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## Survey of Generalized Contact Structures

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# **SURVEY OF GENERALIZED CONTACT STRUCTURES**

by

**JAMES BLAND**

(Under the Direction of Dr. Yi Lin)

## **ABSTRACT**

A generalized complex structure, as introduced by N. Hitchin , is a common generalization of both symplectic and complex structures. Generalized complex geometry provides a natural geometric framework to understand certain recent developments in string physics, and has developed into an active area of research. Very recently, an odd dimensional analogue of a generalized complex structure, namely a generalized contact structure, has been introduced in the works of Vaizman, Poon and Wade. In this thesis, we survey the recent works on generalized contact structures. More importantly, we prove a local normal form theorem of a generalized contact structure. This result, which is a joint work with Yi Lin, is original.

*Key Words:* Contact Structure, Darboux Theorem, Thesis, Mathematics

*2010 Mathematics Subject Classification:* 53D10

**SURVEY OF GENERALIZED CONTACT STRUCTURES**

by

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B.S. in Mathematics

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# **SURVEY OF GENERALIZED CONTACT STRUCTURES**

by

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## DEDICATION

I dedicate this thesis to my parents for their love and support throughout my academic journey.

## ACKNOWLEDGMENTS

I am truly grateful for all the time and effort my advisor Dr. Yi Lin has invested in my education. He has opened my eyes to the most beautiful results of mathematics.

I am fortunate to have been under the tutelage of many great teachers throughout my time in the GSU math department. Without them, I would never have found my path.

*1 \& \forall^{-1} 1 \{James Jason Mike\}*

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# CHAPTER 1

## INTRODUCTION

The goal of this paper is to establish similar results of generalized complex geometry introduced by Nigel Hitchin [10] for generalized contact geometry introduced by Iglesias-ponte and Wade [11]. Our focus is to work toward a working definition of the Darboux Theorem for generalized contact structures. Dr. Marco Gualtieri had a similar goal when working on his PhD thesis. He proved that a generalized version of the Darboux theorem exists for generalized complex structures. We plan to use his approach in [8] as a guideline in showing the Darboux theorem for generalized contact structures.

Beginning in the second chapter and throughout the third, we give a brief overview of common objects and properties. This gives a reader without prior experience in differential geometry a beginning point in the subject matter. Showing the connection between pure spinors and maximal isotropic subspaces, will provide a framework for how we define complex structures. Along, with the  $B$ -transform we can take a generalized complex structure and generate another.

In the next chapter, we show the properties of generalized complex structures. Most importantly, how Dr. Gualtieri approached creating the Darboux theorem for generalized complex structures. Dr. Gualtieri represented generalized complex structures using pure spinors. This representation gives a general form for generalized complex structures and allowed Dr. Gualtieri to finish his proof.

Using the same spinor representation used in [8] we want to expand on the ideas of Poon and Wade in [15]. By defining a generalized contact structure using spinors we can follow [8] and finally create an analog of the Darboux theorem for generalized

contact structures.

## CHAPTER 2

$$V \oplus V^*$$

In this chapter we prepare readers who are unfamiliar with the fundamentals of differential geometry. We will introduce the dual space in order to set up  $V \oplus V^*$ .

### 2.1 Dual Space and B-transforms

The dual space of a vector space  $V$  is defined as the set of all linear functionals on  $V$ . So, we must first define a functional.

**Definition 2.1.1.** *Let  $V$  be a real vector space. A functional  $f : V \rightarrow \mathbb{R}$  is called linear if*

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y), \forall x, y \in V, \text{ and } \forall \lambda, \mu \in \mathbb{R}$$

The set of all linear functionals on  $V$  is called the dual space of  $V$ , denoted as  $V^*$ . Suppose that  $\{e_1, \dots, e_n\}$  is a basis for  $V$ . Then,  $v \in V$  can be written as  $v = a_1 e_1 + \dots + a_n e_n$ . If  $f \in V^*$  then  $f(v) = a_1 f(e_1) + \dots + a_n f(e_n)$ . Assume  $V^*$  has a basis  $\{f_1, \dots, f_m\}$  and that  $f_i(e_j) = 1$  when  $i = j$  and 0 otherwise. This will make  $f_i(v) = a_i$ . We shall prove that  $\{f_1, \dots, f_m\}$  are linearly independent. Now if  $\{f_1, \dots, f_n\}$  is linearly independent we can conclude that it is a basis for  $V^*$ . Given an element  $f \in V^*$  we have  $f = b_1 f_1 + \dots + b_n f_n$ . Applying an arbitrary  $v \in V$  we have  $f(v) = b_1 a_1 + \dots + b_n a_n$ . Since  $v$  is an arbitrary vector the only way for  $b_1 a_1 + \dots + b_n a_n = 0$  to be true for all  $v$  is if  $b_1 = \dots = b_n = 0$ . Thus,  $\{f_1, \dots, f_n\}$  is a basis for  $V^*$ . This result demonstrates that  $V$  and  $V^*$  have the same dimension and, given a basis  $\{e_1, \dots, e_n\}$  for  $V$  there exists a basis  $\{f_1, \dots, f_n\}$  for  $V^*$ , called the dual basis, such that  $f_i(e_j) = \delta_i^j, 1 \leq i, j \leq n$ .

There exists a canonical metric on  $V \oplus V^*$  which is defined by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X)),$$

where  $X, Y \in V$  are vectors and  $\alpha, \beta \in V^*$  are one forms. We need to show that the signature of the metric is  $(n, n)$ . We can show this by letting  $\{e_1, \dots, e_n\}$  be the basis for  $V$  and  $\{e_1^*, \dots, e_n^*\}$  be the basis for  $V^*$ . Let  $f_i = e_i + e_i^*$ ,  $g_i = e_i - e_i^*$ ,  $1 \leq i \leq n$ . Using the metric,  $\{f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n\}$  form an orthogonal basis such that,  $\langle f_i, f_j \rangle = \delta_i^j$ ,  $\langle g_i, g_j \rangle = -\delta_i^j$ ,  $\langle f_i, g_j \rangle = 0$ . This basis shows that the signature is  $(n, n)$ .

A B-transform is a linear map  $e^B$  as follows:

**Definition 2.1.2.** *Let  $B \in \Omega^2(V)$  be a two form.*

$$e^B : V \oplus V^* \rightarrow V \oplus V^*, \quad X + \xi \mapsto X + \xi + \iota_X B$$

Sometimes  $e^B$  will be used to represent  $1 + B + \frac{B \wedge B}{2!} + \frac{B \wedge B \wedge B}{3!} + \dots$  depending on the context in which it is used. This B-transform is an orthogonal automorphism of  $V \oplus V^*$ . The automorphism can be seen if written in the matrix convention used in [1].

$$\mathcal{B} := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

We can also show directly that this B-transform is orthogonal with respect to the canonical metric.

**Lemma 2.1.3.**

$$\langle X + \xi, Y + \eta \rangle = \langle X + \xi + \iota_X B, Y + \eta + \iota_Y B \rangle, \quad \forall X, Y \in V, \quad \forall \xi, \eta \in V^*.$$

*Proof.* For  $X, Y \in V$  and for  $\xi, \eta \in V^*$

$$\begin{aligned}
\langle X + \xi + \iota_X B, Y + \eta + \iota_Y B \rangle &= \frac{1}{2}(\xi(Y) + \iota_X B(Y) + \eta(X) + \iota_Y B(X)) \\
&= \frac{1}{2}(\xi(Y) + B(Y, X) + \eta(X) + B(X, Y)) \\
&= \frac{1}{2}(\xi(Y) + B(Y, X) + \eta(X) - B(Y, X)) \\
&= \frac{1}{2}(\xi(Y) + \eta(X)) \\
&= \langle X + \xi, Y + \eta \rangle
\end{aligned}$$

□

## 2.2 Maximal Isotropic Subspace

Given a subspace  $L$  of  $V \oplus V^*$  if the restriction of  $\langle, \rangle$  on  $L$  vanishes, i.e.,  $\langle X + \alpha, Y + \beta \rangle = 0$ ,  $\forall X + \alpha, Y + \beta \in L$  then  $L$  is isotropic. From before it is shown that the signature of the metric is  $(n, n)$ , so the maximum dimension of the subspace would be dimension  $n$ . Therefore, when  $L$  has dimension  $n$ , we call  $L$  a maximal isotropic subspace of  $V \oplus V^*$ . Sometimes, we will refer to  $L$  as a linear Dirac structure. The two easiest examples of maximal isotropic subspaces are  $V$  and  $V^*$ . Another example would be a subspace and its annihilator.

**Example 2.2.1.** [8] *Let  $E \in V$  be any subspace. Then the subspace*

$$E \oplus \text{Ann}(E) \subset V \oplus V^*$$

*is a maximal isotropic subspace.*

A maximal isotropic subspace under the B-transform is also a maximal isotropic subspace.

**Lemma 2.2.2.** *Suppose that  $L$  is a maximal isotropic subspace of  $V \oplus V^*$ . Then*

$$L_B = \{X + \xi + \iota_X B \mid X + \xi \in L\},$$

*is also a maximal isotropic subspace.*

*Proof.* Let  $X + \xi, Y + \eta \in L$  where  $L$  is a maximal isotropic subspace

$\langle X + \xi + \iota_X B, Y + \eta + \iota_Y B \rangle = \langle X + \xi, Y + \eta \rangle$  because  $L$  is a maximal isotropic subspace  $\langle X + \xi, Y + \eta \rangle = 0$  □

The next example will provide a general form for maximal isotropic subspaces. First we prove it is indeed a maximal isotropic subspace. Then, we will find that all maximal isotropic subspaces can be written in its form.

