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Stabilize Chaotic Flows in a Coupled Triple-Loop Thermosyphon System

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ABSTRACT

This study addresses the control of chaotic dynamic systems represented by three coupled Lorenz systems. In application, Lorenz systems are commonly used to describe the one-dimensional motion of fluids in a tube when heated below and cooled above. This system, in particular, reflects the fluid motion in a coupled triple-loop thermosyphon system. The goal is to derive a system of nonlinear differential equations to help us study various flow patterns governed by such a high-dimensional nonlinear model numerically. Once the driving parameter (Rayleigh number) values are identified corresponding to the chaotic regime, a minimal number of proportional controllers are designed that only depend on the measurable states, which serve as perturbations to the system, so that the system trajectories are stabilized at its equilibrium point even though the Rayleigh numbers are significantly large. The stability property of the control system is then investigated over a large range of the parameter values through simulations. Furthermore, the stability bounds on the controller gains obtained via the Lyapunov Stability Theorem are tested for its feasibility in practice.

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LIST OF NOMENCLATURE

Symbol	Description
x_i	Fourier Coefficient of the fluid velocity for the i^{th} loop
y_i	Fourier Coefficient of the horizontal temperature difference for the i^{th} loop
z_i	Vertical temperature difference for the i^{th} loop
ρ	Fluid density constant
R_i	Rayleigh number for the i^{th} loop
η	Heating coupling intensity coefficient
γ	Momentum coupling intensity coefficient
k_i	Feedback gain for the i^{th} loop
φ	Azimuthal angle
T_i	Fluid Temperature of the i^{th} loop
$f_{i,j}$	Friction force due to shear flow between loops i and j; N/m^3
$h_{i,j}$	Coefficient of heat transfer between loops i and j; W/m^3 K
δ	Dirac delta
ρ_0	Average fluid density
c_p	Heat capacity of fluid

CHAPTER 1

INTRODUCTION

For as long as mankind has been on this Earth, we have attempted to control our surroundings, to make order out of a naturally chaotic world, and mathematics has been a cornerstone in this. By assigning mathematical equations to describe a natural phenomenon, we can analyze the behavior of the system. Once they are known enough to be predictable, we can begin the process of manipulating the system to fit our objectives.

Lorenz equations are a benchmark mathematical dynamical system widely used in studying chaos. The Lorenz system arises in many applications such atmospheric convection, laser beams, and one-dimensional flows in a thermosyphon loop when heated below and cooled above, to name a few. Its key characteristics is that the temperature difference drives the fluids in motion inside the tube. The higher the external heat, the more chaotic the motion. Thermosyphons are relevant in several fields primarily as cooling agents. In nuclear power plants, the natural convection benefits the removal of heat from the system to maintain stability or to depressurize it after a reactor incident.

Previous work has already been done on controlling single and double-looped systems. [1,2,3]. Our goal is to model the flows within three interconnected thermosyphon loops, allowing heat and momentum exchanges at the coupling points, and then design a positive, proportional state feedback controller to stabilize the flows.

CHAPTER 2

DYNAMICAL SYSTEM SETUP

Before controlling a system, it is of course essential that you understand from where it came. In this system we explore the physical dynamical system and how it comes to be represented by a series of ordinary differential equations

2.1 PHYSICAL REPRESENTATION

We are interested in an interconnected triple-loop thermosyphon system in which heat transfer is carried out via fluids in the tubes, see Fig. 2.1.

