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Abstract

In this note, inequalities and bounds for weighted renewal-type integral equations are presented. Some upper and lower bounds for the weighted renewal-type integral equations with monotone weight functions are derived. Some upper and lower bounds for the weighted renewal-type equations with monotone weight functions are derived. Bounds for the difference between two weighted renewal functions as well between the parent and weighted renewal functions are obtained in terms of the parent renewal reliability functions and their first and second moments. Relations for renewal-type integrals of the ruin probability are presented. Some inequalities, bounds and convergence results are also established.

Mathematics Subject Classifications: 62N05, 62B10

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1 Introduction

Renewal-type integral equations are useful in many contexts in applied probability models, including the study of replacement problems in reliability theory, branching processes, insurance ruin theory and demography. The renewal-type integral equation

$$H(t) = G(t) + \int_0^t H(t-s)dF(s), \tag{1}$$

$t \geq 0$, where $H(t)$ is the number of renewals in the interval $[0, t)$, is particularly useful in applied stochastic processes and related areas. In general, closed forms for renewal functions are not known and finding them for life distributions involves the summation of an infinite series of convolution integrals.

There have been considerable interest in obtaining bounds and approximations to the expected number of system failures even for highly reliable systems and for small times. These bounds and approximations are useful in reliability computations and comparisons for systems or units whose sequences of failures can be modeled as a renewal process. Let $N(t)$ denote the number of renewals in the interval $[0, t)$, then the expectation of $N(t)$, $EN(t)$ satisfies the integral equation (1). If it is assumed that X_0, the time to the first event, has the same distribution as $X_1, X_2, ...$, where X_i is the time between the i^{th} and the $(i+1)^{st}$ events, then (1) reduces to

$$M_F(t) = F(t) + \int_0^t M_F(t-s)dF(s), \tag{2}$$

$t \geq 0$, where $M_F(t) = EN(t)$ is the renewal function. See Ross [9] and references therein for details. Let F and G be life distributions, continuous from the right, with $F(0^-) = G(0^-) = 0$ and $F * G(t) = \int_{0^-}^t F(t-y)dG(y)$. Define $F_1 = F$ and for $n > 1$, $F_{n+1} = F_n * F$. Let F_0 be the unit step function with step at 0. The renewal function $M_F(t)$ can be written as

$$M_F(t) = \sum_{n=1}^{\infty} F_n(t). \tag{3}$$

The augmented renewal function with renewal at 0 is given by $M_F^0 = F_0 + M_F$. The limiting behavior of the renewal function is well known. In fact, as $t \to \infty$,

$$M_F(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1), \tag{4}$$

where μ and σ^2 are the mean and variance of X_i, $i \geq 1$, the time between i^{th} and $(i+1)^{st}$ events and X_0 the time to the first event has the same distribution as X_i, $i \geq 1$. The derivative of $M_F(t)$, if it exists is called the renewal density and is given by

$$m(t) = f(t) + \int_0^t m(t-s)f(s)ds, \tag{5}$$

$t \geq 0$, assuming $f(t) = dF(t)/dt$ exists, see Ross [9].

Let $M_F(t)$ and $M_G(t)$ be two renewal functions, where $M_F(t)$ is given by equation (2). Let μ_F and μ_G, and σ_F^2 and σ_G^2, denote the first moments and second central moments of F and G, respectively. Assume these moments are finite and the distribution functions F and G are nonlattice. For large values of t, we know from elementary renewal theorem that if $\mu_F \neq \mu_G$, then

$$M_F(t) - M_G(t) \sim (\mu_F^{-1} - \mu_G^{-1})t, \tag{6}$$

and with the second term in the asymptotic expansion,

$$lim_{t\to\infty}[M_F(t) - M_G(t) - (\mu_F^{-1} - \mu_G^{-1})t] = \frac{\sigma_F^2 - \mu_F^2}{2\mu_F^2} - \frac{\sigma_G^2 - \mu_G^2}{2\mu_G^2}. \tag{7}$$

If $\mu_F = \mu_G = \mu$, then

$$lim_{t\to\infty}[M_F(t) - M_G(t)] = \frac{\sigma_F^2 - \sigma_G^2}{2\mu^2}. \quad (8)$$

It follows therefore that $M_F(t) - M_G(t)$ converges to 0 if and only if $\mu_F = \mu_G$ and $\sigma_F^2 = \sigma_G^2$.