**Example 2.2.3.** *Let  $E \subset V$  be any subspace of  $V$ , and let  $\varepsilon \in \wedge^2 E^*$  be a two form on  $E$ .*

$$L(E, \varepsilon) = \{X + \xi \mid \xi|_E = \iota_X \varepsilon\}$$

*is a maximal isotropic subspace of  $V \oplus V^*$ .*

*Proof.* Suppose  $X + \xi, Y + \eta \in L(E, \varepsilon)$  :

$$\begin{aligned} \langle X + \iota_X \varepsilon, Y + \iota_Y \varepsilon \rangle &= \frac{1}{2}(\eta(X) + \xi(Y)) \\ &= \frac{1}{2}((\iota_X \varepsilon)(Y) + (\iota_Y \varepsilon)(X)) \\ &= \frac{1}{2}(\varepsilon(Y, X) + \varepsilon(X, Y)) \\ &= 0 \end{aligned}$$

Thus,  $L(E, \varepsilon)$  is an isotropic subspace. In order to show that it is maximal, we will show that the dimension is  $n$ . We need to show  $\text{Ann}(E) \subset L(E, \varepsilon)$ .

$$\text{Ann}(E) := \{\alpha \in V^* \mid \alpha(X) = 0, \forall X \in E\}$$

Thus, for all  $\xi \in \text{Ann}(E)$ ,  $\xi|_E = 0 = \iota_O \varepsilon$ , where  $O$  is the zero vector in  $E$ . This proves that  $\text{Ann}(E) \subset L(E, \varepsilon)$ . Choosing a basis  $\{X_1, \dots, X_r\}$  of  $E$  we are going to extend each one form on  $E$  ( $\iota_{X_i} \varepsilon$ ) into one forms on  $V$  ( $\xi_i$ ), using the following linear map:

$$\Phi : E \rightarrow L(E, \varepsilon), X_i \mapsto X_i + \xi_i$$

We can set  $\Phi(\sum_i a_i X_i) = \sum_i (a_i X_i + a_i \xi_i) = 0$ . Then  $\sum_i a_i X_i = 0$ . Since  $\{X_1, \dots, X_r\}$  is basis of  $E$ ,  $a_1, \dots, a_r$  are all forced to equal 0. Therefore, the null space is 0. This shows that  $\Phi$  is an injective map.

Finally, we want to show that the dimension of the image of  $\Phi$  equals  $\dim(E)$ . We can do this by showing  $\text{Im } \Phi \cap \text{Ann}(E) = \{0\}$ . Let  $X + \xi \in \text{Im } \Phi \cap \text{Ann}(E)$ . Since  $X + \xi \in \text{Ann}(E)$  is a one form,  $X = 0$ . With  $\xi \in \text{Im } \Phi$ ,  $\xi = \sum_i a_i X_i + \sum_i a_i \xi_i$ . Since  $X = 0$ , then all the  $a_i$ 's = 0, forcing  $\xi = 0$ . Now we have the following established inequalities:

$$\dim L(E, \varepsilon) \geq \dim \text{Im } \Phi + \dim \text{Ann}(E) = \dim E + \dim \text{Ann}(E) = n$$

However, since  $L(E, \varepsilon)$  is an isotropic subspace of  $V \oplus V^*$ , its dimension can not exceed  $n$ . This will force  $\dim L(E, \varepsilon) = n$ .  $\square$

Now we know  $L(E, \varepsilon)$  is a maximal isotropic subspace. We will show that all maximal isotropic subspaces can be represented in the form  $L(E, \varepsilon)$ .

**Lemma 2.2.4.** [8] *Suppose that  $L$  is a maximal isotropic subspace of  $V \oplus V^*$ , and let  $E = \pi(L)$ . Then there exists a two form  $\varepsilon \in \Omega^2(E)$  such that  $L = L(E, \varepsilon)$ .*

*Proof.* We will define a two form  $\varepsilon$  on  $E \subset V$

$$\varepsilon(X, Y) = \alpha(Y)$$



where  $\alpha \in V^*$  such that  $X + \alpha \in L$ .

To show that  $\varepsilon$  is independant of choice, let  $\alpha_1, \alpha_2, \beta \in V^*$ . With  $X + \alpha_1, X + \alpha_2, Y + \beta \in L$  we have that  $\alpha_1 - \alpha_2 = (X + \alpha_1) - (X + \alpha_2) \in L$ . Since  $L$  is an isotropic subspace,

$$\begin{aligned} 0 &= \langle Y + \beta, \alpha_1 - \alpha_2 \rangle \\ &= \frac{1}{2}(\alpha_1(Y) - \alpha_2(Y)) \\ \frac{1}{2}\alpha_2(Y) &= \frac{1}{2}\alpha_1(Y) \\ \alpha_2 &= \alpha_1 \end{aligned}$$

Given  $X, Y \in E$ , choose  $\alpha, \beta \in V^*$  such that  $X + \alpha, Y + \beta \in L$ . We have  $\varepsilon(Y, X) = \beta(X)$ . Using the canonical metric:

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X)) = 0.$$

This gives  $\alpha(Y) = -\beta(X)$ , which yields  $\varepsilon(X, Y) = -\varepsilon(Y, X)$ . Thus,  $\varepsilon$  is anti-symmetric and shows that  $L \subset L(E, \varepsilon)$ . Since  $L$  is a maximal isotropic subspace,  $L = L(E, \varepsilon)$ .  $\square$

**Definition 2.2.5.** [8] *The type of maximal isotropic  $L(E, \varepsilon)$ , is the codimension  $K$  of its projection onto  $V$ .*

### 2.3 Exterior Algebra

In this section we will review properties on exterior forms. This will be essential to the following section and following chapters. The space of exterior forms of degree  $r$  is denoted as  $\wedge^r(V^*)$ . By design, we have the following conveniences:  $\wedge^1(V) = V$  and  $\wedge^0(V) = \mathbb{F}$ . [4] Where in our practices the field is  $\mathbb{R}$ .

**Definition 2.3.1.** Suppose that  $\xi \in \wedge^p(V^*)$ , and  $\eta \in \wedge^q(V^*)$ . Define

$$\xi \wedge \eta = A_{p+q}(\xi \otimes \eta).$$

Then  $\xi \wedge \eta$  is an exterior  $(p + q)$  – form, called the **exterior (wedge) product** of  $\xi$  and  $\eta$

The wedge product satisfies common properties.

**Theorem 2.3.2.** Let  $\xi_1, \xi_2 \in \wedge^k(V)$ ,  $\eta_1, \eta_2 \in \wedge^l(V)$ ,  $\zeta \in \wedge^h(V)$ . Then:

$$\begin{aligned} 1) \text{DistributiveLaw} \quad & (\xi_1 + \xi_2) \wedge \eta_1 = \xi_1 \wedge \eta_1 + \xi_2 \wedge \eta_1 \\ & \xi_1 \wedge (\eta_1 + \eta_2) = \xi_1 \wedge \eta_1 + \xi_1 \wedge \eta_2 \\ 2) \text{AnticommutativeLaw} \quad & \xi_1 \wedge \eta_1 = (-1)^{kl} \eta_1 \wedge \xi_1 \\ 3) \text{AssociativeLaw} \quad & (\xi_1 \wedge \eta_1) \wedge \zeta = \xi_1 \wedge (\eta_1 \wedge \zeta) \end{aligned}$$

The proofs of the laws can be found in [4]. Using the anticommutative law, we can easily see that if  $\xi, \eta \in V = \wedge^1(V)$ , then  $\xi \wedge \eta = -\eta \wedge \xi$ . This result implies that  $\xi \wedge \xi = \eta \wedge \eta = 0$ .

## 2.4 Spinors

**Definition 2.4.1.** The Spinorial action of elements of  $V \oplus V^*$  on  $\Omega(V)$ , the space of exterior forms, is defined by the following formula:

$$(X + \xi) \cdot \alpha = \iota_X \alpha + \xi \wedge \alpha,$$

where  $X \in V$ ,  $\xi \in V^*$ , and  $\alpha \in \Omega(V)$ .

**Lemma 2.4.2.**  $\forall X + \xi \in V \oplus V^*$ , we have

$$(X + \xi) \cdot ((X + \xi) \cdot \alpha) = \langle X + \xi, X + \xi \rangle \alpha.$$

*Proof.*

$$\begin{aligned} (X + \xi) \cdot ((X + \xi) \cdot \alpha) &= \iota_X \alpha (\iota_X \alpha + \xi \wedge \alpha) + \xi \wedge (\iota_X \alpha + \xi \wedge \alpha) \\ &= (\iota_X \xi) \alpha \\ &= \langle X + \xi, X + \xi \rangle \alpha, \end{aligned}$$

□

**Lemma 2.4.3.** Given any non-zero form  $\varphi \in \Omega(V)$ , define

$$L_\varphi = \{X + \xi, |(X + \xi) \cdot \varphi = 0\}.$$

Then  $L_\varphi$  is an isotropic subspace of  $V \oplus V^*$ .

*Proof.* Using the above lemma we know  $(X + \xi) \cdot ((X + \xi) \cdot \alpha) = \langle X + \xi, X + \xi \rangle \alpha$ .

If  $X + \xi \in L_\varphi$ , we have that  $\langle X + \xi, X + \xi \rangle \varphi = 0$ . Since  $\varphi \neq 0$ ,  $\langle X + \xi, X + \xi \rangle$  must be 0. Now, given  $X + \xi, Y + \eta \in L_\varphi$  we need to show  $\langle x + \xi, Y + \eta \rangle = 0$ .

Using the identity

$$\langle a, b \rangle = \frac{\langle a + b, a + b \rangle - \langle a - b, a - b \rangle}{4},$$

we can rewrite  $\langle x + \xi, y + \eta \rangle \varphi$  as

$$\frac{\langle a + b, a + b \rangle \varphi - \langle a - b, a - b \rangle \varphi}{4},$$

where  $a = X + \xi$ , and  $b = Y + \eta$ .  $\langle a + b, a + b \rangle \varphi = \langle a - b, a - b \rangle \varphi = 0$  so,

$\langle x + \xi, y + \eta \rangle \varphi = 0$ . Since  $\varphi \neq 0$ ,  $\langle x + \xi, Y + \eta \rangle = 0$ . □

**Definition 2.4.4.** *The form  $\varphi$  is called a pure spinor when  $L_\varphi$  is a maximal isotropic subspace.*

Of course, being a maximal isotropic subspace,  $L_\varphi$  would have dimension  $n$ .