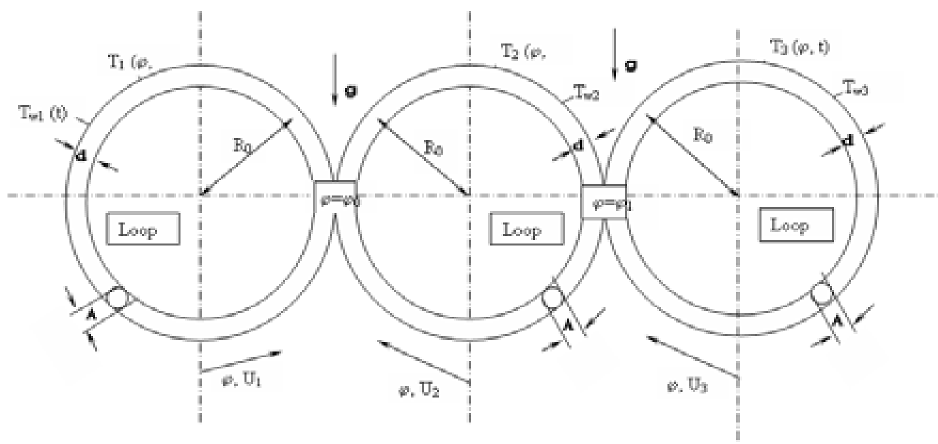

Figure 2.1: The physical form of a coupled triple-loop thermosyphon system

In this system, each loop is heated at the bottom and cooled at the top. It is also important to note the coupling points between the loops, at $\varphi = \varphi_0$ and $\varphi = \varphi_1$. While previous papers on two-loop systems only had one coupling point between the loops [1,2,3], our middle loop has two coupling points. This will affect our differential equations and resulting analysis.

2.2 REPRESENTATION AS PARTIAL DIFFERENTIAL EQUATIONS

When dealing with a dynamical system, partial differential equations are not the easiest with which to work. Regardless, it is still important to understand where these partial differential equations came from before then manipulating them into much more friendly ordinary differential equations. Our system is represented by the following partial differential equations:

$$\frac{\partial u_1}{l\partial\varphi} = \frac{\partial u_2}{l\partial\varphi} = \frac{\partial u_3}{l\partial\varphi} = 0 \quad (1)$$

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial p_1}{l\partial\varphi} - \rho T_1 g \sin\varphi - f_w - \delta(\varphi-\varphi_0) f_{12} \quad (2)$$

$$\rho_0 c_p \left(\frac{\partial T_1}{\partial t} + u_1 \frac{\partial T_1}{l\partial\varphi} \right) = h_w \left(T_{w_1}(\varphi) - T_1 \right) - \delta(\varphi-\varphi_0) h_{12}(T_1 - T_2) \quad (3)$$

$$\rho_0 \frac{\partial u_2}{\partial t} = -\frac{\partial p_2}{l\partial\varphi} - \rho T_2 g \sin\varphi - f_w - \delta(\varphi-\varphi_0) f_{21} - \delta(\varphi-\varphi_1) f_{23} \quad (4)$$

$$\rho_0 c_p \left(\frac{\partial T_2}{\partial t} + u_2 \frac{\partial T_2}{l\partial\varphi} \right) = h_w \left(T_{w_2}(\varphi) - T_2 \right) - \delta(\varphi-\varphi_0) h_{12}(T_2 - T_1) - \delta(\varphi-\varphi_1) h_{23}(T_2 - T_3) \quad (5)$$

$$\rho_0 \frac{\partial u_3}{\partial t} = -\frac{\partial p_3}{l\partial\varphi} - \rho T_3 g \sin\varphi - f_w - \delta(\varphi-\varphi_1) f_{32} \quad (6)$$

$$\rho_0 c_p \left(\frac{\partial T_3}{\partial t} + u_3 \frac{\partial T_3}{l\partial\varphi} \right) = h_w \left(T_{w_3}(\varphi) - T_3 \right) - \delta(\varphi-\varphi_1) h_{32}(T_3 - T_2) \quad (7)$$

In this, each set of three equations governs the flow dynamics in a corresponding loop. The second, fourth, and sixth equations relate to the velocity of each loop. The final term(s) in these equations corresponds to the momentum coupling between loops. Since our middle loop has two coupling points, it has two momentum coupling terms. The third, fifth, and seventh equations relate to each loop’s temperature. Here the final term(s) corresponds to the heat coupling between loops. Similarly, the middle loop has two heat coupling points.

2.3 REPRESENTATION AS ORDINARY DIFFERENTIAL EQUATIONS

Now that we have established a mathematical model for the thermosyphons, as a common practice in modeling, we simplify the equations through a set of idealization assumptions and nondimensionalization process into a form that is easier to handle. First, we introduce a Fourier series for each temperature function T_i in the form:

$$T_i(\varphi,t) = T_{i,0} + \sum_{n=1}^{\infty} S_{i,n}\sin(n\varphi) + C_{i,n}\cos(n\varphi)$$

