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Applying Transportation Forecasting to the Atlanta Area Public Transportation System

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APPLYING TRANSPORTATION FORECASTING TO THE ATLANTA AREA PUBLIC TRANSPORTATION SYSTEM

By

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(Under the Direction of Goran Lesaja)

Abstract

In this thesis, we examine the implementation of volume delay functions to the Metropolitan Atlanta Rapid Transportation Authority (MARTA) system. Volume delay functions are n differentiable functions used to estimate the long-term distribution of user traffic on transportation systems. We will demonstrate the graphical behavior of these functions as well as explain the constraints for these functions. These tasks will be completed by developing a simplistic yet unrealistic model, demonstrate how the integral is used to estimate the total sum of this model, and then introduce the two functions which will be used in the analysis. The final task of this paper will be to develop and implement functions which have properties which allow them to directly relate to the unique behavior of public transportation systems. The final analysis will be the interaction of the functions developed in this paper with the preexisting, well-behaved BPR and Davidson functions.

Index Words: Volume delay functions, user equilibrium, system optimization, Waldrop equilibrium

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Introduction

1.1 The History of MARTA

In 1971, a referendum was passed in Atlanta to provide funding for the construction of a 53-mile subway system, 8 miles of bus way, and an extensive feeder bus system. Although this system was originally scheduled for completion by 1980, lack of sufficient funds meant that only the first phase was completed, 13.7 miles of rail line. Because future funding was uncertain, a study was done to determine the best incremental funding strategy for further construction.

Several factors, including rapid urban development, increasing costs, scarcity of gasoline, and system improvements, had made the analysis done just 6 years earlier outdated. The strategy the analysts had taken was dividing the system into 13 operational segments, and then grouping each segment into four test networks. With their analysis method, planners were able to predict the following:

- *Total increase in transit system patronage*
- *Total patronage on each segment*
- *Total capital cost*
- *Total increase in operating cost*
- *Relative cost-effectiveness*
- *Travel time improvements to major activity centers*
- *Improvements of transit service to special groups*

- *Impacts on land use and development patterns*
- *Environment impacts and energy consumption*

1.2 MARTA Currently

When Marta was initially formed, the counties of Gwinnet, Cobb, and Clayton decided to option out of participating in the project when they were approached about contributing to the program. Although Cobb Community Transit (CCT), Gwinnet County Transit (GCT), and Clayton County Transit (C-Tran) serve their communities and have connecting routes with MARTA, their systems are limited to traditional bus lines and lack direct access to rail lines. The commute between CCT and GCT to MARTA is considered by many users to be timely and undesirable. Also CCT and GCT have limitations within their own systems. Unlike MARTA, GCT does not operate on Sundays or holidays and CCT only serves the lover parts of the county. The exclusion of Cobb and Gwinnet counties from the MARTA system has had severe financial consequences to MARTA. 50% of the MARTA revenue comes from a 1% sales tax paid by all participating counties[*]. The lack of initially expected revenue from these counties slowed development plans severely. Although there have been a number of attempts over the past few decades to fully integrate these counties into the MARTA, all measures have been repeatedly voted down. Even though the most southern MARTA rail terminal, the Airport Station, is located in the most northern part of the county, the Clayton County system does not have a full rail line system. However; C-Tran has been operated by MARTA through contract since 2007. Also, in the July 20th primary of 2010, House Bill 1446 is being proposed to Clayton County voters, a bill which would approve 1% sales

[*] This is based on the FY08 Operating Budget Overview

tax increase in the county. The approval of this bill would be the first step to being fully integrated into the MARTA system.

1.3 Project Basics

In 2007, MARTA compiled a document called Locally Preferred Alternative Report* [13] which was written for the purpose of assessing city transportation needs, goals and objectives. There research consisted mostly of formal discussions with various community groups and stakeholders. In this study, three broad goals were set along with general methods of reaching these goals:

Goal 1: Improve corridor mobility, reliability, and accessibility to employment centers

Objectives:

- Relieve increasing highway congestion in the I-20 corridor by attracting auto users to transit;
- Improve travel times and reliability for all travelers in the I-20;
- Extend high quality public transportation service to employment destinations along Fulton Industrial Boulevard (FIB) by minimizing the number of transfers between different transit lines and routes;
- Improve access to major employment centers in the City of Atlanta and to the FIB area for residents of Fulton and DeKalb counties, the City of Atlanta and others in the region;
- Increase accessibility for the transit dependent population.

* Chapter Two: Basis for Project Alternative; Locally Preferred Alternative Report

Goal 2: Preserve and enhance the environment

Objectives:

- Improve air quality by providing transit alternatives that attract auto users, thereby reducing vehicle miles of travel and air pollution emissions;
- Reduce potential impact on residential areas and the natural and built environment.

Goal 3: Encourage economic development/transit supportive land use

- Encourage continued concentration of development where transportation facilities provide a high level of access, particularly near FIB;
- Concentrate development around transit stations in concert with zoning and related development policies;
- Create public/private collaboration opportunities in real estate development.

The focus of this study was determined by what form of system augmentation would best meet these objectives.

1.4 Transit Technology Options

During the basic screening process, there were three transit technology options considered; bus rapid transit, light rail transit, and heavy rail transit. In choosing these technologies, factors including capital costs, environmental friendliness, carrying capacity and compatibility with the existing system were considered. Below is a rough summary of each technology.

1.4.1 Bus Rapid Transit (BRT)

The BRT has certain characteristics which distinguish it from the traditional bus service. The idea is to combine the service characteristics of rail with the flexibility of buses. The technology is supposed to eliminate on-board fare collection and traffic signal delays in order to increase operating speed and reliability. The BRT system is most efficient in exclusive transit ways or dedicated bus lines, but also works well with High Occupancy Vehicle Lanes (HOV); a system already implemented inner city portions of I-75 and I-85. BRT typically has a capacity of 35-50 persons and operates at an average speed 30-50 miles per hour. With an operating cost of $10-$40 million per mile, it is the least expensive system.

1.4.2 Light Rail Transit (LRT)

LRT systems are typically electric railways with smaller volumes than heavy rail. These facilities are usually operated at-grade (surface level), but can also be grade separated in a tunnel or elevated. In comparison to HRT, light rail is more flexible due to its ability to easily maneuver through existing communities. Approximately 170 persons can be transported per vehicle with an operating speed between 40-60 mph. On average, LRT systems cost $20-$40 million per mile to operate. LRT was eliminated from the West Line expansion project after the Basic Screening Analysis due to cost and connectivity to with the existing transit system. However, I will conclude an analysis of this system within the report.

1.4.3 Heavy Rail Transit

HRT is an electric operated train system which uses exclusive rights-of-way. The capacity ranges from 170-300 persons with train lengths varying from 2 to 10 cars. It can reach speeds up to 70 miles per hour. Of the three systems being evaluated, it is on average the most expensive system, and is implemented only when highly warranted. The cost of these systems ranges from $50-$120 million per mile

1.5 Comparing Systems

In this thesis we are interested in comparing the efficiency of these systems in regards to moving a particular number of people from one point to another. Though this would not be the single factor in determining which system would be integrated into the expansion, it is an important consideration. Utilizing a preexisting field of study in logistics and management sciences called route assignment. We will derive a method which will allow us to calculate the optimal volume distribution between systems.

1.6 Objective

When expanding any systems of roads and highways, the objective of any city planner is to have the most efficient system by building a network which allows people to move from points 'a' to point 'b' in the fastest way possible. For small to medium size cities, this is done by improving existing surface roads and highways through expansions and closures. For larger cities with the populations and funds to support it, government subsidized public transportation is an option.

Due to the potential cost and overall resource investment of public transportation systems, several levels of extensive research is done before such plans even reach the serious design stage. One of the first steps taken in most cases is an environmental impact study. This is usually done by an environmental and/or civil engineer and consists of a basic assessment of whether a proposed system is physically feasible or not. A very basic cost assessment is done at this stage as well.

For systems analysis of large scale transportation improvements, a transit assignment procedure is primarily used. The objective of this paper is to apply this technique to estimating volume delay in systems other than road systems.

Chapter 2
Route Assignment Tools

2.1 The Wardrop Equilibrium

In 1952, J.G. Wardrop of the Road Research Laboratory published a paper on two principles of road network flow distribution; the *user equilibrium principle* and the *system optimum principle* [9]. The user equilibrium principle is based on the assumption that all travelers are making decisions which minimize their personal travel costs with no concern of the total cost to the system. The system optimum principle is based on the assumption that travelers are minimizing the travel cost for the entire system. Let it be noted that travel time and cost are being used interchangeably.

The two most common methods of network distribution are the En-Route and the Equilibrium Assignment method. The resulting optimal distribution of the En-Route Assignment is *user equilibrium*, while the Equilibrium Assignment finds the *system equilibrium*. The only situation in which the user and optimal flows are equal is in the case when no congestion exists. This artificial environment will be used in our models. Because we are assuming a system without points of congestion, at some point in the paper, we will have to use the user equilibrium to find the system optimization.