There are several useful results on bounds for the renewal function, including those given by Brown [2], Daley [4], and Kao [6], and Xie [11] to mention a few. Barlow at al. [1] obtained results of renewal process with increasing failure rate (IFR) distribution function. Kijima [7] also presented results on monotonicity properties of renewal processes. See references therein. Bounds for the differences between two weighted distributions of number of renewals, as well as bounds for differences between the weighted and parent or unweighted distribution of the number of renewals under certain reliability conditions such as increasing failure rate (IFR) or decreasing mean residual life (DMRL) are of particular interest and are presented in this paper.

In section 2, we present basic utility notions and results on the weighted renewal density with monotone weight function. In section 3, we present some upper and lower bounds for the weighted renewal function with monotone weight functions. Bounds for the difference between two weighted renewal functions and those between the weighted and unweighted renewal functions are also presented. In section 4, we present renewal-type inequalities for the ruin probability. Some convergence results are also presented. Concluding remarks are given in section 5.

2 Utility Notions and Basic Results

In this section, we present some basic definitions and utility notions on renewal-type integral equations as well as the weighted distribution and density functions. In many cases, including the non-observability of some events, unequal probability sampling, and damage to the original observations, the recorded observations cannot be considered as a random sample from the original distribution, thus weighted distributions are the appropriate distributions that are applicable in these situations. See Gupta and Keating [5], and Patil and Rao [7], and references therein. In fact, in renewal theory the residual lifetime has a limiting distribution that is a weighted distribution with weight function equal to the reciprocal of the hazard or failure rate function. Also, when observations are selected with probability proportional to their "*length*" the resulting distribution is a weighted distribution referred to as a length-biased distribution. Length-biased distributions occur naturally in a wide variety of settings and finds various applications in reliability, biometry, survival analysis, renewal theory to mention a few areas.

Now consider a process or mechanism that generates a non-negative random variable X with distribution function F and probability density function (pdf) f. Let $W(x)$ be a non-negative weight function with $0 < E_F W(X) < \infty$. The weighted random variable X_w has a reliability function given by

$$\overline{F}_W(x) = \frac{E_F[W(X)|X > x]}{E_F[W(X)]}\overline{F}(x), \tag{9}$$

where $\overline{F}(t) = 1 - F(t)$. Note that the weighted reliability function $\overline{F}_W(x)$ can be expressed as:

$$\overline{F}_W(x) = \overline{F}(x)(W(x) + T_F(x))/E_F(W(X)), \tag{10}$$

where $T_F(x) = \int_x^\infty (\overline{F}(u)W'(u)du)/\overline{F}(x)$, and $W'(u) = dW(u)/du$, assuming that $W(x)\overline{F}(x) \to 0$ as $x \to \infty$. The corresponding probability density function (pdf) of the reliability function given in equation (9) is referred to as a weighted probability density function (wpdf) with weight function $W(x) \geq 0$. In this paper, we assume that the weight function $W(x)$ is monotone. The weighted probability density function $f_W(x)$ of the weighted random variable X_W is given by

$$f_W(x) = W(x)f(x)/\delta^*, \tag{11}$$

$x \geq 0$, where $0 < \delta^* = E_F(W(X)) < \infty$. The hazard function corresponding to the weighted distribution function F_w is given by

$$\lambda_{F_W}(x) = W(x)\lambda_F(x)/(W(x) + T_F(x)), \tag{12}$$

where $T_F(x)$ is given above and $\lambda_F(x) = f(x)/\overline{F}(x)$. When $W(x) = x$, the probability density function (pdf) is called the length-biased pdf and is given by

$$f_l(x) = \frac{xf(x)}{\mu_F}, \tag{13}$$

where $0 < \mu_F = \int_0^\infty \overline{F}(x)dx < \infty$. The length-biased reliability function is given by

$$\overline{F}_l(x) = \frac{\overline{F}(x)V_F(x)}{\mu_F}, \tag{14}$$

where $V_F(x) = E_F(X|X > x)$ is the vitality function.