**Lemma 2.4.5.** *Let  $E \subset V$  have codimension  $k$ . The maximal isotropic  $L(E, 0) = E \oplus \text{Ann}(E) = L_\varphi$ , for any non-zero  $\varphi$  in the one dimensional space  $\wedge^k(\text{Ann}(E)) \in \wedge^k V^*$ .*

*Proof.* For any  $X + \alpha \in L(E, 0)$  as defined before,  $X$  must be in  $E$  and  $\alpha$ , when applied to  $E$ , equals 0. So,  $\alpha \in \text{Ann}(E)$ . Therefore,  $L(E, 0) = E \oplus \text{Ann}(E)$ . To show the rest of the lemma, we will let  $\{v_1^*, \dots, v_j^*\}$  be a basis for  $\text{Ann}(E)$ . Now, we will extend the basis to a basis for  $V^*$   $\{v_1^*, \dots, v_k^*, v_{k+1}^*, \dots, v_n^*\}$ . So,  $\forall X + \alpha \in L(E, 0) = E \oplus \text{Ann}(E)$ , we have that

$$\begin{aligned} (X + \alpha) \cdot (v_1^* \wedge \dots \wedge v_k^*) &= \iota_X(v_1^* \wedge \dots \wedge v_k^*) + \alpha \wedge (v_1^* \wedge \dots \wedge v_k^*) \\ &= \sum_i (-1)^{i-1} (v_i^*(X)) v_1^* \wedge \dots \wedge \hat{v}_i^* \wedge \dots \wedge v_k^* + \alpha \wedge (v_1^* \wedge \dots \wedge v_k^*) \\ &= 0 \end{aligned}$$

If  $\varphi$  is annihilated by the spinorial actions of the elements in  $L(E, \varepsilon)$ , then

$$\varphi = \sum_{0 \leq r \leq n} \sum_{i_1 < \dots < i_r} a_{i_r} v_{i_1}^* \wedge \dots \wedge v_{i_r}^*,$$

where  $a_{i_r}$  are scalars. By the way  $\varphi$  is constructed it is forced to have a common factor of  $v_1^* \wedge \dots \wedge v_k^*$ , where  $v_1^*, \dots, v_k^*$  are all in  $\text{Ann}(E)$ .  $\varphi$  can be represented in a simpler form, once a common factor is pulled out.

$$\varphi = v_1^* \wedge \dots \wedge v_k^* \wedge \beta, \quad \beta = \sum_{r=0}^n \sum_{k+1 \leq j_1 < \dots < j_r \leq n} a_{j_r} v_{j_1}^* \wedge \dots \wedge v_{j_r}^* \in \Omega(V)$$

We have the equation

$$\begin{aligned}
\iota_X \varphi &= \iota_X(v_1^* \wedge \cdots \wedge v_k^* \wedge \beta) \\
&= \iota_X(v_1^* \wedge \cdots \wedge v_k^*) \wedge \iota_X \beta \\
&= \iota_X \beta
\end{aligned}$$

Since  $\forall X \in E$ ,  $\iota_X \varphi = 0$ ,  $\iota_X \beta = 0$ . So  $\beta$  must be of degree zero or, in other words, a constant. This leaves

$$\varphi = \lambda v_1^* \wedge \cdots \wedge v_k^*$$

where  $\lambda$  is a constant. □

**Lemma 2.4.6.** [8] *Let  $L$  be a maximal isotropic subspace of  $V \oplus V^*$ , and let  $B$  be a two form. Suppose that  $\varphi$  is annihilated by the spinorial actions of the elements in  $L$ . Then  $\exp(B) \wedge \varphi$  is annihilated by the spinorial actions of the elements in  $\exp(-B)(L)$ .*

*Proof.* Assume that  $X + \xi \in L$  such that  $(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi = 0$ . By calculation:

$$\begin{aligned}
(X + \xi - \iota_X B) \cdot (\exp(B) \wedge \varphi) &= (\exp(B)) \wedge (\iota_X B \wedge \varphi - \iota_X B \varphi + \iota_X \varphi + \xi \wedge \varphi) \\
&= (\exp(B)) \wedge ((X + \xi) \cdot \varphi) \\
&= 0
\end{aligned}$$

□

## CHAPTER 3

### DIFFERENTIAL GEOMETRY

In this chapter we will define the Lie bracket and show its relation to the Lie derivative using one-parameter groups of diffeomorphisms. The setup will allow us to explain the Frobenius Theorem, which will be used in later chapters. Then, using vector bundles, we will be able to establish a differential structure with tangent bundles and cotangent bundles. We will then extend the Lie bracket to the Courant bracket, allowing us to have a defined bracket on sections of  $T \oplus T^*$ .

#### 3.1 Lie Algebra

A Lie bracket can be described by

$$[X, Y](f) = (XY - YX)(f)$$

The other way to define the Lie bracket is by using local coordinates.

$$\begin{aligned} [X, Y] &= \left[ \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \right] \\ &= \sum_{i,j=1}^n a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{i,j=1}^n b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i,j=1}^n a_j \frac{\partial b_i}{\partial x_j} \frac{\partial}{\partial x_i} - \sum_{i,j=1}^n b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i,j=1}^n \left( a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} \end{aligned}$$

Lie brackets have the following properties.

1.  $[X, Y] = -[Y, X]$

2.  $[X, fY] = (Xf)Y + f[X, Y]$
3.  $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

**Definition 3.1.1.** *Suppose that  $M$  is an  $m$ -dimensional smooth manifold. If there is a smooth map  $\varphi : \mathbb{R} \times M \rightarrow M$ , denoted for any  $(t, p) \in \mathbb{R} \times M$  by*

$$\varphi_t(p) = \varphi(t, p),$$

*such that the following conditions are satisfied:*

1.  $\varphi_0(p) = p$
2.  $\varphi_s \circ \varphi_t = \varphi_{s+t}$  for  $s, t \in \mathbb{R}$

*then we say that  $\mathbb{R}$  acts on the manifold  $M$  (from the left) smoothly, and call  $\varphi_t$  a **one-parameter group of diffeomorphisms** on  $M$ .*

Using the above conditions, we see that  $\varphi_t^{-1} = \varphi_{-t}$ . So every  $\varphi_t$  is invertible and a diffeomorphism from  $M$  to itself. A one-parameter group  $\varphi_t$  induces a vector field  $X$  on  $M$  as follows:

$$X_p f = \frac{df(\varphi_t(p))}{dt}, \quad \forall f \in C^\infty(M), \quad \forall p \in M \quad (3.1)$$

By definition, any smooth function  $f$  on  $M$ ,  $X_p f = \lim_{t \rightarrow 0} \frac{f(\varphi(t, p)) - f(p)}{t}$ .  $\varphi : \mathbb{R} \times M \rightarrow M$  is a smooth function, so  $Xf$  will be a smooth function. Therefore,  $X$  is a smooth vector field.

Now we can choose  $p \in M$  and let

$$\gamma_p(t) = \varphi_t(p).$$

Then  $\gamma_p$  is a parametrized curve through  $p$  on  $M$ , called the orbit of  $\varphi_t$  through  $p$ .  $\gamma_p$  is the integral curve of the tangent vector field  $X$ . For any point  $q = \gamma_p(s)$  on the orbit  $\gamma_p$ ,  $X_q$  is the tangent vector of  $\gamma_p$  at  $t = s$ . In fact, since  $\gamma_q(t) = \varphi_t(q) = \varphi_t \circ \varphi_s(p) = \varphi_{t+s}(p) = \gamma_p(t + s)$ , we have

$$\begin{aligned} X_p f &= \lim_{t \rightarrow 0} \frac{f(\varphi_t(q)) - f(q)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\varphi_t \circ \varphi_s(p)) - f(\varphi_s(p))}{t} \\ &= X_p(f \circ \varphi_s) \\ &= ((\varphi_s)_* X_p) f. \end{aligned}$$

That is

$$(\varphi_s)_* X_p = X_{\gamma_p(s)}.$$

We have shown global one-parameter group of diffeomorphisms, now we will show local representation of one-parameter groups.

**Definition 3.1.2.** *Suppose that  $U$  is an open set in the smooth manifold  $M$ . If there is a smooth map  $\varphi : (-\epsilon, \epsilon) \times U \rightarrow M$ , denoted by  $\varphi_t(p) = \varphi(t, p)$  for any  $p \in U, |t| < \epsilon$ , which satisfies*

1. for any  $p \in U, \varphi_0(p) = p$
2. if  $|s| < \epsilon, |s + t| < \epsilon$  and  $p, \varphi_t(p) \in U$ , then  $\varphi_{t+s}(p) = \varphi_s \circ \varphi_t(p)$ ,

then  $\varphi_t$  is called a **local one-parameter group of diffeomorphisms** acting on  $U$ .