2.2 En-Route Assignment

It is a fair enough assumption to say that at any given time, with a certain number of routes, not every driver has the same quality or quantity of information. For example, let us observe two individuals (John and Jacob) that leave from the same place (they are next door neighbors) and head to the same destination (they work together as well). They have both lived in the city for the same number of years; both worked at the same job for an equal number of years and have comparable vehicles. It is fair assumption that if both leave at the same time, both will arrive at the same time. There are other factors which could be considered which could make this assumption invalid, such as one is a Sunday driver while the other is a speed demon, but controlling these factors, their total travel time will be on average the same. Both should have a comparable understanding of traffic patterns and both would probably know the quickest way. These individuals have reached system optimization, in terms of the greatest utility in respect to time saved. System optimization is a point of Pareto Equilibrium, a point where no action can benefit one individual without hurting another.

Now suppose that John is a technophile. He likes to get all the newest technology as soon as it comes out. So, as soon as GPS systems became available for public use, John could not snatch it from the shelves fast enough. Meanwhile, Jacob is a luddite, he does not even have a car radio. Where each of these individuals are at a given time, because of asymmetric knowledge, would no longer be so easy to predict. Even with the assumption that they are both rational actors who are trying to take the minimum amount of time to get from one point to another, John may take an alternative route to avoid the traffic jam his GPS told him about while Jacob would run right into the confusion.

Such problems are best handled by the En-Route Assignment model. In this type of model the exogenous inputs to optimize include information strategy (information provided) and penetration rate. It is possible to design an iterative algorithm to determine the optimum values of these inputs. With this technique, the user equilibrium is found for the individual with disproportionate knowledge. Given the available knowledge, no possible choice will improve well being. Being this model lacks perfect information, this point is not necessarily the system optimization.

Although it would be possible to apply this method to the public transportation analysis problem, the benefits are outweighed by its drawbacks. This method adds needless complexity in regards to the overall objective. There is also a large amount of computing power required to run the micro simulation. Users' choices within the framework of public transportation are fairly inelastic, so thinking in terms of the aggregate in the long run is more appropriate.

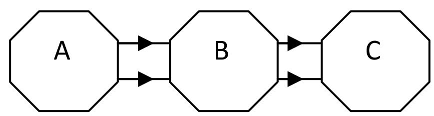
2.3 Equilibrium Assignment

There are a number of procedures and techniques which can be used to perform this task of equilibrium assignment for the system optimization. In most cases when this method is used, there are two main components: a procedure to determine a new set of time dependent path flows given the experienced path travel times on the previous iteration, and a method to determine the actual travel times that result from a given set of path flow rates.

There are two ways in which a system optimal flow can be achieved. One in which there is a centralized control over route decisions and one which there is a penalty, such as a toll, with taking a particular route. The models we develop later will consist of routes comprised of public transportation systems. Such systems are centrally controlled. Also, one could think of the difference between vehicle operation and the cost of a ticket.

2.3.1 Shortest Route Algorithm & System Optimization

In order to better understand the principals of system optimization and link updating, we will examine this very simple system consisting of three nodes and four edges.



$$\boldsymbol{Figure\ 2.1}\text{: } \boldsymbol{Graphical\ Model\ of\ a\ Simple\ System}$$

For person 1, the system has the following travel time in minutes,

<u>Person 1:</u>

$$\overrightarrow{AB} = 5, \qquad \overrightarrow{AB'} = 8, \qquad \overrightarrow{BC} = 6, \qquad \overrightarrow{BC'} = 9$$

the shortest path is $\overrightarrow{AB} \rightarrow \overrightarrow{BC}$, with a travel time of 11 minutes.

For person 2, the routes have a different weight,

<u>Person 2:</u>

$$\overrightarrow{AB} = 9, \qquad \overrightarrow{AB'} = 8, \qquad \overrightarrow{BC} = 7, \qquad \overrightarrow{BC'} = 9$$

the shortest path is now $\overrightarrow{AB'} \rightarrow \overrightarrow{BC}$, with a travel time of 15 minutes.

In a system optimal model, a shortest path algorithm, such as Dijkstra's algorithm is implemented and followed. As the individuals move through the system, each route that is used is updated using a route assignment function. The following individual now finds the shortest route of the newly updated system and performs the process with the updated system. The optimized time is the sum of each person travel time.

2.4 Constructing an Approach

Before proceeding to the specific analysis of assignment functions, it will help to take a step back and restate the problem. There are three possible expansion choices, each with their own advantages and disadvantages. Whatever system, or combination of systems, is eventfully constructed, it will coexist parallel to the present existing system. Thus, a user will find themselves with at least two possible routes and at least two feasible choices. For the purpose of determining the minimized time required, we propose to use a slightly modified view of route assignment problem. The route assignment function uses the maximum capacity of a system within a certain frame of time and the time required for an individual to move through the system depending on their order. By its very nature it is easier to measure or at least estimate the parameters within these systems in comparison to one composed of personal owned vehicles (POV).

There are other factors which must be considered that are associated with traditional route assignment. The shortest path (temporally speaking) is not the only factor which will determine a most desired route. We must now take into account other cost, such as fuel expenditures and long term vehicle care. We will even try to account for the basic and raw displeasure individuals may feel towards driving POV's.

2.5 The Basics

The first step of route assignment is finding a way to calculate the time required for a given number of individuals to move through a particular path. One basic assumption is that as more people travel through a given path, the time required for each successive person increases.

Let us start with the observations of individuals traveling through a particular path connecting points A and B.

$$\boldsymbol{Table\ 2.1\text{: } Travel\ Times\ for\ \overrightarrow{AB}}$$

Numerical Order	Length of Trip in Minutes
1	12
2	15
3	18
4	21
5	24

From this artificial example, it is easy to see the increase in time is linear, and it is rather simple to construct a function expressing the relationship between order and travel times. Later in this paper, we will cover why a linear function does not fit the requirements for a good time delay function.

Using point-slope form,

$$m(v_2 - v_1) = (t_2 - t_1) \Rightarrow m(2 - 1) = (12 - 15)$$

$$m = 3 \Rightarrow f_1(v) = 3v + 9 = t_v$$

where v is the order of the individual and t_v is the time it takes the v order individual to travel through the path, it is possible to estimate travel time for any number of individuals.

The total travel time is,

$$Z = \sum_{i=1}^{5} t_i = 90 \tag{2.1}$$

With en-route assignment, instead of summing the total time to get the estimate, this value is approximated by integrating the travel time function.

$$Z \approx \int_0^x f_1(v)dv = \int_0^5 (3v + 9)dv = 82.5 \tag{2.2}$$

This is clearly a very rough approximation.

To make this a true route assignment problem, there has to be an alternative route available which begins and ends at the same points. With this new route, the idea is to place some of drivers on this second route for the purpose of reducing the total time. The second route is observed to have the following travel times.

$\boldsymbol{Table\ 2.2:\ Travel\ Times\ for \overrightarrow{AB}'}$

Numerical Order	Length of Trip in Minutes
1	10
2	14
3	18
4	22
5	26

One again the increase in time is linear, and function $f_2(v') = 4v' + 6$ is derived.

The total travel time for this alternative path would be 90 minutes.

$$Z' = \sum_{i=1}^{5} t_i{}' = 90 \tag{2.3}$$

The approximation using the integral is,

$$Z \approx \int_0^x f_2(v)dv = \int_0^5 (4v + 6)dv = 80 \tag{2.4}$$

Although the travel times for each of the routes are the same, we can decrease this travel time by allowing some individuals to take route $\overrightarrow{AB}$ and others to take route $\overrightarrow{AB}'$,

$$\textbf{Table 2.3: Combined Travel Times for } \overrightarrow{AB} \textbf{ and } \overrightarrow{AB}'$$

Numerical Order	Route	Total Travel Time
1	$\overrightarrow{AB}'$	10
2	$\overrightarrow{AB}$	22
3	$\overrightarrow{AB}'$	36
4	$\overrightarrow{AB}$	51
5	$\overrightarrow{AB}'$	69

Thus, by distributing the flow of traffic between the two routes, the total time was decreased from 90 to 69 minutes, more than a fifth.

If this were a much more complicated system, one with hundreds of nodes and edges, it would be unreasonable to go through each edge, compare, and update. Instead, we can minimize the assignment time functions. In our simple case we have,

$$\min Z = \int_0^x (3v + 9)dv + \int_0^{x'} (4v' + 6)dv', \tag{2.4}$$

$$s.t.\ x + x' = 5.$$

After completing the integral, the problem becomes a two variable, nonlinear minimization with a linear equality constraint.

$$F(x, x') = \frac{3}{2}x^2 + 9x + 2(x')^2 + 6x', \tag{2.5}$$

$$s.t.\ x + x' = 5,$$

This function can be maximized by using the constraint to eliminate one of the variables,

$$F[x,(5-x)] = \frac{3}{2}x^2 + 9x + (5-x)^2 + 6(5-x)$$

$$\frac{dF}{dx} = 5x - 7 \tag{2.6}$$

$$\frac{d^2F}{dx^2} = 5 \tag{2.7}$$

The extreme point of this function can be found by calculating the root of the derivative (2.6). The second derivative (2.7), being positive across the range, shows the function is strictly convex. Because the function is strictly convex, the extreme point is a global minimum. Finding the root of the first derivative, we get the following results,

$$v_1 = \frac{17}{7} \approx 2.4286 \tag{2.8}$$

$$v_2 = 5 - \frac{17}{7} \approx 2.5714 \tag{2.9}$$

The interpretation of (2.8) and (2.9) literally tells us the system time is minimized when 2.4286 people are assigned to the first route and 2.5714 are assigned to the second route. These results are rounded; we get the result of assigning 2 people to the first route and 3 to the second, the same result we got by inspection on pages 17 and 18.