Then using a Galerkin method, we truncate the sum to n=1 and substitute this into our equations. We define our T_W as simply T_H in the upper half of each loop and T_C in the bottom half. In addition, we define a constant η relating the contact areas to the overall wall surface in each loop:

$$\eta = \frac{Acont}{Awall} << 1$$

By integrating each of our momentum equations with respect to φ, knowing that the velocity functions are independent of φ due to (1), we can match the term coefficients for our equations. Finally, we use a known nondimensionalization method to make all state variables without units. The final result is the following systems of ordinary differential equations:

$$\begin{aligned}
\dot{x}_1 &= p\{(y_1 - x_1) - \gamma(x_1 - x_2)\} \\
\dot{y}_1 &= R_1 x_1 - y_1 - x_1 z_1 \\
\dot{z}_1 &= x_1 y_1 - z_1 - \eta(z_1 - z_2) \\
\dot{x}_2 &= p\{(y_2 - x_2) - \gamma(x_2 - x_1) - \gamma(x_2 - x_3)\} \\
\dot{y}_2 &= R_2 x_2 - y_2 - x_2 z_2 \\
\dot{z}_2 &= x_2 y_2 - z_2 - \eta(z_2 - z_1) - \eta(z_2 - z_3) \\
\dot{x}_3 &= p\{(y_3 - x_3) - \gamma(x_3 - x_2)\} \\
\dot{y}_3 &= R_3 x_3 - y_3 - x_3 z_3 \\
\dot{z}_3 &= x_3 y_3 - z_3 - \eta(z_3 - z_2)
\end{aligned}$$

where the x's represent fluid velocity in each loop, the y's represent the horizontal temperature difference for each loop, and the z's represent the vertical temperature difference for each loop.

CHAPTER 3

SYSTEM MODELING (WITHOUT CONTROLLER)

Before testing the results of our controller, we first observe the system's natural behavior. As the heating parameters, commonly referred to as Rayleigh numbers, rise, the system's behavior falls into three categories: stable, periodic, and chaotic.

3.1 SINGLE LOOP HEATING

When our heating parameters are small enough, the system is stable on its own and doesn't require a controller; however, once the heating parameters cross a certain threshold, the system begins exhibiting periodic and then chaotic motion. Below are the velocities for each loop when loop 1 is heated (Figure 3.1).

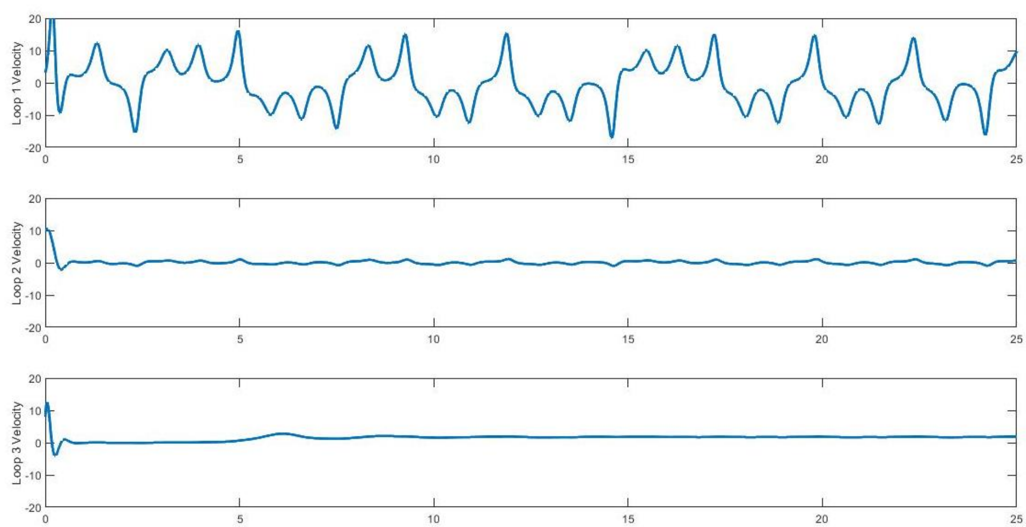

Figure 3.1.1: Velocities with Loop 1 Heated

Intuitively, it makes sense that the system will behave similarly when loop 3 is heated. One might expect a different scenario when only loop 2 is heated; however, the result is generally the same with the heated loop being chaotic and the adjacent loops having minor disturbances. It is significant to note the differences between heating a single loop in a 3-loop system and heating a single loop in a 1-loop system. While in both systems, the heated loop behaves chaotically, their patterns are noticeably different. In a 3-loop system, the other two loops provide interference (Figure 3.1.2). This will affect our controller's response time.

