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Discrete Fourier Restriction Associated with KdV Equations

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DISCRETE FOURIER RESTRICTION ASSOCIATED WITH KDV EQUATIONS

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ABSTRACT. In this paper, we consider a discrete restriction associated with KdV equations. Some new Strichartz estimates are obtained. We also establish the local well-posedness for the periodic generalized Korteweg-de Vries equation with nonlinear term $F(u)\partial_x u$ provided $F \in C^5$ and the initial data $\phi \in H^s$ with $s > 1/2$.

1. INTRODUCTION

The discrete restriction problem associated with KdV equations is a problem asking the best constant $A_{p,N}$ satisfying

$$\sum_{n=-N}^{N} \left| \widehat{f}(n, n^3) \right|^2 \leq A_{p,N} \|f\|_{p'}^2 \,, \tag{1.1}$$

where f is a periodic function on $\mathbb{T}^2$, $\widehat{f}$ is Fourier transform of f on $\mathbb{T}^2$, $p \geq 2$ and $p' = p/(p-1)$. It is natural to pose a conjecture asserting that for any $\varepsilon > 0$, $A_{p,N}$ satisfies

$$A_{p,N} \leq \begin{cases} C_p N^{1-\frac{8}{p}+\varepsilon} & \text{for } \; p \geq 8 \\ C_p & \text{for } \; 2 \leq p < 8 \,. \end{cases} \tag{1.2}$$

It was proved by Bourgain that $A_{6,N} \leq N^{\varepsilon}$. The desired upper bound for $A_{8,N}$ is not yet obtained, however, we are able to establish an affirmative answer for large p cases.

Theorem 1.1. *Let $A_{p,N}$ be defined as in (1.1). If $p \geq 14$, then for any $\varepsilon > 0$, there exists a constant C_p independent of N such that*

$$A_{p,N} \leq C_p N^{1-\frac{8}{p}+\varepsilon} \,. \tag{1.3}$$

The periodic Strichartz inequality associated to KdV equations is the inequality seeking for the best constant $K_{p,N}$ satisfying

$$\left\| \sum_{n=-N}^{N} a_n e^{2\pi i t n^3 + 2\pi i x n} \right\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T})} \leq K_{p,N} \left(\sum_{n=-N}^{N} |a_n|^2 \right)^{\frac{1}{2}} \,. \tag{1.4}$$

By duality, we see immediately

$$K_{p,N} \sim \sqrt{A_{p,N}} \,.$$

Henceforth, Theorem 1.1 is equivalent to Strichartz estimates,

$$K_{p,N} \leq C N^{\frac{1}{2}-\frac{4}{p}+\varepsilon} \,, \; \text{ for } \; p \geq 14 \,. \tag{1.5}$$

It was observed by Bougain that the periodic Strichartz inequalities (1.4) for $p = 4, 6$ are crucial for obtaining the local well-posedness of periodic KdV (mKdV or gKdV). The local

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(global) well-posedness of periodic KdV for $s \geq 0$ was first studied by Bourgain in [2]. Via a bilinear estimate approach, Kenig, Ponce and Vega in [9] established the local well-posedness of periodic KdV for $s > -1/2$. The sharp global well-posedness of the periodic KdV was proved by Colliander, Keel, Staffilani, Takaoka, and Tao in [5], by utilizing the I-method.

Inspired by Bourgain's work, we can obtain the following theorem on gKdV. Here the gKdV is the generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0 \\ u(x,0) = \phi(x), \qquad x \in \mathbb{T},\ t \in \mathbb{R}\,, \end{cases} \tag{1.6}$$

where $k \in \mathbb{N}$ and $k \geq 3$.

Theorem 1.2. *The Cauchy problem (1.6) is locally well-posed if the initial data $\phi \in H^s$ for $s > 1/2$.*

Theorem 1.2 is not new. It was proved by Colliander, Keel, Staffilani, Takaoka, and Tao in [4]. However, our method is different from the method in [4]. Let us point out the difference here. The method used in [4] is based on a rescaling argument and the bilinear estimates, proved by Kenig, Ponce and Vega [9]. Our method is more straightforward and does not need to go through the rescaling argument, the bilinear estimates in [9] or the multilinear estimates in [4]. This allows us to extend Theorem 1.2 to a very general setting. More precisely, consider the Cauchy problem for periodic generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} u_t + u_{xxx} + F(u) u_x = 0 \\ u(x,0) = \phi(x), \qquad x \in \mathbb{T},\ t \in \mathbb{R}\,. \end{cases} \tag{1.7}$$

Here F is a suitable function. Then the following theorem can be established.

Theorem 1.3. *The Cauchy problem (1.7) is locally well-posed provided F is a C^5 function and the initial data $\phi \in H^s$ for $s > 1/2$.*

For sufficiently smooth F, say $F \in C^{15}$, the existence of a local solution of (1.7) for $s \geq 1$ and the global well-posedness of (1.7) for small data $\phi \in H^s$ with $s > 3/2$ were proved by Bourgain in [3]. The index $1/2$ is sharp because the ill-posedness of (1.6) for $s < 1/2$ is known (see [4]). In order to make (1.7) well-posed for the initial data $\phi \in H^s$ with $s > 1/2$, the sharp regularity condition for F perhaps is C^4. But the method utilized in this paper, with a small modification, seems to be only able to reach an affirmative result for $F \in C^{\frac{9}{2}+}$ and $s > 1/2$. Moreover, the endpoint $s = 1/2$ case could be possibly done by combining the ideas from [4] and this paper. But we would not pursue this endpoint result in this paper.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need to introduce a level set. Since $\sqrt{A_{p,N}} \sim K_{p,N}$, it suffices to prove the Strichartz estimates (1.4). Let F_N be a periodic function on $\mathbb{T}^2$ given by

$$F_N(x,t) = \sum_{n=-N}^{N} a_n e^{2\pi i n x} e^{2\pi i n^3 t}\,, \tag{2.1}$$

where $\{a_n\}$ is a sequence with $\sum_n |a_n|^2 = 1$ and $(x,t) \in \mathbb{T}^2$. For any $\lambda > 0$, set a level set E_λ to be

$$E_\lambda = \left\{(x,t) \in \mathbb{T}^2 : |F_N(x,t)| > \lambda\right\} . \tag{2.2}$$

To obtain the desired estimate for the level set, let us first state a lemma on Weyl's sums.

Lemma 2.1. *Suppose that $t \in \mathbb{T}$ satisfies $|t - a/q| \leq 1/q^2$, where a and q are relatively prime. Then if $q \geq N^2$,*

$$\left|\sum_{n=1}^{N} e^{2\pi i(tn^3+bn^2+cn)}\right| \leq CN^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}} . \tag{2.3}$$

Here b and c are real numbers, and the constant C is independent of b, c, t, a, q and N.

The proof of Lemma 2.1 relies on Weyl's squaring method. See [8] or [10] for detail. Also we need the following lemma proved in [1].

Lemma 2.2. *For any integer $Q \geq 1$ and any integer $n \neq 0$, and any $\varepsilon > 0$,*

$$\sum_{Q\leq q<2Q} \left|\sum_{a\in\mathcal{P}_q} e^{2\pi i \frac{a}{q} n}\right| \leq C_\varepsilon d(n,Q) Q^{1+\varepsilon} .$$

Here $\mathcal{P}_q$ is given by

$$\mathcal{P}_q = \{a \in \mathbb{N} : 1 \leq a \leq q \ \text{ and } \ (a,q) = 1\} \tag{2.4}$$

and $d(n,Q)$ denotes the number of divisors of n less than Q and C_ε is a constant independent of Q, n.

Lemma 2.2 can be proved by observing that the arithmetic function defined by $f(q) = \sum_{a\in\mathcal{P}_q} e^{2\pi i \frac{a}{q} n}$ is multiplicative, and then utilize the prime factorization for q to conclude the lemma.

Proposition 2.1. *Let K_N be a kernel defined by*

$$K_N(x,t) = \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} . \tag{2.5}$$

For any given positive number Q with $N^2 \leq Q \leq N^3$, the kernel K_N can be decomposed into $K_{1,Q} + K_{2,Q}$ such that

$$\|K_{1,Q}\|_\infty \leq C_1 N^{\frac{1}{4}+\varepsilon} Q^{1/4} . \tag{2.6}$$

and

$$\|\widehat{K_{2,Q}}\|_\infty \leq \frac{C_2 N^\varepsilon}{Q} . \tag{2.7}$$

Here the constants C_1, C_2 are independent of Q and N.

Proof. We can assume that Q is an integer, since otherwise we can take the integer part of Q. For a standard bump function φ supported on $[1/200, 1/100]$, we set

$$\Phi(t) = \sum_{Q\leq q\leq 5Q} \sum_{a\in\mathcal{P}_q} \varphi\left(\frac{t - a/q}{1/q^2}\right) . \tag{2.8}$$

Clearly Φ is supported on $[0,1]$. We can extend Φ to other intervals periodically to obtain a periodic function on $\mathbb{T}$. For this periodic function generated by Φ, we still use Φ to denote it. Then it is easy to see that

$$\widehat{\Phi}(0)=\sum_{q\sim Q}\sum_{a\in\mathcal{P}_q}\frac{\mathcal{F}_{\mathbb{R}}\varphi(0)}{q^2}=\sum_{q\sim Q}\frac{\phi(q)}{q^2}\mathcal{F}_{\mathbb{R}}\varphi(0) \tag{2.9}$$

is a constant independent of Q. Here ϕ is Euler phi function, and $\mathcal{F}_{\mathbb{R}}$ denotes Fourier transform of a function on $\mathbb{R}$. Also we have

$$\widehat{\Phi}(k)=\sum_{q\sim Q}\sum_{a\in\mathcal{P}_q}\frac{1}{q^2}e^{-2\pi i\frac{a}{q}k}\mathcal{F}_{\mathbb{R}}\varphi(k/q^2)\,. \tag{2.10}$$

Applying Lemma 2.2 and the fact that $Q\leq N^3$, we obtain

$$\left|\widehat{\Phi}(k)\right|\leq\frac{N^{\varepsilon}}{Q}\,, \tag{2.11}$$

if $k\neq 0$.

We now define that

$$K_{1,Q}(x,t)=\frac{1}{\widehat{\Phi}(0)}K_N(x,t)\Phi(t),\ \text{ and }\ K_{2,Q}=K_N-K_{1,Q}\,.$$

(2.6) follows immediately from Lemma 2.1 since intervals $J_{a/q}=[\frac{a}{q}+\frac{1}{100q^2},\frac{a}{q}+\frac{1}{50q^2}]$'s are pairwise disjoint for all $Q\leq q\leq 5Q$ and $a\in\mathcal{P}_q$.

We now prove (2.7). In fact, represent Φ as its Fourier series to get

$$K_{2,Q}(x,t)=-\frac{1}{\widehat{\Phi}(0)}\sum_{k\neq 0}\widehat{\Phi}(k)e^{2\pi ikt}K_N(x,t)\,.$$

Thus its Fourier coefficient is

$$\widehat{K_{2,Q}}(n_1,n_2)=-\frac{1}{\widehat{\Phi}(0)}\sum_{k\neq 0}\widehat{\Phi}(k)\mathbf{1}_{\{n_2=n_1^3+k\}}(k)\,.$$

Here $(n_1,n_2)\in\mathbb{Z}^2$ and $\mathbf{1}_A$ is the indicator function of a measurable set A. This implies that $\widehat{K_{2,Q}}(n_1,n_2)=0$ if $n_2=n_1^3$, and if $n_2\neq n_1^3$,

$$\widehat{K_{2,Q}}(n_1,n_2)=-\frac{1}{\widehat{\Phi}(0)}\widehat{\Phi}(n_2-n_1^3)\,.$$

Applying (2.11), we estimate $\widehat{K_{2,Q}}(n_1,n_2)$ by

$$\left|\widehat{K_{2,Q}}(n_1,n_2)\right|\leq\frac{CN^{\varepsilon}}{Q}\,,$$

since $N^2\leq Q\leq N^3$. Henceforth we obtain (2.7). Therefore we complete the proof. □

Now we can state our theorem on the level set estimates.

Theorem 2.1. *For any positive numbers ε and $Q\geq N^2$, the level set defined as in (2.2) satisfies*

$$\lambda^2\left|E_\lambda\right|^2\leq C_1N^{\frac14+\varepsilon}Q^{\frac14}\left|E_\lambda\right|^2+\frac{C_2N^{\varepsilon}}{Q}\left|E_\lambda\right| \tag{2.12}$$

for all $\lambda>0$. Here C_1 and C_2 are constants independent of N and Q.

Proof. Notice that if $Q \geq N^3$, (2.12) becomes trivial since $E_\lambda = \emptyset$ if $\lambda \geq CN^{1/2}$. So we can assume that $N^2 \leq Q \leq N^3$. For the function F_N and the level set E_λ given in (2.1) and (2.2) respectively, we define f to be

$$f(x,t) = \frac{F_N(x,t)}{|F_N(x,t)|}\mathbf{1}_{E_\lambda}(x,t)\,.$$

Clearly

$$\lambda |E_\lambda| \leq \int_{\mathbb{T}^2} \overline{F_N(x,t)} f(x,t)dxdt\,.$$

By the definition of F_N, we get

$$\lambda |E_\lambda| \leq \sum_{n=-N}^{N} \overline{a_n}\widehat{f}(n,n^3)\,.$$

Utilizing Cauchy-Schwarz's inequality, we have

$$\lambda^2 |E_\lambda|^2 \leq \sum_{n=-N}^{N} \left|\widehat{f}(n,n^3)\right|^2\,.$$

The right hand side can be written as

$$\langle K_N * f, f\rangle\,. \tag{2.13}$$

For any Q with $N^2 \leq Q \leq N^3$, we employ Proposition 2.1 to decompose the kernel K_N. We then have

$$\lambda^2 |E_\lambda|^2 \leq |\langle K_{1,Q} * f, f\rangle| + |\langle K_{2,Q} * f, f\rangle| \tag{2.14}$$

From (2.6) and (2.7), we then obtain

$$\lambda^2 |E_\lambda|^2 \leq C_1 N^{\frac{1}{4}+\varepsilon} Q^{\frac{1}{4}} \|f\|_1^2 + \frac{C_2 N^\varepsilon}{Q}\|f\|_2^2 \leq C_1 N^{\frac{1}{4}+\varepsilon} Q^{\frac{1}{4}} |E_\lambda|^2 + \frac{C_2 N^\varepsilon}{Q}|E_\lambda|\,,$$

as desired. Therefore, we finish the proof of Theorem 2.1. □

Corollary 2.1. *If $\lambda \geq 2C_1 N^{\frac{3}{8}+\varepsilon}$, then*

$$|E_\lambda| \leq \frac{CN^{1+\varepsilon}}{\lambda^{10}}\,. \tag{2.15}$$

Here C_1 is the constant C_1 in Theorem 2.1 and C is a constant independent of N and λ.