We can show that a local one-parameter group also induces a smooth vector field on  $U$ . Suppose that  $p \in U$ , and choose a local coordinate system  $(V, x^i), V \subset U$ , at



$p$ . Due to the smoothness of  $\varphi$ , for sufficiently small positive  $\epsilon_0 < \epsilon$ , if  $|t| < \epsilon_0$  then we have  $\varphi_t(p) \in V$ . Using the equation 3.1 we have

$$X_p = \sum_{i=1}^m X_q^i \left( \frac{\partial}{\partial x^i} \right)_p,$$

where

$$X_p^i = \frac{dx^i(\gamma_p(t))}{dt} \Big|_{t=0}.$$

When  $p$  and  $q = \gamma_p(s)$  are both in  $V$ , we also have

$$X_q = \sum_{i=1}^m X_q^i \left( \frac{\partial}{\partial x^i} \right)_q,$$

where

$$X_q^i = \frac{dx^i(\gamma_p(t))}{dt} \Big|_{t=s}.$$

**Theorem 3.1.3.** *Suppose that  $X$  is a smooth vector field on  $M$ . Then for any point  $p \in M$  there exist a neighborhood  $U$  and a local one-parameter group  $\varphi_t$  of diffeomorphisms on  $U$ ,  $|t| < \epsilon$ , such that  $X|_U$  is precisely the vector field induced by  $\varphi_t$  on  $U$ .*

If  $X_p \neq 0$  at the point  $p$ , then there exists local coordinates  $u^i$  near  $p$  such that  $X = \frac{\partial}{\partial u^1}$ . Then  $\varphi_t$  has the very simple expression:

$$\varphi_t(u^1, \dots, u^m) = (u^1 + t, u^2, \dots, u^m),$$

in other words,  $\varphi_t$  manifests itself as a displacement along the  $u^1$ -axis.

**Corollary 3.1.4.** *Suppose that  $X$  is a smooth vector field on a smooth compact manifold  $M$ . Then  $X$  determines a one-parameter group of diffeomorphisms on  $M$ .*

**Theorem 3.1.5.** *Suppose that  $X, Y$  are any two smooth vector fields on a manifold  $M$ . If the local one-parameter group of diffeomorphisms generated by  $X$  is  $\varphi_t$ , then*

$$[X, Y] = \lim_{t \rightarrow 0} \frac{Y - (\varphi_t)_* Y}{t}$$

using a change of variable  $t = -t$  we see:

$$[X, Y] = \lim_{t \rightarrow 0} \frac{(\varphi_t^{-1})_* Y - Y}{t}$$

**Definition 3.1.6.** *The Lie Derivative of the tangent vector field  $Y$  with respect to  $X$  is denoted by  $L_X Y$  and is equal to  $[X, Y]$ .*

This can be shown by supposing that  $\gamma_p$  is the orbit through  $p$  of the one-parameter group of  $\varphi_t$ . Because  $\varphi_t^{-1}$  maps the point  $q = \gamma_p(t) = \varphi_t(p)$  in  $\gamma_p$  to the point  $p$ ,  $(\varphi_t^{-1})_*$  establishes a homomorphism from the tangent space  $T_q M$  to the tangent space  $T_p M$ . If  $Y$  is a vector field on  $M$  defined on the orbit  $\gamma_p$ , then  $(\varphi_t^{-1})_* Y_{\varphi_t(p)}$  is a curve on the tangent space  $T_p(M)$ . We already know that  $[X, Y]_p$  is precisely the tangent space of this curve at  $t = 0$ , hence it is the rate of change of the tangent vector  $Y$  along the orbit of  $X$ .

We can generalize the Lie derivative to any tensor field on  $M$ . The map  $(\varphi_t)^*$  establishes a homomorphism from the cotangent space  $T_q^* M$  to the cotangent space  $T_p^* M$ . This map and  $(\varphi_t^{-1})_*$  together then induce a homomorphism  $\Phi_t : T_s^r(\varphi_t(p)) \rightarrow T_s^r(p)$  between tensor spaces so that for any  $v_1, \dots, v_r \in T_{\varphi_t(p)}(M)$ , and  $v_1^*, \dots, v_r^* \in T_{\varphi_t(p)}^*(M)$ , we have

$$\Phi_t(v_1 \otimes \dots \otimes v_r \otimes v_1^* \otimes \dots \otimes v_s^*) = (\varphi_t^*)_* v_1 \otimes \dots \otimes (\varphi_t^{-1})_* v_r \otimes \varphi_t^* v_1^* \otimes \dots \otimes \varphi_t^* v_s^*.$$

Thus given a type  $(r, s)$  tensor field  $\xi$ , the Lie Derivative of  $\xi$  with respect to  $X$  is defined by

$$L_X \xi = \lim_{t \rightarrow 0} \frac{\Phi_t(\xi) - \xi}{t}$$

This shows that  $L_X\xi$  is also a type  $(r, s)$  tensor field.

**Definition 3.1.7.** For a smooth vector  $X$  on  $M$ , we define a linear operator  $\iota_X : \Omega^r(M) \rightarrow \Omega^{r-1}(M)$  as follows:

1. If  $r = 0$ , then  $\iota_X$  acts on  $\Omega^0(M)$  as the zero map.
2. If  $r = 1$ ,  $\omega \in \Omega^1(M)$ , then define

$$\iota_X\omega = \langle X, \omega \rangle .$$

3. If  $r > 1$ , then for any  $r - 1$  smooth vector fields  $Y_1, \dots, Y_{r-1}$ , we have

$$\langle Y_1 \wedge \dots \wedge Y_{r-1}, \iota_X\omega \rangle = \langle X \wedge Y_1 \wedge \dots \wedge Y_{r-1}, \omega \rangle .$$

Suppose  $X$  is a smooth vector field on the manifold  $M$ , and that  $\alpha$  and  $\beta$  are smooth differential forms of degree  $p$  and  $q$  respectively. We can use the definitions of Lie derivative and the definition of the linear operator  $\iota_X$  to obtain the following common properties:

1.  $L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$
2.  $\iota_X(\alpha \wedge \beta) = (\iota_X\alpha) \wedge \beta + (-1)^P \alpha \wedge (\iota_X\beta)$

**Theorem 3.1.8. Frobenius Theorem:** An  $r$ -distribution  $\Delta$  on an  $m$ -manifold  $M$  is involutive if and only if  $\Delta$  is completely integrable.

We say that  $\Delta$  is involutive if for any  $X, Y \in \Delta$ ,  $[X, Y] \in \Delta$ .  $\Delta$  is completely integrable if there exists a local coordinate system  $\{x_1, \dots, x_m\}$ , such that  $\Delta = \text{span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ .

### 3.2 Vector Bundles

A vector bundle is a topological construction.

**Definition 3.2.1.** *Suppose that  $E$  and  $M$  are two smooth manifolds, and  $\pi : E \rightarrow M$  is a smooth surjective map. Let  $V = \mathbb{R}^q$  be a  $q$ -dimensional vector space. If an open covering  $U_\alpha$  of  $M$  and a set of maps  $\varphi_\alpha$  satisfy all of the following conditions, then  $(E, \pi, M)$  is called a (real)  $q$ -dimensional vector bundle on  $M$ , where  $E$  is called the **total space**,  $M$  is called the **base space**,  $\pi$  is called the **bundle projection**, and  $V = \mathbb{R}^q$  is called the **typical fiber**:*

1. Each map  $\varphi_\alpha$  is a diffeomorphism from  $U_\alpha \times \mathbb{R}^q \rightarrow \pi^{-1}(U_\alpha)$ , and for any  $p \in U_\alpha$ ,  $y \in \mathbb{R}^q$ ,

$$\pi \circ \varphi_\alpha(p, y) = p.$$

2. For any fixed  $p \in U_\alpha$ , let

$$\varphi_{\alpha,p}(y) = \varphi_\alpha(p, y), y \in \mathbb{R}^q.$$

Then  $\varphi_{\alpha,p} : \mathbb{R}^q \rightarrow \pi^{-1}(p)$  is a homeomorphism. When  $U_\alpha \cap U_\beta \neq \emptyset$ , for any  $p \in U_\alpha \cap U_\beta$ ,

$$g_{\alpha\beta}(p) = \varphi_{\beta,p}^{-1} \varphi_{\alpha,p} : \mathbb{R}^q \rightarrow \mathbb{R}^q$$

is a linear isomorphism of  $V = \mathbb{R}^q$ , i.e.,  $g_{\alpha\beta} \in GL(V)$ .

3. When  $U_\alpha \cap U_\beta \neq \emptyset$ , the map  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(V)$  is smooth.

The simplest example of a vector bundle is the trivial vector bundle which is constant.

**Example 3.2.2.** Let  $M$  be a smooth manifold. Let  $E = M \times \mathbb{R}^q$  and let

$$\pi : E \rightarrow M, (p, v) \rightarrow p, \forall (p, v) \in M \times \mathbb{R}^q.$$

While the previous example is the simplest vector bundle, the two most useful vector bundles are the tangent bundle and cotangent bundle. The tangent bundle is the collection of all the tangent spaces at every point in a differential manifold. Similarly, we can define the cotangent bundle.

**Definition 3.2.3.** Suppose that  $M$  is an  $n$ -dimensional differentiable manifold, and that  $T_p M$  and  $T_p^* M$  are the tangent spaces and cotangent spaces of  $M$  at a point  $p$ . We can define the tangent bundle ( $TM$ ) and the cotangent bundle ( $T^*M$ ) as,

$$TM = \bigcup_{p \in M} T_p M, \quad T^*M = \bigcup_{p \in M} T_p^* M.$$

We will need to define a topology on  $TM$  in order to define a  $C^\infty$  differentiable structure on  $TM$  to make it a smooth manifold. First, we will suppose that  $V$  is a  $n$ -dimensional vector space. Denote the group of linear automorphisms of  $V$  by  $GL(V)$ . We will choose a basis  $\{e_1, \dots, e_n\}$  then,  $V$  is isomorphic to  $\mathbb{R}^n$ . We will represent an element  $y \in V$  as a coordinate row

$$y = (y^1, \dots, y^n).$$

Now,  $GL(V)$  is a multiplicative group of  $n \times n$  matrices, i.e.,  $GL(V)$  is the general linear group  $GL(n; \mathbb{R})$ . We can define the action of  $GL(V)$  on  $V$  as a multiplication on the right, with the matrix representation given by

$$y \cdot a = (y^1, \dots, y^n) \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

where  $\det(a) = \det(a_{ij}) \neq 0$ . Define a map  $\pi$  as follows:

$$\pi : TM \rightarrow M, X_p \mapsto p, \forall X_p \in T_p M.$$

For any coordinate neighborhood  $U_\alpha$  with local coordinates  $x_1, \dots, x_n$ , define

$$\varphi_\alpha : U_\alpha \times V \rightarrow TM, (p, y^1, \dots, y^n) \mapsto \sum_{i=1}^n y^i \left( \frac{\partial}{\partial x_i} \right)_p.$$

The map  $\varphi_\alpha$  is a one-to-one map from  $U_\alpha \times V$  onto  $\pi^{-1}(U_\alpha)$ . Now consider all such coordinate neighborhoods  $U_\alpha$  and maps  $\varphi_\alpha$ , and define

$$S = \{O_\alpha | O_\alpha = \varphi_\alpha(U_\alpha \times W_\alpha), W_\alpha \text{ is an open set in } \mathbb{R}^n\}.$$

Then  $S$  generates a topology  $\Xi$  on  $TM$ .

