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Emily C. Chizmar  
*Georgia Southern, ec02591@georgiasouthern.edu*

Colton Magnant  
*Georgia Southern University, cmagnant@georgiasouthern.edu*

Pouria Salehi Nowbandegani  
*Vanderbilt University*

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Note on vertex and total proper connection numbers

Emily Chizmar, Colton Magnant *, Pouria Salehi Nowbandegani

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, USA

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Abstract

This note introduces the vertex proper connection number of a graph and provides a relationship to the chromatic number of minimally connected subgraphs. Also a notion of total proper connection is introduced and a question is asked about a possible relationship between the total proper connection number and the vertex and edge proper connection numbers.

Keywords: Colored connectivity; Proper connection; Vertex proper connection

1. Introduction

All graphs considered in this work are simple, finite and undirected. Unless otherwise noted, by a coloring of a graph, we mean a vertex-coloring, not necessarily proper.

Now well studied, the (edge) rainbow k-connection number of a graph is the minimum number of colors c such that the edges of the graph can be colored so that between every pair of vertices, there exist k internally disjoint rainbow edge-colored paths. See [1,2] for surveys of results about the rainbow connection number. Note that the rainbow 1-connection number is related, at least conceptually, to the diameter of the graph.

The total rainbow k-connection number, defined in [3], is defined to be the minimum number of colors c such that the edges and vertices of the graph can be colored with c colors so that between every pair of vertices, there exist k internally disjoint rainbow paths where here rainbow means all interior vertices and edges have distinct colors. Note that we cannot require the end-vertices of the paths to also have distinct colors as that would reduce the problem to edge rainbow k-connectivity since every vertex would then be required to have a distinct color.

The edge proper connection number \( pc_k(G) \), defined in [4] and further studied in [5], is defined to be the minimum number of colors c such that the edges of the graph \( G \) can be colored with c colors such that between each pair of vertices, there exist k internally disjoint, properly edge-colored paths. One feature of edge-proper connection that makes the results extremely complicated is that proper edge-colored paths are not transitive in the sense that if there is a proper path from \( u \) to \( v \) and a proper path from \( v \) to \( w \), there may not be a proper path from \( u \) to \( w \). For example, let \( G \) be a path on three vertices, \( uvw \) and color both edges red.
In this work, we consider a vertex version of the edge proper connection number. For a positive integer \( k \), a colored graph \( G \) is called \( (\text{vertex}) \) properly \( k \)-connected if, between every pair of vertices, there exist at least \( k \) internally disjoint properly colored paths. Note that each path, including end-vertices, must be properly colored. Given a graph \( G \), the vertex proper \( k \)-connection number of the graph \( G \), denoted \( \text{vpc}_k(G) \), is the minimum number of colors needed to produce a properly \( k \)-connected coloring of \( G \). For ease of notation, let \( \text{vpc}(G) = \text{vpc}_1(G) \).

The function \( \text{vpc}_k(G) \) is clearly well defined if and only if \( \kappa(G) \geq k \). Also note that \( \text{vpc}_k(G) \leq \chi(G) \) for every \( k \)-connected graph \( G \). Furthermore, the following fact is immediate.

**Fact 1.** For all \( k \geq 2 \) and every \( k \)-connected graph \( G \), \( \text{vpc}_k(G) \geq \text{vpc}_{k-1}(G) \).

A graph \( G \) is called \( \text{minimally} \) \( k \)-connected if \( G \) is \( k \)-connected but the removal of any edge from \( G \) leaves a graph that is not \( k \)-connected. A classical result of Mader [6] (also found in [7]) will immediately give us one of our upper bounds.

**Theorem 1** (\([6,7]\)). A minimally \( k \)-connected graph is \( k+1 \) colorable and this bound is sharp.

2. General classification

Our first observation demonstrates the transitivity of the vertex proper connection, a fact that is not true in the case of edge proper connection.

**Fact 2.** In a colored graph \( G \), if there is a proper path from \( u \) to \( v \) and a proper path from \( v \) to \( w \), then there is a proper path from \( u \) to \( w \).

**Proof.** The proof is trivial if the \( u-v \) path and the \( v-w \) path intersect only at \( v \) so suppose the paths intersect elsewhere and let \( x \) be the first vertex on the path from \( u \) to \( v \) that is also on the \( v-w \) path. Note that we may have \( x = u \). Then the subpath of the \( u-v \) path that goes from \( u \) to \( x \) and the subpath of the \( v-w \) path that goes from \( x \) to \( w \) is a properly colored path and completes the proof. \( \blacksquare \)

Clearly the addition of edges cannot increase the vertex proper connection number of a graph so the following fact is trivial.

**Fact 3.** Given a positive integer \( k \) and a \( k \)-connected graph \( G \), if \( H \) is a spanning \( k \)-connected subgraph of \( G \), then \( \text{vpc}_k(G) \leq \text{vpc}_k(H) \).