The estimated total time is $F(2,3) = 51$, compared to an actual time of 69. Much of this error is associated with being a rough approximation of the sum.

2.5.1 Defining the Functional Constraints

A general form representing two nodes connected by n edges is,

$$\min \boldsymbol{Z} = \sum_{i=1}^{n} \int_{0}^{x_i} f_i(v_i) dv_i$$

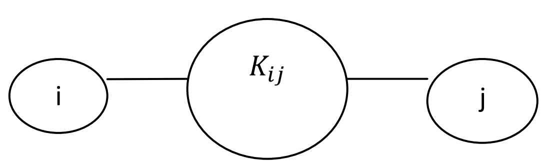
$$s.t. \sum_{i=1}^{n} x_i = T,$$

where T is the total number of travelers.

This description for the objective function is adequate for a system of any number of nodes or edges, but the constraint only defines a system consisting of two nodes with any number of edges. Therefore, the final step for defining a system of any number of bounded nodes and edges is to improve upon the constraint definition.

In order to build the idea of the constraint function, we must start with a system consisting of more than two nodes. For the constraint of this system, the first assumption is that no one enters or leaves this system. So, after all iterations of movement have taken place, the total number of individuals at every node will not change.

We start with two points, i and j. In between these points are all possible edges and nodes which connect these points. This set is referred to as K_{ij}.



$\boldsymbol{Figur\ 2.2\text{: }Simplified\ Multi-Node\ Model}$

Between nodes i and j, there is a finite number of edges which connect them. Of these finite edges, we will designate an unspecified path as between$i$ and j, $k \in K_{ij}$. In the process of moving between these nodes, x_{ij}^k represents the number of individuals which move from nodes i and j on a path k.

With this notation, we can set up the first equality constraint. Let T_{ij} be the total number of people that move from points i to j. Then we have,

$$\sum_{k \in K_{ij}} x_{ij}^k = T_{ij},$$

$$x_{ij}^k \geq 0 \ , \ \forall k \epsilon K_{ij}$$

The final step to defining the constraints is to isolate individual edges as they relate to a particular path. If an edge α is used in a path k, we set $\sigma_{ij}^{\alpha k}$=1; otherwise it is 0.

Taking all the conditions into account, a general form of route assignment problem consisting of n edges is,

$$\min \boldsymbol{Z} = \sum_{i=1}^{n} \int_0^{x_i} f_i(v_i) dv_i,$$

$$v_\alpha = \sum_i \sum_j \sum_k \sigma_{ij}^{\alpha k} x_{ij}^k, \forall \alpha \in A, \tag{2.10}$$

$$\sum_{k \in K_{ij}} x_{ij}^k = T_{ij},$$

$$v_\alpha \geq 0, x_{i,j}^k \geq 0, \forall k \epsilon K_{ij.}$$

2.6 Link Performance Functions

In most traffic assignment methods, the effect of road capacity on travel times is specified by means of a volume-delay function which is used to express the travel time and/or cost on a road link as a function of the traffic volume. The functions are the product of the free flow time, t_o, multiplied by delay function $f(*)$ where the argument is some form of the ratio of v and c , c being a measure of the capacity of the road, i.e.,

$$t(v) = t_0 * f\left(\frac{v}{c}\right).$$

There are three properties desired for a well-behaved function. [11]

1. $f(x)$ is strictly increasing. This is necessary to have a unique solution.
2. $f^{'}(x)$ exist and is strictly increasing. This ensures convexity. Although this is not necessary, it is very desirable.
3. $f'(0) > 0$. This guarantees uniqueness of link volumes and distributes volumes on competing uncongested paths proportional to their capacity.
4. $f'(0) < M$, where M is a positive constant. The steepness of the congestion curve is limited.

2.7 Frank-Wolfe Algorithm

The Frank-Wolfe method (FW) is one of the most widely used algorithms for solving routing problems because of simplicity and low memory requirements. Because the rate of convergence for FW decreases as the algorithm approaches the optimal point, it is often used with some modifications. Two of these methods are developed by Saida, Rachid [10], and Fukushima. [3]

Summary of Standard Frank-Wolfe Method

Frank-Wolfe is used for minimizing nonlinear functions of a set of linear constraints. At iteration k, FW approximates f by linearizing at the current iteration x_k. With x_k, we have the linear function $f_k(y) \cong f(x_k) + \nabla f(x_k)(y - x_k)$.

First step: The direction is found by solving linear program:

$$\text{LP(k)}\begin{bmatrix} \min f_k(y) \\ y \in X, \end{bmatrix}$$

Where X is a feasible set.

Let y_k be the optimal solution of LP(k). The direction of FW is defined by:

$d_k = y_k - x_k$. In the traffic application, LP(k) decomposes into a set of shortest path problems.

Second step: The objective function is minimized along the line segment passing by the point x_k and the direction d_k. The step-size α_k is then used to find the update, $x_{k+1} = x_k + \alpha_k d_k$.

$\alpha_k = \arg min_{0 \leq \alpha \leq 1} f(x_k + \alpha d_k)$.

The stopping criteria is $\|\nabla f(x_k)\| < \mathcal{E}$ for some $\mathcal{E} > 0$.

The Franke-Wolfe method is one of the most widely used algorithms for solving routing problems; its popularity is attributed to its simplicity and modest memory requirements. Because FW converges slowly, it is more favorable to use it with modifications. Two popular modifications include the Saida-Rachid and Fukushima methods. Without these modifications, the Franke-Wolfe method may zigzag as it approaches the convergence point.

Chapter 3
The Simplified Routing Problem

3.1 Introducing a Problem

Imagine a scenario in which there is an apartment complex housing five hundred individuals who all live in the same apartment complex and all work in the same factory. Now, assume all these individuals have to be at work between 8:00 and 9:00 a.m. To make matters even more difficult for these employees, imagine there are only two methods which can be used to make the commute. Because there is only a single road connecting these points, they can either decide to drive to work or use a company subsidized bus system which uses an exclusive and dedicated lane. The question which follows is how many people will decide to drive and how many people will decide to take the bus system.

In the perfect world of nonbiased, non prejudiced rational actors, we can expect equilibrium to be reached in the "long run". By "long run" we mean an unspecified time in which every commuter takes the means of travel which minimizes the time of their commute. Once this equilibrium is reached, we can assume there is no exchange between different systems.

In order to solve the optimum volume distribution of passengers, we will review the BPR as well as the Davidson functions

3.2 Volume Delay Functions

3.2.1 BPR Volume Delay Function

The BPR volume delay function was developed by civil engineers at the Bureau of Public Roads (later renamed the Federal Highway Administration) in 1962. It was developed from empirical analysis with the previously mentioned conditions in mind. It is the most commonly used function in traffic assignment and intersection delay problems.

It is defined as follows:

$$t_a^{BPR} = t_a^0\left[1 + \alpha\left(\frac{v_a}{c_a}\right)^{\beta}\right], \tag{3.1}$$

$$0 \leq v_a \leq c_\alpha{}^*.$$

For this function, t_a^0 is the time it takes for the first vehicle to move through the edge; α and β are adjustable parameters which are determined by the type of road represented by the edge. For interstates $\alpha\epsilon(0,.3)$, for highways $\alpha\epsilon[.3,.6)$, and for surface roads $\alpha\epsilon[.6,.9)$. Parameter β is an adjustable parameter usually set at 4.

It is easy to show by the first and second derivative tests that this function is both increasing and convex.

$$\frac{d}{dv_a} t_a^{BPR} = \alpha\beta t_a^0\left(\frac{1}{{c_a}^{\beta}}\right){v_a}^{\beta-1} > 0 \tag{3.2}$$

$$\frac{d^2}{d{v_a}^2} t_a^{BPR} = \alpha(\beta^2 - \beta) t_a^0\left(\frac{1}{{c_a}^{\beta}}\right){v_a}^{\beta-2} > 0, \beta > 1 \tag{3.3}$$

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* With many models using the BPR volume delay function, although it is not required, to keep the function within well behaved parameters, the volume never exceeds capacity.

