}\quad\omega(h,k)}. \tag{3.39}$$

$$\tilde{p}_{3.15}(n) = \sum_{\substack{k=1 \\ (4,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{4}\right)}{k\sqrt{8n-1}\quad\omega(h,k)}. \tag{3.40}$$

$$\begin{aligned}\tilde{p}_{3.16}(n) = &\sum_{\substack{k=1 \\ (6,k)=6}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(h,\frac{k}{6}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n-1}\quad\omega\left(h,\frac{k}{3}\right)^2} \\ &+ \sum_{\substack{k=1 \\ (6,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right)\omega\left(3h,\frac{k}{2}\right)\omega\left(h,\frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1}\quad\omega(3h,k)^2}.\end{aligned} \tag{3.41}$$

$$\begin{aligned}\tilde{p}_{3.17}(n) = &\sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(3h,\frac{k}{4}\right)^2\omega(3h,k)^2\omega\left(h,\frac{k}{2}\right)^2}{\sqrt{3}k\sqrt{8n-1}\omega(h,k)\omega\left(3h,\frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{4e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2\sqrt{3}k}\right)\omega(3h,k)^2\omega\left(6h,\frac{k}{2}\right)^2\omega\left(h,\frac{k}{2}\right)^2}{3k\sqrt{8n-1}\omega(h,k)\omega\left(3h,\frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}}\pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right)\omega\left(h,\frac{k}{12}\right)^2\omega\left(h,\frac{k}{3}\right)^2\omega\left(h,\frac{k}{2}\right)^2}{k\sqrt{8n-1}\omega\left(h,\frac{k}{6}\right)^5\omega(h,k)}\end{aligned} \tag{3.42}$$

$$\tilde{p}_{3.18}(n) = \sum_{\substack{k=1 \\ (2,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{24n-1}\quad\omega(h,k)}. \tag{3.43}$$

$$\tilde{p}_{3.19}(n) = \sum_{\substack{k=1 \\ (2,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{6\sqrt{2}k}\right)\omega(h,k)}{k\sqrt{24n+1}\quad \omega(2h,k)}. \tag{3.44}$$

$$\tilde{p}_{3.20}(n) = \sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{3\sqrt{2}k}\right)\omega\left(h,\frac{k}{2}\right)^2}{k\sqrt{24n+1}\quad \omega(h,k)\omega\left(2h,\frac{k}{2}\right)}. \tag{3.45}$$

$$\begin{aligned}\tilde{p}_{3.21}(n) = &\sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{3\sqrt{2}k}\right)\omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n+1}\quad \omega\left(2h,\frac{k}{2}\right)} \\ &+ \sum_{\substack{k=1 \\ (4,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{6\sqrt{2}k}\right)\omega(2h,k)}{2k\sqrt{12n+1}\quad \omega(4h,k)}.\end{aligned} \tag{3.46}$$

$$\begin{aligned}\tilde{p}_{3.22}(n) = &\sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2}k}\right)\omega\left(h,\frac{k}{3}\right)^2\omega(2h,k)^2\omega\left(4h,\frac{k}{3}\right)^2}{4k\sqrt{n}\quad \omega\left(2h,\frac{k}{3}\right)^5\omega(4h,k)} \\ &+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{2}{3}}\sqrt{n}\pi}{k}\right)\omega\left(h,\frac{k}{2}\right)^2\omega(3h,k)^2\omega\left(6h,\frac{k}{2}\right)^2}{3k\sqrt{n}\quad \omega\left(2h,\frac{k}{2}\right)\omega\left(3h,\frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0\le h<k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2}k}\right)\omega(2h,k)^2\omega(3h,k)^2\omega(12h,k)^2}{4\sqrt{3}k\sqrt{n}\quad \omega(4h,k)\omega(6h,k)^5}.\end{aligned} \tag{3.47}$$