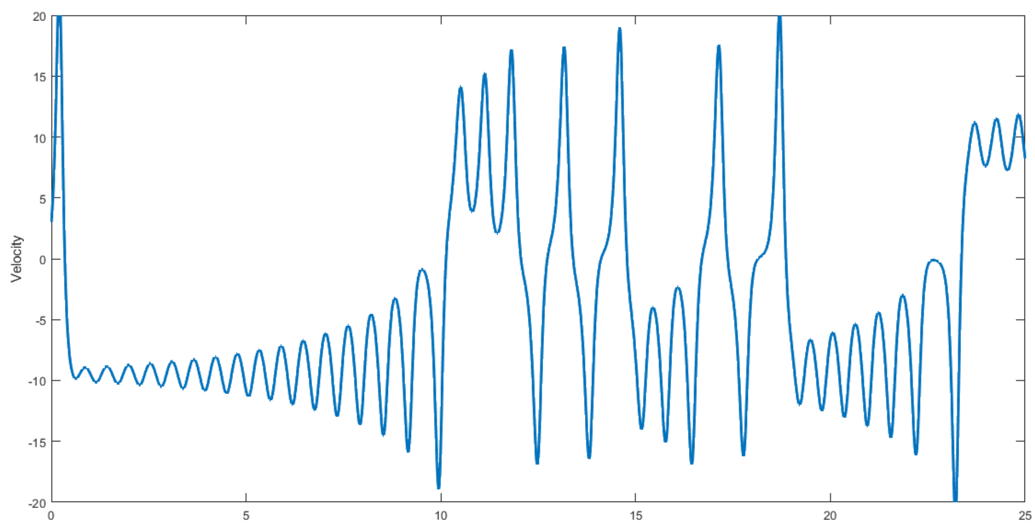

Figure 3.1.2: 1-Loop Heated System

3.2 DOUBLE LOOP HEATING

When two loops are heated, the results are similar. The heated loops are chaotic and the adjacent loop in only minorly excited. When loops one and three are heated, the unheated loop only shows minor disturbances even when the other two loops are heated at very high levels. Changing the amount of heating in the outer loops doesn't seem to affect the behavior of the center loop (Figure 3.2).

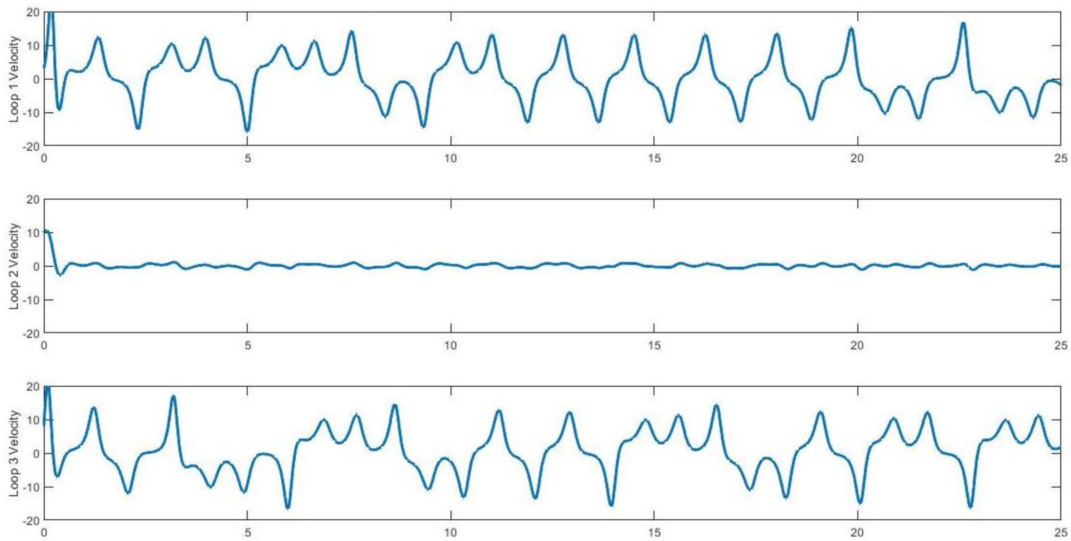

Figure 3.2: Velocities with Loop 2 Unheated

Instinctively, one might conclude that the reason that the middle loop isn't highly active is because the activities in loops one and three "cancel out." However, even when loop 1 is unheated, it also only shows minor motion.