Proof. Since $\lambda \geq 2C_1 N^{\frac{3}{8}+\varepsilon}$, we simply take Q satisfies $2C_1 N^{\frac{1}{4}+\varepsilon} Q^{1/4} = \lambda^2$. Then Corollary 2.1 follows from Theorem 2.1. □

We now are ready to finish the proof of Theorem 1.1. In fact, let $p \geq 14$ and write $\|F\|_p^p$ as

$$p\int_0^{2C_1 N^{\frac{3}{8}+\varepsilon}} \lambda^{p-1}|E_\lambda| d\lambda + p\int_{2C_1 N^{\frac{3}{8}+\varepsilon}}^{2N^{1/2}} \lambda^{p-1}|E_\lambda| d\lambda\,. \tag{2.16}$$

Observe that $A_{6,N} \leq N^\varepsilon$ implies

$$|E_\lambda| \leq \frac{N^\varepsilon}{\lambda^6}\,. \tag{2.17}$$

Thus the first term in (2.16) is bounded by

$$CN^{\frac{3(p-6)}{8}+\varepsilon} \leq CN^{\frac{p}{2}-4+\varepsilon}\,, \tag{2.18}$$

since $p \geq 14$. From (2.15), the second term is majorized by

$$CN^{\frac{p}{2}-4+\varepsilon}\,. \tag{2.19}$$

Putting both estimates together, we complete the proof of Theorem 1.1.

3. A Lower bound of $A_{p,N}$

In this section we show that $N^{1-8/p}$ is the best upper bound of $A_{p,N}$ if $p \geq 8$. Hence (1.3) can not be improved substantially, and it is sharp up to a factor of N^ε.

For $b \in \mathbb{N}$, let $S(N;b)$ be defined by

$$S(N;b) = \int_{\mathbb{T}^2} \left| \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right|^{2b} dxdt\,. \tag{3.1}$$

Proposition 3.1. *Let $S(N;b)$ be defined as in (3.1). Then*

$$S(N;b) \geq C\left(N^b + N^{2b-4}\right)\,. \tag{3.2}$$

Here C is a constant independent of N.

Proof. Clearly $S(N;b)$ is equal to the number of solutions of

$$\begin{cases} n_1 + \cdots + n_b = m_1 + \cdots + m_b \\ n_1^3 + \cdots + n_b^3 = m_1^3 + \cdots + m_b^3 \end{cases} \tag{3.3}$$

with $n_j, m_j \in \{-N, \cdots, N\}$ for all $j \in \{1, \cdots, b\}$. For each $(m_1, \cdots, m_b)$, we may obtain a solution of (3.3) by taking $(n_1, \cdots, n_b) = (m_1, \cdots, m_b)$. Thus

$$S(N;b) \geq N^b\,. \tag{3.4}$$

To derive a further lower bound for $S(N;b)$, we set Ω to be

$$\Omega = \left\{ (x,t) : |x| \leq \frac{1}{60N}\,, \quad |t| \leq \frac{1}{60N^3} \right\}\,. \tag{3.5}$$

If $(x,t) \in \Omega$ and $|n| \leq N$, then

$$\left| tn^3 + xn \right| \leq \frac{1}{30}\,. \tag{3.6}$$

Henceforth if $(x,t) \in \Omega$,

$$\left| \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right| \geq \left| \mathrm{Re} \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right| \geq \sum_{n=-N}^{N} \cos\left(2\pi(tn^3 + xn)\right) \geq CN\,. \tag{3.7}$$

Consequently, we have

$$S(N;b) \geq \int_{\Omega} \left| \sum_{n=-N}^{N} e^{2\pi i t n^3 + 2\pi i x n} \right|^{2b} dxdt \geq CN^{2b}|\Omega| \geq CN^{2b-4}\,. \tag{3.8}$$

□

Proposition 3.2. *Let $p \geq 2$ be even. Then $A_{p,N}$ satisfies*

$$A_{p,N} \geq C(1 + N^{1-\frac{8}{p}}) . \tag{3.9}$$

Here C is a constant independent of N.

Proof. Let $p = 2b$ since p is even. Setting $a_n = 1$ for all n in the definition of $K_{p,N}$, we get

$$S(N; b) \leq K^p_{p,N}(2N)^b . \tag{3.10}$$

By Proposition 3.1, we have

$$K_{p,N} \geq C\left(1 + N^{\frac{1}{2}-\frac{4}{p}}\right) . \tag{3.11}$$

Consequently, we conclude (3.9) since $A_{p,N} \sim K^2_{p,N}$. □

4. An estimate of Hua

The following theorem was proved by Hua in [8] by an arithmetic argument. Here we utilize our method to provide a different proof.

Theorem 4.1. *Let $S(N; b)$ be defined as in (3.1). Then*

$$S(N; 5) \leq CN^{6+\varepsilon} . \tag{4.1}$$

By Proposition 3.1, we see that the estimate (4.1) is (almost) sharp. $S(N; 4) \leq N^{4+\varepsilon}$ is still open. We now prove Theorem 4.1.

Proof. Let G_λ be the level set given by

$$G_\lambda = \left\{(x, t) \in \mathbb{T}^2 : |K_N(x, t)| \geq \lambda\right\} . \tag{4.2}$$

Here K_N is the function defined as in (2.5).

let $f = \mathbf{1}_{G_\lambda} K_N/|K_N|$ and we then have

$$\lambda|G_\lambda| \leq \sum_{n=-N}^{N} \widehat{f}(n, n^3) = \langle f_N, K_N\rangle , \tag{4.3}$$

where f_N is a rectangular Fourier partial sum defined by

$$f_N(x, t) = \sum_{\substack{|n_1|\leq N \\ |n_2|\leq N^3}} \widehat{f}(n_1, n_2)e^{2\pi n_1 x}e^{2\pi i n_2 t} . \tag{4.4}$$

Employing Proposition 2.1 for K_N, we estimate the level set G_λ by

$$\lambda|G_\lambda| \leq |\langle f_N, K_{1,Q}\rangle| + |\langle f_N, K_{2,Q}\rangle| \tag{4.5}$$

for any $Q \geq N^2$. From (2.6) and (2.7), $\lambda|G_\lambda|$ can be bounded further by

$$C\left(N^{\frac{1}{4}+\varepsilon}Q^{1/4}\|f_N\|_1 + \sum_{\substack{|n_1|\leq N \\ |n_2|\leq N^3}} \left|\widehat{K_{2,Q}}(n_1, n_2)\widehat{f}(n_1, n_2)\right|\right) . \tag{4.6}$$

Thus from the fact that L^1 norm of Dirichlet kernel D_N is comparable to $\log N$, (2.7), and Cauchy-Schwarz inequality, we have

$$\lambda|G_\lambda| \leq CN^{\frac{1}{4}+\varepsilon}Q^{1/4}|G_\lambda| + \frac{CN^{2+\varepsilon}}{Q}|G_\lambda|^{1/2}\,, \tag{4.7}$$

for all $Q \geq N^2$. For $\lambda \geq 2CN^{\frac{3}{4}+\varepsilon}$, take Q to be a number satisfying $2CN^{\frac{1}{4}+\varepsilon}Q^{1/4} = \lambda$ and then we obtain

$$|G_\lambda| \leq \frac{CN^{6+\varepsilon}}{\lambda^{10}}\,. \tag{4.8}$$

Notice that

$$\|K_N\|_6 \leq N^{\frac{1}{2}}K_{6,p} \leq N^{\frac{1}{2}+\varepsilon}\,. \tag{4.9}$$

Henceforth, by (4.3), we majorize $|G_\lambda|$ by

$$|G_\lambda| \leq \frac{CN^{3+\varepsilon}}{\lambda^6}\,. \tag{4.10}$$

We now estimate $S(N;5)$ by

$$S(N;5) \leq C\int_{2CN^{\frac{3}{4}+\varepsilon}}^{2N} \lambda^{10-1}|G_\lambda|d\lambda + C\int_0^{2CN^{\frac{3}{4}+\varepsilon}} \lambda^{10-1}|G_\lambda|d\lambda\,. \tag{4.11}$$

From (4.8), the first term in the right hand side of (4.11) can be bounded by $CN^{6+\varepsilon}$. From (4.10), the second term is clearly bounded by $N^{6+\varepsilon}$. Putting both estimates together,

$$S(N;5) \leq CN^{6+\varepsilon}\,, \tag{4.12}$$

as desired. Therefore, we complete the proof. □

5. Estimates for the nonlinear term and Local well-posedness of (1.6)

For any measurable function u on $\mathbb{T}\times\mathbb{R}$, we define the space-time Fourier transform by

$$\widehat{u}(n,\lambda) = \int_{\mathbb{R}}\int_{\mathbb{T}} u(x,t)e^{-inx}e^{-i\lambda t}dx\,dt \tag{5.1}$$

and set

$$\langle x\rangle := 1 + |x|\,.$$

We now introduce the $X_{s,b}$ space, initially used by Bourgain.

Definition 5.1. *Let I be an time interval in $\mathbb{R}$ and $s, b \in \mathbb{R}$. Let $X_{s,b}(I)$ be the space of functions u on $\mathbb{T}\times I$ that may be represented as*

$$u(x,t) = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{u}(n,\lambda)e^{inx}e^{i\lambda t}d\lambda \ \text{ for } \ (x,t)\in\mathbb{T}\times I \tag{5.2}$$

with the space-time Fourier transform $\widehat{u}$ satisfying

$$\|u\|_{X_{s,b}(I)} = \left(\sum_n \int \langle n\rangle^{2s}\langle\lambda - n^3\rangle^{2b}|\widehat{u}(n,\lambda)|^2 d\lambda\right)^{1/2} < \infty\,. \tag{5.3}$$

Here the norm should be understood as a restriction norm.

We should take the time interval to be $[0,\delta]$ for a small positive number δ, and abbreviate $\|u\|_{X_{s,b}(I)}$ as $\|u\|_{s,b}$ for any function u restricted to $\mathbb{T}\times[0,\delta]$. In this section, we always restrict the function u to $\mathbb{T}\times[0,\delta]$. Let w be the nonlinear function defined by

$$w=\left(u^k-\int u^k dx\right)u_x\,. \tag{5.4}$$

We also define

$$\|u\|_{Y_s}:=\|u\|_{s,\frac{1}{2}}+\left(\sum_n\langle n\rangle^{2s}\left(\int|\widehat{u}(n,\lambda)|\,d\lambda\right)^2\right)^{\frac{1}{2}}\,. \tag{5.5}$$

We need the following estimate on the nonlinear function w, in order to establish a contraction on the space $\{u:\|u\|_{Y_s}\le M\}$ for some $M>0$.

Proposition 5.1. *For $s>1/2$, there exists $\theta>0$ such that, for the nonlinear function w given by (5.4),*

$$\|w\|_{s,-\frac{1}{2}}+\left(\sum_n\langle n\rangle^{2s}\left(\int\frac{|\widehat{w}(n,\lambda)|}{\langle\lambda-n^3\rangle}d\lambda\right)^2\right)^{\frac{1}{2}}\le C\delta^\theta\|u\|_{Y_s}^{k+1}. \tag{5.6}$$

Here C is a constant independent of δ and u.

The proof of Proposition 5.1 will appear in Section 6. We now start to derive the local well-posedness of (1.6). For this purpose, we only need to consider the well-posedness of the Cauchy problem:

$$\begin{cases}u_t+u_{xxx}+\left(u^k-\int_{\mathbb{T}}u^k dx\right)u_x=0\\ u(x,0)=\phi(x), \qquad x\in\mathbb{T},\ t\in\mathbb{R}\end{cases}. \tag{5.7}$$

This is because if v is a solution of (5.7), then the gauge transform

$$u(x,t):=v\left(x-\int_0^t\int_{\mathbb{T}}v^k(y,\tau)dyd\tau,t\right)\,. \tag{5.8}$$

is a solution of (1.6) with the same initial value ϕ. Notice that this transform is invertible and preserves the initial data ϕ. The inverse transform is

$$v(x,t):=u\left(x+\int_0^t\int_{\mathbb{T}}u^k(y,\tau)dyd\tau,t\right)\,. \tag{5.9}$$

It is easy to see that for any solution u of (1.6), this inverse transform of u defines a solution of (5.7). Hence to establish well-posedness of (1.6), it suffices to obtain the well-posedness of (5.7). This gauge transform was used in [4].

By Duhamel principle, the corresponding integral equation associated to (5.7) is

$$u(x,t)=e^{-t\partial_x^3}\phi(x)-\int_0^t e^{-(t-\tau)\partial_x^3}w(x,\tau)d\tau, \tag{5.10}$$

where w is defined as in (5.4).

Since we are only seeking for the local well-posedness, we may use a bump function to truncate time variable. Let ψ be a bump function supported in $[-2,2]$ with $\psi(t)=1,|t|\le 1$, and let ψ_δ be

$$\psi_\delta(t)=\psi(t/\delta)\,.$$

Then it suffices to find a local solution of

$$u(x,t) = \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)d\tau.$$

Let T be an operator given by

$$Tu(x,t) := \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)d\tau. \tag{5.11}$$

We denote the first term (the linear term) in (5.11) by $\mathcal{L}u$ and the second term (the nonlinear term) by $\mathcal{N}u$. Henceforth we represent Tu as $\mathcal{L}u + \mathcal{N}u$.

Lemma 5.1. *The linear term $\mathcal{L}$ satisfies*

$$\|\mathcal{L}u\|_{Y_s} \leq C\|\phi\|_{H^s}\,. \tag{5.12}$$

Here C is a constant independent of δ.