$$\tilde{p}_{3.23}(n) = \sum_{\substack{k=1\\(12,k)=3}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right) \omega(2h,k)\omega\left(4h,\frac{k}{3}\right)^2 \omega\left(h,\frac{k}{3}\right)^2}{k\sqrt{12n-1}\quad \omega\left(2h,\frac{k}{3}\right)^5}$$
$$+ \sum_{\substack{k=1\\(12,k)=12}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h,\frac{k}{12}\right)^2 \omega\left(h,\frac{k}{2}\right) \omega\left(h,\frac{k}{3}\right)^2}{k\sqrt{12n-1}\quad \omega\left(h,\frac{k}{6}\right)^5}$$
$$+ \sum_{\substack{k=1\\(12,k)=4}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h,\frac{k}{2}\right) \omega\left(3h,\frac{k}{4}\right)^2 \omega(3h,k)^2}{\sqrt{3}k\sqrt{12n-1}\quad \omega\left(3h,\frac{k}{2}\right)^5}$$
$$+ \sum_{\substack{k=1\\(12,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right) \omega(2h,k)\omega(3h,k)^2\omega(12h,k)^2}{\sqrt{3}k\sqrt{12n-1}\quad \omega(6h,k)^5}. \tag{3.48}$$

$$\tilde{p}_{3.24}(n) = \sum_{\substack{k=1\\(6,k)=6}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h,\frac{k}{6}\right) \omega\left(h,\frac{k}{2}\right)}{k\sqrt{12n-1}\quad \omega\left(h,\frac{k}{3}\right)^2}$$
$$+ \sum_{\substack{k=1\\(6,k)=2}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(3h,\frac{k}{2}\right) \omega\left(h,\frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1}\quad \omega(3h,k)^2}. \tag{3.49}$$

$$\tilde{p}_{3.25}(n) = \sum_{\substack{k=1\\(2,k)=1}}^{\infty} \sum_{\substack{0\le h<k\\(h,k)=1}} \frac{e^{-\frac{2ihn\pi}{k}} \left(2\cosh\left(\frac{\sqrt{n}\pi}{k}\right) - \frac{2k\sinh\left(\frac{\sqrt{n}\pi}{k}\right)}{\sqrt{n}\pi}\right) \omega(h,k)^2}{8\sqrt{k}n\omega(2h,k)}. \tag{3.50}$$

BIBLIOGRAPHY

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's lost notebook, part I*, Springer (2005).

[2] G. E. Andrews and B. C. Berndt, *Ramanujan's lost notebook, part II*, Springer (2009).

[3] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, 2d. edition, Springer (1990).

[4] J. Bruinier, K. Ono, *An algebraic formula for the partition function*, http://www.aimath.org/news/partition/brunier-ono.pdf.

[5] D. Bowman, J. McLaughlin, A. V. Sills, *Some more identities of the Rogers-Ramanujan type*, Ramanujan J. 18 (2009), 307–325.

[6] L. Euler, *Introduction to Analysis of the Infinite*, Book I, Springer (1988).

[7] L. R. Ford, *Fractions*, Amer. Math Monthly 45 (1938), 586–601.

[8] L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) 54 (1952), 147–167.

[9] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) 17 (1918, 75–115).

[10] H. Rademacher, *On the expansion of the partition function in a series*, Ann. of Math. (2) 44 (1943), 416–422. MR 5, 35a.

[11] H. Rademacher, *On the partition function $p(n)$*, Proc. London Math Soc. (2) 43 (1937), 241–254.

[12] G. W. Starcher, *On identities arising from solutions to q-difference equations and some interpretations in number theory*, Amer. J. Math. 53 (1931), 801–816.

[13] G. N. Watson, *A Treatise on The Theory of Bessel Functions*, 2d. edition, Cambridge, 1944.