3.3 CHAOTIC MOTION AT A SMALLER AMPLITUDE

An interested phenomenon to notice is the behavior of loops adjacent to heating. Although the motion of an adjacent loop is small, a closer look reviews that its motion is still chaotic. The exact nature of its chaos depends on the parameters of all three loops, not just those that have high heating parameters. For instance, below are the velocities of the middle loop when loops 1 and 3 are heated at Rayleigh numbers 25, 0, and 50 and 25, 15, and 50 (Figures 3.3). Notice that in the second figure, the velocity of loop two has shifted to be centered below the axis.

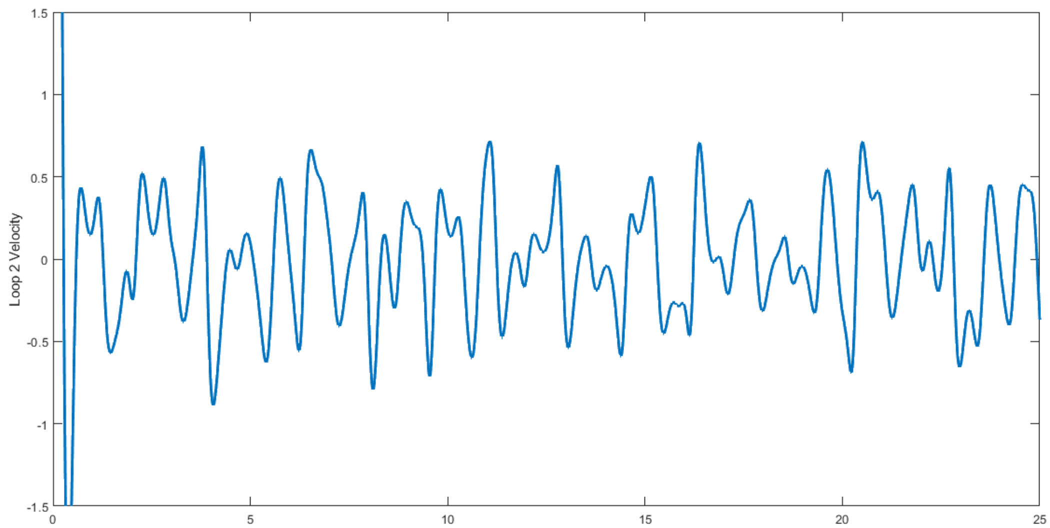

Figure 3.3.1: Loop 2 Velocity for R_2 set to 0

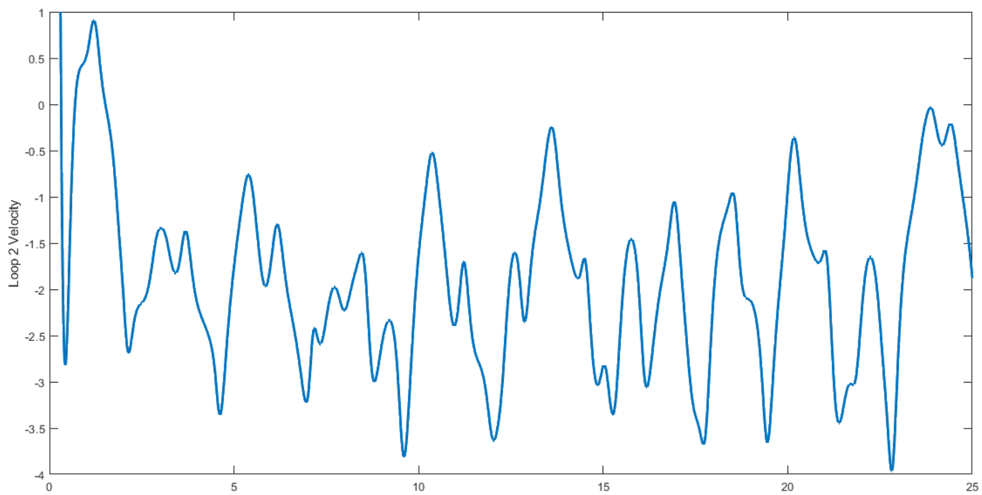

Figure 3.3.2: Loop 2 Velocity for R_2 set to 15

CHAPTER 4

INTRODUCTION OF THE SYSTEM CONTROLLER

To make this system non-chaotic, we introduce a positive state feedback controller. This is done by simply adding on a controller term to each of our $\dot{y}_i$ equations in the form of $-k_i y_i$, where k_i is known as the state feedback gain. The reason for choosing the *y*-state is that it is a state that is proportional to the easily measurable/observable temperature state. This design considers real implementation of the controller design. The question then, is what are the values of the k_i s? By applying Lyapunov's Stability Theorem [4], we find that our system is globally stable if the k_i s are large enough. However, large gains imply greater cost for building the controllers. We opt to seek smaller bounds or optimal bounds that are sufficient for stabilizing the systems.