**Lemma 3.2.4.** *For any coordinate neighborhood  $U_\alpha \subset M$ , let  $\psi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$  be the coordinate map on  $U_\alpha$ . Define a one-to-one map*

$$F_\alpha : \pi^{-1}(U_\alpha) \rightarrow V_\alpha \times \mathbb{R}^n, (x, y) \mapsto \varphi_\alpha(\psi_\alpha^{-1}(x), y), \forall x \in V_\alpha, y \in \mathbb{R}^n,$$

and define  $\Phi_\alpha = F_\alpha^{-1}$ . Then,  $(\pi^{-1}(U_\alpha), \Phi_\alpha)$  defines a differentiable structure on  $TM$ .

*Proof.* It suffices to show that for any two coordinate neighborhoods  $U_\alpha \cap U_\beta \neq \emptyset$  of  $M$ , the transition function

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^n \rightarrow (V_\alpha \cap V_\beta) \times \mathbb{R}^n$$

is a smooth map. Now suppose that  $\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (y, w)$ , where  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n) = \psi_\beta \circ \psi_\alpha^{-1}(x_1, \dots, x_n) \in V_\alpha \cap V_\beta$ ,  $v = (v_1, \dots, v_n)$ ,  $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ . So,  $y = \psi_\beta \circ \psi_\alpha^{-1}(x)$  is a smooth function of  $x$ . Also, by definition we have that

$$\sum_{i=1}^n w_i \frac{\partial}{\partial y_i} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}.$$

Since  $\frac{\partial}{\partial x_j} = \sum \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial y_i}$ , we get that

$$\sum_{i=1}^n w_i \frac{\partial}{\partial y_i} = \sum_{j=1}^n v_j \frac{\partial}{\partial x_j} = \sum_{i=1}^n \left( \sum_{j=1}^n v_j \frac{\partial x_j}{\partial y_i} \right) \frac{\partial}{\partial y_i}$$

It follows

$$(w_1, \dots, w_n) = (v_1, \dots, v_n) \cdot \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

because of the Jacobi matrix we can conclude that  $\Phi_\beta \circ \Phi_\alpha^{-1}$  is a smooth map.  $\square$

Similarly, we can find a topology on  $T^*M$ . Then we would be able to define a differentiable structure on  $T^*M$  by making only a few adjustments.

### 3.3 $T \oplus T^*$

The Lie bracket is not defined on  $T \oplus T^*$ , so we will introduce the Courant bracket which fails the Jacobi identity.

**Definition 3.3.1.** *The Courant bracket is a skew-symmetric bracket defined on smooth sections of  $T \oplus T^*$ , where  $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$ ,*

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi). \quad (3.2)$$

From the definition of the Courant bracket we can see how vector fields are affected. In this instance, the Courant bracket will reduce to the Lie bracket. We would like for the  $B$  – transform to preserve Courant bracket. We can show that it indeed is preserved but only when  $B$  is closed.

**Theorem 3.3.2.** *[8] The map  $e^B$  is an automorphism of the Courant bracket if and only if  $B$  is closed, i.e.  $dB = 0$ .*

*Proof.* Let  $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$  and let  $B$  be a smooth 2-form. Then,

$$\begin{aligned}
[e^B(X + \xi), e^B(Y + \eta)] &= [X + \xi + \iota_X B, Y + \eta + \iota_Y B] \\
&= [X + \xi, Y + \eta] + [X, \iota_Y B] + [\iota_X B, Y] \\
&= [X + \xi, Y + \eta] + L_X \iota_Y B - \frac{1}{2} d\iota_X \iota_Y B - L_Y \iota_X B + \frac{1}{2} d\iota_Y \iota_X B \\
&= [X + \xi, Y + \eta] + L_X \iota_Y B - \iota_Y L_X B + \iota_Y \iota_X dB \\
&= [X + \xi, Y + \eta] + \iota_{[X, Y]} B + \iota_Y \iota_X dB \\
&= e^B([X + \xi, Y + \eta]) + \iota_Y \iota_X dB
\end{aligned}$$

We see that  $e^B$  is an automorphism of the Courant bracket if and only if  $\iota_Y \iota_X dB = 0$  for all  $X, Y$ , which happens only when  $dB = 0$ . □



**CHAPTER 4**  
**GENERALIZED COMPLEX GEOMETRY**

**4.1 Linear structure**

Let  $V$  be a vector space. A generalized complex structure on  $V$  is an orthogonal linear automorphism  $\mathcal{J} : V \oplus V^* \rightarrow V \oplus V^*$  such that  $\mathcal{J}^2 = -id$  (identity)[1].  $\mathcal{J}$  extends naturally to the complexification of  $V \oplus V^*$ . This extension will also be denoted as  $\mathcal{J}$ .

**Definition 4.1.1.** *Let  $L \subseteq V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$  be a maximal isotropic subspace. Then  $L \cap \bar{L}$  is real, i.e. the complexification of a real space:  $L \cap \bar{L} = K \otimes \mathbb{C}$ , for  $K \subseteq V \oplus V^*$ . The real index  $r$  of the maximal isotropic  $L$  is defined by*

$$r = \dim_{\mathbb{C}} L \cap \bar{L} = \dim_{\mathbb{R}} K.$$

**Proposition 4.1.2.** *A generalized complex structure on  $V$  is equivalent to the specification of a maximal isotropic complex subspace  $L \subseteq V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$  of real index zero, i.e. such that  $L \cap \bar{L} = \{0\}$ .*

*Proof.* If  $\mathcal{J}$  is a generalized complex structure, then let  $L$  be its  $+i$ -eigenspace in  $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ . Then if  $x, y \in L$ ,  $\langle x, y \rangle = \langle \mathcal{J}x, \mathcal{J}y \rangle$  by orthogonality  $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle ix, iy \rangle = -\langle x, y \rangle$ , implying that  $\langle x, y \rangle = 0$ . Therefore,  $L$  is isotropic and half-dimensional, i.e. maximally isotropic. Also,  $L$  is the  $+i$ -eigenspace of  $\mathcal{J}$  and thus  $L \cap \bar{L} = \{0\}$ . Conversely, given such an  $L$ , simply define  $\mathcal{J}$  to be the multiplication of  $i$  on  $L$  and by  $-i$  on  $\bar{L}$ . This real transformation then defines a generalized complex structure on  $V$ . □

**Proposition 4.1.3.** [8] *The maximal isotropic  $L(E, \varepsilon)$  has real index zero if and*

only if  $E + \bar{E} = V \otimes \mathbb{C}$  and  $\varepsilon$  such that the real skew 2-form  $\omega_{\nabla} = \text{Im}(\varepsilon|_{E \cap \bar{E}})$  is non-degenerate on  $E \cap \bar{E} = \Delta \otimes \mathbb{C}$ .

*Proof.* Let  $L$  have real index zero. Then, since  $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* = L \oplus \bar{L}$ , we see that  $E + \bar{E} = V \otimes \mathbb{C}$ . Also, if  $0 \neq X \in \Delta$  such that  $(\varepsilon - \bar{\varepsilon})(X) = 0$ , then there exists  $\xi \in V_{\mathbb{C}}^*$  such that  $X + \xi \in L \cap \bar{L}$ , which is a contradiction. Hence,  $\omega_{\nabla}$  is non-degenerate. Conversely, assume  $E + \bar{E} = V \otimes \mathbb{C}$  and that  $\omega_{\nabla}$  is non-degenerate. Suppose  $0 \neq X + \xi \in L \cap \bar{L}$ ; then  $\xi|_E = \xi|_{\bar{E}} = 0$ , hence  $\varepsilon = 0$  as well, a contradiction. So it follows that  $L \cap \bar{L} = \{0\}$ .  $\square$

**Corollary 4.1.4.** *Suppose that a real vector space  $V$  admits a generalized complex structure  $\mathcal{J}$ . Then  $V$  must be even dimensional.*

*Proof.* Let  $L = L(E, \varepsilon)$  be the  $i$ -eigenspace of the generalized complex structure  $\mathcal{J} : V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* \rightarrow V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ . It follows from 4.1.2 and 4.1.3 that  $E \cap \bar{E}$  is even dimensional. Let  $W$  be any subspace of  $E$  complement to  $E \cap \bar{E}$ . Then it follows that

$$V \otimes \mathbb{C} = (E \cap \bar{E}) \oplus W \oplus \bar{W}.$$

Consequently  $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim(E \cap \bar{E}) + \dim(W) + \dim(\bar{W}) = \dim(E \cap \bar{E}) + 2\dim(W)$ . So,  $V$  must be even dimensional.  $\square$

Let  $\omega$  be a two form on  $V$ . It induces a linear map

$$\omega : V \rightarrow V^*, \quad X \mapsto \iota_X \omega = \omega(X, \cdot). \quad (4.1)$$

If 4.1 is an isomorphism, then  $\omega$  is called a **symplectic** form on  $V$ . On the other hand, a complex structure on a vector space  $V$  is a linear automorphism  $I : V \rightarrow V$  such that  $I^2 = -1$ .