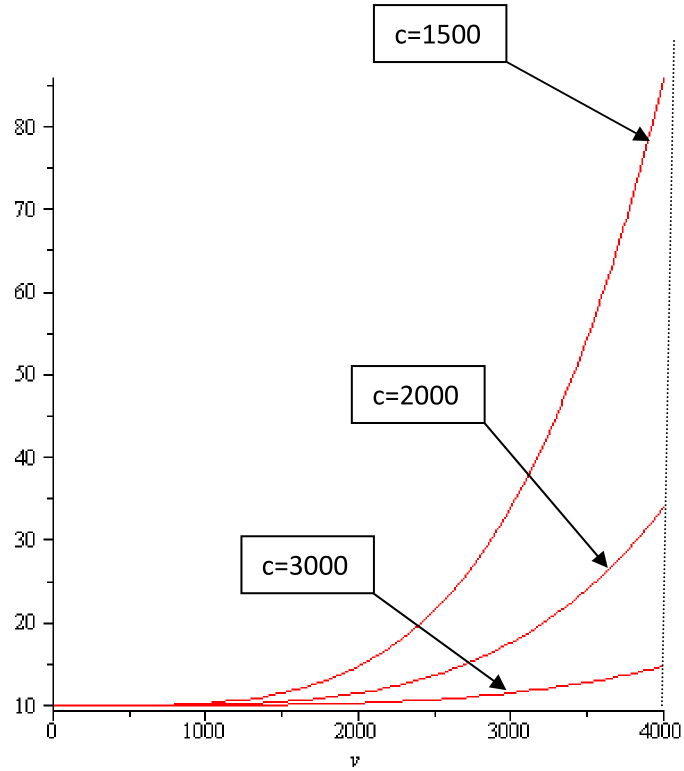


Figure* 3.1: *Graphical Representation of BPR Function

$$10\left(1+.15\left(\frac{v}{c}\right)^4\right).\ c = 1500, 2000, and\ 3000.$$

3.2.2 Davidson Volume Delay Function

The Davidson function was first proposed by K.B. Davidson in 1966. It is similar to the BPR function in the sense that it has similar calibration parameters and uses traffic volume as a sole input. Unlike the BPR function, travel time in the Davidson function is asymptotic to flow capacity.

$$t_a^D = t_a^0\left[1+\alpha\left(\frac{v_a}{c_a - v_a}\right)\right],$$

$$0 \leq v_a < c_a$$

The Davidson function is shown to be increasing and convex across the domain as well

$$\frac{d}{dx_a} t_a^D = \alpha t_a^0 \frac{v_a}{(c_a - v_a)^2} > 0, \tag{3.4}$$

$$\frac{d^2}{d{x_a}^2} t_a^D = \alpha t_a^0 \frac{2v_a}{(c_a - v_a)^3} > 0, \tag{3.5}$$

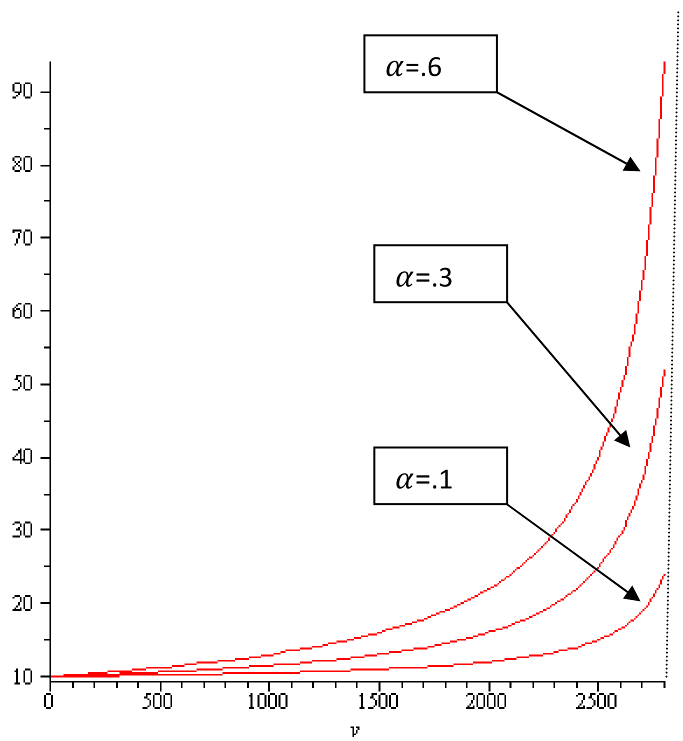


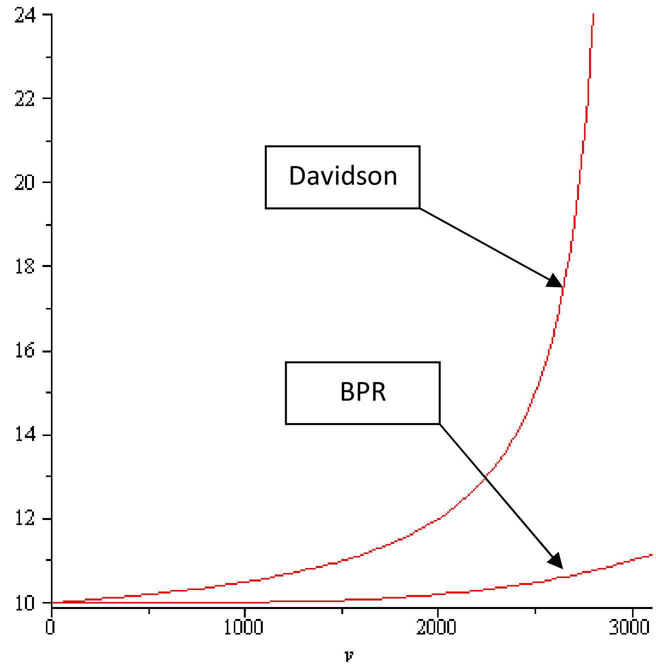
Figure 3.2: Graphical Representation of Davidson Function

In the figure 3.2 we graph the Davidson function,

$$10\left(1 + \alpha\left(\frac{v}{3000 - v}\right)\right) \ \alpha = .1, .3, and\ .6.$$

3.2.3 Graphical Comparison of Functions

The Davidson and BPR functions have different properties which we can use to reflect the different aspects of the system being modeled.



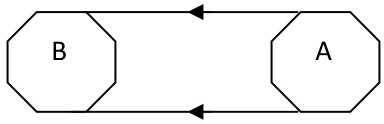
$\boldsymbol{Figure\ 3.3}$: $\boldsymbol{Comparing\ the\ BPR\ and\ Davidson\ Functions}$

In the Figure 3.3 we graph the BPR and Davidson functions for comparison purposes.

3.3 Solving a Simple Example

In this model the two alternatives are represented by two directed edges with identical departure and destination points. The edges will be designated as $\overrightarrow{AB_r}$ and $\overrightarrow{AB_b}$ designating the privately owned vehicle route and the bus route respectively.

Within an hour, 500 people will be distributed between the two paths. With the information given, we can form a simple graphical model below.



The simple distribution constraint is given by,

$$v_{\overrightarrow{AB_r}} + v_{\overrightarrow{AB_b}} = 500.$$

For this model, this is the only constraint which will be considered. Hidden in this distribution constraint is a not so obvious interpretation of the population decision making. Essentially, it is being assumed all employees may be willing to either drive or take the bus. The alternative to this, and inarguably more reasonable assumption, is that some individuals will always drive or always take the bus unless presented with extreme incentives or disincentives.

Both route assignment methods being analyzed in this report, the BPR method as well as the Davidson method, require an estimate of free flow travel time , t_i, and capacity of the link, c_i.

To find capacity, we will assume there are ten complete bus trips with a capacity of 32 individuals. So for the bus line, given the unit of time is an hour, the capacity 320 individuals in an hour. We can assume there is a capacity of 400 POVs in an hour.

Hence,

$$c_b = 320,$$

$$c_r = 400.$$

Then it takes the first bus 24 minutes to move through the link and the car 15 minutes. Because time is being measured in hours, we have the following values for t_0.

$$t_{\overrightarrow{AB}_r} = 15\left(\frac{1}{60}\right) = .25$$

$$t_{\overrightarrow{AB}_b} = 24\left(\frac{1}{60}\right) = .4$$

The BPR volume delay function with parameters α=.6 and β=4 is given below.

$$t^{BPR}\left(x_{\overrightarrow{AB}}\right) = .4\left(1 + .6\frac{{v_b}^4}{320^4}\right) + .25\left(1 + .6\frac{{v_r}^4}{400^4}\right).$$

The optimum value is found by solving the following optimization problem.

$$min\ Z = \int_0^x .4\left(1 + .6\frac{{v_b}^4}{320^4}\right)dv_b + \int_0^y .25\left(1 + .6\frac{{v_\alpha}^4}{400^4}\right)dv_\alpha \tag{3.6}$$

$$s.t.\ x + y = 500,$$

$$0 \leq x, 0 \leq y,$$

which leads to the following problem,

$$\min Z = .4x + .048\frac{x^5}{320^4} + .25y + .03\frac{y^5}{400^4}, \tag{3.7}$$

$$s.t. x + y = 500,$$

$$0 \leq x, 0 \leq y.$$

It was established in equations (2.12) and (2.13) that the BPR function is convex over the domain. From this, we know there is a minimizer. Because volume delay functions are nonlinear yet still have linear constraints, the Frank-Wolfe algorithm is a viable option for solving the optimization problem. However, because the specific problem (3.7) is relatively simple, the solution can be obtained by solving for the Karush-Kuhn-Tucker (KKT) conditions.