Proof. Notice that

$$\widehat{\mathcal{L}u}(n,\lambda) = \widehat{\phi}(n)\mathcal{F}_{\mathbb{R}}\psi_\delta(\lambda - n^3) = \widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi\left(\delta(\lambda-n^3)\right),$$

Thus from the definition of Y_s norm,

$$\begin{aligned}\|\mathcal{L}u\|_{Y_s} = &\left(\sum_n \int \langle n\rangle^{2s}\langle \lambda - n^3\rangle \left|\widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi\left(\delta(\lambda-n^3)\right)\right|^2 d\lambda\right)^{\frac12}\\ &+\left(\sum_n \langle n\rangle^{2s}\left(\int\left|\widehat{\phi}(n)\delta\mathcal{F}_{\mathbb{R}}\psi\left(\delta(\lambda-n^3)\right)\right|d\lambda\right)^2\right)^{\frac12}.\end{aligned}$$

Since ψ is a Schwartz function, its Fourier transform is also a Schwartz function. Using the fast decay property for the Schwartz function, we have

$$\|\mathcal{L}u\|_{Y_s} \leq C\left(\sum_n \langle n\rangle^{2s}\left|\widehat{\phi}(n)\right|^2\right)^{\frac12} = C\|\phi\|_{H^s}.$$

□

Lemma 5.2. *The nonlinear term $\mathcal{N}$ satisfies*

$$\|\mathcal{N}u\|_{Y_s} \leq C\left(\|w\|_{s,-\frac12} + \left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}d\lambda\right)^2\right)^{\frac12}\right), \tag{5.13}$$

where C is a constant independent of δ.

Proof. Represent w as its space-time inverse Fourier transform so that we write

$$\mathcal{N}u(x,t) = -\psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3}\left(\sum_n\int \widehat{w}(n,\lambda)e^{inx}e^{i\lambda\tau}d\lambda\right)d\tau\,, \tag{5.14}$$

which is equal to

$$\begin{aligned}&-\psi_\delta(t)\sum_n\int\widehat{w}(n,\lambda)\int_0^t e^{-(t-\tau)(in)^3}e^{inx}e^{i\lambda\tau}d\tau d\lambda\\ =&-\psi_\delta(t)\sum_n\int\widehat{w}(n,\lambda)e^{inx}e^{in^3t}\,\frac{e^{i(\lambda-n^3)t}-1}{i(\lambda-n^3)}\,d\lambda\,.\end{aligned}$$

We decompose the nonlinear term $\mathcal{N}u$ into three parts, denoted by $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ respectively.

$$\begin{aligned}\mathcal{N}u(x,t) = &- \psi_\delta(t)\sum_n \int_{|\lambda-n^3|\le\frac{1}{100\delta}} \widehat{w}(n,\lambda)e^{inx}e^{in^3t}\sum_{k\ge1}\frac{(it)^k}{k!}(\lambda-n^3)^{k-1}d\lambda \\ &+ i\psi_\delta(t)\sum_n\int_{|\lambda-n^3|>\frac{1}{100\delta}} \frac{\widehat{w}(n,\lambda)}{\lambda-n^3}e^{inx}e^{i\lambda t}d\lambda \\ &- i\psi_\delta(t)\sum_n\left(\int_{|\lambda-n^3|>\frac{1}{100\delta}}\frac{\widehat{w}(n,\lambda)}{\lambda-n^3}d\lambda\right)e^{inx}e^{in^3t} \\ :=&\mathcal{N}_1u+\mathcal{N}_2u+\mathcal{N}_3u.\end{aligned}$$

First we estimate $\mathcal{N}_2$. Using Fourier series expansion for ψ, we get

$$\psi_\delta(t) = \sum_{m\in\mathbb{Z}} C_m e^{imt/\delta}\,.$$

Here the coefficients C_m's satisfy

$$C_m \le C(1+|m|)^{-100}\,.$$

Hence $\mathcal{N}_2u$ can be represent as

$$\mathcal{N}_2u = i\sum_m C_m\sum_n e^{inx}\int_{|\lambda-n^3|>\frac{1}{100\delta}}\frac{\widehat{w}(n,\lambda)}{\lambda-n^3}e^{i(\lambda+m/\delta)t}d\lambda \tag{5.15}$$

By a change of variables $(\lambda+m/\delta)\mapsto\lambda$,

$$\mathcal{N}_2u = i\sum_m C_m\sum_n e^{inx}\int_{|\lambda-\frac{m}{\delta}-n^3|>\frac{1}{100\delta}}\frac{\widehat{w}(n,\lambda-m/\delta)}{\lambda-\frac{m}{\delta}-n^3}e^{i\lambda t}d\lambda \tag{5.16}$$

Thus we estimate

$$\|\mathcal{N}_2u\|^2_{s,\frac12} \le C\sum_m(1+|m|)^{-50}\sum_n\langle n\rangle^{2s}\int_{|\lambda-\frac{m}{\delta}-n^3|>\frac{1}{100\delta}}\frac{\langle\lambda-n^3\rangle\,|\widehat{w}(n,\lambda-m/\delta)|^2}{|\lambda-\frac{m}{\delta}-n^3|^2}d\lambda\,. \tag{5.17}$$

Changing variables again, we obtain

$$\|\mathcal{N}_2u\|^2_{s,\frac12} \le C\sum_m(1+|m|)^{-50}\sum_n\langle n\rangle^{2s}\int_{|\lambda-n^3|>\frac{1}{100\delta}}\frac{\langle\lambda+\frac{m}{\delta}-n^3\rangle\,|\widehat{w}(n,\lambda)|^2}{\langle\lambda-n^3\rangle^2}d\lambda\,. \tag{5.18}$$

Notice that $|\lambda-n^3|>\frac{1}{100\delta}$ implies

$$\langle\lambda+\frac{m}{\delta}-n^3\rangle \le 200m\langle\lambda-n^3\rangle\,. \tag{5.19}$$

We obtain immediately

$$\|\mathcal{N}_2u\|_{s,\frac12} \le C\|w\|_{s,-\frac12}\,. \tag{5.20}$$

On the other hand,

$$\sum_n\langle n\rangle^{2s}\left(\int|\widehat{\mathcal{N}_2u}(n,\lambda)|d\lambda\right)^2 \le C\sum_m\langle m\rangle^{-5}\sum_n\langle n\rangle^{2s}\left(\int_{|\lambda-\frac{m}{\delta}-n^3|>\frac{1}{100\delta}}\frac{|\widehat{w}(n,\lambda-m/\delta)|d\lambda}{|\lambda-\frac{m}{\delta}-n^3|}\right)^2,$$

which is clearly bounded by

$$\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|d\lambda}{\langle \lambda - n^3\rangle}\right)^2 . \tag{5.21}$$

Putting (5.20) and (5.21) together, we have

$$\|\mathcal{N}_2 u\|_{Y_s} \leq C\left(\|w\|_{s,-\frac{1}{2}} + \left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}d\lambda\right)^2\right)^{\frac{1}{2}}\right). \tag{5.22}$$

Let A_n be defined by

$$A_n = \int_{|\lambda - n^3|\leq \frac{1}{100\delta}} \widehat{w}(n,\lambda)(\lambda - n^3)^{k-1}d\lambda . \tag{5.23}$$

Then $\mathcal{N}_1 u$ can be written as

$$\mathcal{N}_1 u(x,t) = -\sum_{k\geq 1}\frac{i^k}{k!}t^k\psi_\delta(t)\sum_n A_n e^{inx}e^{in^3 t} . \tag{5.24}$$

Hence the space-time Fourier transform of $\mathcal{N}_1 u$ satisfies

$$\left|\widehat{\mathcal{N}_1 u}(n,\lambda)\right| \leq \sum_{k\geq 1}\frac{1}{k!}|A_n|\left|\mathcal{F}_{\mathbb{R}}(\tilde{\psi_\delta})(\lambda - n^3)\right| , \tag{5.25}$$

where $\tilde{\psi_\delta}(t) = t^k\psi_\delta(t)$. Using the definition of Fourier transform, we have

$$\left|\mathcal{F}_{\mathbb{R}}(\tilde{\psi_\delta})(\lambda - n^3)\right| \leq C\delta^{k+1}k^3\langle \delta(\lambda - n^3)\rangle^{-3} .$$

Thus

$$\begin{aligned}\|\mathcal{N}_1 u\|_{Y_s}^2 &\leq \sum_{k\geq 1}\frac{C}{k^5}\sum_n \langle n\rangle^{2s}|A_n|^2\delta^{2k}\int \delta^2\langle \lambda - n^3\rangle\langle \delta(\lambda - n^3)\rangle^{-6}d\lambda \\ &\quad + \sum_{k\geq 1}\frac{C}{k^5}\sum_n \langle n\rangle^{2s}|A_n|^2\delta^{2k}\left(\int \delta\langle \delta(\lambda - n^3)\rangle^{-3}d\lambda\right)^2 \\ &\leq \sum_{k\geq 1}\frac{C}{k^5}\sum_n \langle n\rangle^{2s}|A_n|^2\delta^{2k} .\end{aligned}$$

Clearly A_n is bounded by

$$|A_n| \leq C\delta^{-k}\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}d\lambda . \tag{5.26}$$

Henceforth, we obtain

$$\|\mathcal{N}_1 u\|_{Y_s} \leq C\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}d\lambda\right)^2\right)^{\frac{1}{2}} . \tag{5.27}$$

Similarly, we may obtain

$$\|\mathcal{N}_3 u\|_{Y_s} \leq C\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle}d\lambda\right)^2\right)^{\frac{1}{2}} . \tag{5.28}$$

Therefore we complete the proof.

□

Proposition 5.2. *Let $s > 1/2$ and T be the operator defined as in (5.11). Then there exits a positive number θ such that*

$$\|Tu\|_{Y_s} \leq C\left(\|\phi\|_{H^s} + \delta^\theta \|u\|_{Y_s}^{k+1}\right) . \tag{5.29}$$

Here C is a constant independent of δ.

Proof. Since $Tu = \mathcal{L}u + \mathcal{N}u$, Proposition 5.2 follows from Lemma 5.1, Lemma 5.2 and Proposition 5.1. □

Proposition 5.2 yields that for δ sufficiently small, T maps a ball in Y_s into itself. Moreover, we write

$$\begin{aligned} & \left(u^k - \int_{\mathbb{T}} u^k dx\right) u_x - \left(v^k - \int_{\mathbb{T}} v^k dx\right) v_x \\ = & \left(u^k - \int_{\mathbb{T}} u^k dx\right)(u-v)_x + \left((u^k - v^k) - \int_{\mathbb{T}} (u^k - v^k) dx\right) v_x \end{aligned}$$

which equals to

$$\left(u^k - \int_{\mathbb{T}} u^k dx\right)(u-v)_x + \sum_{j=0}^{k-1} \left((u-v)u^{k-1-j}v^j - \int_{\mathbb{T}} (u-v)u^{k-1-j}v^j dx\right) v_x . \tag{5.30}$$

For $k+1$ terms in (5.30), repeating similar argument as in the proof of Proposition 5.1, one obtains, for $s > 1/2$,

$$\|Tu - Tv\|_{Y_s} \leq C\delta^\theta \left(\|u\|_{Y_s}^k + \sum_{j=1}^{k-1} \|u\|_{Y_s}^{k-1-j} \|v\|_{Y_s}^{j+1}\right) \|u-v\|_{Y_s} . \tag{5.31}$$

Henceforth, for $\delta > 0$ small enough, T is a contraction and the local well-posedness follows from Picard's fixed-point theorem.

6. Proof of Proposition 5.1

From the definition of w in (5.4), we may write $\widehat{w}(n,\lambda)$ as

$$\sum_{\substack{m+n_1+\cdots+n_k=n \\ n_1+\cdots+n_k\neq 0}} m \int \widehat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)\widehat{u}(n_1,\lambda_1)\cdots\widehat{u}(n_k,\lambda_k) d\lambda_1 \cdots d\lambda_k . \tag{6.1}$$

By duality, there exists a sequence $\{A_{n,\lambda}\}$ satisfying

$$\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |A_{n,\lambda}|^2 d\lambda \leq 1 , \tag{6.2}$$

and $\|w\|_{s,-\frac{1}{2}}$ is bounded by

(6.3)
$$\sum_{\substack{m+n_1+\cdots+n_k=n \\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s |m|}{\langle \lambda - n^3\rangle^{\frac{1}{2}}} |\widehat{u}(m, \lambda-\lambda_1-\cdots-\lambda_k)|\cdot|\widehat{u}(n_1,\lambda_1)|\cdots|\widehat{u}(n_k,\lambda_k)||A_{n,\lambda}| d\lambda_1\cdots d\lambda_k d\lambda .$$

Since the $X_{s,b}$ is a restriction norm, we may assume that u is supported in $\mathbb{T}\times[0,\delta]$. However, the inverse space-time Fourier transform $|\widehat{u}|^\vee$ in general may not be a function with compact support. The following standard trick allows us to assume $|\widehat{u}|^\vee$ has a compact support too. In fact, let η be a bump function supported on $[-2\delta, 2\delta]$ and with $\eta(t) = 1$ in $|t| \leq \delta$. Also $\widehat{\eta}$ is positive. Then $u = u\eta$ and $\widehat{u} = \widehat{u} * \widehat{\eta}$. Thus $|\widehat{u}| \leq |\widehat{u}| * \widehat{\eta} = (|\widehat{u}|^\vee \eta)^\wedge$. Whenever we need to make $|\widehat{u}|^\vee$ to be supported in a small time interval, we replace $|\widehat{u}|$ by $(|\widehat{u}|^\vee \eta)^\wedge$ since $|\widehat{u}|^\vee \eta$ clearly is supported on $\mathbb{T} \times [-2\delta, 2\delta]$. This will help us gain a positive power of δ in our estimates. Moreover, without loss of generality we can assume $|n_1| \geq |n_2| \geq \cdots \geq |n_k|$.