**Example 4.1.5.** Let  $\omega$  be a symplectic structure on  $V$ . Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} : V \oplus V^* \rightarrow V \oplus V^*$$

is a generalized complex structure. The  $i$ -eigenspace of  $\mathcal{J}_\omega$  is given by

$$L = \{X - i\iota_X\omega \mid X \in V_{\mathbb{C}}\}.$$

**Example 4.1.6.** Let  $I$  be a complex structure on  $V$ . Then

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix} : V \oplus V^* \rightarrow V \oplus V^*$$

defines a generalized complex structure. The  $i$ -eigenspace of  $\mathcal{J}_I$  is given by

$$L = V_{0,1} \oplus V^{*1,0}.$$

## 4.2 Complex Structures on Sub-bundles

Let  $M$  be a smooth manifold. A sub-bundle  $L$  of  $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M$  is called an **almost Dirac** structure if  $\forall x \in M$ ,  $L_x$  is a maximal isotropic subspace of  $T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$ . An almost Dirac structure  $L$  is said to be an **integrable Dirac** structure, or simply a Dirac structure, if its smooth sections are closed under the Courant bracket (3.2).

**Proposition 4.2.1.** [8] Let  $E \subset T_{\mathbb{C}}$  be a sub-bundle and  $\varepsilon \in C^\infty(\wedge^2(E^*))$ . Then the maximal isotropic  $L(E, \varepsilon)$  defines an integrable Dirac structure if and only if  $E$  is involutive and  $d_E\varepsilon = 0$ .

*Proof.* Let  $i : E \rightarrow T_{\mathbb{C}}$  be the inclusion. Then  $d_E : C^\infty(\wedge^k E^*) \rightarrow C^\infty(\wedge^{k+1} E^*)$  is defined by  $i^* \circ d = d_E \circ i^*$ . Now let  $\sigma \in C^\infty(\wedge^2 T \otimes \mathbb{C})$  be a smooth extension of  $\varepsilon$ ,

i.e.  $i^*\sigma = \varepsilon$ . Suppose that  $X + \xi, Y + \eta \in C^\infty(L)$ , which means that  $\xi|_E = \iota_X\varepsilon$  and  $\eta|_E = \iota_Y\varepsilon$ . Consider the bracket  $Z + \zeta = [X + \xi, Y + \eta]$ ; if  $L$  is Courant involutive, then  $Z \in C^\infty(E)$ , showing  $E$  is involutive, and the difference

$$\begin{aligned}
\zeta|_E - \iota_Z\varepsilon &= i^*(L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi)) - \iota_{[X,Y]}i^*\sigma \\
&= \iota_Xd_Ei^*\eta - \iota_Yd_Ei^*\xi + \frac{1}{2}d_E(\iota_X\iota_Y\varepsilon - \iota_Y\iota_X\varepsilon) - i^*[L_X, \iota_Y]\sigma \\
&= \iota_Xd_Ei^*\eta - \iota_Yd_Ei^*\xi + d_E\iota_X\iota_Y\varepsilon - i^*(\iota_Xd\iota_Y\sigma + d\iota_X\iota_Y\sigma - \iota_Yd\iota_X\sigma - \iota_Y\iota_Xd\sigma) \\
&= \iota_Y\iota_Xd_E\varepsilon
\end{aligned}$$

must vanish for all  $X + \xi, Y + \eta \in C^\infty(L)$ , showing that  $d_E\varepsilon = 0$ . Reversing the argument, we see that the converse holds as well.  $\square$

Let  $M$  be an  $n$ -dimensional manifold. There is a natural metric of type  $(n, n)$  on  $TM \oplus T^*M$  given by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$

which extends naturally to  $T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M = (TM \oplus T^*M) \otimes \mathbb{C}$ . A generalized almost complex structure on a manifold  $M$  is a bundle map  $\mathcal{J} : TM \oplus T^*M \rightarrow TM \oplus T^*M$  which is orthogonal with respect to the natural metric defined above so that  $\mathcal{J}^2 = -1$ . A **generalized complex structure** on a manifold  $M$  is an almost generalized complex structure  $\mathcal{J}$  so that the sections of its  $i$ -eigenbundle is closed under the Courant bracket.

A two form  $\omega$  on a manifold  $M$  is called a symplectic form if  $\forall p \in M$ , the restriction of  $\omega$  to each tangent space  $T_pM$  is a symplectic structure, and  $d\omega = 0$ .

**Example 4.2.2.** *Let  $\omega$  be a symplectic form on a manifold  $M$ , and let  $\mathcal{J}_\omega$  be the almost generalized complex structure defined as in 4.1.5. Then the  $i$ -eigenbundle of*

$\mathcal{J}_\omega$  is given by

$$L = \{X - i\iota_X\omega \mid X \in T_{\mathbb{C}}M\}.$$

It follows immediately from 4.2.1 that  $\mathcal{J}_\omega$  is a generalized complex structure on  $M$ .

### 4.3 Darboux Theorem

**Theorem 4.3.1. Darboux Theorem.** *Let  $\omega$  be a symplectic form on a  $2n$ -dimensional manifold  $M$ . Then for any  $p \in M$ , there exists a coordinate neighborhood  $(U, x_1, \dots, x_{2n})$  such that*

$$\omega = \sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}.$$

Using Moser's method we can show a slightly stronger result. Suppose that  $\omega_{0 \leq t \leq 1}$  is a family of symplectic forms on  $M$  which depend smoothly on  $t$ , and that  $\frac{d\omega_t}{dt} = d\sigma_t$  for a family of one forms  $\sigma_t$  on  $M$  which depend smoothly on  $T$ . We would like to find a family of diffeomorphisms  $\varphi_t$  which depend smoothly on  $t$  such that

$$\varphi_t^* \omega_t = \omega_0. \tag{4.2}$$

Suppose that  $X_t$  is the flow on  $M$  generated by  $\varphi_t$ , i.e., a family of vector fields on  $M$  depending smoothly on  $t$  such that

$$\frac{d(f \circ \varphi_t)}{dt} = X_t(f), \quad \forall f \in C^\infty(M) \tag{4.3}$$

Differentiate both sides of 4.2 to get

$$\begin{aligned}
0 &= \frac{d}{dt} \varphi_t^* \omega_t \\
&= \frac{d\varphi_t^*}{dt} \omega_t + \varphi_t^* \frac{d\omega_t}{dt} \\
&= \varphi_t^* L_{X_t} \omega_t + d\varphi_t^* \sigma_t \\
&= \varphi_t^* (d\iota_{X_t} + \iota_{X_t} d) \omega_t + \varphi_t^* d\sigma_t \\
&= d\varphi_t^* \iota_{X_t} \omega_t + d\varphi_t^* \sigma_t \\
&= d\varphi_t^* (\iota_{X_t} \omega_t + \sigma_t)
\end{aligned}$$

Since  $\omega_t$  are symplectic forms, one can always find a smooth family of vector fields  $X_t$  such that

$$\iota_{X_t} \omega_t + \sigma_t = 0.$$

Let  $\varphi_t$  be the diffeomorphisms generated by the flow  $X_t$  by 4.3, then we must have 4.2.

There exists an analogous Darboux theorem for generalized complex structures. Shown in [8] by Dr. Gualtieri.

**Theorem 4.3.2.** [8] *Any regular point in a generalized complex manifold has a neighborhood which is equivalent, via a diffeomorphism and a B-field transformation, to the product of an open set in  $\mathbb{C}^k$  with an open set in the standard symplectic space  $(\mathbb{R}^{2n-2k}, \omega_0)$ .*

*Proof.* From 4.2.1 in a regular neighborhood, a generalized complex structure can be expressed by  $L(E, \varepsilon)$  where  $E \subset T_{\mathbb{C}}$  is an involutive sub-bundle and  $\varepsilon \in C^\infty(\wedge^2 E^*)$  satisfies  $d_E \varepsilon = 0$ .  $E$  determines a foliation of the neighborhood with transverse

complex structure isomorphic to an open set in  $\mathbb{R}^{2n-2k} \times \mathbb{C}^k$ , where  $E$  is spanned by  $\{\partial/\partial x_1, \dots, \partial/\partial x_{2n-2k}, \partial/\partial z_1, \dots, \partial/\partial z_k\}$ , where  $x_i$  are coordinates for the leaves  $\mathbb{R}^{2n-2k}$  and  $z_i$  are transverse complex coordinates. Therefore, by choosing  $B + i\omega \in \mathbb{C}^\infty(\wedge^2 T^* \otimes \mathbb{C})$  such that  $i^*(B + i\omega) = \varepsilon$ , we may write a generator for the canonical bundle defining  $L(E, \varepsilon)$  as follows:

$$\rho = e^{B+i\omega}\Omega,$$

where  $\Omega = dz_1 \wedge \dots \wedge dz_k$ ; note  $\rho$  is independent of the choice of extension for  $\varepsilon$ . It is shown that

$$i^*d(B + i\omega) = d_E i^*(B + i\omega) = d_E \varepsilon = 0,$$

which means that  $d(B + i\omega) \in \text{Ann}(E)$ , implying finally that

$$d\rho = e^{B+i\omega}d(B + i\omega) \wedge \Omega = 0.$$

Every maximal isotropic in  $V \oplus V^*$  corresponds to a pure spinor line in the sense of 2.4.5 generated by

$$\varphi_L = e^{B+i\omega}\Omega, \tag{4.4}$$

where  $B, \omega$  are real 2-forms and  $\Omega = \theta_1 \wedge \dots \wedge \theta_k$  for some linearly independent complex 1-forms  $(\theta_1, \dots, \theta_k)$ .  $k$  refers to the type of maximal isotropic subspace. We know that  $\dim L \cap \bar{L} = 0$  if and only if  $(\varphi_L, \bar{\varphi}_L) \neq 0$ , in other words

$$\begin{aligned} 0 &\neq (e^{B+i\omega}\Omega, e^{B-i\omega}\Omega) \\ &= (e^{2i\omega}\Omega, \bar{\Omega}) \\ &= \frac{(-1)^{2n-k}(2i)^{n-k}}{(n-k)!} \omega^{n-k} \wedge \Omega \wedge \bar{\Omega}, \end{aligned}$$

So, it suffices to say the maximal isotropic subspace is of real index zero if and only if

$$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0.$$

According to Weinstein's proof of the Darboux normal coordinate theorem for a family of symplectic structures, we can find a leaf-preserving local diffeomorphism  $\varphi$  taking  $\omega$  to a 2-form whose pullback to each leaf preserves the standard Darboux theorem. So,

$$\varphi^*\omega|_{\mathbb{R}^{2n-2k} \times \{pt\}} = \omega_0 = \sum_1^{2n-2k} dx_{2n-1} \wedge dx_{2n}.$$

Applying the diffeomorphism, we obtain the new 2-forms  $\varphi^*B + i\varphi^*\omega$ . Since,  $z_i$  are constant along the leaves,  $\Omega$  is unaffected by the diffeomorphism.