We start by writing the Lagrangian function for (3.7) and the K-K-T conditions,

$$\mathcal{L}(x,y,\lambda) = .4x + .048\frac{x^5}{320^4} + .25y + .03\frac{y^5}{400^4} + \lambda(x + y - 500), \tag{3.8}$$

$$\frac{\partial \mathcal{L}}{\partial x} = .4 + .24\frac{x^4}{320^4} + \lambda = 0, \tag{3.9}$$

$$\frac{\partial \mathcal{L}}{\partial y} = .25 + .15\frac{y^4}{400^4} + \lambda = 0, \tag{3.10}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 500 = 0. \tag{3.11}$$

From (3.9) and (3.10) we derive the following equality at minimization:

$$.4 + .24\frac{x^4}{320^4} = .25 + .15\frac{y^4}{400^4} \tag{3.12}$$

Using the condition (3.11), the result is $y = 500 - x$. Substituting this into equation (3.12), we have

$$.4 + .24\frac{x^4}{320^4} - \left(.25 + .15\frac{(500 - x)^4}{400^4}\right) = 0. \tag{3.13}$$

Using a subroutine for finding the solutions an equation in MATLAB, we find the following result

$$x = 98.5674,$$

implying that

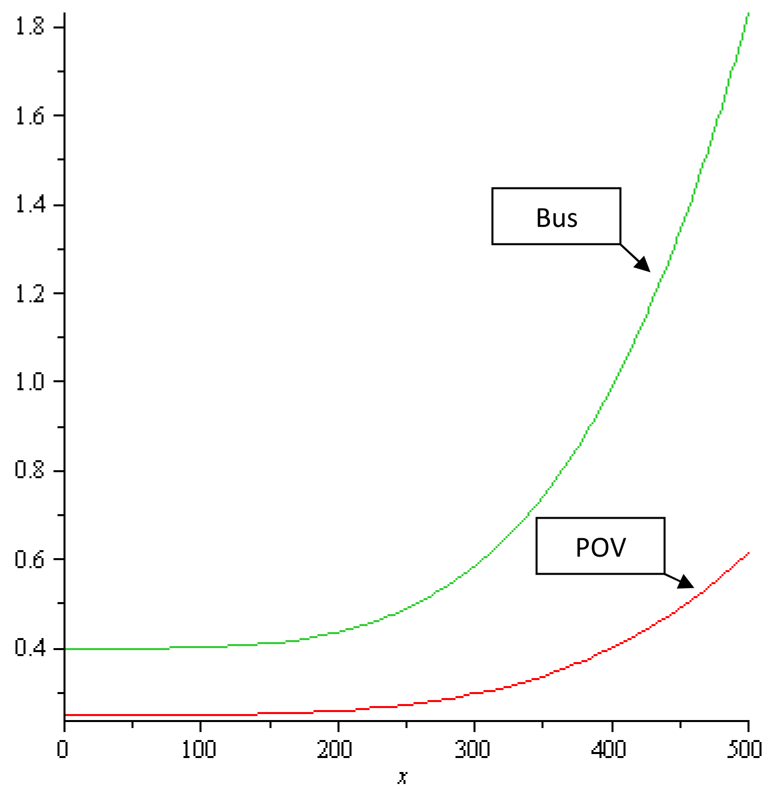
$$y = 500 - x = 401.4326.$$

The model predicts that the system will reach equilibrium when 99 individuals use buses and 401 people drive. With each bus with a capacity of 32 individuals, 4 buses will be used. However, this means three would be full and fourth would only have three passengers. From this model, planners would probably choose a suboptimal system.

In the next section we generalize the example and use the Lagrangian method to find the solution.

3.3.1 Developing the General Optimization Technique

The alternative way of obtaining the solution is described below. First, we present the graphs of the two BPR functions we used in the example.



$\boldsymbol{Figure\ 3.5}$: $\boldsymbol{POV\ and\ Bus\ Travel\ Times}$

It is clear from the graph that the volume delay for the bus users is greater than that of the POVs.

The following graph shows the interaction of passengers between the two systems where $f_b(*)$ is the function for buses and $f_r(*)$ is the function for POVs.

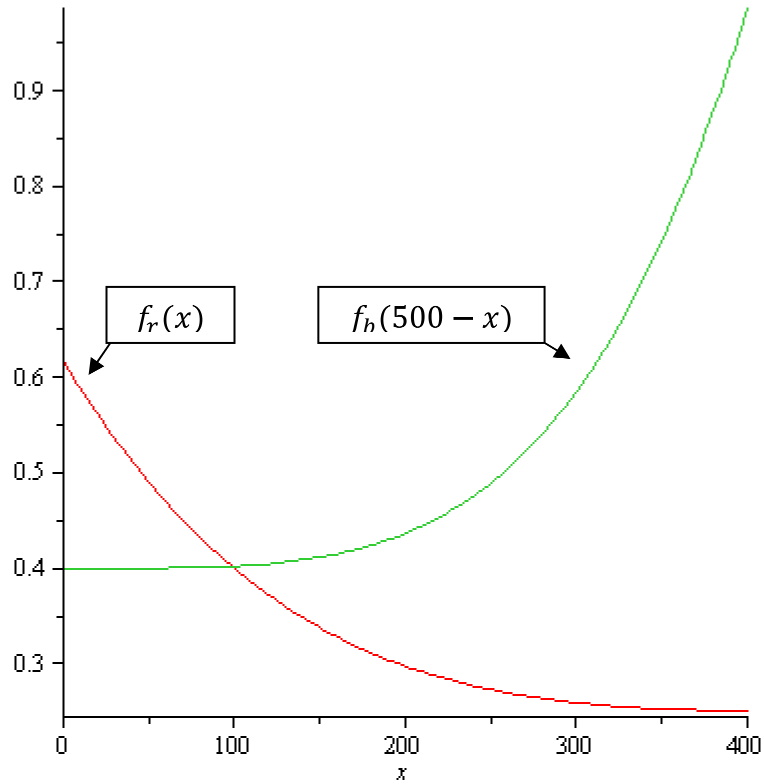


***Figure* 3.6: *POV and Bus Travel Times*(*Graphic Intersection*)**

By inspection, it appears the graphs of the functions intersect at the point of optimization, $x = 99$. We will start with a generalized form of the problem (3.6),

$$\min Z = \int_0^x f_1(v_a)dv_a + \int_0^y f_2(v_b)dv_b, \tag{3.14}$$

$$s.t.\ x + y = T,$$

$$0 \leq x, 0 \leq y.$$

where T is the number of passengers moving through v_b and v_r.

Taking the integral gives the following result,

$$\min Z = F_1(x) + F_2(y), \tag{3.15}$$

$$s.t.\ x + y = T,$$

$$0 \leq x, 0 \leq y$$

The next steps will be to develop a Lagrange function and set up the K-K-T conditions required to find the optimal values.

$$\mathcal{L}(x, y, \lambda) = F_1(x) + F_2(y) + \lambda(x + y - T), \tag{3.16}$$

$$\frac{\partial \mathcal{L}}{\partial x} = f_1(x) + \lambda = 0, \tag{3.17}$$

$$\frac{\partial \mathcal{L}}{\partial y} = f_2(y) + \lambda = 0, \tag{3.18}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y - T = 0, \tag{3.19}$$

Note that the results of (3.17) and (3.18) follow from the Fundamental Theorem of Calculus.

From (3.17) and (3.18) we get that the following equality holds at the point of minimization

$$f_1(x) = f_2(y). \tag{3.20}$$

Based on (3.19), we get the following identity

$$y = T - x. \tag{3.21}$$

Substituting this into (3.20), the equality holds if

$$f_1(x) = f_2(T - x). \tag{3.22}$$

By using this property, the optimal point can be found without the use of an iterative optimization method. In this simple case it is enough to solve eq. (3.22). Thus taking the integral is not necessary.

3.4 Applying Volume Delay Function to Public Transit

Having established a strong foundation of the volume delay function, we are almost ready to apply it to the assessment of the three public transportation systems proposed for the West Line expansion. However, there is a critical yet easily rectifiable error with the use of a pre-existing volume delay functions to the analysis of this system.

The implicit assumption of the route assignment function is that the volume and feed of traffic is continuous. A characteristic of all three proposed systems is that the systems have both continuous and discrete periods of traffic flow. As more people board each system, the longer the delay. This is the component that changes. The fixed is the actual travel time from one stop (node) to the next. The construction an implementation of this function will be done in the next chapter. We will modify the volume delay function by incorporating boarding time.