Appendix A

RADEMACHER.M MATHEMATICA PACKAGE

```
Print["The Rademacher Package"]

Print["Version of January 19, 2011"]

Print["V. Kiria and A. Sills"]


FindL[ simplifiedinput_List]:=

Apply[LCM,PowersOfQ[simplifiedinput]]


PowersOfQ[simplifiedinput_List]:=

Table[ simplifiedinput[[i,1]] ,{i,1,Length[simplifiedinput]}]


PowersOfP[simplifiedinput_List]:=

Table[ simplifiedinput[[i,2]] ,{i,1,Length[simplifiedinput]}]


BigOmega[d_Integer,h_Symbol,k_Symbol,

omega_Symbol,simplifiedinput_List]:=

Module[{numfactors,ppowers,qpowers},

numfactors=Length[simplifiedinput];

ppowers=PowersOfP[simplifiedinput ];

qpowers=PowersOfQ[simplifiedinput];

Product[ (omega[h,k ]/.{

k-> k/GCD[d,qpowers[[i]]],

h->  qpowers[[i]]*h/GCD[d,qpowers[[i]]]

 }
```

```
)^ppowers[[i]]    ,{i,1,numfactors}]
]

BigZ[ d_Integer, z_Symbol, simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[ (Sqrt[z]/.{
z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]]
 }
)^ppowers[[i]]    ,{i,1,numfactors}]
]

BigExp[d_Integer,z_Symbol,k_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Collect[Simplify[
Sum[
ppowers[[i]]*(Pi (1/z - z)/12/k)/.
{k-> k/GCD[d,qpowers[[i]]],
 z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]]}
,{i,1,numfactors}
```

```
]
]
,{z,z^(-1)}
]
]

BigP[d_Integer,P_Symbol,qprime_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[  P[qprime^(GCD[d,qpowers[[i]]]^2/qpowers[[i]])]^ppowers[[i]]
  ,{i,1,numfactors}]
]

OnePieceModularRHS[d_Integer,h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
{BigOmega[d,h,k,omega,simplifiedinput],BigZ[ d, z, simplifiedinput],
BigExp[d,z,k,simplifiedinput],BigP[d,P,qprime,simplifiedinput]}

ModularRHS[h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
Module[{d,L},
L=FindL[ simplifiedinput];
d=Divisors[L];
```

```
Table[{d[[i]],OnePieceModularRHS[ d[[i]],  h,k,z,omega,
P,qprime,simplifiedinput]},{i,1, Length[d]}]
]

ZToZetaIntegrandComponents[n_,h_,k_,z_,
zeta_,omega_,simplifiedinput_]:=
Module[{extractchange,transstuff,P,Q},
transstuff=ModularRHS[h,k,z,omega,P,Q,simplifiedinput];
extractchange =
Table[
{transstuff[[i,1]],
{transstuff[[i,2,1]],
transstuff[[i,2,2]],2 Pi n z/k + transstuff[[i,2,3]]
}/.{z-> zeta/k}
} ,{i,Length[transstuff]}]
]

ExtractNonTrivialCases[n_,h_,k_,z_,zeta_,
omega_,simplifiedinput_]:=Module[{tmp},
tmp=ZToZetaIntegrandComponents[n,h,k,z,
zeta,omega,simplifiedinput];
Complement[
Table[
If[ Coefficient[tmp[[i,2,3]], zeta,-1]>0,
tmp[[i]]
```

```
]
, {i,1,Length[tmp]}],
{Null}
]
]

ZetaToTTransformation[ expo_, zeta_, t_]:=
zeta-> Coefficient[ expo, zeta,-1]/t

DZetaToDt[expo_,zeta_,dzeta_,t_,dt_]:=
dzeta-> D[Extract[ZetaToTTransformation[
expo, zeta, t],2],t]dt