The trouble occurs mainly because of the factor $|m|$ resulted from $\partial_x u$. The idea is that either the factor $\langle \lambda - n^3 \rangle^{-\frac{1}{2}}$ can be used to cancel $|m|$, or $|m|$ can be distributed to some of $\widehat{u}$'s. More precisely, we consider three cases.

$$|m| < 1000k^2 |n_2| \, ; \tag{6.4}$$

$$1000k^2 |n_2| \leq |m| \leq 100k|n_1| \, ; \tag{6.5}$$

$$|m| > 100k|n_1| \, . \tag{6.6}$$

6.1. **Case (6.4).** This is the simplest case. In fact, In this case, it is easy to see that

$$\langle n \rangle^s |m| \leq C \langle n_1 \rangle^s \langle n_2 \rangle^{\frac{1}{2}} \langle m \rangle^{\frac{1}{2}} . \tag{6.7}$$

Let

$$F_1(x,t) = \sum_n \int \frac{|A_{n,\lambda}|}{\langle \lambda - n^3 \rangle^{\frac{1}{2}}} e^{i\lambda t} e^{inx} d\lambda \, ; \tag{6.8}$$

$$G(x,t) = \sum_n \int \langle n \rangle^{\frac{1}{2}} |\widehat{u}(n,\lambda)| e^{i\lambda t} e^{inx} d\lambda \tag{6.9}$$

$$H(x,t) = \sum_n \int \langle n \rangle^{s} |\widehat{u}(n,\lambda)| e^{i\lambda t} e^{inx} d\lambda \tag{6.10}$$

$$U(x,t) = \sum_n \int |\widehat{u}(n,\lambda)| e^{i\lambda t} e^{inx} d\lambda \tag{6.11}$$

Then using (6.7), we can estimate (6.3) by

$$C \sum_{m+n_1+\cdots+n_k=n} \int \widehat{F_1}(n,\lambda) \widehat{G}(m, \lambda - \lambda_1 - \cdots - \lambda_k) \widehat{H}(n_1,\lambda_1) \widehat{G}(n_2,\lambda_2) \prod_{j=3}^{k} \widehat{U}(n_j,\lambda_j) d\lambda_1 \cdots d\lambda_k d\lambda \, , \tag{6.12}$$

which clearly equals

$$C \int_{\mathbb{T}\times\mathbb{R}} F_1(x,t) G(x,t)^2 H(x,t) U(x,t)^{k-2} dx dt \, .$$

Apply Hölder inequality to majorize it by

$$C \|F_1\|_4 \|G\|_{6+}^2 \|H\|_4 \|U\|_{6(k-2)-}^{k-2} \, .$$

Since U is supported on $\mathbb{T} \times [-2\delta, 2\delta]$, one more use of Hölder inequality yields

$$(6.3) \leq C\delta^\theta \|F_1\|_4 \|G\|_{6+}^2 \|H\|_4 \|U\|_{6(k-2)}^{k-2} \, . \tag{6.13}$$

Let us recall some useful local embedding facts on $X_{s,b}$.

$$X_{0,\frac{1}{3}} \subseteq L^4_{x,t}\,, \quad X_{0+,\frac{1}{2}+} \subseteq L^6_{x,t}\,, \quad (t \text{ local}) \tag{6.14}$$

$$X_{\alpha,\frac{1}{2}} \subseteq L^q_{x,t}, \quad 0<\alpha<\frac{1}{2},\ 2\leq q<\frac{6}{1-2\alpha} \quad (t \text{ local}), \tag{6.15}$$

$$X_{\frac{1}{2}-\alpha,\frac{1}{2}-\alpha} \subseteq L^q_t L^r_x, \quad 0<\alpha<\frac{1}{2},\ 2\leq q,r<\frac{1}{\alpha}. \tag{6.16}$$

The two embedding results in (6.14) are consequences of the discrete restriction estimates on L^4 and L^6 respectively. (6.15) and (6.16) follow by interpolation (see [4] for details). (6.14) yields

$$\|F_1\|_4 \leq C\|F_1\|_{0,\frac{1}{3}} \leq C\left(\sum_n \int |A_{n,\lambda}|^2 d\lambda\right)^{1/2} \leq C\,,$$

and

$$\|H\|_4 \leq C\|H\|_{0,\frac{1}{3}} \leq C\|u\|_{s,\frac{1}{2}} \leq C\|u\|_{Y_s}\,.$$

(6.15) implies

$$\|G\|_{6+} \leq C\|G\|_{0+,\frac{1}{2}} \leq C\|u\|_{s,\frac{1}{2}} \leq C\|u\|_{Y_s}\,.$$

Using (6.16), we get

$$\|U\|_{6(k-2)} \leq C\|U\|_{\frac{1}{2}-,\frac{1}{2}-} \leq C\|u\|_{s,\frac{1}{2}} \leq C\|u\|_{Y_s}.$$

Henceforth, we have, for the case (6.4),

$$(6.3) \leq C\delta^\theta \|u\|^{k+1}_{Y_s}\,. \tag{6.17}$$

6.2. **Case (6.5).** In this case, we should further consider two subcases.

$$|m+n_1| \leq 1000k^2|n_2| \tag{6.18}$$

$$|m+n_1| > 1000k^2|n_2| \tag{6.19}$$

In the subcase (6.18), we use the triangle inequality to get

$$|n| = |m+n_1+n_2+\cdots+n_k| \leq C|n_2| \tag{6.20}$$

Hence, we have

$$\langle n\rangle^s |m| \leq C\langle n_2\rangle^s \langle m\rangle^{\frac{1}{2}} \langle n_1\rangle^{\frac{1}{2}}\,. \tag{6.21}$$

Thus this subcase can be treated exactly the same as the case (6.4). We omit the details.

For the subcase (6.19), the crucial arithmetic observation is

$$n^3 - (m^3+n_1^3+\cdots+n_k^3) = 3(m+n_1)(m+a)(n_1+a) + a^3 - (n_2^3+\cdots+n_k^3)\,, \tag{6.22}$$

where $a = n_2+\cdots+n_k$. This observation can be easily verified since $n = m+n_1+\cdots+n_k$. From (6.5) and (6.19), we get

$$\left|n^3 - (m^3+n_1^3+\cdots+n_k^3)\right| \geq Ck^2\langle n_2\rangle |m||n_1| \geq Ck|m|^2\,. \tag{6.23}$$

This implies at least one of following statements holds:

$$\left|\lambda - n^3\right| \geq C|m|^2\,, \tag{6.24}$$

$$\left|(\lambda-\lambda_1-\cdots-\lambda_k) - m^3\right| \geq C|m|^2\,, \tag{6.25}$$

$$\exists i \in \{1,\cdots,k\} \text{ such that } |\lambda_i - n_i^3| \geq C|m|^2\,. \tag{6.26}$$

For (6.24), (6.3) can be bounded by

$$\sum_{m+n_1+\cdots+n_k=n}\int \langle n_1\rangle^s|\widehat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)||\widehat{u}(n_1,\lambda_1)|\cdots|\widehat{u}(n_k,\lambda_k)||A_{n,\lambda}|d\lambda_1\cdots d\lambda_k d\lambda. \tag{6.27}$$

Let F_2 be defined by

$$F_2(x,t)=\sum_n\int |A_{n,\lambda}|e^{i\lambda t}e^{inx}d\lambda\,. \tag{6.28}$$

Then we represent (6.27) as

$$\sum_{m+n_1+\cdots+n_k=n}\int \widehat{F_2}(n,\lambda)\widehat{U}(m,\lambda-\lambda_1-\cdots-\lambda_k)\widehat{H}(n_1,\lambda_1)\prod_{j=2}^k\widehat{U}(n_j,\lambda_j)d\lambda_1\cdots d\lambda_k d\lambda\,. \tag{6.29}$$

Here H and U are functions defined in (6.10) and (6.11) respectively. Clearly (6.29) equals

$$\int_{\mathbb{T}\times\mathbb{R}}F_2(x,t)H(x,t)U(x,t)^k dxdt\,. \tag{6.30}$$

Utilizing Hölder inequality, we estimate it further by

$$\|F_2\|_2\|H\|_4\|U\|_{4k}^k\leq C\delta^\theta\|u\|_{Y_s}^{k+1}\,. \tag{6.31}$$

This yields the desired estimate for the subcase (6.24).

For the subcase of (6.25), (6.3) is estimated by

$$\sum_{m+n_1+\cdots+n_k=n}\int\frac{\langle n_1\rangle^s|A_{n,\lambda}|}{\langle\lambda-n^3\rangle^{\frac12}}\,\langle(\lambda-\lambda_1-\cdots-\lambda_k)-m^3\rangle^{\frac12}|\widehat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)|$$
$$\cdot|\widehat{u}(n_1,\lambda_1)|\cdots|\widehat{u}(n_k,\lambda_k)|d\lambda_1\cdots d\lambda_k d\lambda\,,$$

which is equal to

$$\int_{\mathbb{T}\times\mathbb{R}}F_1(x,t)G(x,t)H(x,t)U^{k-1}(x,t)dxdt\,. \tag{6.32}$$

Apply Hölder inequality to control (6.32) by

$$\|F_1\|_4\|G\|_4\|H\|_4\|U\|_{4(k-1)}^{k-1}\leq C\delta^\theta\|u\|_{Y_s}^{k+1}\,. \tag{6.33}$$

This completes the estimate for the subcase (6.25).

For the contribution of (6.26), we only consider $|\lambda_2-n_2^3|\geq C|m|^2$ without loss of generality for $i\in\{2,\cdots,k\}$. This is because the $|\lambda_1-n_1^3|\geq C|m|^2$ case can be handled similarly as (6.25). Hence, in this case, (6.3) can be bounded by

$$\sum_{m+n_1+\cdots+n_k=n}\int\frac{\langle n_1\rangle^s|A_{n,\lambda}|}{\langle\lambda-n^3\rangle^{\frac12}}\,\langle\lambda_2-n_2^3\rangle^{\frac12}|\widehat{u}(m,\lambda-\lambda_1-\cdots-\lambda_k)|\prod_{j=1}^k|\widehat{u}(n_j,\lambda_j)|d\lambda_1\cdots d\lambda_k d\lambda.$$

Now set a function I by

$$I(x,t)=\sum_n\int\langle\lambda-n^3\rangle^{\frac12}|\widehat{u}(n,\lambda)|e^{i\lambda t}e^{inx}d\lambda\,. \tag{6.34}$$

Then we estimate (6.3) by

$$\int_{\mathbb{T}\times\mathbb{R}} F_1(x,t)H(x,t)I(x,t)U^{k-1}(x,t)dxdt\,, \tag{6.35}$$

which is majorized by

$$\|F_1\|_4\|H\|_4\|I\|_2\|U\|_\infty^{k-1}\,. \tag{6.36}$$

Notice this time we cannot simply use Hölder's inequality to get δ as we did before because there is no way of making any above 4 or 2 even a little bit smaller. But this can be fixed as follows.

First observe that

$$\|u\|_{0,0} \leq \delta^{1/2}\|u\|_{L^2_xL^\infty_t} \leq C\delta^{1/2}\|u\|_{0,\frac{1}{2}+}\,,$$

for u is supported in a δ-sized interval in time variable. Thus by interpolation, we get

$$\|u\|_{0,\frac{1}{3}} \leq C\delta^{\frac{1}{6}-}\|u\|_{0,\frac{1}{2}}\,. \tag{6.37}$$

Since U can be assumed to be a function supported in a δ-sized time interval, we may put the same assumption to H. Henceforth, we have

$$\|H\|_4 \leq C\|H\|_{0,\frac{1}{3}} \leq C\delta^{\frac{1}{6}-}\|H\|_{0,\frac{1}{2}} \leq C\delta^{\frac{1}{6}-}\|u\|_{Y_s}\,. \tag{6.38}$$

Also note that

$$\|I\|_2 \leq \|u\|_{0,\frac{1}{2}} \leq \|u\|_{Y_s}\,. \tag{6.39}$$

and

$$\|U\|_\infty \leq C\|u\|_{Y_s}\,. \tag{6.40}$$

From (6.38), (6.39) and (6.40), we can estimate (6.3) by $C\delta^{\frac{1}{6}-}\|u\|_{Y_s}^{k+1}$ as desired. Therefore we finish our discussion for the case (6.5).

6.3. **Case (6.6).** The arithmetic observation (6.22) again plays an important role. In this case, let us further consider two subcases.

$$|m|^2 \leq 1000k^2|n_2|^2|n_3| \tag{6.41}$$

$$|m|^2 > 1000k^2|n_2|^2|n_3| \tag{6.42}$$

For the contribution of (6.41), we observe that from (6.41),

$$|m|^2 \leq C|n_1||n_2||n_3|\,,$$

since $|n_2| \leq |n_1|$. Henceforth we have

$$|m| = |m|^{\frac{1}{3}}|m|^{\frac{2}{3}} \leq C|m|^{\frac{1}{3}}|n_1|^{\frac{1}{3}}|n_2|^{\frac{1}{3}}|n_3|^{\frac{1}{3}}\,. \tag{6.43}$$

This implies immediately

$$\langle n\rangle^s|m| \leq C|m|^{s+1} \leq \langle m\rangle^{\frac{s+1}{3}}\langle n_1\rangle^{\frac{s+1}{3}}\langle n_2\rangle^{\frac{s+1}{3}}\langle n_3\rangle^{\frac{s+1}{3}}. \tag{6.44}$$

Introduce a new function H_1 defined by

$$H_1(x,t) = \sum_n\int_{\mathbb{R}}\langle n\rangle^{\frac{s+1}{3}}|\widehat{u}(n,\lambda)|e^{i\lambda t}e^{inx}d\lambda\,. \tag{6.45}$$

As before, in this case, we bound (6.3) by

$$\int_{\mathbb{T}\times\mathbb{R}} F_1(x,t)H_1^4(x,t)U^{k-3}(x,t)dxdt\,. \tag{6.46}$$

Then Hölder inequality yields

$$(6.3) \leq C\delta^\theta \|F_1\|_4 \|H_1\|_{6+}^4 \|U\|_{12(k-3)}^{k-3}\,. \tag{6.47}$$

$\|H_1\|_{6+} \leq C\|u\|_{Y_s}$ because $\frac{s+1}{3} < s$ for $s > 1/2$. Hence we obtain the desired estimate for the subcase (6.41).

We now turn to the contribution of (6.42). Clearly we have

$$|(n_2 + \cdots + n_k)^3 - (n_2^3 + \cdots + n_k^3)| \leq 10k|n_2|^2|n_3|\,, \tag{6.48}$$

since $|n_2| \geq |n_3| \geq \cdots \geq |n_k|$. From the crucial arithmetic observation (6.22), (6.48), and (6.42), we have

$$\left|n^3 - \left(m^3 + n_1^3 + \cdots + n_k^3\right)\right| \geq Ck|m|^2\,. \tag{6.49}$$

This is same as (6.23). Hence again we reduce the problems to (6.24), (6.25), and (6.26), which are all done in Subsection 6.2. Therefore we finish the case of (6.6).