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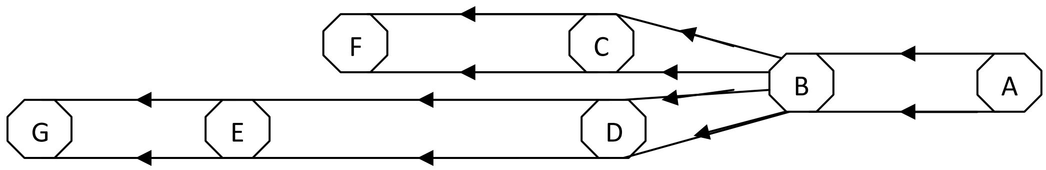
Chapter 4
Application to MARTA

In this chapter we will develop several volume delay functions which will be applied to specific public transportation networks. Using a model which approximates the West Line expansions, we will use these functions as well as the BPR and Davidson functions to compare the possible expansion. The Figure 4.1 represents the part of MARTA under construction.



$\boldsymbol{Figure\ 4.1}$: $\boldsymbol{Graph\ of\ West\ Line\ Expansion}$

Routes $\overrightarrow{\boldsymbol{AB}}$, $\overrightarrow{\boldsymbol{BC}}$, and $\overrightarrow{\boldsymbol{CF}}$ correspond to Interstate 20. Routes $\overrightarrow{\boldsymbol{BD}}$ and $\overrightarrow{\boldsymbol{DE}}$ represent a four lane section of Martin Luther King Jr. Blvd., with $\overrightarrow{\boldsymbol{EG}}$ representing the portion where MLK Jr. Blvd. reduces to two lanes. Each node, with the exceptions of **A** and **B,** represents a potential location for a mass transit station. Nodes **A** and **B** represent West Lake and Hamilton E. Holmes train stations respectively. Although $\overrightarrow{\boldsymbol{AB}}$ is a currently existing route, we will still perform analysis on it.

As mentioned in section 1.4, we will be analyzing three public transportation systems; bus rapid transit (BRT), light rail transit (LRT), and heavy rail transit (HRT).

Because MARTA does not have definitive plans of how the West Line will be expanded, this model is based on the suggestions made by planners of where stations can be placed as well as existing stations. Our objective is to demonstrate how an analysis can be done, not to get an analysis with the most detailed or accurate information. The focus is more on the process than the results.

4.1 Deriving the Public Transportation Function

The objective of this function is to approximate the time it takes for a train or bus to move a certain number of people from one point to another. Our first observation is that a certain time is required for boarding a bus or train dependent of number of occupants. We will assume the time delay for boarding has some similar patterns as traffic time delay. We assume that as more people enter the system, the rate of boarding increases. Considering the two functions that have already been mentioned in this report, the BPR has the closest desired result. We do not want a function that approaches the rate of capacity asymptotically, such as the Davidson function. In fact, even though the rate of volume delay is not perfectly linear, we believe it is a fair enough assumption that it takes individuals about the same average time to board.

There are four values required to construct functions to represent the volume delay times for the three public transportation systems.

Capacity

The capacities are taken from the research of the MARTA West Line project.[13] This is the total number of people which can fit on each system. The BRT, LRT, and HRT capacities are 32, 170, and 300 respectively.

Boarding Times

For each system, the time it takes for the first person to board each system is assumed to be .1 minutes (6 seconds). The total boarding time represents the time it takes for people to fill each system to capacity. The boarding times are estimated based on the capacity of each system and the number of entrances. The BRT, LRT, and HRT boarding times are minutes are 3, 5, and 6 respectively.

Delay Times

The delay times are the total times required for a single unit in each system (i.e. a single bus) to move through the edges. The delay time is calculated by measuring the distance between stations and estimating travel speeds. The assumed average speeds for the BRT, LRT, and HRT systems are dependent on location of the system and the distance between edges. For example, the route represented by edge $\overrightarrow{BD}$ has a length of 1.2 miles. Going an average speed of 60 mph, it takes the BRT 1.2 minutes to move through the edge.

The parameters used to construct the public transportation volume delay functions (PTF) are listed in the table below.

$\textit{Table 4.1}: \textit{Public Transportation Parameters}$

	BRT	LRT	HRT
Capacity	$32(c_{BRT})$	$170(c_{LRT})$	$300(c_{HRT})$
Int. Boarding Time	.1	.1	.1
Total Boarding Time	3	5	6
Delay Time $(\overrightarrow{AB})$	1.5	1.2	1.1
Delay Time $(\overrightarrow{BC})$	.6	.5	.4
Delay Time $(\overrightarrow{BD})$	1.2	1.2	.8
Delay Time $(\overrightarrow{CF})$	2.4	1.8	1.4
Delay Time $(\overrightarrow{DE})$	1.8	1.5	1.3
Delay Time $(\overrightarrow{EG})$	1.2	1	.85

4.1.1 Calculating the Delay Component

In the first component of the PTF we will calculate the volume delay associated with passengers boarding an individual unit of the system. The function being used will be based on the BPR volume delay function.

Two things which should be noted is that the β parameter is set at 1 and the α parameter does not have the restriction of $0 < \alpha < 1$ like in the BPR and Davidson functions. In order to distinguish the α used for the PTF from that used in BPR and Davidson functions, we use the notation α_{PTF}.

For each of the systems in the model, Figure 4.2 contains the parameters t_0, c, and $T_B(c)$. Using these values, we can calculate α_{PTF} for each of the systems. This will be illustrated for the BRT.

We start with the following function:

$$T_B(v) = t_0\left[1 + \alpha_{PTF}\left(\frac{v}{c}\right)\right].$$

This is the form of the volume delay function we will use for passengers boarding an individual unit in the system. It should be noted that the relationship between passengers and time is assumed to be linear. Substituting the parameters for the BRT taken from Figure 4.2, we get the following result

$$.1\left[1 + \alpha_{PTF}\left(\frac{32}{32}\right)\right] = 3 \Rightarrow \alpha_{PTF} = 29.$$

Now we can perform the same calculations for the LRT and HRT. The results are listed in Table 4.2.

***Table* 4.2: Calculated α_{PTF} Values**

ROUTE	BRT	LRT	HRT
$\boldsymbol{\alpha_{PTF}}$	29	49	59

Now that we have the α_{PTF} parameter for each of the three systems, we can form the volume delay functions for a single unit within each system.

$$\left.\begin{aligned} T_{BRT}(v) &= .1\left(1 + 29\left(\frac{v}{32}\right)\right) \\ T_{LRT}(v) &= .1\left(1 + 49\left(\frac{v}{170}\right)\right) \\ T_{HRT}(v) &= .1\left(1 + 29\left(\frac{v}{32}\right)\right) \end{aligned}\right] \; time\ to\ board\ v\ passengers \tag{4.1}$$

With an adequate method of calculating loading time for each system, we can now focus on the time it takes for an individual unit to move between stations. There are a few characteristics which make this time much easier to estimate. We can assume there is only an individual unit moving through a route at a time. Each route is short enough that multiple units would either result in increased delays or accidents. Also, travel times for BRT and rail systems are easy to predict; with a well managed transportation network the time of departure and arrival can be accurately predicted.

For any given number of people using a particular system, it can be expected that a full number of units will not be needed to move all individuals from one station to next. For example, to move 46 individuals from station A to station B using BRT will require two buses. However, only one bus will be completely full, the second bus will have 14 passengers. Using the parameters from Figure 4.2, we know the first bus will take 3 minutes to fill to capacity and 1.5 minutes to move between A and B, or a total time of 4.5 minutes. Although the travel time (delay time) will be 1.5 minutes for the second bus too, the loading time will not be the same. Assuming the second bus is not completely full, we can calculate the total number on this bus as following

$$Number\ of\ people\ on\ second\ bus = v - c\left(\left\lfloor\frac{v}{c}\right\rfloor\right)$$

Applying this to the example of 46 bus customers, we get the following result

$$46 - 32\left(\left\lfloor \frac{46}{32} \right\rfloor\right) = 46 - 32(\lfloor 1.4375 \rfloor) = 14.$$

Implementing this into the volume delay function, we get the following result for the total boarding and travel time

Calculates the number of people who do not fill an entire bus

Travel time

$$t_{\overrightarrow{AB}}^{BRT}(v) = .1\left[1 + 29\,\frac{\left\lceil v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right\rceil}{32}\right] + 1.5 \tag{4.2}$$

$$s.t. \, 32 > \ v > 0, v \in \mathbb{Z} \, ,$$

If there are n buses that move through the route $\overrightarrow{AB}$, this function shows the delay for the n^{th} bus. The total travel time for the previous $n-1$ buses (that are full) will be the sum of boarding times(T_B) and the travel times(T_D) for each of the units

$$[T_D + T_B]\left(\left\lfloor \frac{v}{32} \right\rfloor\right).$$

Note that $\left\lfloor \frac{v}{32} \right\rfloor$ is the number of full units. For example, with BRT on edge $\overrightarrow{\mathrm{AB}}$ we have from Table 4.1, $T_D + T_B = 3 + 1.5 = 4.5$. Thus, to get the volume delay for both full and partially filled units for BRT on route $\overrightarrow{AB}$, we get

$$t_{\overrightarrow{AB}}^{BRT}(v) = [4.5]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\,\frac{\left\lceil v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right\rceil}{32}\right] + 1.5,$$

$$s.t. \ v > 0, v \in \mathbb{Z} \, ,$$

where the first term is the time for the full units and the second one is for partially full units. The functions are listed in the table below.

