Calculus Blues: Unlocking Mysteries in Related Rates

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“Calculus Blues: Unlocking mysteries in related rates”

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Students and Calculus performance

• About half of all students enrolled in introductory calculus courses either fail or withdraw (Ferrini-Mundy & Gaudard, 1992; Ferrini-Mundy & Graham, 1991; Peterson, 1986)

• Students tend to view mathematics as a collection of algorithmic procedures to be mastered (Dreyfus, 1990; Schoenfeld, 1994; Silver & Marshall, 1990)

• Students at the college level are proficient at performing algorithms, but they lack the ability to connect the algorithms to their underlying conceptual bases (Orton 1983a, 1983b).
Students and the Calculus performance (cont.)

• The inability to link the conceptual with the procedural is thought to be at the root of students' difficulties with higher-level mathematics (Dreyfus, 1990).
What is a “Related rates problem”

- The type of calculus problem that requires the determination of "the rate of change (with respect to time) of some variables based on their relationship to other variables whose rates of change are known" (Dick & Patton, 1992, p. 270)
Related Rates problem: Two examples

A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm$^3$/s, how fast is the water level rising when the water is 2 cm deep?

1) Changing dimensions in a rectangle. The length $l$ of a rectangle is decreasing at the rate of 2 cm/sec while the width $w$ is increasing at a rate of 2 cm/sec. When $l = 12 cm$ and $w = 5 cm$, at what rate is the area of the rectangle changing?
Research done on related rates bottlenecks

• Students have a procedural approach to solving related rates problems
  (Clark et al., 1997; Martin, 1996, 2000; White & Mitchelmore, 1996)

• (Martin 2000) investigated students’ difficulties with geometric related rates problems. She identified a multi step model for solving related rates problems and classified each step as procedural or conceptual.

• Difficulties seem to stem from the students’ inability to build a conceptual model of the situation, identify the relevant relationships and coordinate the changes to the objects in their mind (Engelke 2004)
Related Rates bottlenecks

• Students’ difficulties appear to stem from their misconceptions about variable, function, and derivative – particularly the chain rule (Carlson, 1998; Clark et al., 1997; Engelke, 2004; White & Mitchelmore, 1996)
# Seven Steps by Martin

## Description and Classification of Steps in Solving Geometric Related-Rates Problems

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
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</tr>
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<tbody>
<tr>
<td>1</td>
<td>Sketch the situation and label the sketch with variables or constants</td>
<td>Conceptual</td>
</tr>
<tr>
<td>2</td>
<td>Summarise the problem statement by defining the variables and rates involved in the problem (words to symbols translation) and identifying the given and requested information</td>
<td>Conceptual</td>
</tr>
<tr>
<td>3</td>
<td>Identify the relevant geometric equation</td>
<td>Procedural</td>
</tr>
<tr>
<td>4</td>
<td>Implicitly differentiate the geometric equation to transform a statement relating measurements to a statement relating rates</td>
<td>Procedural</td>
</tr>
<tr>
<td>5</td>
<td>Substitute specific values of the variables into the related-rates equation and solve for the desired rate</td>
<td>Procedural</td>
</tr>
<tr>
<td>6</td>
<td>Interpret and report results</td>
<td>Conceptual</td>
</tr>
<tr>
<td>7</td>
<td>Solve an auxiliary geometry problem</td>
<td>(varies)</td>
</tr>
</tbody>
</table>
Example of use of the steps in problem solving

1) Changing dimensions in a rectangle. The length $l$ of a rectangle is decreasing at the rate of $2 \text{ cm/sec}$ while the width $w$ is increasing at a rate of $2 \text{ cm/sec}$. When $l = 12 \text{ cm}$ and $w = 5 \text{ cm}$, at what rate is the area of the rectangle changing?

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Area of rectangle is increasing at the rate of $14 \text{ cm/sec}$
Example of the use of all seven steps in problem solving

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A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm³/s, how fast is the water level rising when the water is 2 cm deep?

\[
V = \frac{1}{3} \pi r^2 h
\]

\[
\text{Step 2: height of water at time } t \text{ seconds } = h
\]

\[
\text{Rate at which the water level is rising after } t \text{ seconds is } \frac{dh}{dt}
\]

\[
\text{Water volume at time } t \text{ seconds } = V
\]

\[
\text{Step 4: } \frac{dV}{dt} = \frac{3\pi}{100} \cdot 2h^2 \frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \frac{3\pi}{100} h^2 \frac{dh}{dt}
\]

\[
\text{Step 5: } h = 2 \frac{dh}{dt} = 2 \Rightarrow 2 = \frac{9\pi}{100} \cdot 4 \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{50}{9\pi} \text{ cm/sec}
\]

Step 6: When the water is 2 cm deep, the water level is rising at \( \frac{50}{9\pi} \) cm/sec.
Think Aloud on related rates

• In 2011, 6 think aloud sessions were conducted in two UW-two campuses, where students “thought aloud” through 3 problems.
• Results of the think aloud corroborated existing research.
• One problem area was the differentiation step (Step 4). Click to see students having difficulties in problem 1 and problem 3.
• We felt that a possible gap in instruction could be one reason for the student difficulties.
Professors assume that students may be able to combine knowledge from the chain rule lesson and implicit differentiation lesson to differentiate equations in related rates.
The study

• A special lesson was introduced after the implicit differentiation lesson. The lesson involved animations of related rates scenarios as well as several examples of equations where several variables were differentiated with respect to the time $t$.

• This intervention was done at two year UW campus.
The study (continued)

- A related rates quiz was given to students in both treatment sections as well as to a control section in another 4 year UW institution.

- The performance of the students in the “differentiation” step was assessed in all three section.
Related Rates Quiz

1. A police cruiser, approaching a right angled intersection from the north, is chasing a speeding car that has turned the corner is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

2. A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft. from the lift-off point. At the moment the range finders elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

3. Suppose that the edge lengths $x$, $y$, and $z$ of a closed rectangular box are changing at the following rates:

Find the rate at which the box’s volume is changing at the instant when $x=4$, $y=3$, and $z=2$.

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$
Results

The average scores for each problem are shown. The first score is for setting up the equation relating the variables and the second score is for differentiating the equation.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Treatment (n=31)</th>
<th>Control (n=25)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Related Equation</td>
<td>Related Rates</td>
</tr>
<tr>
<td>1</td>
<td>94%</td>
<td>85%</td>
</tr>
<tr>
<td>2</td>
<td>47%</td>
<td>57%</td>
</tr>
<tr>
<td>3</td>
<td>97%</td>
<td>56%</td>
</tr>
</tbody>
</table>
Conclusions

• The additional lesson improves overall performance.

• Problems that require auxiliary information like trigonometry, product rule affect performance.
References


References (Contd)