Choose the Lyapunov Candidate:

$$V = \frac{1}{2}\left(\frac{R_1}{\rho}x_1^2 + y_1^2 + z_1^2 + \frac{R_2}{\rho}x_2^2 + y_2^2 + z_2^2 + \frac{R_3}{\rho}x_3^2 + y_3^2 + z_3^2\right)$$

Then

$$\dot{V} = \frac{R_1}{\rho}x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1 + \frac{R_2}{\rho}x_2\dot{x}_2 + y_2\dot{y}_2 + z_2\dot{z}_2 + \frac{R_3}{\rho}x_3\dot{x}_3 + y_3\dot{y}_3 + z_3\dot{z}_3 = -X^T AX$$

where $X = \begin{bmatrix} x_1 & y_1 & z_1 & x_2 & y_2 & z_2 & x_3 & y_3 & z_3 \end{bmatrix}$ and the symmetric matrix A is:

$$A=\begin{bmatrix} R_1(1+\gamma) & -R_1 & 0 & -\frac{\gamma}{2}(R_1+R_2) & 0 & 0 & 0 & 0 & 0 \\ -R_1 & 1+k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\eta & 0 & 0 & -\eta & 0 & 0 & 0 \\ -\frac{\gamma}{2}(R_1+R_2) & 0 & 0 & R_2(1+2\gamma) & -R_2 & 0 & -\frac{\gamma}{2}(R_2+R_3) & 0 & 0 \\ 0 & 0 & 0 & -R_2 & 1+k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 & 1+2\eta & 0 & 0 & -\eta \\ 0 & 0 & 0 & -\frac{\gamma}{2}(R_2+R_3) & 0 & 0 & R_3(1+\gamma) & -R_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_3 & 1+k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta & 0 & 0 & 1+\eta \end{bmatrix}$$

Then it suffices to show that A is positive definite. We do this by setting the determinant of each principle minor to be greater than zero. Each principal minor yields conditions on k_1, k_2, k_3 that can be simplified to:

$$1+k_1 > \frac{R_1^2 R_2(1+2\gamma)}{\psi_1}$$

$$1+k_2 > \frac{R_1 R_2^2[(1+\gamma)(1+k_1)-R_1]}{(1+k_1)\psi_1 - R_1^2 R_2(1+2\gamma)}$$

$$1+k_3 > \frac{R_3^2\{(1+k_1)(1+k_2)\psi_1 - R_1^2 R_2(1+k_2)(1+2\gamma) - R_1 R_2^2[(1+\gamma)(1+k_1)-R_1]\}}{R_1(1+k_2)[(1+\gamma)(1+k_1)-R_1]\psi_2 - R_3(1+k_1)(1+k_2)(1+\gamma)(R_1+R_2)^2\frac{\gamma^2}{4} - R_1R_2^2R_3(1+\gamma)[(1+\gamma)(1+k_1)-R_1]}$$

where

$$\psi_1 = R_1R_2(1+3\gamma+\gamma^2) - (R_1-R_2)^2\frac{\gamma^2}{4}$$

$$\psi_2 = R_2R_3(1+3\gamma+\gamma^2) - (R_2-R_3)^2\frac{\gamma^2}{4}$$

CHAPTER 5

SYSTEM MODELING (WITH CONTROLLER)

We now introduce the controllers into the system and observe the results. Given various Rayleigh numbers and initial starting conditions, is the controller universally effective? When solving for our controller bounds, it was required that our Rayleigh numbers are similar in magnitude, so for this next chapter, Rayleigh numbers are within 10 units of each other.

5.1 STABILIZING THE CHAOTIC SYSTEM

First we consider a scenario where the starting conditions of each loop are similar and the Rayleigh numbers are large enough to result in a chaotic system. At 40 seconds into the simulation, the controller in implemented (Figures 5.1).