Putting all cases together, we obation

$$\|w\|_{s,-\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}\,. \tag{6.50}$$

Finally we need to estimate

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle} d\lambda\right)^2\right)^{\frac{1}{2}}. \tag{6.51}$$

Let $\{A_n\}$ be a sequence $\{A_n\}$ with $\left(\sum_n |A_n|^2\right)^{\frac{1}{2}} \leq 1$. By duality, it suffices to estimate

$$\sum_{\substack{m+n_1+\cdots+n_k=n\\ n_1+\cdots+n_k\neq 0}} \int \frac{\langle n\rangle^s |m|}{\langle \lambda - n^3\rangle} |\widehat{u}(m, \lambda - \lambda_1 - \cdots - \lambda_k)||\widehat{u}(n_1,\lambda_1)|\cdots|\widehat{u}(n_k,\lambda_k)||A_n| d\lambda_1\cdots d\lambda_k d\lambda\,. \tag{6.52}$$

Again, without loss of generality, we can assume $|n_1| \geq \cdots \geq |n_k|$. We still go through the cases used previously. Almost all cases are similar and there are only two exceptions. In fact, we only need to replace F_1 by F_3 in each case where $\|F_1\|_4$ is employed. Here F_3 is given by

$$\sum_n \int_{\mathbb{R}} \frac{|A_n|}{\langle \lambda - n^3\rangle} e^{i\lambda t} e^{inx} d\lambda\,. \tag{6.53}$$

Then all those cases can be done because

$$\|F_3\|_4 \leq C\|F_3\|_{0,\frac{1}{3}} = \left(\sum_n |A_n|^2 \int \frac{1}{\langle \lambda - n^3\rangle^{\frac{4}{3}}} d\lambda\right)^{\frac{1}{2}} \leq C\,. \tag{6.54}$$

The only exceptions are

$$|\lambda - n^3| \geq C|n_1||m| \;\text{ and }\; |n_2| \ll |m| \leq C|n_1| \tag{6.55}$$

$$|\lambda - n^3| \geq Cm^2 \;\text{ and }\; |m| \gg |n_1| \tag{6.56}$$

For the case of (6.55), we define

$$F_4(x,t) = \sum_n \int_{\mathbb{R}} \frac{\langle n\rangle^{\frac{1}{2}} \mathbf{1}_{\{|\lambda - n^3| \geq C\langle n\rangle\}}}{|\lambda - n^3|} e^{i\lambda t} e^{inx} d\lambda \tag{6.57}$$

A direct calculation gives

$$\|F_4\|_2 \leq \left(\sum_n \int_{|\lambda-n^3|\geq C\langle n\rangle} \frac{\langle n\rangle |A_n|^2}{|\lambda-n^3|^2} d\lambda\right)^{1/2} \leq C\,. \tag{6.58}$$

In this case, clearly

$$\langle n\rangle^s |m| \leq \langle n\rangle^{\frac{1}{2}} \langle n_1\rangle^s \langle m\rangle^{\frac{1}{2}}\,. \tag{6.59}$$

Then (6.52) is dominated by

$$\int_{\mathbb{T}\times\mathbb{R}} F_4(x,t)G(x,t)H(x,t)U^{k-1}(x,t)dxdt\,. \tag{6.60}$$

By a use of Hölder inequality and (6.58), one gets

$$(6.52) \leq C\|F_4\|_2 \|H\|_4 \|G\|_6 \|U\|_{12(k-1)}^{k-1} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}\,. \tag{6.61}$$

This finishes the proof for the case (6.55).

For the contribution of (6.56), we set

$$F_5(x,t) = \sum_n \int_{\mathbb{R}} \frac{\langle n\rangle \mathbf{1}_{\{|\lambda-n^3|\geq C\langle n\rangle^2\}}}{|\lambda-n^3|} e^{i\lambda t} e^{inx} d\lambda\,. \tag{6.62}$$

Clearly

$$\|F_5\|_2 \leq \left(\sum_n \int_{|\lambda-n^3|\geq C\langle n\rangle^2} \frac{\langle n\rangle^2 |A_n|^2}{|\lambda-n^3|^2} d\lambda\right)^{1/2} \leq C\,. \tag{6.63}$$

In this case, we have $|\lambda - n^3| \geq C\langle n\rangle^2$ since $|n| \sim |m|$, henceforth, by the observation of

$$\langle n\rangle^s |m| \leq C\langle m\rangle^s \langle n\rangle\,,$$

we estimate (6.52) by

$$\int_{\mathbb{T}\times\mathbb{R}} F_5(x,t)H(x,t)U^k(x,t)dxdt\,. \tag{6.64}$$

Using Hölder inequality and (6.63), we have

$$(6.52) \leq C\|F_5\|_2 \|H\|_4 \|U\|_{4k}^{4k} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}\,, \tag{6.65}$$

as desired. Hence

$$\left(\sum_n \langle n\rangle^{2s} \left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle} d\lambda\right)^2\right)^{\frac{1}{2}} \leq C\delta^\theta \|u\|_{Y_s}^{k+1}. \tag{6.66}$$

Therefore we complete the proof of Proposition 5.1 by combining (6.50) and (6.66).

7. Proof of Theorem 1.3

The argument is similar to those in Section 5. By using a gauge transform as in (5.8) with v^k replaced by $F(v)$, the well-posedness of (1.7) is equivalent to the well-posedness of the following equation:

$$\begin{cases} u_t + u_{xxx} + \left(F(u) - \fint_{\mathbb{T}} F(u)dx\right) u_x = 0 \\ u(x,0) = \phi(x), \qquad x \in \mathbb{T},\ t \in \mathbb{R}\,. \end{cases} \tag{7.1}$$

Now the nonlinear function w is defined by

$$w = \partial_x u \left(F(u) - \int_{\mathbb{T}} F(u)dx \right). \tag{7.2}$$

Let T_F be an operator given by

$$T_F u(x,t) := \psi_\delta(t) e^{-t\partial_x^3}\phi(x) - \psi_\delta(t)\int_0^t e^{-(t-\tau)\partial_x^3} w(x,\tau)d\tau. \tag{7.3}$$

As in Section 5, the local well-posedness relies on the following proposition.

Proposition 7.1. *Let $s > 1/2$. There exists $\theta > 0$ such that, for the nonlinear function w given by (7.2) and any u satisfying $\|u\|_{Y_s} \leq C_0\|\phi\|_{H^s}$,*

$$\|w\|_{s,-\frac{1}{2}} + \left(\sum_n \langle n\rangle^{2s} \left(\int \frac{|\widehat{w}(n,\lambda)|}{\langle \lambda - n^3\rangle} d\lambda \right)^2 \right)^{\frac{1}{2}} \leq C(\|\phi\|_{H^s}, F)\delta^\theta \|u\|_{Y_s}^4, \tag{7.4}$$

provided $F \in C^5$. Here C_0 is a suitably large constant, and $C(\|\phi\|_{H^s}, F)$ is a constant independent of δ and u, but may depend on $\|\phi\|_{H^s}$ and F.

The constant $C(\|\Phi\|_{H^s}, F)$ will be specified in the proof of Proposition 7.1. We postpone the proof of Proposition 7.1 to Section 8, and return to the proof of Theorem 1.3. Proposition 7.1 implies that for δ sufficiently small, T_F maps a ball $\{u \in Y_s : \|u\|_{Y_s} \leq C_0\|\phi\|_{H^s}\}$ into itself. Moreover, using Lemma 5.2 and repeating similar argument as in the proof of Proposition 7.1, one obtains, for $s > 1/2$ and $F \in C^5$,

$$\|T_F u - T_F v\|_{Y_s} \leq \delta^\theta C(\|\phi\|_{H^s}, F)\|u - v\|_{Y_s}\,. \tag{7.5}$$

for all u, v in the ball $\{u \in Y_s : \|u\|_{Y_s} \leq C_0\|\phi\|_{H^s}\}$. Therefore, for $\delta > 0$ small enough, T_F is a contraction on the ball and the local well-posedness again follows from Picard's fixed-point theorem. This completes the proof of Theorem 1.3.

8. Proof of Proposition 7.1

First we introduce a decomposition of $F(u)$, which was used by Bourgain. Let K be a dyadic number, and define a Fourier multiplier operator P_K by setting

$$P_K u(x,t) = \int \psi_K(y) u(x-y,t)dy\,. \tag{8.1}$$

Here the Fourier transform of ψ_K is a standard bump function supported on $[-2K, 2K]$ and $\widehat{\psi_K}(x) = 1$ for $x \in [-K, K]$. Let u_K denote the Littlewood-Paley Fourier multiplier, that is,

$$u_K = P_K u - P_{K/2} u\,. \tag{8.2}$$

Then we may decompose $F(u)$ by

$$\begin{aligned} F(u) &= \sum_K \big(F(P_K u) - F(P_{K/2}u)\big) \\ &= \sum_K F_1(P_K u, P_{K/2}u)u_K + R_1\,, \end{aligned}$$

where R_1 is a function independent of the space variable x. Repeating this procedure for F_1, we obtain

$$\begin{aligned} F(u) &= \sum_{K_1\geq K_2} F_2(P_{2K_2}u,\cdots,P_{K_2/4}u)u_{K_1}u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \\ &= \sum_{K_1\geq K_2\geq K_3} F_3(P_{4K_3}u,\cdots,P_{K_3/8}u)u_{K_1}u_{K_2}u_{K_3} \\ &\quad + \sum_{K_1\geq K_2} R_3 u_{K_1}u_{K_2} + \sum_{K_1} R_2 u_{K_1} + R_1 \end{aligned}$$

where R_1, R_2, R_3 are functions independent of the space variable.
Set

$$G_{K_3}(x,t) = F_3(P_{4K_3}u,\cdots,P_{K_3/8}u)\,. \tag{8.3}$$

Hence we represent w defined in (7.2) as

$$\begin{aligned} w &= \sum_{K_0,K_1\geq K_2\geq K_3} \partial_x u_{K_0}\left(u_{K_1}u_{K_2}u_{K_3}G_{K_3} - \int_{\mathbb{T}} u_{K_1}u_{K_2}u_{K_3}G_{K_3}dx\right) \\ &\quad + \sum_{K_0,K_1\geq K_2} \partial_x u_{K_0}\left(u_{K_1}u_{K_2} - \int_{\mathbb{T}} u_{K_1}u_{K_2}dx\right)R_3 \\ &\quad + \sum_{K_0,K_1} \partial_x u_{K_0}\left(u_{K_1} - \int_{\mathbb{T}} u_{K_1}dx\right)R_2\,. \end{aligned}$$

The main contribution of w is from the first term. The remaining terms can be handled by the method presented in Section 6 because R_2, R_3 are functions independent of the space variable x (actually they only depend on the conserved quantity $\int_{\mathbb{T}} udx$). Hence in what follows we will only focus on estimating the first term–the most difficult one. Denote the first term by w_1, i.e.,

$$w_1 = \sum_{K_0,K_1\geq K_2\geq K_3} \partial_x u_{K_0}\left(u_{K_1}u_{K_2}u_{K_3}G_{K_3} - \int_{\mathbb{T}} u_{K_1}u_{K_2}u_{K_3}G_{K_3}dx\right)\,. \tag{8.4}$$

We should prove

$$\|w_1\|_{s,-\frac{1}{2}} + \left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}_1(n,\lambda)|}{\langle\lambda-n^3\rangle}d\lambda\right)^2\right)^{1/2} \leq \delta^\theta C(\|\phi\|_{H^s},F)\|u\|_{Y_s}^4\,. \tag{8.5}$$

In order to specify the constant $C(\|\phi\|_{H^s},F)$, we define $\mathfrak{M}$ by setting
(8.6)

$$\mathfrak{M} = \sup\left\{|D^\alpha F_3(u_1,\cdots,u_6)| : u_j \text{ satisfies } \|u_j\|_{Y_s}\leq C_0\|\phi\|_{H^s} \text{ for all } j=1,\cdots,6;\;\alpha\right\}\,.$$

Here $D^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_6}^{\alpha_6}$ and α is taken over all tuples $(\alpha_1,\cdots,\alpha_6)\in(\mathbb{N}\cup\{0\})^6$ with $0\leq\alpha_j\leq 2$ for all $j\in\{1,\cdots,6\}$. $\mathfrak{M}$ is a real number. This is because, for $s>1/2$, $\|u\|_{Y_s}\leq 2\|\phi\|_{H^s}$

yields that u is bounded by $C\|\phi\|_{H^s}$, and the previous claim follows from $F_3 \in C^2$.

In order to bound $\|w_1\|_{s,-\frac{1}{2}}$, by duality, it suffices to bound

$$\sum_{\substack{K_0,K_1\geq K_2\geq K_3\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \frac{A_{n,\lambda}\langle n\rangle^s n_0}{\langle \lambda - n^3\rangle^{\frac{1}{2}}} \widehat{u}_{K_0}(n_0, \lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \cdot \prod_{j=1}^{3} \widehat{u}_{K_j}(n_j,\lambda_j)\widehat{G_{K_3}}(m,\mu)d\lambda_1\cdots d\lambda_4 d\lambda d\mu\,, \tag{8.7}$$

where $A_{n,\lambda}$ satisfies

$$\sum_n \int |A_{n,\lambda}|^2 d\lambda = 1\,.$$

The trouble maker is G_{K_3} since there is no way to find a suitable upper bound for its $X_{s,b}$ norm. Because of this, the method in Section 6 is no more valid, and we have to treat m and μ differently from n and λ respectively. A delicate analysis must be done for overcoming the difficulty caused by G_{K_3}. For simplicity, we assume that $\delta = 1$. One can modify the argument to gain a decay of δ^θ by using the technical treatment from Section 6.