Table 4.3 Calculated PTF

ROUTE	BRT	LRT	HRT
$\overrightarrow{AB}$	$[4.5]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right]}{32}\right] + 1.5$	$[6.2]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{v}{170} \right\rfloor\right)\right]}{170}\right] + 1.2$	$[7.1]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + 1.1$
$\overrightarrow{BC}$	$[3.6]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right]}{32}\right] + .6$	$[5.5]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{v}{170} \right\rfloor\right)\right]}{170}\right] + .5$	$[6.4]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + .4$
$\overrightarrow{BD}$	$[4.2]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right]}{32}\right] + 1.2$	$[6.2]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{v}{170} \right\rfloor\right)\right]}{170}\right] + 1.2$	$[6.8]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + .8$
$\overrightarrow{CF}$	$[5.4]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right]}{32}\right] + 2.4$	$[6.8]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{v}{170} \right\rfloor\right)\right]}{170}\right] + 1.8$	$[7.4]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + 1.4$
$\overrightarrow{DE}$	$[4.8]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{v}{32} \right\rfloor\right)\right]}{32}\right] + 1.8$	$[6.5]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{v}{170} \right\rfloor\right)\right]}{170}\right] + 1.5$	$[7.3]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + 1.3$
$\overrightarrow{EG}$	$[4.2]\left(\left\lfloor \frac{v}{32} \right\rfloor\right) + .1\left[1 + 29\frac{\left[v - 32\left(\left\lfloor \frac{xv}{32} \right\rfloor\right)\right]}{32}\right] + 1.2$	$[6]\left(\left\lfloor \frac{v}{170} \right\rfloor\right) + .1\left[1 + 49\frac{\left[v - 170\left(\left\lfloor \frac{xv}{170} \right\rfloor\right)\right]}{170}\right] + 1$	$[6.85]\left(\left\lfloor \frac{v}{300} \right\rfloor\right) + .1\left[1 + 59\frac{\left[v - 300\left(\left\lfloor \frac{v}{300} \right\rfloor\right)\right]}{300}\right] + .85$

The next task will be to define the parameters used for the BPR and Davidson function.

4.2 Deriving the BPR & Davidson Functions

Once again, looking at the BPR and Davidson functions we have

$$t^{BPR}(v) = t^0 \left[1 + \alpha \left(\frac{v}{c}\right)^{\beta}\right]$$

$$t^{D}(v) = t^0 \left[1 + \alpha \left(\frac{v}{c-v}\right)\right]$$

There are four parameters needed for formulation of both of these functions.

The t^0 parameter

The t^0 parameter is the time it takes for a POV to move through the respective route. The values used to calculate this time are in the table below. The time is expressed as minutes.

$Table\ 4.4$: $Road\ Information$

ROUTE	POV Travel Speeds (mph)	Speeds at Capacity (mph)	Route Distances
$\overrightarrow{AB}$	65	45	1.2
$\overrightarrow{BC}$	65	45	.325
$\overrightarrow{BD}$	35	20	.7
$\overrightarrow{CF}$	65	45	1.84
$\overrightarrow{DE}$	35	20	1.17
$\overrightarrow{EG}$	45	30	.75

The speeds at capacity are the speeds of a POV when a road has reached the point of capacity. We interpret capacity to be the threshold for when someone can still drive at legal speed. For example, routes $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CF}$ correspond to areas on Interstate 20 where the minimum driving speed is 45mph. These speeds will become relevant when calculating the α parameter for the volume delay function.

The α, c, and β parameters

In previous examples used in this paper, we estimated the capacity values to be used in the volume delay functions. This value will now be determined via a look-up table that relates these variables to the type of link and the area type surrounding the link. The values we use in this paper come from a table used in the Urban Transportation Planning Software distributed by the Urban Mass Transportation in the 1970s and 1980s.

We will deviate from the somewhat vague method of defining the α parameter used earlier in this paper. Instead, we will use the formula,

$$\alpha = (S_0/S_c) - 1,$$

where S_0 is the free traveling speed and S_c is the speed at capacity.

Instead of using 4 for the β parameter, we will use values taken from a FHA paper. All four parameters for each route are listed in the table below.

$\boldsymbol{Table\ 4.5}$: $\boldsymbol{BPR\ and\ Davidson\ Function\ Parameters}$

ROUTE	α	c	β	t^0
$\overrightarrow{AB}$	.44	2000	9.8	1.1
$\overrightarrow{BC}$	.44	2000	9.8	.3
$\overrightarrow{BD}$	.75	870	2.1	1.2
$\overrightarrow{CF}$	.44	2000	9.8	1.7
$\overrightarrow{DE}$	.75	870	2.1	2
$\overrightarrow{EG}$	.5	1000	2.1	1.0

4.2.1 Calculating the Hidden Cost of Driving

The rational driver will take into account not only the time required to drive when deciding to either drive or take public transportation, but the cost of operation. There are several methods of calculating costs.

$$\textbf{\textit{Table 4.6: Operation Costs}}$$

Factor	Cost per Mile
Insurance/registration	$.094
Depreciation	$.286
Fuel/oil	$.059
Maitenance/tires	$.059
Total	**$.498**

This will be included into the volume delay function by interpreting it as a constant time and interpreting the time as monetary value with the assumption of a $20/hour wage.

For example, in order calculate the cost in terms of time for a POV operator on route $\overrightarrow{AB}$, we first find the cost in terms of dollars. This is done by multiplying route length by cost per miles,

$$Operating\ Costs = \ Miles \times \left(\frac{Cost}{Miles}\right) = 1.2 \times \$.498 = \$.6,$$

To calculate cost in minutes, we calculate the time required to earn the amount of the operating costs (with a $20/hour wage),

$$Cost\ in\ terms\ of\ time = \frac{Operating\ Costs}{Wage} = \frac{\$.6}{\$20/hour} = .03\ hours,$$

which in terms of minutes is 1.8 minutes. The cost for each route is listed in Table 4.7 below.

$Table\ 4.7$: $Cost\ in\ Minutes$

Route	Miles	Cost	Cost in minutes
$\overrightarrow{AB}$	1.2	.6	1.8
$\overrightarrow{BC}$	.325	.16	.48
$\overrightarrow{BD}$	.7	.35	1.05
$\overrightarrow{CF}$	1.84	.9	2.7
$\overrightarrow{DE}$	1.17	.58	1.74
$\overrightarrow{EG}$	.75	.37	1.11

The final parameter to be added to the BPR and Davidson functions is what we will refer to as a "preference" component. The purpose of this parameter is to reflect the inflexibility of certain passengers regarding their decision to drive. For whatever reason, there will always be people who make the decision to drive and would continue to drive unless face with either extreme disincentives (i.e. gas prices that exceed $4.00/gallon or unimaginable commute times) or unreasonable incentives (i.e. they are paid not to drive). Since these people will always be people on the road, they instantly increase travel time. Quite arbitrarily, we have chosen 5 to be the parameter value for all BPR and Davidson functions. Though we did not include this parameter in the public transportation functions, it is a very reasonable assumption to make that there would also be a similar group who will always take public transportation for their own reason (i.e. they cannot afford to own or operate a car).

The final BPR functions is

$$t_0\left[1+\alpha\left(\frac{v}{c}\right)^{\beta}\right]+o+p$$

and the final Davidson function is

$$t_0\left[1+\alpha\left(\frac{v}{c-v}\right)\right]+o+p.$$

The specific BPR and Davidson functions for each route are listed I Table 4.8.

Table 4.8: BPR and Davidson Functions For Each Route

Route	BPR	Davidson
$\overrightarrow{AB}$	$1.1\left[1+.44\left(\frac{v}{2000}\right)^{9.8}\right]$+1.8+5	$1.1\left[1+.44\left(\frac{v}{2000-v}\right)\right]$+1.8+5
$\overrightarrow{BC}$	$.3\left[1+.44\left(\frac{v}{2000}\right)^{9.8}\right]$+.48+5	$.3\left[1+.44\left(\frac{v}{2000-v}\right)\right]$+.48+5
$\overrightarrow{BD}$	$1.2\left[1+.75\left(\frac{v}{870}\right)^{2.1}\right]$+1.05+5	$1.2\left[1+.75\left(\frac{v}{870-v}\right)\right]$+1.05+5
$\overrightarrow{CF}$	$1.7\left[1+.44\left(\frac{v}{2000}\right)^{9.8}\right]$+2.7+5	$1.7\left[1+.44\left(\frac{v}{2000-v}\right)\right]$+2.7+5
$\overrightarrow{DE}$	$2\left[1+.75\left(\frac{v}{870}\right)^{2.1}\right]$+1.74+5	$2\left[1+.75\left(\frac{v}{870-v}\right)\right]$+1.74+5
$\overrightarrow{EG}$	$1\left[1+.5\left(\frac{v}{1000}\right)^{2.1}\right]$+1.11+5	$1\left[1+.5\left(\frac{v}{1000-v}\right)\right]$1.11+5

4.3 Finding Free Flow Travel Times

Using a search subroutine in MATLAB, we are able to calculate the optimum route assignments in a similar fashion as in section 3.3.1. We have results for both the BPR and Davidson volume delay functions. The constraint used for each route was the route's capacity. For example, route $\overrightarrow{AB}$ has a capacity 2,000, so the system was optimized for having a total of 2,000 individuals moving through it.