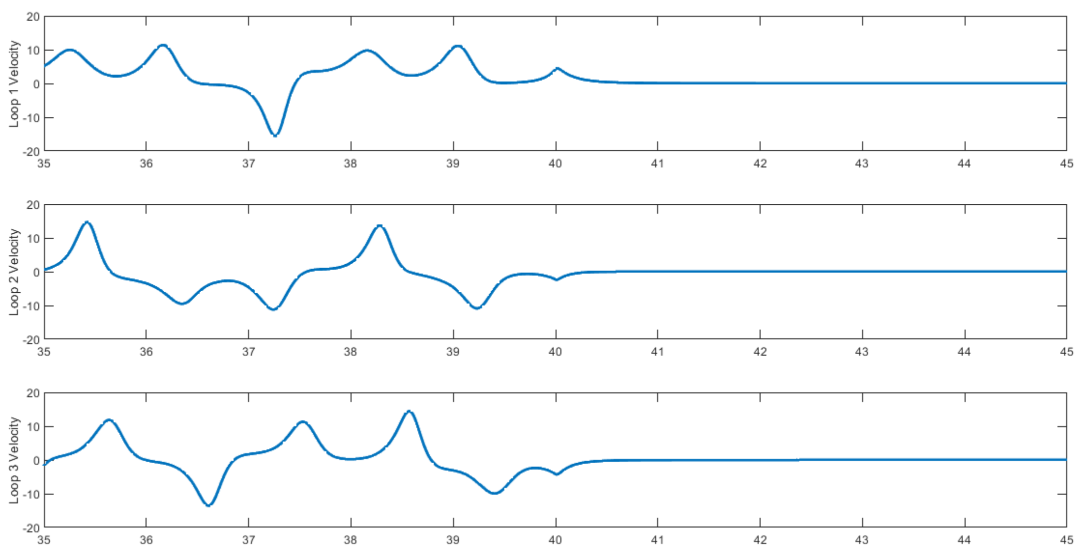

Figure 5.1.1: Stabilized Chaotic System

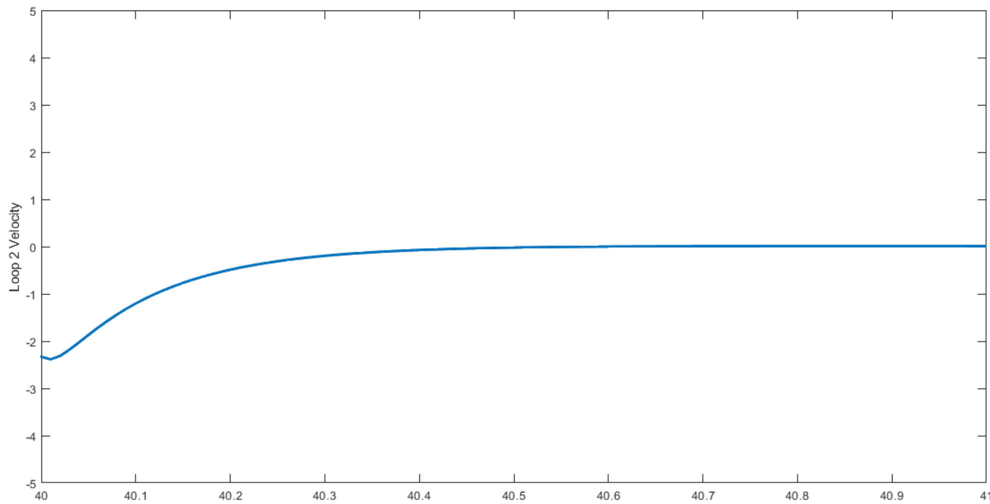

Figure 5.1.2: Stabilized Loop 2 Velocity

Even with drastically high Rayleigh numbers, our system still stabilizes in under one second. This suggests that our controllers are in fact an effective means of stabilizing our system.

CHAPTER 6

CLOSING REMARKS

Stabilizing a three-loop thermosyphon system is perhaps the last hurdle in stabilizing an n-loop thermosyphon system. For any number of loops in a row, no loop has any more than two coupling points, so we can expect the results for any increase in loops to be similar. We also intend to address controlling a system with minor disturbances. Our next step after finding the bounds for the k_i s is to study a four-loop system. Once this is completed, we can begin the process of exploring the general n-loop thermosyphon system.

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