We assume that the distribution functions F and F_W are absolutely continuous and both $F(t)$ and $F_W(t)$ are zero for $t < 0$. We let $H(t)$ denote the solution to the general renewal-type integral equation (1) and let $M_F(t)$ and $m(t)$ be the renewal function and renewal density respectively. Similarly, we let M_{F_W} and m^W be the renewal function and renewal density corresponding to the weighted distribution function F_W. Following Xie [11], starting with an arbitrary bounded function $H_j^W(t)$, we define the recursive relation:

$$H_{j+1}^W(t) = G(t) + \int_0^t H_j^W(t-s)dF_W(s), \tag{15}$$

$t \geq 0$ for $j = 1, 2, ...$, as the weighted version of (1).

Theorem 2.1 . *For any bounded $H_j^W(t)$, assume $H_1^W(t) \leq H_2^W(t)$ for all $t \leq T$, then $H_1^W(t) \leq .... \leq H_j^W(t) \leq H_{j+1}^W(t) \leq .... \leq H^W(t)$. If $H_1^W(t) \geq H_2^W(t)$ for all $t \leq T$, the inequalities are reversed.*

Proof: The result follows directly from Xie [11] with $H_j^W(t)$ in place of $H_j(t)$.

For the ease of reference, we give some basic definitions that are useful in the results presented in sections 3 and 4.

Definition 2.2 . *Let X and Y be random variables with distribution functions F and G respectively. We say X is larger than Y in stochastic ordering ($X \geq_{st} Y$) if $\overline{F}(t) \geq \overline{G}(t)$ for all $t \geq 0$.*

Definition 2.3 . *A distribution function F is said to have increasing (decreasing) hazard rate or failure rate on $[0, \infty)$, denoted by IHR (DHR) or IFR (DFR), if $F(0-) = 0$, $F(0) < 1$ and $P(X > x + t|X > t) = \overline{F}(x + t)/\overline{F}(t)$ is decreasing (increasing) in $t \geq 0$ for each $x > 0$.*

Definition 2.4 . *A distribution function F with probability density function (pdf) f is said to have increasing (decreasing) mean residual life on $(0, \infty)$, denoted by IMRL (DMRL), if $\mu_F = \int_0^\infty \overline{F}(x)dx < \infty$, $F(0) < 1$ and $E(X - x|X > x)$ is increasing (decreasing) in $x \geq 0$.*

Note that if F has DHR and $\mu_F = \int_0^\infty \overline{F}(x)dx < \infty$, then F has IMRL.

3 Bounds for Weighted Renewal-Type Equations

In this section, we present some bounds and inequalities for the weighted renewal-type integral equations with monotone weight function. In theorem 3.3, an upper bound for the difference between the parent and weighted renewal functions corresponding to the distribution functions F and F_W respectively is obtained in terms of the parent reliability function, the mean of the weighted distribution function and the expectation of the weight function. In particular, when the weighted distribution is length-biased, the bound and approximation is expressed in terms of the parent reliability function $\overline{F}$, the first and the second moments of the distribution function F.