Our main result solidifies the link between the \( \text{vpc}_k \) function and the chromatic number of the graph. It turns out that \( \text{vpc}_k(G) \) always equals the chromatic number of a particular subgraph of \( G \). Let

\[
\ell \chi_k(G) = \min\{\chi(H) : H \text{ is a } k \text{-connected spanning subgraph of } G\}.
\]

**Theorem 2** (Classification). Given a \( k \)-connected graph \( G \), \( \text{vpc}_k(G) = \ell \chi_k(G) \).

**Proof.** Given a \( k \)-connected spanning subgraph \( H \) of \( G \) with chromatic number \( \ell \), color this subgraph properly with \( \ell \) colors. Then between every pair of vertices in \( H \), there are at least \( k \) internally disjoint properly colored paths. Thus, using **Fact 3**, \( \text{vpc}_k(G) \leq \text{vpc}_k(H) = \ell \) so \( \text{vpc}_k(G) \leq \ell \chi_k(G) \).

Now let \( \ell = \text{vpc}_k(G) \) and consider an \( \ell \)-coloring of \( G \) which is properly \( k \)-connected. Let \( \mathcal{P} \) be the set of all proper paths between pairs of vertices (\( k \) paths for each pair of vertices). Then the subgraph \( H \) of \( G \) induced on all the edges of \( \mathcal{P} \) spans \( G \), is \( k \)-connected and has chromatic number at most \( \ell \). This means \( \text{vpc}_k(G) \geq \ell \chi_k(G) \), completing the proof. \( \blacksquare \)

3. Consequences of Theorem 2

**Theorem 2** shows that every statement about \( \text{vpc}_k \) is a statement about the chromatic number of a minimally \( k \)-connected subgraph. Particularly, if \( G \) is minimally \( k \)-connected, then \( \text{vpc}_k(G) = \chi(G) \). When the graph is bipartite, we get the following easy observation.
Corollary 3. If $G$ is $k$-connected and bipartite, then for all $t \leq k$, we have $vpc_t(G) = 2$.

In light of the classification theorem, we immediately get equivalent colored “fan lemma” and “disjoint paths between $k$-sets” versions of the definition of vertex proper connectivity.

Corollary 4. A colored graph $G$ is properly $k$-connected if and only if for every vertex $v$ and $k$-set of vertices $\{u_1, u_2, \ldots, u_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where $P_i$ goes from $v$ to $u_i$ and $P_i \cap P_j = \{v\}$ for all $i, j$.

Corollary 5. A colored graph $G$ is properly $k$-connected if and only if for every $2k$-set of vertices $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where $P_i$ goes from $u_i$ to $v_j$ for some $j$ and $P_i \cap P_\ell = \emptyset$ for all $i, \ell$.

Theorem 2, along with Theorem 1, also gives us the following general upper bound. The sharpness of Theorem 1 and Corollary 3 yield the sharpness of both bounds here.

Corollary 6. If $G$ is $k$-connected, then for $t \leq k$, we have $2 \leq vpc_t(G) \leq t + 1$ and both bounds are sharp.

When $k = 1$, Corollary 6 reduces to the following.

Corollary 7. For every connected graph $G$ on at least 2 vertices, $vpc(G) = 2$.

4. Total proper connection

A natural definition of a total proper connection number is the following. Let $tpc(G)$ be the minimum number of colors needed to color the vertices and edges of $G$ so that between every pair of vertices $u, v$, there is a path $P = P_{u,v}$ such that the vertices of $P$ induce a properly (vertex-)colored path and the edges of $P$ also induce a properly (edge-)colored path. Furthermore, we define $tpc_k(G)$ to be the minimum number of colors needed to produce $k$ internally disjoint such paths between every pair of vertices.

One might think that $tpc_k(G)$ might simply be the maximum of $pc_k(G)$ and $vpc_k(G)$ but this is not obvious even when $k = 1$ since the edge path (for $pc$) and the vertex path (for $vpc$) must be the same path. Indeed, in Question 1, we ask whether this equality holds in general. Our results concerning the function $tpc$ support a positive answer to this question.

Question 1. Is it true that $tpc_k(G) = \max\{pc_k(G), vpc_k(G)\}$?

First we recall a result of Borozan et al. [4] which was originally stated in a stronger form.

Theorem 8 ([4]). If $G$ is bipartite and 2-connected, then $pc(G) = 2$.

Proposition 1. If $\kappa(G) \geq 3$, then $tpc(G) = 2$.

Proof. With $\kappa(G) \geq 3$, there is a spanning 2-connected bipartite subgraph $B$. Color the vertices with two colors according to this subgraph. By Theorem 8, the proper connection number of $B$ is 2. Color the edges of $B$ with 2 colors to be properly connected. For any pair of vertices in $B$, there is a properly edge-colored path between them which induces a properly vertex-colored path as well since the vertices are properly colored. This means $tpc(B) = 2$. Since $B \subseteq G$, we must also have $tpc(G) = 2$ as well. □

Using a similar argument and Corollary 3, we easily get the following result.

Corollary 9. If $G$ is $k$-connected and bipartite, then for $t \leq k$, $tpc_t(G) = \max\{pc_t(G), vpc_t(G)\} = pc_t(G)$.
References