IntegrandInT[n_,h_,k_,zeta_,dzeta_,omega_,
t_,dt_, simplifiedinput_]:= Module[{z, zetacases,tmp},
zetacases=ExtractNonTrivialCases[
n,h,k,z,zeta,omega,simplifiedinput];
(*Print[zetacases];*)
tmp=
Table[
{zetacases[[i,1]],
{I/k^2 *Exp[-2 Pi I n h/k ],
zetacases[[i,2,3]]/.ZetaToTTransformation[
 zetacases[[i,2,3]], zeta, t] ,
zetacases[[i,2,1]],
```

```
zetacases[[i,2,2]]/.ZetaToTTransformation[
 zetacases[[i,2,3]], zeta, t],
dzeta/.DZetaToDt[zetacases[[i,2,3]],zeta,dzeta,t,dt]
}
},{i,1,Length[zetacases]}
];
(*Print[tmp];*)
Table[
{tmp[[i,1]] ,
{tmp[[i,2,1]]*(tmp[[i,2,5]]*tmp[[i,2,4]])/.{t->1,dt-> 1},
Collect[Together[tmp[[i,2,2]]],{t,t^(-1)}],
Exponent[ tmp[[i,2,5]] ,t]+
Exponent[PowerExpand[tmp[[i,2,4]] ],t],
tmp[[i,2,3]],
dt
}
}
,{i,1,Length[tmp]}]
]

RawFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,temp,zeta,dzeta,t,dt},temp=IntegrandInT[
n,h,k,zeta,dzeta,omega,t,dt, simplifiedinput];
(*Print[temp];*)
Table[{temp[[i,1]],
```

```
PowerExpand[
temp[[i,2,1]]*2*Pi*I/
PowerExpand[Simplify[(Sqrt[Coefficient[temp[[i,2,2]],
t^(-1)]]])^(-(1+temp[[i,2,3]]))]]
*temp[[i,2,4]]*
BESSELI[(-(1+temp[[i,2,3]])),(2PowerExpand[Simplify[
Sqrt[Coefficient[temp[[i,2,2]],t^(-1)]]]]])]
]
}
,{i,1,Length[temp]}]
]

FormattedSummand[ n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,tmp,L},
L=FindL[simplifiedinput];
tmp=RawFormula[n,h,k,omega,simplifiedinput];
Sum[
tmp[[i,2]]*Floor[1/GCD[h,k]]*\[Chi][GCD[k,L]==tmp[[i,1]] ]
,{i,1,Length[tmp]}]//PowerExpand
]

DedekindSum[h_,k_]:=
Sum[(\[Mu]/k-Floor[\[Mu]/k]-1/2)*(h*\[Mu]/k-Floor[
h*\[Mu]/k]-1/2),{\[Mu],1,k-1}]
```

```
Omega[h_,k_]:=
Exp[Pi*I*DedekindSum[h,k]]


NearlyHeadlessNick[simplifiedinput_,q_,MaxN_]:=
Series[Product[(1/QPochhammer[q^(simplifiedinput[
[i,1]])])^simplifiedinput[[i,2]],
{i,1,Length[simplifiedinput]}],{q,0,MaxN}]


TestFormula[ n_,h_, k_,omega_,
simplifiedinput_, MaxK_,MaxN_]:=
Module[{calcvals,i,tmp,series,
summand,q,seriesvals},
summand=FormattedSummand[n,h,k,omega,
simplifiedinput];
tmp=Sum[  summand  ,{k,1,MaxK},{h,0,k-1}];
tmp=tmp/.{BESSELI->
BesselI,omega->Omega, \[Chi][True]-> 1,
 \[Chi][False]-> 0 };
(*Print[tmp];*)
calcvals=Table[ N[tmp/.n-> i],{i,1,MaxN}];
series= NearlyHeadlessNick[simplifiedinput,q,MaxN];
seriesvals=Table[ Coefficient[series,q,i] ,{i,1,MaxN}];
{Re[calcvals]-seriesvals, Im[calcvals]}
]
```

```
PrettyFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{fs,rf,L},
L=FindL[simplifiedinput ];
rf=RawFormula[n,h,k,omega,simplifiedinput]/.BESSELI-> BesselI;
(*Sum[
Sum[ HoldForm [TraditionalForm[Evaluate[rf[[i,2]] ] *
\[Chi][ GCD[h,k]==1 && GCD[k,L]==rf[[i,1]] ] ]]
,{k,1,Infinity},{h,0,k-1}]
,{i,Length[rf]} ]*)
(*fs=ToString[FormattedSummand[