For a dyadic number M, define the Littlewood-Paley Fourier multiplier by

$$g_{K_3,M} = P_M G_{K_3} - P_{M/2}G_{K_3} = (G_{K_3})_M\,. \tag{8.8}$$

Let v be defined by

$$v(x,t) = \sum_n \int \frac{A_{n,\lambda}}{\langle\lambda - n^3\rangle^{\frac{1}{2}}} e^{i\lambda t}e^{inx}d\lambda\,. \tag{8.9}$$

To estimate (8.7), it suffices to estimate

$$\sum_{\substack{K,K_0,K_1\geq K_2\geq K_3,M\\ n_0+n_1+n_2+n_3+m=n\\ n_1+n_2+n_3+m\neq 0}} \int \widehat{\partial_x^s v_K}(n,\lambda)\widehat{\partial_x u_{K_0}}(n_0,\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) \prod_{j=1}^{3}\widehat{u}_{K_j}(n_j,\lambda_j)\widehat{g_{K_3,M}}(m,\mu)d\lambda_1\cdots d\lambda_4 d\lambda d\mu\,. \tag{8.10}$$

Here K is a dyadic number.

As we did in Section 6, we consider three cases:

$$K_0 < 2^{100}K_2\,; \tag{8.11}$$

$$2^{100}K_2 \leq K_0 \leq 2^{10}K_1\,; \tag{8.12}$$

$$K_0 > 2^{10}K_1\,. \tag{8.13}$$

The rest part of the paper is devoted to a proof of these three cases. In what follows, we will only provide the details for the estimates of $\|w_1\|_{s,-\frac{1}{2}}$ with $1/2 < s < 1$ (the case $s \geq 1$ is easier). For the desired estimate of

$$\left(\sum_n \langle n\rangle^{2s}\left(\int \frac{|\widehat{w}_1(n,\lambda)|}{\langle\lambda - n^3\rangle}d\lambda\right)^2\right)^{1/2}\,,$$

simply replace v by

$$v_1(x,t)=\sum_n\int\frac{C_{n,\lambda}A_n}{\langle\lambda-n^3\rangle}e^{i\lambda t}e^{inx}d\lambda\,, \tag{8.14}$$

and then the desired estimate follows similarly. Here $C_{n,\lambda}\in\mathbb{C}$ satisfies $\sup_\lambda|C_{n,\lambda}|\leq 1$ and $\{A_n\}$ satisfies $\sum_n|A_n|^2\leq 1$.

9. Proof of Case (8.11)

In this case, we should consider further two subcases:

$$M\leq 2^{10}K_1\,. \tag{9.1}$$

$$M>2^{10}K_1\,. \tag{9.2}$$

For the contribution of (9.1), noticing $K\leq CK_1$ in this subcase, we then estimate (8.10) by

$$\sum_{K_1\geq K_2\geq K_3}\int_{\mathbb{T}\times\mathbb{R}}\left|\left(\sum_{K\leq CK_1}\partial_x^s v_K\right)\left(\sum_{K_0\leq CK_2}\partial_x u_{K_0}\right)u_{K_1}u_{K_2}u_{K_3}\left(P_{2^{10}K_1}G_{K_3}\right)\right|dxdt\,, \tag{9.3}$$

which is bounded by

$$\sum_{K_3}\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\int_{\mathbb{T}\times\mathbb{R}}\sum_{K_1}\sum_{K\leq CK_1}K^s v_K^*|u_{K_1}|\sum_{K_2}\sum_{K_0\leq CK_2}K_0u_{K_0}^*|u_{K_2}|dxdt\,, \tag{9.4}$$

where f^* stands for the Hardy-Littlewood maximal function of f. By the Schür test, (9.4) can be estimated by

$$\begin{aligned}&\sum_{K_3}K_3^{-\frac{2s-1}{2}}\|u\|_{Y_s}\mathfrak{M}\int\left(\sum_K|v_K^*|^2\right)^{\frac12}\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{\frac12}\\&\qquad\cdot\left(\sum_{K_0}K_0|u_{K_0}^*|^2\right)^{\frac12}\left(\sum_{K_2}K_2|u_{K_2}|^2\right)^{\frac12}dxdt\,.\end{aligned} \tag{9.5}$$

Since $s>1/2$, we then obtain, by a use of Hölder inequality, that (9.4) is majorized by

$$\begin{aligned}&C\mathfrak{M}\|u\|_{Y_s}\left\|\left(\sum_K|v_K^*|^2\right)^{\frac12}\right\|_4\left\|\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{\frac12}\right\|_4\\&\qquad\left\|\left(\sum_{K_0}K_0|u_{K_0}^*|^2\right)^{\frac12}\right\|_4\left\|\left(\sum_{K_2}K_2|u_{K_2}|^2\right)^{\frac12}\right\|_4\,.\end{aligned} \tag{9.6}$$

Observe that

$$\left\|\left(\sum_K|v_K^*|^2\right)^{\frac12}\right\|_4\leq\left\|\left(\sum_K|v_K|^2\right)^{\frac12}\right\|_4\leq C\|v\|_4\leq C\|v\|_{0,\frac13}\leq C\,. \tag{9.7}$$

Here the first inequality is obtained by using Fefferman-Stein's vector-valued inequality on the maximal function, and the second one is a consequence of classical Littlewood-Paley theorem. Similarly,

$$\left\| \left(\sum_{K_0} K_0 |u^*_{K_0}|^2 \right)^{\frac{1}{2}} \right\|_4 \leq \left\| \left(\sum_{K_0} K_0 |u_{K_0}|^2 \right)^{\frac{1}{2}} \right\|_4 \leq C \|\partial_x^{1/2} u\|_4 \leq C\|u\|_{\frac{1}{2},\frac{1}{3}} \leq C\|u\|_{Y_s} , \tag{9.8}$$

and

$$\left\| \left(\sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{\frac{1}{2}} \right\|_4 \leq C \|\partial_x^s u\|_4 \leq C\|u\|_{s,\frac{1}{3}} \leq C\|u\|_{Y_s} . \tag{9.9}$$

Hence from (9.7), (9.8) and (9.9), we have

$$(8.10) \leq C\mathfrak{M}\|u\|^4_{Y_s} . \tag{9.10}$$

For the contribution of (9.2), since in this subcase $K \leq CM$, we estimate (8.10) by

$$\sum_{K_1} \|u_{K_1}\|_\infty \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3\leq K_1} |u_{K_3}| \sum_M \sum_{K\leq CM} K^s v^*_K |g_{K_3,M}| \sum_{K_2} \sum_{K_0 \leq CK_2} K_0 u^*_{K_0} |u_{K_2}| dxdt , \tag{9.11}$$

which is bounded by

$$\begin{split} \sum_{K_1} K_1^{-\frac{2s-1}{2}} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \sum_{K_3\leq K_1} |u_{K_3}| \left(\sum_K |v^*_K|^2 \right)^{1/2} \left(\sum_M M^{2s} |g_{K_3,M}|^2 \right)^{1/2} \\ \left(\sum_{K_0} K_0 |u^*_{K_0}|^2 \right)^{1/2} \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} dxdt . \end{split} \tag{9.12}$$

By a use of Cauchy-Schwarz inequality, (9.12) is estimated by

$$\begin{split} \sum_{K_1} K_1^{-\frac{2s-1}{2}} \|u\|_{Y_s} \int_{\mathbb{T}\times\mathbb{R}} \left(\sum_K |v^*_K|^2 \right)^{1/2} \left(\sum_{K_0} K_0 |u^*_{K_0}|^2 \right)^{1/2} \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \\ \left(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \right)^{1/2} \left(\sum_{K_3\leq K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \right)^{1/2} dxdt . \end{split} \tag{9.13}$$

Using Hölder inequality, we then bound it further by

$$\begin{split} \sum_{K_1} K_1^{-\frac{2s-1}{2}} \|u\|_{Y_s} \left\| \left(\sum_K |v^*_K|^2 \right)^{1/2} \right\|_4 \left\| \left(\sum_{K_2} K_0 |u^*_{K_0}|^2 \right)^{1/2} \right\|_6 \left\| \left(\sum_{K_2} K_2 |u_{K_2}|^2 \right)^{1/2} \right\|_6 \\ \left\| \left(\sum_{K_3} K_3^{2s} |u_{K_3}|^2 \right)^{1/2} \right\|_4 \left\| \left(\sum_{K_3\leq K_1} \sum_M \frac{M^{2s}}{K_3^{2s}} |g_{K_3,M}|^2 \right)^{1/2} \right\|_6 , \end{split} \tag{9.14}$$

which is majorized by

$$\sum_{K_1} K_1^{-\frac{2s-1}{2}} \|u\|_{Y_s}^4 \sum_{K_3\leq K_1} K_3^{-s} \left\| \left(\sum_M M^{2s} |g_{K_3,M}|^2 \right)^{1/2} \right\|_6$$
$$\leq \sum_{K_1} K_1^{-\frac{2s-1}{2}} \|u\|_{Y_s}^4 \sum_{K_3\leq K_1} K_3^{-s} \|\partial_x^s G_{K_3}\|_\infty \,.$$

From the definition of G_{K_3} , we have

$$\partial_x G_{K_3}(x,t) = \mathrm{O}\left(\mathfrak{M} K_3\right) \|u\|_{Y_s} = \mathrm{O}\left(\mathfrak{M} K_3\right) \|\phi\|_{H^s} \,. \tag{9.15}$$

Hence, for $s < 1$,

$$\|\partial_x^s G_{K_3}\|_\infty \leq C\mathfrak{M} K_3^s \|\phi\|_{H^s} \,. \tag{9.16}$$

Since $s > 1/2$, we then have

$$(9.14) \leq C\mathfrak{M} \|\phi\|_{H^s} \sum_{K_1} K_1^{-\frac{2s-1}{2}+\varepsilon} \|u\|_{Y_s}^4 \leq C\mathfrak{M} \|\phi\|_{H^s} \|u\|_{Y_s}^4 \,. \tag{9.17}$$

This completes our discussion on Case (8.11).

10. Proof of Case (8.12)

In this case, it suffices to consider the following subcases:

$$K \leq 2^{10} K_2 \,; \tag{10.1}$$

$$K \leq 2^{10} M \,; \tag{10.2}$$

$$K > 2^9 (K_2 + M) \ \text{ and } \ K_3 \geq K_0^{1/2}; \tag{10.3}$$

$$K > 2^9 (K_2 + M) \,, \ \ K_3 \leq K_0^{1/2} \ \text{ and } \ M \geq 2^{-10} K_0^{2/3}; \tag{10.4}$$

$$K > 2^9 (K_2 + M) \,, \ \ K_3 \leq K_0^{1/2} \ \text{ and } \ M < 2^{-10} K_0^{2/3} \,. \tag{10.5}$$

(10.1) and (10.2) can be proved exactly the same as the case (9.1) and the case (9.2) respectively. We omit the details.

For the case of (10.3), observe that (8.12) and (10.3) imply

$$K \leq C K_1 \tag{10.6}$$

and

$$K_0^{1/2} \leq K_2^{1/2} K_3^{1/2} \,. \tag{10.7}$$

Hence (8.10) is bounded by

$$\int \sum_{K_1} \sum_{K\leq CK_1} K^s v_K^* |u_{K_1}| \sum_{\substack{K_0\geq K_2\geq K_3 \\ K_0\leq K_3^2}} K_0 u_{K_0}^* |u_{K_2}| |u_{K_3}| \|G_{K_3}\|_\infty dxdt \,. \tag{10.8}$$

Applying Hölder inequality, we estimate (10.8) by

$$C\mathfrak{M} \int \left(\sum_K |v_K^*|^2 \right)^{\frac12} \left(\sum_{K_1} K_1^{2s} |u_{K_1}|^2 \right)^{\frac12} \prod_{j=0,2,3} \left(\sum_{K_j} K_j^{1+\varepsilon} |u_{K_j}|^2 \right)^{\frac12} dxdt \,. \tag{10.9}$$

One more use of Hölder inequality yields that (10.8) is bounded by

$$C\mathfrak{M}\left\|\left(\sum_K |v_K|^2\right)^{\frac12}\right\|_4 \left\|\left(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\right)^{\frac12}\right\|_4 \prod_{j=0,2,3}\left\|\left(\sum_{K_j} K_j^{1+\varepsilon}|u_{K_j}|^2\right)^{\frac12}\right\|_6 .$$

Hence we obtain

$$(10.8) \leq C\mathfrak{M}\|u\|_{Y_s}^4 . \tag{10.10}$$

This finishes the proof of (10.3).

For the case of (10.4), we estimate (8.10) by

$$\sum_{K_2,K_3}\int \sum_{K_1}\sum_{K\leq CK_1} K^s v_K^* |u_{K_1}| \sum_{K_0} K_0 |u_{K_0}^*||u_{K_2}||u_{K_3}| \sum_{M\geq CK_0^{2/3}} |g_{K_3,M}|\, dxdt\,, \tag{10.11}$$

which is dominated by

$$\begin{aligned} &C\sum_{K_2,K_3}\int \left(\sum_K |v_K^*|^2\right)^{1/2}\left(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\right)^{1/2} |u_{K_2}||u_{K_3}| \\ &\qquad \left(\sum_{K_0} K_0 |u_{K_0}^*|^2\right)^{1/2}\left(\sum_M M^{3/2}|g_{K_3,M}|^2\right)^{1/2} dxdt\,. \end{aligned} \tag{10.12}$$

By Hölder inequality with L^4 norms for the first two functions in the integrand, L^{6+} for the next three functions, and L^p norm (very large p) for the last one, (10.12) is dominated by

$$C\|u\|_{Y_s}\sum_{K_2,K_3}\|u_{K_2}\|_{6+}\|u_{K_3}\|_{6+}\left\|\left(\sum_{K_0} K_0|u_{K_0}^*|^2\right)^{1/2}\right\|_{6+}\|\partial_x^{3/4}G_{K_3}\|_\infty\,. \tag{10.13}$$

Applying (9.16), we estimate (10.12) by

$$\begin{aligned} & C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2 \prod_{j=2}^3\sum_{K_j} K_j^{3/8}\|u_{K_j}\|_{6+} \\ \leq\ & C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^2 \prod_{j=2}^3\sum_{K_j} K_j^{3/8}\|u_{K_j}\|_{0+,\frac12} \\ \leq\ & C\mathfrak{M}\|\phi\|_{H_s}\|u\|_{Y_s}^4\,, \end{aligned}$$

as desired. This completes the discussion of (10.4).