Theorem 3.1 . *If $f_W(t)$ is a non-decreasing weighted probability density function (wpdf) and $f_W^{'}(t)$ exists for $0 \leq t \leq t_0 < \infty$, then $m^W(t)$ is monotone and in the same direction.*

Proof: Suppose $f_W(t)$ is non-decreasing on $[0, t_0]$. We show that $\frac{dm^W(t)}{dt} \geq 0$ for $0 \leq t \leq t_0$. Assume $f_W^{'}$ exists, then

$$\begin{aligned} \frac{dm^W(t)}{dt} &= f_W^{'}(t) + m^W(0) f_W^{'}(t) + \int_0^t \frac{\partial}{\partial t} m^W(t-s) dF_W(s) \\ &= g(t) + \int_0^t \frac{\partial}{\partial t} m^W(t-s) dF_W(s), \end{aligned} \tag{16}$$

where $g(t) = f_W^{'}(t) + m^W(0) f_W^{'}(t)$. Note that, since $f_W(t)$ is non-decreasing, we have $g(t) \geq 0$ for $0 \leq t \leq t_0$, and $\frac{dm^W(t)}{dt} = g(t) + \int_0^t \frac{\partial}{\partial t} m^W(t-s) dF_W(s)$ is a renewal-type integral equation. Consequently, $\frac{dm^W(t)}{dt} \geq 0$, in view of the fact that $m^W(0) = f_W(0)$, and $f_W^{'}(t) \geq 0$ for $0 \leq t \leq t_0 < \infty$.

Theorem 3.2 . *Let $f_W(t)$ be a wpdf with monotone weight function $W(t)$ and $\overline{F}_W(t)$ the corresponding weighted reliability function. If the $\lambda_{F_W}(t) \geq b$ for all $t \leq t_0 < \infty$, then the weighted renewal density $m^W(t) \geq b$ for all $t \leq t_0 < \infty$.*

Proof: Let Let $m_1^W(t) = b$, and applying Theorem 1, we have $m_{j+1}^W(t) \geq m_j^W(t)$ for all $t \leq t_0$. Infact,

$$\begin{aligned} m_{j+1}^W(t) &\geq f_W(t) + \int_0^t b dF_W(x) \\ &= f_W(t) + bF_W(t) \\ &\geq b, \end{aligned} \tag{17}$$

for $t \leq t_0$. Consequently, $m^W(t) \geq b$ for all $t \leq t_0$. Similarly, $m^W(t) \leq b$ for all $t \leq t_0$ if $\lambda_{F_W}(t) \leq b$.

It is well known that if G_1 and G_2 are absolutely continuous with respect to a σ-finite measure ν, with Radon-Nikodym derivative g_1 and g_2, then

$$\int |g_2 - g_1| d\nu = 2Sup_\gamma |G_1(\Delta) - G_2(\Delta)|, \tag{18}$$

where γ is the collection of Borel subsets of $[0, \infty)$, see [2]. Indeed if $P(X = Y)$ is small then g_1 and g_2 are close in $L_1(\nu)$ norm, where the distributions of X and Y are given by G_1 and G_2 respectively. The following results are due in part to an application of the lemma given by Brown [3].

Theorem 3.3 . *Let F and F_W be the parent and weighted life distribution functions with monotone weight function $W(t) \geq 0$. If $W(x)$ is not a linear function of x, then for every $T^* > 0$ satisfying $F(T^*) < 1$, $F_W(T^*) < 1$,*

$$|M_{F_W}(t) - M_F(t)| \leq (\overline{F}(t))^{-2}(1 - \delta^*/\mu_{_{F_W}}),$$

where $0 < \delta^* < \mu_{_{F_W}} = \int_0^\infty \overline{F}_W(y)dy$, and $M_F(t)$ is given by equation (2).

Proof: Note that (See Tortorella[10])

$$|M_{F_W}(t) - M_F(t)| \leq (\overline{F}(t)\overline{F}_W(t))^{-1}Sup\{|F(t) - F_W(t)|, 0 \leq t \leq T^*\}.$$

Since $\delta^*\overline{F}_W(t)/\overline{F}(t)$ is non-decreasing, we have $\overline{F}_W(t) \geq \overline{F}(t)$ for all $t \geq 0$ and

$$Sup\{|F(t) - F_W(t)|, 0 \leq t \leq T^*\} \leq 1 - \delta^*/\mu_{_{F_W}}.$$

Also, since the weighted function $W(t)$ is non-decreasing, the reliability functions $\overline{F}(t)$ and $\overline{F}_W(t)$ are stochastically ordered, so that

$$\begin{aligned} |M_{F_W}(t) - M_F(t)| &\leq (\overline{F}(t)\overline{F}_W(t))^{-1}Sup\{|F(t) - F_W(t)|, 0 \leq t \leq T^*\} \\ &\leq (\overline{F}(t))^{-2}(1 - \delta^*/\mu_{_{F_W}}). \end{aligned} \tag{19}$$