n,h,k,omega,simplifiedinput]/.BESSELI->BesselI,
TraditionalForm]*)

fs = PowerExpand[ FormattedSummand[n,h,k,omega,
simplifiedinput]/.BESSELI->BesselI]//TraditionalForm;
(*SUM[ fs, {k,1,Infinity},{h,0,k-1}]*)
fs
]

ProductToList[ expr_, q_:q,f_:f,
 psi_:\[Psi], phi_:\[Phi] ]:=
Module[{tmp,tmplist,a,b,c,p,r,numer,
denom,ex,subs2,subs3,subs4,subs6},
tmp=expr;
tmp=tmp/. f[a_,b_]->  QPochhammer[-a, a*b]
```

```
 QPochhammer[-b, a*b] QPochhammer[a*b];
tmp= tmp/. psi[-q^a_.]->
QPochhammer[q^(2a)]/QPochhammer[-q^a,q^(2a)];
tmp=tmp/. psi[q^a_.]->
 QPochhammer[q^(2a)]/QPochhammer[q^a,q^(2a)];
tmp=tmp/. phi[-q^a_.]->
QPochhammer[q^a]/QPochhammer[-q^a];
tmp=tmp/. phi[q^a_.]->
QPochhammer[-q,q^(2a)]^2 QPochhammer[q^(2a)];
tmp=tmp/. QPochhammer[ a_,-q^p_]->
QPochhammer[ a,q^(2p)]*QPochhammer[-a*q^p,q^(2p)];
tmp = tmp/. QPochhammer[-q^r_.,q^p_.]->
QPochhammer[q^(2r),q^(2p)]/QPochhammer[q^r,q^p];
numer=Numerator[tmp]/. QPochhammer[a_,b_]^c_->
 QPochhammer[a,b];
If[numer[[0]]=== Times,
numer=numer/.Times-> List,
numer={numer}];
If[verbose,Print["numer=",numer]];
denom=Denominator[tmp]/. QPochhammer[a_,b_]^c_->
QPochhammer[a,b];
If[denom[[0]]===Times,
denom=denom/.Times-> List,
denom={denom}];
If[verbose,Print["denom=",denom]];
```

```
ex=Union[Table[Exponent[numer[[i,2]] ,q],{i,1,Length[numer]}],
Table[Exponent[denom[[i,2]],q],{i,1,Length[denom]}]
];
If[verbose,Print[ex]];
ex=Union[Flatten[Table[ Divisors[ex[[i]]],{i,1,Length[ex]}]]];
If[verbose,Print[ex]];
subs2= Table[
QPochhammer[q^(ex[[i]]/2),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/2) ]/QPochhammer[q^ex[[i]] ]
,{i,1,Length[ex]}];
If[verbose,Print["subs2 = ",subs2]];
tmp=tmp/.subs2;
If[verbose,Print["after subs 2, tmp = ",tmp ] ];
subs3= Table[
QPochhammer[q^(ex[[i]]/3),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/3) ]/QPochhammer[q^ex[[i]] ]/
QPochhammer[q^(2ex[[i]]/3),q^ex[[i]]]
,{i,1,Length[ex]}];
If[verbose,Print["subs3 = ",subs3]];
tmp=tmp/.subs3;
If[verbose,Print["after subs3, tmp = ",tmp ] ];
subs4= Table[
QPochhammer[q^(ex[[i]]/4),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/4) ,q^(ex[[i]]/2)]
QPochhammer[q^(3ex[[i]]/4),q^ex[[i]]]
```

```
,{i,1,Length[ex]}];
If[verbose,Print["subs4 = ",subs4]];
tmp=(tmp/.subs4)/.subs2;
If[verbose,Print["after subs4 & subs2, tmp = ",tmp ] ];
subs6= Table[
QPochhammer[q^(ex[[i]]/6),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/6) ,q^(ex[[i]]/3)]/
QPochhammer[q^(5ex[[i]]/6),q^ex[[i]]]/
QPochhammer[ q^(ex[[i]]/2) ,q^(ex[[i]])]
,{i,1,Length[ex]}];
If[verbose,Print["subs6 = ",subs6]];
tmp=(tmp/.subs6)/.subs2;
If[verbose,Print["after subs6 & subs2, tmp = ",tmp ] ];
(*tmp=(((tmp/.subs2)/.subs3)/.subs4)/.subs6;