We now turn to the case (10.5). In this case, we have

$$|n_0+n_1| + 2K_2 + M \geq |n| \geq K/2 \geq 2^8(K_2+M)\,, \tag{10.14}$$

which implies

$$|n_0+n_1| \geq 2^5(K_2+M)\,. \tag{10.15}$$

Notice that
(10.16)
$$(n_0+n_1+n_2+n_3+m)^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3 =$$
$$3(n_0+n_1)(n_0+n_2+n_3+m)(n_1+n_2+n_3+m) + (n_2+n_3+m)^3 - n_2^3 - n_3^3 - m^3\,.$$

From (10.15), (10.16) and (10.5), we obtain

$$\left|n^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3\right| \geq C(K_2+M)K_0K_1 \geq CK_0K_1 \geq CK_0^2\,. \tag{10.17}$$

Henceforth one of the following four statements must be true:

$$\left|\lambda - n^3\right| \geq K_0^2\,, \tag{10.18}$$

$$\left|(\lambda-\lambda_1-\lambda_2-\lambda_3-\mu) - n_0^3\right| \geq K_0^2\,, \tag{10.19}$$

$$\exists i \in \{1,2,3\} \text{ such that } |\lambda_i - n_i^3| \geq K_0^2\,, \tag{10.20}$$

$$|\mu| \geq K_0^2\,. \tag{10.21}$$

For the case of (10.18), we set

$$\tilde{v}(x,t) = \left(\widehat{v}\mathbf{1}_{|\lambda-n^3|\geq K_0^2}\right)^{\vee}(x,t)\,. \tag{10.22}$$

We then estimate (8.10) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int |\partial_x u_{K_0}| \sum_{K_1} \sum_{K\leq CK_1} K^s \tilde{v}_K^* |u_{K_1}| dxdt\,. \tag{10.23}$$

This is clearly bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \sum_{K_0} \int K_0 |u_{K_0}^*| \left(\sum_K |\tilde{v}_K^*|^2\right)^{1/2} \left(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\right)^{1/2} dxdt\,. \tag{10.24}$$

Using Cauchy-Schwarz inequality, we bound (10.24) by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \int \left(\sum_{K_0} K_0^{\varepsilon}|u_{K_0}^*|^2\right)^{\frac12} \left(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde{v}_K^*|^2\right)^{\frac12} \left(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\right)^{\frac12} dxdt\,. \tag{10.25}$$

By Hölder inequality, (10.25) is majorized by

$$C\mathfrak{M}\|u\|_{Y_s}^2 \left\|\left(\sum_{K_0} K_0^{\varepsilon}|u_{K_0}^*|^2\right)^{\frac12}\right\|_4 \left\|\left(\sum_{K_0} K_0^{2-\varepsilon} \sum_K |\tilde{v}_K^*|^2\right)^{\frac12}\right\|_2 \left\|\left(\sum_{K_1} K_1^{2s}|u_{K_1}|^2\right)^{\frac12}\right\|_4\,,$$

which is controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^{\varepsilon} u\|_4 \left(\sum_{K_0} K_0^{2-\varepsilon}\|\tilde{v}\|_2^2\right)^{1/2} \leq C\mathfrak{M}\|u\|_{Y_s}^3 \|\partial_x^{\varepsilon} u\|_4 \sum_{K_0} K_0^{-\varepsilon/2} \leq C\mathfrak{M}\|u\|_{Y_s}^4\,. \tag{10.26}$$

This finishes the proof of the case (10.18).

For the case of (10.19), let $\tilde{u}$ be defined by

$$\tilde{u} = (\widehat{u}\mathbf{1}_{|\lambda-n^3|\geq K_0^2})^{\vee}\,. \tag{10.27}$$

Then (8.10) can be estimated by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_0}\int|\partial_x\tilde u_{K_0}|\sum_{K_1}\sum_{K\le CK_1}K^sv_K^*|u_{K_1}|dxdt\,. \tag{10.28}$$

By Schür test and Hölder inequality, we control (10.28) by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_0}\|\partial_x\tilde u_{K_0}\|_2\left\|\left(\sum_K|v_K|^2\right)^{1/2}\right\|_4\left\|\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)\right\|_4, \tag{10.29}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s}^3\sum_{K_0}\|u_{K_0}\|_{0,\frac12}\le C\mathfrak{M}\|u\|_{Y_s}^4\,. \tag{10.30}$$

This completes the proof of the case (10.19).

For the case of (10.20), if $j=1$, then we dominate (8.10) by

$$\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_0}\int|\partial_x u_{K_0}|\sum_{K_1}\sum_{K\le CK_1}K^sv_K^*|\tilde u_{K_1}|dxdt\,. \tag{10.31}$$

As we did in the case (10.19), we bound (10.31) by

$$C\mathfrak{M}\|u\|_{Y_s}^2\sum_{K_0}\|\partial_x u_{K_0}\|_4\|v\|_4\left\|\left(\sum_{K_1}K_1^{2s}|\tilde u_{K_1}|^2\right)^{1/2}\right\|_2. \tag{10.32}$$

This can be futher controlled by

$$C\mathfrak{M}\|u\|_{Y_s}^3\sum_{K_0}\frac{1}{K_0}\|\partial_x u_{K_0}\|_4\|v\|_4\le C\mathfrak{M}\|u\|_{Y_s}^3\sum_{K_0}\frac{1}{K_0}\|u_{K_0}\|_{1,\frac13}\le C\mathfrak{M}\|u\|_{Y_s}^4\,, \tag{10.33}$$

as desired.

We now consider $j=2$ or $j=3$. Without loss of generality, assume $j=2$. In this case, we estimate (8.10) by

$$\sum_{K_3}\|u_{K_3}\|\|G_{K_3}\|_\infty\sum_{K_0}\int|\partial_x u_{K_0}|\sum_{K_1}\sum_{K\le CK_1}K^sv_K^*|u_{K_1}|\sum_{K_2\le CK_0}|\tilde u_{K_2}|dxdt\,, \tag{10.34}$$

which is bounded by

$$C\mathfrak{M}\|u\|_{Y_s}\sum_{K_0}\|\partial_x u_{K_0}\|_\infty\sum_{K_2\le K_0}\|\tilde u_{K_2}\|_2\|v\|_4\left\|\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{1/2}\right\|_4.$$

Notice that

$$\begin{aligned}\sum_{K_0}\|\partial_x u_{K_0}\|_\infty \sum_{K_2\leq K_0}\|\tilde{u}_{K_2}\|_2 &\leq C\sum_{K_0}\frac{1}{K_0}\|\partial_x u_{K_0}\|_\infty\|u\|_{Y_s}\\ &\leq C\sum_n\int|\widehat{u}(n,\lambda)|d\lambda\|u\|_{Y_s}\\ &\leq C\|u\|_{Y_s}^2\,.\end{aligned}$$

Henceforth (10.34) is dominated by

$$(10.34)\leq C\mathfrak{M}\|u\|_{Y_s}^4\,. \tag{10.35}$$

This completes the case of (10.20).

We now turn to the most difficult case (10.21) in Case (8.12). We should decompose G_{K_3}, with respect to the t-variable, into Littlewood-Paley multipliers in the same spirit as before. More precisely, for any dyadic number L, let Q_L be

$$Q_L u(x,t)=\int\psi_L(\tau)u(x,t-\tau)d\tau\,. \tag{10.36}$$

Here the Fourier transform of ψ_L is a bump function supported on $[-2L,2L]$ and $\widehat{\psi_L}(x)=1$ if $x\in[-L,L]$. Let

$$\Pi_L u=Q_L u-Q_{L/2}u\,. \tag{10.37}$$

Then $\Pi_L u$ gives a Littlewood-Paley multiplier with respect to the time variable t. Using this multiplier, we represent

$$u_K=\sum_L u_{K,L}\,. \tag{10.38}$$

Here $u_{K,L}=\Pi_L(u_K)$. We decompose G_{K_3} as

(10.39)

$$\begin{aligned}G_{K_3}&=C+\sum_L\big(F_3(Q_LP_{4K_3}u,\cdots,Q_LP_{K_3/8}u)-F_3(Q_{L/2}P_{4K_3}u,\cdots,Q_{L/2}P_{K_3/8}u)\big)\\ &=C+\sum_{\substack{j=4,2,1,\frac12,\frac14,\frac18\\ L}}H_{K_3,L}\,u_{jK_3,L}\,,\end{aligned}$$

where $H_{K_3,L}$ is given by

$$H_{K_3,L}=F_4\left(Q_{\ell L}P_{4K_3}u,\cdots,Q_{\ell L}P_{K_3/8}u;\ell=1,\frac12\right)\,. \tag{10.40}$$

Let $\mathfrak{M}_1$ be defined by

(10.41)

$$\mathfrak{M}_1=\sup\left\{|D^\alpha F_4(u_1,\cdots,u_{12})|: u_j \text{ satisfies } \|u_j\|_{Y_s}\leq C_0\|\phi\|_{H^s} \text{ for all } j=1,\cdots,12;\ \alpha\right\}\,.$$

Here $D^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_{12}}^{\alpha_{12}}$ and α is taken over all tuples $(\alpha_1,\cdots,\alpha_{12})\in(\mathbb{N}\cup\{0\})^{12}$ with $0\leq\alpha_j\leq1$ for all $j\in\{1,\cdots,12\}$. $\mathfrak{M}_1$ is a real number because $F_4\in C^1$.

In order to finish the proof, we need to consider further three subcases:

$$L \le 2^{10}K_3^3\,, \tag{10.42}$$

$$2^{10}K_3^3 < L \le 2^{-5}K_0^2\,, \tag{10.43}$$

$$L > 2^{-5}K_0^2\,. \tag{10.44}$$

For the contribution of (10.42), we set

$$h_{K_0,jK_3,L} = \left(\widehat{H_{K_3,L}u_{jK_3,L}}\mathbf{1}_{|\mu|\ge K_0^2}\right)^{\vee}\,. \tag{10.45}$$

Here $j = 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. From the definition of $H_{K_3,L}$, we get

$$\|h_{K_0,jK_3,L}\|_4 \le C\mathfrak{M}_1\|\phi\|_{H^s}\frac{L}{K_0^2}\|u_{jK_3,L}\|_4\,. \tag{10.46}$$

Then (8.10) is bounded by

$$\begin{aligned}&\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_0}\int K_0u^*_{K_0}\sum_{K_3\le CK_0^{1/2}}\|u_{K_3}\|_\infty\\ &\quad\cdot\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\sum_{K_1}\sum_{K\le CK_1}K^sv^*_K|u_{K_1}|dxdt\,,\end{aligned} \tag{10.47}$$

which is majorized by

$$\begin{aligned}&\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_0}K_0\sum_{K_3\le CK_0^{1/2}}\|u_{K_3}\|_\infty\int u^*_{K_0}\\ &\quad\cdot\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\left(\sum_K|v^*_K|^2\right)^{1/2}\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{1/2}dxdt\,.\end{aligned} \tag{10.48}$$

Using Hölder inequality with L^4 norms for four functions in the integrand, we estimate (10.48) by

$$\begin{aligned}&C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}^2\sum_{K_0}K_0\|u_{K_0}\|_4\sum_{K_3\le K_0^{1/2}}\|u_{K_3}\|_\infty\sum_{L\le CK_3^3}\frac{L}{K_0^2}\|u_{jK_3,L}\|_4\\ &\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^3\sum_{K_0}K_0^{1/2}\|u_{K_0}\|_{0,\frac{1}{3}}\\ &\le C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4\,.\end{aligned} \tag{10.49}$$

This finishes the case of (10.42).

For the contribution of (10.43), we bound (8.10) by

$$\begin{aligned}&\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_xu_{K_0}|\sum_{2^{10}K_3^3<L\le 2^{-10}K_0^2}|h_{K_0,jK_3,L}|\\ &\quad\cdot\sum_{K_1}\sum_{K\le CK_1}K^sv^*_K|u_{K_1}|dxdt\,,\end{aligned} \tag{10.50}$$

which is dominated by

$$\begin{aligned}(10.51)\qquad & C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-10}\\ \Delta \text{ dyadic}}}\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq\Delta K_0^2}}|h_{K_0,jK_3,L}| \\ & \cdot\left(\sum_K |v_K^*|^2\right)^{1/2}\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{1/2} dxdt\,,\end{aligned}$$

By Cauchy-Schwarz inequality, we estimate (10.51) further by

(10.52)

$$\begin{aligned}& C\|u\|_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-10}\\ \Delta \text{ dyadic}}}\Delta^{-1/2}\int\sum_{K_0}\frac{|\partial_x u_{K_0}|}{K_0} \\ & \cdot\left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\right)^{1/2}\left(\sum_K |v_K^*|^2\right)^{1/2}\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{1/2} dxdt\,,\end{aligned}$$

Applying Hölder inequality with L^∞ norm for the first function in the integrand, L^2 norm for the second one, and L^4 norms for the last two functions, we then majorize (10.52) by

(10.53)

$$C\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-10}\\ \Delta \text{ dyadic}}}\Delta^{-1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\left\|\left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\right)^{1/2}\right\|_2 .$$

Notice that if $L\sim\Delta K_0^2$, then

$$(10.54)\qquad \|h_{K_0,jK_3,L}\|_2\leq C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3,L}\|_2\,.$$

Thus we have

$$\begin{aligned}(10.55)\qquad & \left\|\left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq\Delta K_0^2}}L|h_{K_0,jK_3,L}|^2\right)^{1/2}\right\|_2 \\ & \leq C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq\Delta K_0^2}}L\|u_{jK_3,L}\|_2^2\right)^{1/2} \\ & \leq C\mathfrak{M}_1\|\phi\|_{H^s}\Delta\|u_{jK_3}\|_{0,\frac12} \\ & \leq C\mathfrak{M}_1\|\phi\|_{H^s}^2\Delta\,.\end{aligned}$$

From (10.55), (10.53) is bounded by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\leq 2^{-10}\\ \Delta\text{ dyadic}}}\Delta^{1/2}\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\,, \tag{10.56}$$

which is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4\,. \tag{10.57}$$

This finishes the case of (10.43).