Remark: If $W(x)$ is a linear function of x with $W(0) > 0$ and X is stochastically small, then $\delta^* > \mu_{_{F_W}}$. Note also that if the weight function is known and/or additional information on $\overline{F}_W$ and $\overline{F}$ are available, the bound may be improved.

Theorem 3.4 . *Under length-biased distribution, $W(x) = x$ and*

$$|M_{F_l}(t) - M_F(t)| \leq (\overline{F}(t))^{-2}(1 - \mu^2/\mu_2) = \frac{\sigma_F^2}{(\overline{F}(t))^2(\sigma_F^2 + \mu_F^2)},$$

where $\mu_2 = \int_0^\infty x^2 f(x)dx$, $\delta^* = \mu_F = \int_0^\infty \overline{F}(x)dx$, σ_F^2 is the variance of the distribution function F and

$$\overline{F}_l(t) = \overline{F}(t)\{t + \int_t^\infty \overline{F}(y)dy/\overline{F}(t)\}/\mu_{_F}$$

is the length-biased reliability function.

Theorem 3.5 . *Let F and F_W be the parent and weighted life distribution functions with non-decreasing weight function $W(t) \geq 0$. If $|\overline{F}_W(t)) - \overline{F}(t)|$ is non-decreasing on $[0, T^*]$, then for every $T^* > 0$ satisfying $F(T^*) < 1$, $F_W(T^*) < 1$,*

$$|M_{F_W}(t) - M_F(t)| \leq (\overline{F}(t))^{-1}\{E[W(X)|X > t] - \delta^*\},$$

where δ^* is given in (11), $E[W(X)|X > t] = \frac{\delta^*\overline{F}_W(t)}{\overline{F}(t)}$ and $M_F(t)$ is given by (2).

Proof: Note that, since $|\overline{F}_W(t)) - \overline{F}(t)|$ is non-decreasing on $[0, T^*]$,

$$Sup_{0\leq t\leq T^*}|\overline{F}_W(t)) - \overline{F}(t)| = |\overline{F}_W(T^*)) - \overline{F}(T^*)|,$$

for all $0 \leq t \leq T^*$, so that

$$\begin{aligned} |M_{F_W}(t) - M_F(t)| &\leq (\overline{F}(t)\overline{F}_W(t))^{-1} Sup\{|F(t) - F_W(t)|, 0 \leq t \leq T^*\} \\ &= (\overline{F}(t)\overline{F}_W(t))^{-1}|F(T^*) - F_W(T^*)| \\ &= (\overline{F}(t)\overline{F}_W(t))^{-1}|\overline{F}_W(T^*) - \overline{F}(T^*)| \\ &\leq (\overline{F}(t))^{-2}|\overline{F}_W(T^*) - \overline{F}(T^*)| \\ &\leq (\overline{F}(t))^{-1}\{\frac{\delta^*\overline{F}_W(t)}{\overline{F}(t)} - \delta^*\} \\ &= \delta^*(\overline{F}(t))^{-2}\{\overline{F}_W(t) - \overline{F}(t)\}, \end{aligned} \tag{20}$$

for all $0 \leq t \leq T^*$, by using the fact that $E[W(X)|X > t] = \frac{\delta^*\overline{F}_W(t)}{\overline{F}(t)}$ is non-decreasing so that $\overline{F}(t)$ and $\overline{F}_W(t)$ are stochastically ordered.