Now, we can set up a tri-degree  $(p, q, r)$ , by letting  $K = \mathbb{R}^{2n-2k}$  and  $N = \mathbb{C}^K$ . All components can be separated into three parts  $(\wedge^p K^* \otimes \wedge^q N_{1,0}^* \otimes \wedge^r N_{0,1}^*)$ . The exterior derivative will decompose into the sum of three operators.

$$d = d_f + \partial + \bar{\partial}$$

Each component is of degree 1.  $\Omega$  is of type  $(0, k, 0)$ , while the complex 2-form  $A = \varphi^*B + i\varphi^*\omega$  decomposes into six components:

$$\begin{array}{l} A^{200} \\ A^{110} \quad A^{101} \\ A^{020} \quad A^{011} \quad A^{002} \end{array}$$

Only components  $A^{200}$ ,  $A^{101}$ ,  $A^{002}$  act nontrivially on  $\Omega$  since each of the other components would have an element in common with  $\Omega$  causing  $e^A\Omega = 0$ . So, we can modify  $A^{110}$ ,  $A^{020}$ , and  $A^{011}$  without loss of generality. The imaginary part of  $A^{200}$  is  $\omega_0$ . Since  $\omega_0$  is in constant Darboux form,  $d(A^{200} - \overline{A^{200}}) = 0$ .

Using the condition that  $d(B + i\omega) \wedge \Omega = 0$ , we obtain the following equations by computing  $(0, k, 3)$ ,  $(1, k, 2)$ ,  $(2, k, 1)$ , and  $(3, k, 0)$  respectively.

$$\bar{\partial}A^{002} = 0 \tag{4.5}$$



$$\bar{\partial}A^{101} + d_f A^{002} = 0 \quad (4.6)$$

$$\bar{\partial}A^{200} + d_f A^{101} = 0 \quad (4.7)$$

$$d_f A^{200} = 0. \quad (4.8)$$

Now we can work on modifying  $A$ , so that  $\varphi^*\rho = e^A\Omega$  is unchanged, but  $A$  is replaced with  $\tilde{A} = \tilde{B} + \frac{1}{2}(A^{200} - \overline{A^{200}})$ .  $\tilde{B}$  is a real closed 2-form. This gives

$$\varphi^*\rho = e^{\tilde{B} + i\omega_\circ}\Omega,$$

or, that  $\rho$  is equivalent, via the composition of a  $B$ -field transformation and a diffeomorphism, to the product of a symplectic with a complex structure.

The most general form for  $\tilde{B}$  is represented as

$$\tilde{B} = \frac{1}{2}(A^{200} + \overline{A^{200}}) + A^{101} + \overline{A^{101}} + A^{002} + \overline{A^{002}} + C,$$

where  $C$  is a real 2-form of type  $(0,1,1)$ . Requiring  $d\tilde{B} = 0$  will give the following constraints:

$$(d\tilde{B})^{012} = \partial A^{002} + \bar{\partial}C = 0 \quad (4.9)$$

$$(d\tilde{B})^{111} = \partial A^{101} + \bar{\partial}\overline{A^{101}} + d_f C = 0 \quad (4.10)$$

Now, we will use these condition to find an appropriate real  $(0,1,1)$ -form  $C$ . First, Dr. Gualtieri uses the Dolbeault lemma and 4.5 to obtain that  $A^{002} = \bar{\partial}\alpha$  for some  $(0,0,1)$ -form  $\alpha$ . Then 4.9 is equivalent to  $\bar{\partial}(C - \partial\alpha) = 0$ , whose general solution is

$$C = \partial\alpha + \bar{\partial}\bar{\alpha} + i\partial\bar{\partial}\chi,$$

for any real function  $\chi$ . Now, all we need to do is find a suitable  $\chi$  to satisfy 4.10. Using 4.6 we obtain that  $\bar{\partial}(A^{101} - d_f\alpha) = 0$ , implying that  $A^{101} = d_f\alpha + \bar{\partial}\beta$  for some  $(1,0,0)$ -form  $\beta$ . Condition 4.10 then is equivalent to

$$-id_f\partial\bar{\partial}\chi = \partial\bar{\partial}(\beta - \bar{\beta}),$$

which can be solved if and only if the right hand side is  $d_f$ -closed. Using 4.7 we see that  $\bar{\partial}(A^{200} - d_f\beta) = 0$ , showing that  $A^{200} = d_f\beta + \delta$ , where  $\delta$  is a  $\bar{\partial}$ -closed (2,0,0)-form. Hence,

$$d_f\partial\bar{\partial}(\beta - \bar{\beta}) = \partial\bar{\partial}(A^{200} - \overline{A^{200}}),$$

and the right hand side vanishes precisely because  $A^{200} - \overline{A^{200}} = 2\omega_0$ , which is closed. So,  $\chi$  can be chosen to satisfy 4.10, and we obtain a closed 2-form  $\tilde{B}$ .  $\square$

**CHAPTER 5**  
**GENERALIZED CONTACT GEOMETRY**

**5.1 Review of Generalized Contact Structures**

**Definition 5.1.1.** *Suppose  $V$  is a  $2n+1$  dimensional real vector space. A generalized contact structure on  $V$  is a triple  $(X, \eta, \Phi)$ , where  $X \in V, \eta \in V^*$ , and  $\Phi : V \oplus V^* \rightarrow V \oplus V^*$ , is a linear map such that*

1.  $\eta(X) = 1, \Phi(X) = 0,$  and  $\Phi(\eta) = 0$ ;
2.  $\Phi + \Phi^* = 0$ ;
3.  $\forall Y + \xi \in L_X^0 \oplus L_\eta^0$ , we have  $\Phi^2(Y + \xi) = -(Y + \xi)$ , where  $L_\eta^0 = \{Y \in V | \eta(Y) = 0\}$ , and  $L_X^0 = \{\xi \in V^* | \xi(X) = 0\}$ .

Now set

$$L_X = \text{span}\{X\}, \quad L_\eta = \text{span}\{\eta\},$$

and set  $E^{0,1}$  to be the  $i$ -eigen-space of  $\Phi$ . Define

$$L = L_X \oplus E^{0,1}. \tag{5.1}$$

**Lemma 5.1.2.** [15] *Both  $E^{0,1}$  and  $L$  are isotropic subspaces of  $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ .*

*Proof.* First note that we have the natural isomorphism

$$L_X^0 \rightarrow (L_\eta^0)^*, \quad \alpha \mapsto \alpha|_{L_\eta^0}, \quad \alpha \in L_X^0.$$

It gives rise to the following metric preserving natural identification

$$L_\eta^0 \oplus L_X^0 \cong (L_\eta^0) \oplus (L_\eta^0)^*.$$

Using the above identification,  $\Phi|_{L_\eta}$  defines a generalized complex structure on  $L_\eta$ , and the  $i$ -eigenspace of the generalized complex structure is exactly  $E^{0,1}$ . It follows immediately that  $E^{0,1}$  is an isotropic subspace of  $L_\eta^0 \oplus L_X^0$ , and so an isotropic subspace of  $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ . Now a check of the definition of  $L$  shows that it is an isotropic subspace of  $(V_{\mathbb{C}}/L_X) \oplus (V_{\mathbb{C}}/L_X)^*$  as well.  $\square$

In [15] an alternate proof can be found which also shows that  $E^{1,0}$ ,  $L^*$ , and  $\overline{L^*}$  are also isotropic.

A generalized almost contact structure on a  $(2n + 1)$  dimensional manifold  $M$  is a triple  $(X, \eta, \Phi)$  such that on the tangent space  $T_x M$ , for any point  $x \in M$ , it creates a generalized contact structure.

**Definition 5.1.3.** [15] *Given a generalized almost contact structure, if the space  $\Gamma(L)$  of sections of the associated bundle  $L$  is closed under the Courant bracket, then the generalized almost contact structure is simply called a generalized contact structure.*

However, when  $\Gamma(L)$  is closed under the Courant bracket,  $\Gamma(L^*)$  is not guaranteed to be closed.

## 5.2 Pure Spinor Representation

Using pure spinors we can give a representation of generalized contact structures which allows us to use results from generalized complex structures.