If[verbose,Print["after repeated subs, tmp = ",tmp ] ];*)
numer=Numerator[tmp];
denom=Denominator[tmp];

If[numer[[0]]===Times,numer= numer/.Times->List,numer={numer}];
If[numer==={1},numer={}];
If[verbose,Print["numer = ",numer ] ];
If[denom[[0]]===Times,denom=denom/.Times->List,denom={denom}];
If[denom==={1},denom={}];
If[verbose,Print["denom = ",denom ] ];
Join[(numer/. QPochhammer[a_]^b_. ->
```

```
{Log[a]/Log[q],-b})//PowerExpand,
(denom/. QPochhammer[a_]^b_. ->
{Log[a]/Log[q],b})//PowerExpand
]
]

RademacherTypeFormula[
prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=PrettyFormula[
n,h,k,omega,ProductToList[ prodexpr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodexpr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]

LaTeXSummand[prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,
omega_:\[Omega]]:=Module[{pf,simplifiedinput},
simplifiedinput=ProductToList[prodexpr];
TeXForm[Subscript[a,eqcode][n]==
FormattedSummand[ n,h,k,omega,
  simplifiedinput]/.{BESSELI->BesselI,Floor[GCD[h, k]^(-1)]->1}
]
]
```

```
(*
RTF[prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=
 Style[\[CapitalSigma]\[CapitalSigma],Large]
 PrettyFormula[n,h,k,omega,ProductToList[
  prodexpr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodexpr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]
*)

ListToProduct[simplifiedinput_List,q_:q,n_:n]:=
Product[(1-q^(
simplifiedinput[[i,1]]n))^(-simplifiedinput[[i,2]]),
{i,1,Length[simplifiedinput]} ]

MakeBoxes[MySum[s_,{k_, k0_, k1_},
{n_, n0_, n1_}],TraditionalForm]:=
RowBox[{UnderoverscriptBox["\[Sum]",RowBox[
{MakeBoxes[k,TraditionalForm],"=",
MakeBoxes[k0,TraditionalForm]}],
MakeBoxes[k1,TraditionalForm]],
```

```
RowBox[{UnderoverscriptBox["\[Sum]",RowBox[
{MakeBoxes[n,TraditionalForm],"=",
MakeBoxes[n0,TraditionalForm]}],
MakeBoxes[n1,TraditionalForm]],
MakeBoxes[s,TraditionalForm]}]}]

MakeBoxes[MyProduct[s_,{k_, k0_, k1_}],TraditionalForm]:=
RowBox[{UnderoverscriptBox["\[
Product]",RowBox[{MakeBoxes[k,TraditionalForm],"=",
MakeBoxes[k0,TraditionalForm]}],
MakeBoxes[k1,TraditionalForm]],
MakeBoxes[s,TraditionalForm]}]

The Rademacher Package

Version of January 19, 2011

V. Kiria and A. Sills

RTF[prodexpr_,a_:a, eqcode_:{},
 n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=
 TraditionalForm[
 MySum[
```

```
   PrettyFormula[n,h,k,omega,ProductToList[
   prodexpr, q,f, \[Psi], \[Phi] ]],
  {k,1,Infinity},{h,0,k-1}
  ]
]
;
Print["If ", TraditionalForm[
 MyProduct[ ListToProduct[ProductToList[prodexpr] ],
 {n,1,Infinity}] ==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[", then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]

verbose=False;
```