Theorem 3.6 . *Let the weight function $W(t)$ be non-decreasing in $t \geq 0$. Then $M_F(t) \geq M_{F_W}(t)$, for all $0 \leq t \leq T^*$.*

Proof: Since $W(t)$ is non-decreasing, $\delta^*\overline{F}_W(t)/\overline{F}(t)$ is non-decreasing, so that $\overline{F}_W(t)$ and $\overline{F}(t)$ are stochastically ordered, that is, $F(t) \geq F_W(t)$ for all $t \geq 0$.

Clearly, for every $n < \infty$,

$$M_F(t) - M_{F_W}(t) \geq \sum_{k=1}^{n}[F_k(t) - F_{W_k}(t)] \geq 0, \tag{21}$$

due to the fact that

$$M_F(t) - M_{F_W}(t) = [F(t) - F_W(t)] + \sum_{k=1}^{\infty}[F_k(t) - F_{W_k}(t)]. \tag{22}$$

Consequently,

$$M_F(t) - M_{F_W}(t) \geq F(t) - F_W(t) \geq 0. \tag{23}$$

The result follows.

Corollary 3.7 . *Under length-biased distribution, and for $0 \leq t \leq T^*$, we have $M_F(t) \geq M_{F_l}(t)$.*

Proof: Since $\lambda_{F_l}(x) = f_l(x)/\overline{F}_l(x) \geq \lambda_F(x)$ for all $t \geq 0$, we have $\overline{F}_l(t) \geq \overline{F}(t)$ for all $t \geq 0$, so that the result follows immediately.

Theorem 3.8 . *Let $f_{W_i}(t) = W_i(t)f_i(t)/E[W_i(T)]$ be weighted probability density functions with $0 < E[W_i(T)] < \infty$, $i = 1, 2$. If $F_{W_2}(t)/F_{W_1}(t)$ is non-decreasing for all $t \geq 0$, and $\overline{F}_{W_2}$ or $\overline{F}_{W_1}$ are DHR reliability functions, then $M_{F_{W_1}}(t) \geq M_{F_{W_2}}(t)$, for all $0 \leq t \leq T^*$.*

Proof: Since $B(t) = F_{W_2}(t)/F_{W_1}(t)$ is non-decreasing, it follows that $\overline{F}_{W_2}$ and $\overline{F}_{W_1}$ are stochastically ordered, that is, $\overline{F}_{W_2}(t) \geq \overline{F}_{W_1}(t)$ for all $t \geq 0$. Note that $B(t)$ is non-decreasing for all $t \geq 0$ and F_{W_2} or F_{W_1} are DHR distribution functions implies that

$$\lambda_{F_{W_2}}(t) \geq \lambda_{F_{W_1}}(t), \tag{24}$$

for all $t \geq 0$. It follows from Theorem 7 that

$$M_{F_{W_1}}(t) - M_{F_{W_2}}(t) \geq F_{W_2}(t) - F_{W_1}(t) \geq 0.$$

for all $0 \leq t \leq T^*$.

Consequently, $M_{F_{W_1}}(t) \geq M_{F_{W_2}}(t)$, for all $0 \leq t \leq T^*$.

Theorem 3.9 . *If $F_W(t)/F_l(t)$ is non-decreasing for $t \geq 0$ and $\overline{F}_W$ or $\overline{F}_l$ are DHR reliability functions, then we have*

$$|M_{F_W}(t) - M_{F_l}(t)| \leq \frac{\sigma_F^2 + \mu_F(\mu_F - \delta^*)}{(\overline{F}(t))^2(\sigma_F^2 + \mu_F^2)},$$

and

$$M_{F_W}(t) - M_{F_l}(t) \geq F_W(t) - F_l(t),$$

for all $0 \leq t \leq T^*$, provided $\mu_F \geq \delta^*$.