**Proposition 5.2.1.** *Let  $(X, \eta, \Phi)$  be a generalized contact structure on a vector space  $V$ , and let  $L$  be a maximal isotropic subspace of  $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ . Then  $L$  corresponds to a*

pure spinor line generated by

$$e^{B+i\omega}\Omega,$$

where  $B, \omega \in \wedge^2 L_X^0$  are real two forms,  $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$  for some linearly independent complex one forms  $\theta_1, \cdots, \theta_k \in L_X^0$ , and

$$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0.$$

*Proof.* From the proof of 5.1.2, we know that  $E^{0,1}$  is the  $i$ -eigenspace of a generalized complex structure. This generalized complex structure corresponds to a pure spinor as seen from above.

$$e^{\omega+iB}\Omega,$$

where  $B, \omega \in \wedge^2(L_\eta^0)^* = \wedge^2 L_X^0$ ,  $\omega = \theta_1 \wedge \cdots \wedge \theta_k$  for some linearly independent complex one forms  $\theta_1, \cdots, \theta_k \in L_X^0$ . Now, from the isomorphism of 2.2.4 proposition 5.2.1 follows.  $\square$

The direct product of two generalized complex manifolds is itself a complex manifold. However, the direct product of two generalized contact manifolds cannot create a generalized contact manifold since the dimension would be even. The following construction shows how the direct product of a generalized complex manifold and a generalized contact manifold is a generalized contact manifold. Suppose that  $(X, \eta, \Phi)$  is a generalized contact structure on an odd dimensional manifold  $M$ , and that  $I$  is a complex structure on an even dimensional manifold  $N$ . Let

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$$

be the generalized complex structure induced by the complex structure  $I$ . Now using the natural identification

$$T(M \times N) \oplus T^*(M \times N) \cong (TM \oplus T^*M) \bigoplus (TN \oplus T^*N),$$

we define an automorphism

$$\Psi = \Phi \oplus \mathcal{J}_I : T(M \times N) \oplus T^*(M \times N) \rightarrow T(M \times N) \oplus T^*(M \times N)$$

Define a vector field  $\bar{X}$  on  $M \times N$  by the formula

$$\bar{X}(p, q) = (i_q)_*(X_p), \quad \forall (p, q) \in M \times N$$

where

$$i_q : M \rightarrow M \times N, \quad x \mapsto (x, q)$$

is the inclusion map. Then the triple  $(\bar{X}, \pi^*\eta, \Gamma)$  defines a generalized contact structure on  $M \times N$ , where  $\pi : M \times N \rightarrow M$  is the natural projection map.

**Example 5.2.2.** Let  $\{x_0, x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$  be the standard coordinates on  $\mathbb{R}^{2n+1} = \mathbb{R} \times \mathbb{R}^{2n}$ , let  $\omega_0 = \sum_i^n dx_i \wedge dy_i$  be the standard symplectic form on  $\mathbb{R}^{2n}$ , and let  $X = \frac{\partial}{\partial x_0}$ . Choose  $\eta$  to be any one form such that  $\eta(X) = 1$ . Note the map

$$\# : T\mathbb{R}^{2n+1} \rightarrow T^*\mathbb{R}^{2n+1}, \quad Y \mapsto \iota_Y \omega_0 - \eta(Y)\eta$$

gives us an isomorphism from the tangent bundle to the cotangent bundle. Define a bivector

$$\pi(\alpha, \beta) = \omega_0(\#^{-1}(\alpha), \#^{-1}(\beta)),$$

and an automorphism

$$\Phi = \begin{pmatrix} 0 & \pi \\ \omega_0 & 0 \end{pmatrix} : T\mathbb{R}^{2n+1} \oplus T^*\mathbb{R}^{2n+1} \rightarrow T\mathbb{R}^{2n+1} \oplus T^*\mathbb{R}^{2n+1}.$$

Then,  $(X, \eta, \Phi)$  forms a generalized contact structure on  $\mathbb{R}^{2n+1}$ .

**Example 5.2.3.** Let  $(X, \eta, \Phi)$  be the generalized contact structure on an open subset  $U \subset \mathbb{R}^{2n+1}$  as constructed in example 5.2.2. Let  $I$  be the standard complex structure on an open subset  $V \subset \mathbb{C}^m$ . Then there is a generalized contact structure on  $U \times V$ .

**Theorem 5.2.4.** [8] Let  $L$  be a complex Lie algebroid on the real  $n$ -manifold  $M$  with anchor  $a$ , and such that  $E + \bar{E} = T_{\mathbb{C}}$ , where  $E = a(L)$ . Let  $m \in M$  be a regular point for the Lie algebroid, i.e. a point where  $k = \dim(E \cap \bar{E})$  is locally constant. Then in some neighborhood  $U$  of  $m$ , there exist complex functions  $z_1, \dots, z_k \in C^\infty(U, \mathbb{C})$  such that  $\{dz_1, \dots, dz_k\}$  are linearly independent at each point in  $U$  and annihilate all complex vector fields lying in  $E$ , i.e. we have a transverse complex structure to the foliation, at regular points.

**Lemma 5.2.5.** Let  $(X, \eta, \Phi)$  be a generalized contact structure on a  $(2n+1)$ -dimensional manifold  $M$ , and let  $L$  be the Dirac structure as defined in 5.2.1. Suppose that  $p$  is a regular point of  $M$ . Then there exists an open neighborhood  $U$  of  $p$  which is isomorphic to an open set in  $\mathbb{R}^{2m+1} \times \mathbb{C}^k$  such that

1. The pure spinor  $\rho$  on  $U$  determined by the Dirac structure  $L$  can be expressed as  $\rho = e^{B+i\omega}\Omega$ , where  $\Omega$  is decomposable of degree  $0 \leq k \leq n$  and such that

$$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0.$$

2. For the real coordinates  $\{x_i\}$  on  $\mathbb{R}^{2m+1}$  and complex coordinates  $\{z_i\}$  on  $\mathbb{C}^{2k}$ , we have that

$$X = \frac{\partial}{\partial x_0}, \quad E = \pi(L) = \text{span}\left\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}\right\},$$

$$\omega = \sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}, \quad \text{and} \quad \Omega = dz_1 \wedge \dots \wedge dz_n,$$

where  $\pi : TM \oplus T^*M \rightarrow TM$  is the natural projection map.

*Proof.* By Lemma 2.2.4, around an open neighborhood  $U$  of a regular point  $p \in M$ ,  $L$  may be expressed as  $L(E, \varepsilon)$ , where  $E \subseteq T_{\mathbb{C}}$  is an involutive sub-bundle and  $\varepsilon \in C^\infty(\wedge^2 E^*)$  satisfies  $d_E \varepsilon = 0$ . Therefore, by choosing  $B + i\omega \in C^\infty(\wedge^2 T_{\mathbb{C}}^*)$  such that  $(B + i\omega)|_E = \varepsilon$ , we have a generator for the canonical bundle defining  $L(E, \varepsilon)$  as follows:

$$\rho = e^{B+i\omega} \Omega,$$

where  $\Omega$  is a decomposable complex  $k$ -form. Let  $i : E \rightarrow T_{\mathbb{C}}M$  be the inclusion map. Observe that  $i^*d(B + i\omega) = d_E i^*(B + i\omega) = d_E \varepsilon = 0$ . This implies that  $d(B + i\omega) \in \text{Ann}^* E$ . As a result, we have that

$$d\rho = d(B + i\omega) \wedge e^{B+i\omega} \wedge \Omega = 0.$$

Now by Theorem 5.2.4 and using the algebroid  $L$  from equation 5.1, we may assume that  $U$  is of the form  $U_1 \times U_2 \times U_3$ , where  $U_1$  is an open set in  $\mathbb{R}^1$ ,  $U_2$  is an open set in  $\mathbb{R}^{2m}$ , and  $U_3$  is an open set in  $\mathbb{C}^{2k}$ ; moreover, there exists real coordinates  $\{x_0, x_1, y_1, \dots, x_m, y_m\}$  on  $\mathbb{R}^{2m+1} = \mathbb{R} \times \mathbb{R}^{2m}$  and complex coordinates  $\{z_1, \dots, z_k\}$  on  $\mathbb{C}^k$ , such that

1.  $E$  is spanned by  $\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_k}\}$ ;
2.  $X = \frac{\partial}{\partial x_0}$ ,  $\Omega = dz_1 \wedge \dots \wedge dz_k$ ;
3. At point  $p$ ,  $\omega$  coincides with the closed two form

$$\omega_0 = \sum_{i=0}^k dx_i \wedge dy_i.$$

Note that  $\omega|_{U_1}$  is a symplectic form. From [13], the usual argument of the Darboux theorem for a family of symplectic forms can be used to produce a local



diffeomorphism  $\varphi$  such that  $\varphi^*\omega|_{\{pt\}\times U_2\times\{pt\}} = \omega_0$  and such that  $\varphi|_{U_1\times\{pt\}\times U_3}$  is the identity map on each leaf  $U_1 \times \{pt\} \times U_3$ .  $\square$

**Theorem 5.2.6. *Darboux Theorem for Generalized Contact structures*** Any regular point in a generalized contact manifold  $M$  has a neighborhood which is equivalent, via a diffeomorphism and a  $B$ -transform, to the generalized contact structure constructed in 5.2.3.

*Proof.* Let  $U_1, U_2, U_3$  and  $\varphi$  be the same as in the proof of Lemma 5.2.5. This proof will give us  $\varphi^*(e^{B+i\omega_0}\Omega) = e^{\varphi^*B+i\omega_0}\Omega$ .

Set  $K = \mathbb{R}^{2n+1}$  and  $N = \mathbb{C}^m$ . Then differential forms now have tri-degree  $(p, q, r)$  for components in  $\wedge^p K^*, \wedge^q N_{1,0}^*, \wedge^r N_{0,1}^*$ . The exterior derivative will decompose into the sum of three operators

$$d = d_K + \partial + \bar{\partial},$$

each of degree 1 in the respective component of the tri-grading. Now following straight from Dr. Gualtieri's paper [8], we can modify  $\varphi^*B$  to a real closed two form  $\tilde{B}$  such that

$$e^{\varphi^*B+i\omega_0}\Omega = e^{\tilde{B}+i\omega_0}\Omega.$$

$\square$

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