4.3.1 Solving for a Volume Delay Function which is not Continuously Differentiable

In section 3.3.1, we proved a method of finding the equilibrium distribution using K-K-T conditions. One of the requirements for the use of the K-K-T conditions is a continuously differentiable function. The function

$$\boldsymbol{t^{PTF}(v) = [T_D + T_B]\left(\left\lfloor \frac{v}{c} \right\rfloor\right) + t_0\left[1 + \alpha_{PTF}\frac{v - c\left(\left\lfloor \frac{v}{c} \right\rfloor\right)}{c}\right] + T_D},$$

is not differentiable with the values of v which are divisible by c.

In order to show equilibrium is reached at the point where the two functions intersect graphically, as in figure 3.6, another method must be used.

The first step will be to show that the two functions only intersect at one point. The Davidson and BPR functions have already been shown to be strictly increasing with equations (2.12) and (2.14). The result,

$$\frac{d}{dx_a}\boldsymbol{t^{PTF}} = \alpha_{PTF}\frac{1}{c} > 0,$$

where the derivative is defined, is enough to show the function is non decreasing. Any points of intersection are unique.

Because we are assuming the function has no points of congestion, we can assume the user equilibrium and system optimization are equal.

Though the concepts were developed independently, the Wardrop concept of user equilibrium is the same concept as the Nash equilibrium. We can treat the route assignment as a game where players are choosing the strategy which gives them the shortest time.

4.3.2 Finding the Nash Equilibrium

In game theory, we let N={1,…,n} denote the set of players, with a strategy for player I being represented by an element $s^i \epsilon S$, where S is the set of all strategies and $F^i(s)$ is a set of all possible strategies for player i in response to s.

A Nash Equilibrium is a strategy profile $s^* \epsilon\ S$ such that for all $i \epsilon N$ and all $s \epsilon F^i(s)$, $u_i(s^*) \geq u_i(s)$.

This is a very technical way of saying that no user's unilateral choice will improve their level of success. The Nash Equilibrium is not necessarily the same as the system optimization.

There is only one case when that the PTF and volume delay functions intersect, represented by points *z*. This point is a Nash equilibrium. Point *y* is not an intersection point, but gives a optimum volume distribution.

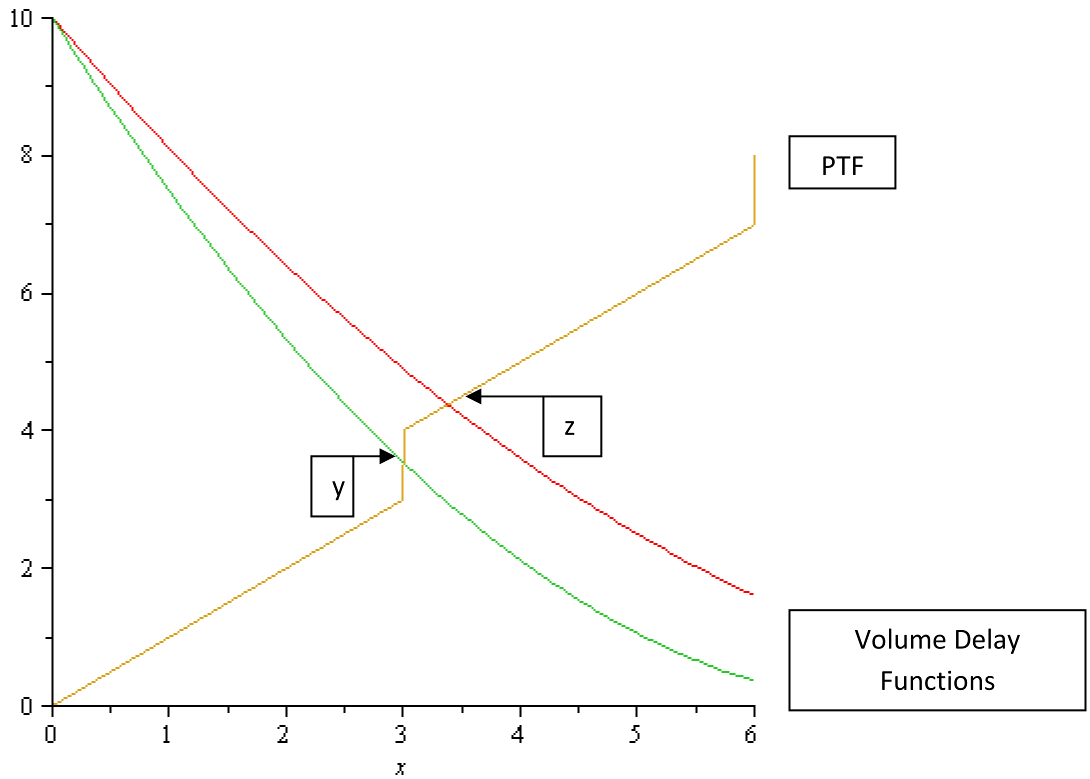


Figure 4.2: Graphical Demonstration of Nash Equilibrium

4.4 Optimization Results

Below are tables showing the results of using the MATLAB "fzero" subroutine to find the system optimum points for the BPR, Davidson functions, and the PTFs.

$\boldsymbol{Table\ 4.9}$: $\boldsymbol{Optimum\ Route\ Assignment\ Values\ For\ the\ BPR\ Function}$

ROUTE	BPR:BRT	BPR:LRT	BPR:HRT
$\overrightarrow{AB}$	1936:64	1769:251	1660:340
$\overrightarrow{BC}$	1951:49	1830:170	1730:270
$\overrightarrow{BD}$	810:60	690:180	570:300
$\overrightarrow{CF}$	1945:55	1797:203	1668:332
$\overrightarrow{DE}$	806:64	659:211	540:330
$\overrightarrow{FG}$	944:56	816:184	830:270

$\boldsymbol{Table\ 4.10}$: $\boldsymbol{Optimum\ Route\ Assignment\ Values\ For\ the\ Davidson\ Function}$

ROUTE	DAVIDSON:BRT	DAVIDSON:LRT	DAVIDSON:HRT
$\overrightarrow{AB}$	1882:118	1685:315	1574:426
$\overrightarrow{BC}$	1923:77	1800:200	1700:300
$\overrightarrow{BD}$	770:100	1758:242	1654:346
$\overrightarrow{CF}$	1879:121	1673:327	1545:455
$\overrightarrow{DE}$	752:118	576:294	476:394
$\overrightarrow{FG}$	909:91	787:213	700:300

The first number of each cell is the assignment of people to POVs, while the second number is the assignment of people to the respective public transportation system.

Chapter 5
Conclusion

5.1 Interpreting the Results

Looking at the results, it becomes clear that the BPR function over assigns as compared to the Davidson Function. This is a result of the universal greater rate of increase of the Davidson function to the BPR function, as shown in figure 3.3. Because the Davidson function increases at a faster rate, it reaches an equilibrium point before the BPR, resulting in a lower volume assignment. By calculating the results of both functions, it gives us a wider, yet still consistent, estimation of the system. It is consistent regarding the order in which the technologies are capable of handling a quantity of passengers (BRT, LRT, and then HRT).

A strict interpretation of the results gives varying results between systems and routes. For example, The BPR function for route $\overrightarrow{AB}$ tells us the system will reach equilibrium when 64 individuals decide to use the BRT system, while the Davidson predicts equilibrium at 118 individuals. This means the system is optimized when 2 units (with a capacity of 32) are used under the BPR, but at 4 with the Davidson function.

Instead of reading the results in this manner, it is better to understand them as a high/ low comparison between systems. For example, for the same route under the BPR 251 individuals are assigned to LRT and 315 with Davidson. Because LRT has a capacity of 170, either estimate would tell us to implement one unit for the LRT. However, if a planner knows that in order to optimize the system they will need at least two units and at most four, it creates one more dimension to assessing a system. This data would help a planner judge not only whether or not an objective is obtainable, but if the system required to achieve the objective is within budget.

5.2 Examining for Multiple Routes

Compared to most route assignment problems, the number of possible paths is very limited. It is reasonable to assume that given a system consisting of POVs and a single system, there will be two routes users would choose, either strictly POV or strictly public transportation. However, if we wanted to maximize a system using multiple systems or even alternating between POVs and public transportation, the public transportation functions developed in this paper are not effective. The time required to change between systems is not being taken into account. These functions are reasonable in the examination of single routes or a model consisting only of paths made up only of public transportation systems in which a large number of people are exiting and entering every unit at each stop. In a model in which most of the people are moving through multiple stops on a single unit, our public transportation delay functions will overestimate the total boarding time.

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