Proof: Since $K(t) = F_W(t)/F_l(t)$ is non-decreasing, it follows that $\overline{F}_W(t) \geq \overline{F}_l(t)$ for all $t \geq 0$. Note that $K(t)$ is non-decreasing for all $t \geq 0$ and $F_W(t)$ or $F_l(t)$ are DHR distribution functions implies that

$$\lambda_{F_W}(t) \geq \lambda_{F_l}(t), \tag{25}$$

for all $t \geq 0$. Now, under length-biased distribution, the weight function $W(t) = t$ and the reliability function is

$$\overline{F}_l(t) = \overline{F}(t)\{t + \int_t^\infty \overline{F}(y)dy/\overline{F}(t)\}/\mu_F,$$

so that

$$|M_{F_W}(t) - M_{F_l}(t)| \leq (\overline{F_l}(t))^{-2}(1 - \delta^*/\mu_{F_l}).$$

That is,

$$\begin{aligned} |M_{F_W}(t) - M_{F_l}(t)| &\leq (\overline{F_l}(t))^{-2}(1 - \delta^*\mu^2/\mu_2) \\ &\leq \frac{\sigma_F^2 + \mu_F(\mu_F - \delta^*)}{(\overline{F}(t))^2(\sigma_F^2 + \mu_F^2)}, \end{aligned} \tag{26}$$

where $\mu_2 = \int_0^\infty x^2 f(x)dx$, δ^* is given by (11), and σ_F^2 is the variance of the distribution function F. Note that since $\lambda_{F_W}(t) \geq \lambda_{F_l}(t)$, we have $\overline{F}_W(t) \geq \overline{F}_l(t)$, so that

$$M_{F_W}(t) - M_{F_l}(t) \geq F_W(t) - F_l(t) \geq 0,$$

for all $0 \leq t \leq T^*$. Consequently, $M_{F_W}(t) \geq M_{F_l}(t)$, for all $0 \leq t \leq T^*$.

4 Renewal-Type Inequalities for Ruin Probability

In this section, we establish stochastic inequalities, bounds and relations for renewal-type inequalities for the ruin probability. Let the claim sizes $\{Y_j\}$ for $j \geq 1$ be independent an identically distributed (i.i.d) with common distribution function $G(x)$ and mean $\mu > 0$. The arrival times of claims follow a Poisson process with rate λ, independent of $\{Y_j\}$. It is well known that for an initial capacity x and premium rate $\delta > 0$ the ruin probability $\Psi(x)$ is given by

$$\begin{aligned}\Psi(x) &= P(\sum_{i=1}^{N} X_i > x) \\ &= \frac{1}{1+\theta}\sum_{j=1}^{\infty}(\frac{1}{1+\theta})^j \overline{F}^{(j)}(x),\end{aligned} \tag{27}$$

where $\theta = (\delta/\lambda\mu) - 1 > 0$ is the relative safety loading factor, $\{X_i\}$ are i.i.d random variables following the stationary distribution

$$F(x) = \frac{1}{\mu_F}\int_0^x \overline{G}(y)dy, \tag{28}$$

$F^{(j)}(x)$ is the j-convolution of $F(x)$, N is a geometric random variable with parameter $\frac{\theta}{1+\theta}$, which is independent of $\{X_i\}$, and $\overline{F}^{(j)}(x) = 1 - F^{(j)}(x)$.

There is usually no explicit expression for $\Psi(x)$, however if $\overline{F}(x) = \overline{G}(x) = exp\{-x/\mu\}$, then

$$\Psi(x) = \frac{1}{1+\theta} exp\{-\frac{\theta x}{(1+\theta)\mu}\}. \tag{29}$$

In this context, we consider the following renewal-type integral equation satisfied by $\Psi(x)$ and given by

$$\Psi(x) = \frac{1}{1+\theta}\overline{F}(x) + \frac{1}{1+\theta}\int_0^x \Psi(x-y)dF(y), \tag{30}$$

for $x \geq 0$. Assume that the distribution function F is absolutely continuous and $F(x) = 0$ for $x < 0$. Let $\Psi(x)$ denote the solution to the general renewal-type integral equation. If we write

$$\Psi_n(x) = \sum_{i=1}^{n} (\frac{1}{1+\theta})^j [F^{(j)}(x) - F^{(j+1)}(x)], \tag{31}$$

then $\Psi_n(x)$ satisfies the recursive equation

$$\Psi_{n+1}(x) = \frac{1}{1+\theta}\overline{F}(x) + \frac{1}{1+\theta}\int_0^x \Psi_n(x-y)dF(y), \tag{32}$$

for $n \geq 1$, with $\Psi_0(x) = \frac{1}{1+\theta}\overline{F}(x)$.

Theorem 4.1 . *If $\Psi_1(x) \leq (\geq)\Psi(x)$ for all $x \leq X$, where X may be infinite, then $\Psi_n(x) \leq (\geq)\Psi(x)$ for all n and $x \leq X$, where $\Psi_n(x)$ is given by equation* (31).

Proof: Suppose $\Psi_n(x) \leq \Psi(x)$ for all $x \leq X$, then

$$\begin{aligned} \Psi_{n+1}(x) &\leq \frac{1}{1+\theta}\overline{F}(x) + \frac{1}{1+\theta}\int_0^x \Psi(x-y)dF(y) \\ &= \Psi(x). \end{aligned} \tag{33}$$

Since $\Psi_1(x) \leq \Psi(x)$ for all $x \leq X$, the proof follows by induction.

Theorem 4.2 . *For any bounded $\Psi_1(x)$, assume $\Psi_1(x) \leq \Psi_2(x)$. Then $\Psi_n(x)$ monotonically increases to $\Psi(x)$ as $n \to \infty$.*

Proof: We have

$$\Psi_{n+1}(x) - \Psi_n(x) = \int_0^x \{\Psi_n(x-y) - \Psi_{n-1}(x-y)\}dF(y), \tag{34}$$

so that $\Psi_{n+1}(x) \geq \Psi_n(x)$ for all $n \geq 1$ and $x \leq X$. Consequently, $\Psi_n(x)$ satisfies $\Psi_1(x) \leq \Psi_2(x) \leq .... \leq \Psi_n(x) \leq \Psi_{n+1}(x) \leq .... \leq \Psi(x)$ and the result follows. The inequalities are reversed if $\Psi_2(x) \leq \Psi_1(x)$ for all $x \leq X$.

Theorem 4.3 . *Let $\Psi_n(x)$ be bounded and defined recursively as above for for all x and $n \geq 1$. Then $|\Psi_n(x) - \Psi(x)| \to 0$ as $n \to \infty$, provided $F(x) < 1$ for all x.*

Proof:

$$
\begin{aligned}
|\Psi(x)-\Psi_n(x)| &= \Big|\frac{1}{1+\theta}\{\overline{F}(x)+\int_0^x \Psi(x-y)dF(y)\} \\
&- \frac{1}{1+\theta}\{\overline{F}(x)+\int_0^x \Psi_n(x-y)dF(y)\}\Big| \\
&= \Big|\int_0^x \Psi(x-y)-\Psi_n(x-y)dF(y)\Big| \\
&\leq \int_0^x |\Psi(x-y)-\Psi_n(x-y)|dF(y) \\
&\leq \sup_{0\leq y\leq x} |\Psi(y)-\Psi_n(y)|F(x) \\
\leq \ldots\ldots\ldots \leq &\sup_{0\leq y\leq x} |\Psi(y)-\Psi_n(y)|(F(x))^n.
\end{aligned}
\tag{35}
$$

Consequently, $\Psi_n(x)$ converges to $\Psi(x)$ as $n \to \infty$, since $\Psi_1(x)$ is bounded and by virtue of the fact that $F(x) < 1$ for all x. It is clear that the rate of convergence to zero is of order $(F(x))^n$.

5 Some Concluding Remarks

In this paper, we have presented some useful results on the bounds for weighted renewal-type integral equations including renewal-type integral for the ruin probability. Results on the upper and lower bounds for the renewal density with monotone weight functions are presented. Also, upper bounds for the difference between two weighted renewal functions and those between the weighted and the parent renewal functions are presented in terms of the the parent reliability function and the first two moments, when the weighted distribution is a length or size-biased distribution.

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