For the contribution of (10.44), we estimate (8.10) by

$$\begin{aligned}\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}|\partial_x u_{K_0}|\sum_{L>2^{-5}K_0^2}|h_{K_0,jK_3,L}|\\ \cdot\sum_{K_1}\sum_{K\leq CK_1}K^s v_K^*|u_{K_1}|dxdt\,,\end{aligned} \tag{10.58}$$

which is bounded by

$$\begin{aligned}\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\left(\sum_{K_0}\frac{|\partial_x u_{K_0}|^2}{K_0^2}\right)^{1/2}\left(\sum_{L>2^{-5}K_0^2}L|h_{K_0,jK_3,L}|^2\right)^{1/2}\\ \cdot\left(\sum_K|v_K^*|^2\right)^{1/2}\left(\sum_{K_1}K_1^{2s}|u_{K_1}|^2\right)^{1/2}.\end{aligned} \tag{10.59}$$

Applying Hölder inequality, we estimate (10.59) further by

$$C\mathfrak{M}_1\|u\|_{Y_s}^2\sum_{K_3}\|u_{K_3}\|_\infty\sum_{K_0}\frac{\|\partial_x u_{K_0}\|_\infty}{K_0}\left\|\left(\sum_{L>2^{-5}K_0^2}L|u_{jK_3,L}|^2\right)^{1/2}\right\|_2. \tag{10.60}$$

This is clearly majorized by

$$C\mathfrak{M}_1\|\phi\|_{H_s}\|u\|_{Y_s}^4\,. \tag{10.61}$$

Hence we complete the case of (10.44).

11. Proof of Case (8.13)

In this case, it suffices to consider the following subcases:

$$M\geq 2^{-10}K_0^{2/3}\,; \tag{11.1}$$

$$M<2^{-10}K_0^{2/3}\ \text{ and }\ K_2^2K_3\geq 2^{-10}K_0^2; \tag{11.2}$$

$$M<2^{-10}K_0^{2/3}\ \text{ and }\ K_2^2M\geq 2^{-10}K_0^2; \tag{11.3}$$

$$M<2^{-10}K_0^{2/3}\,,\ \ K_2^2K_3<2^{-10}K_0^2\ \text{ and }\ K_2^2M<2^{-10}K_0^2\,. \tag{11.4}$$

For the case of (11.1), notice that, in this case, we have

$$K\leq CM^{3/2}\,. \tag{11.5}$$

Henceforth we estimate (8.10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M \sum_{K\leq CM^{3/2}} K^s v_K^* \sum_{K_0\leq CM^{3/2}} K_0 u_{K_0}^* |g_{K_3,M}| dxdt\,, \tag{11.6}$$

which is bounded by

(11.7)

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_M M^{\frac{3}{2}(1-s)}|g_{K_3,M}| \sum_{K\leq CM^{3/2}} K^s v_K^* \left(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\right)^{1/2} dxdt\,,$$

since $1/2 < s < 1$. Applying Schür test, we estimate (11.7) by

$$\begin{aligned}&\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \left(\sum_M M^3 |g_{K_3,M}|^2\right)^{1/2}\\ &\qquad \left(\sum_K |v_K^*|^2\right)^{1/2} \left(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\right)^{1/2} dxdt\,.\end{aligned} \tag{11.8}$$

By Hölder inequality and $s > 1/2$, (11.8) is majorized by

(11.9)

$$\begin{aligned}&C \sum_{K_1\geq K_2\geq K_3} \|\partial_x^{3/2} G_{K_3}\|_\infty \left(\prod_{j=1}^3 \|u_{K_j}\|_{6+}\right) \left\|\left(\sum_K |v_K|^2\right)^{1/2}\right\|_4 \left\|\left(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\right)^{1/2}\right\|_4\\
&\leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2)\|u\|_{Y_s} \sum_{K_1\geq K_3\geq K_3} K_3^{3/2} \prod_{j=1}^3 \|u_{K_j}\|_{6+}\\
&\leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2)\|u\|_{Y_s} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2}\|u_{K_j}\|_{0+,\frac{1}{2}}\\
&\leq C\mathfrak{M}(\|\phi\|_{H^s} + \|\phi\|_{H^s}^2)\|u\|_{Y_s}^4\,.\end{aligned}$$

This finishes the case of (11.1).

For the case of (11.2), observe that, in this case,

$$K_0 \leq C K_1^{1/2} K_2^{1/2} K_3^{1/2}\,. \tag{11.10}$$

We estimate (8.10) by

$$\int \sum_{K_1\geq K_2\geq K_3} |u_{K_1}||u_{K_2}||u_{K_3}| \sum_{K\leq CK_0} K^s v_K^* \sum_{K_0\leq C(K_1K_2K_3)^{1/2}} K_0 u_{K_0}^* \|G_{K_3}\|_\infty dxdt\,, \tag{11.11}$$

which is bounded by

$$C\mathfrak{M}\int \left(\sum_K |v_K^*|^2\right)^{1/2} \left(\sum_{K_0} K_0^{2s}|u_{K_0}^*|^2\right)^{1/2} \prod_{j=1}^3 \sum_{K_j} K_j^{1/2}|u_{K_j}|dxdt\,. \tag{11.12}$$

Using Hölder inequality with L^4 norms for first two functions and L^6 norms for the last three functions in the integrand, we obtain

$$C\mathfrak{M}\|u\|_{Y_s}\prod_{j=1}^{3}\left\|\sum_{K_j}K_j^{1/2}|u_{K_j}|\right\|_6 \leq C\mathfrak{M}\|u\|_{Y_s}^4\,. \tag{11.13}$$

This completes the case of (11.2).

For the case of (11.3), we have, in this case,

$$K_0 \leq CK_1^{1/2}K_2^{1/2}M^{1/2}\,. \tag{11.14}$$

Hence we dominate (8.10) by

$$\int \sum_{K_1\geq K_2\geq K_3}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_M |g_{K_3,M}|\sum_{K\leq CK_0}K^s v_K^* \sum_{K_0\leq C(K_1K_2M)^{1/2}}K_0u_{K_0}^*dxdt\,, \tag{11.15}$$

which is bounded by

$$\begin{aligned} &C\sum_{K_3}\int\left(\sum_K |v_K^*|^2\right)^{1/2}\left(\sum_{K_0}K_0^{2s}|u_{K_0}^*|^2\right)^{1/2}|u_{K_3}| \\ &\cdot\left(\sum_M M|g_{K_3,M}|^2\right)^{1/2}\prod_{j=1}^{2}\sum_{K_j}K_j^{1/2}|u_{K_j}|dxdt\,. \end{aligned} \tag{11.16}$$

Using Hölder inequality with L^4 norms for first two functions, L^6 norms for the third one, L^p norm with p very large for the fourth one, and L^{6+} for the last two functions in the integrand, we obtain

$$C\|u\|_{Y_s}\prod_{j=1}^{2}\left\|\sum_{K_j}K_j^{1/2}|u_{K_j}|\right\|_{6+}\sum_{K_3}\|u_{K_3}\|_6\|\partial_x^{1/2}G_{K_3}\|_\infty\,. \tag{11.17}$$

Clearly (11.17) is dominated by

$$C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^3\sum_{K_3}K_3^{1/2}\|u_{K_3}\|_6 \leq C\mathfrak{M}\|\phi\|_{H^s}\|u\|_{Y_s}^4\,. \tag{11.18}$$

Hence the case of (11.3) is done.

For the case of (11.4), we observe that, in this case,

$$M^2K_2 \leq 2^{-10}K_0^2\,. \tag{11.19}$$

In fact, if (11.19) does not hold, then from (11.4),

$$M^2K_2 > 2^{-10}K_0^2 > K_2^2M\,.$$

Thus $M > K_2$, which yields immediately

$$M^3 > M^2K_2 > 2^{-10}K_0^2\,,$$

contradicting to $M < 2^{-10}K_0^{2/3}$. Hence (11.19) must be true. From (11.19), $K_2^2K_3+K_2^2M < 2^{-9}K_0^2$, we get

$$\left|(n_2+n_3+m)^3-n_2^3-n_3^3-m^3\right| \leq 2^{-5}K_0^2\,. \tag{11.20}$$

Since $n_1+n_2+n_3+m \neq 0$, from (8.13), (11.4) and (11.20), the crucial arithmetic observation (10.16) then yields

$$|n^3 - n_0^3 - n_1^3 - n_2^3 - n_3^3 - m^3| \geq 2K_0^2 \,. \tag{11.21}$$

Henceforth one of the following four statements must be true:

$$\left|\lambda - n^3\right| \geq K_0^2 \,, \tag{11.22}$$

$$\left|(\lambda - \lambda_1 - \lambda_2 - \lambda_3 - \mu) - n_0^3\right| \geq K_0^2 \,, \tag{11.23}$$

$$\exists i \in \{1,2,3\} \text{ such that } |\lambda_i - n_i^3| \geq K_0^2 \,, \tag{11.24}$$

$$|\mu| \geq K_0^2 \,. \tag{11.25}$$

For the case of (11.22), we estimate (8.10) by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |u^*_{K_0}| \left| \sum_{K\leq CK_0} \partial_x^s \tilde{v}_K \right| dxdt \,. \tag{11.26}$$

Then Cauchy-Schwarz inequality yields

$$\begin{aligned} & C\mathfrak{M}\|u\|^3_{Y_s} \left\| \left(\sum_{K_0} K_0^{2-2s} \left| \sum_{K\leq CK_0} \partial_x^s \tilde{v}_K \right|^2 \right)^{1/2} \right\|_2 \left\| \left(\sum_{K_0} K_0^{2s} |u^*_{K_0}|^2 \right)^{1/2} \right\|_2 \\ & \leq C\mathfrak{M}\|u\|^4_{Y_s} \left(\sum_{K_0} K_0^{2-2s} \sum_{K\leq CK_0} \|\partial_x^s \tilde{v}_K\|_2^2 \right)^{1/2} \;\leq\; C\mathfrak{M}\|u\|^4_{Y_s} \,. \end{aligned} \tag{11.27}$$

This finishes the proof of the case (11.22).

For the case of (11.23), (8.10) can be estimated by

$$\sum_{K_1,K_2,K_3} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_0} \int K_0 |\tilde{u}^*_{K_0}| \sum_{K\leq CK_0} K^s v^*_K dxdt \,. \tag{11.28}$$

By Schür test and Hölder inequality, we control (11.28) by

$$C\mathfrak{M}\|u\|^3_{Y_s} \left\| \left(\sum_{K} |v^*_K|^2 \right)^{1/2} \right\|_2 \left\| \left(\sum_{K_0} K_0^{2s+2} |\tilde{u}_{K_0}|^2 \right)^{1/2} \right\|_2 \,, \tag{11.29}$$

which is clearly bounded by

$$C\mathfrak{M}\|u\|^3_{Y_s} \left(\sum_{K_0} K_0^{2s} \|u_{K_0}\|^2_{0,\frac{1}{2}} \right)^{1/2} \;\leq\; C\mathfrak{M}\|u\|^4_{Y_s} \,. \tag{11.30}$$

This completes the proof of the case (11.23).

For the case of (11.24), without loss of generality, assume $j = 1$. We then dominate (8.10) by

$$\sum_{K_2,K_3} \|u_{K_2}\|_\infty \|u_{K_3}\|_\infty \|G_{K_3}\|_\infty \sum_{K_1} \sum_{K_0} \int K_0 |u^*_{K_0}| |\tilde{u}_{K_1}| \sum_{K\leq CK_0} K^s v^*_K dxdt \,. \tag{11.31}$$

By Hölder inequality, we bound (11.31) by

$$\begin{aligned}(11.32)\qquad &\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\sum_{K_0}\sum_{K\le CK_0}K^sK_0\|u_{K_0}\|_4\|\tilde u_{K_1}\|_2\|v_K\|_4\\ &\le\sum_{K_2,K_3}\|u_{K_2}\|_\infty\|u_{K_3}\|_\infty\|G_{K_3}\|_\infty\sum_{K_1}\|u_{K_1}\|_{0,\frac12}\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\|v_K\|_4\,.\end{aligned}$$

By Schür test, we dominate (11.32) by

$$\begin{aligned}(11.33)\qquad &C\mathfrak{M}\|u\|_{Y_s}^2\sum_{K_1}\|u_{K_1}\|_{0,\frac12}\left(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_4^2\right)^{1/2}\left(\sum_K\|v_K\|_4^2\right)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^3\left(\sum_{K_0}K_0^{2s}\|u_{K_0}\|_{0,\frac13}^2\right)^{1/2}\left(\sum_K\|v_K\|_{0,\frac13}^2\right)^{1/2}\\ &\le C\mathfrak{M}\|u\|_{Y_s}^4\,.\end{aligned}$$

Hence the case of (11.24) is done.

In order to finish the proof, as before we need to consider further three subcases:

$$(11.34)\qquad L\le 2^{10}K_3^3\,,$$
$$(11.35)\qquad 2^{10}K_3^3<L\le 2^{-5}K_0^2\,,$$
$$(11.36)\qquad L>2^{-5}K_0^2\,.$$

For the contribution of (11.34), notice that

$$(11.37)\qquad \|h_{K_0,jK_3,L}\|_6\le C\mathfrak{M}_1\|\phi\|_{H^s}\frac{L}{K_0^2}\|u_{jK_3,L}\|_6\,.$$

Here $h_{K_0,jK_3,L}$ is defined as in (10.45). In this particular case we also have $K_3\le K_0^{2/3}$ from $K_2^2K_3\le 2^{-10}K_0^2$. Then (8.10) is bounded by

$$(11.38)\qquad \int\sum_{K_0}K_0u_{K_0}^*\sum_{K\le CK_0}K^sv_K^*\sum_{\substack{K_1\ge K_2\ge K_3\\ K_3\le K_0^{2/3}}}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dxdt\,.$$

Write (11.38) as

$$(11.39)\qquad \sum_{\substack{\Delta\text{ dyadic}\\ \Delta\le1}}\int\sum_{K_0}K_0u_{K_0}^*\sum_{K\le CK_0}K^sv_K^*\sum_{\substack{K_1\ge K_2\ge K_3\\ \Delta K_0^{2/3}/2<K_3\le\Delta K_0^{2/3}}}|u_{K_1}||u_{K_2}||u_{K_3}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|\,dxdt\,.$$

Observe that if $\Delta K_0^{2/3}/2<K_3\le\Delta K_0^{2/3}$, then we have

$$(11.40)\qquad K_0\le\Delta^{-3/2}K_1^{1/2}K_2^{1/2}K_3^{1/2}\,.$$

Henceforth, (11.39) is bounded by

$$
\begin{aligned}
&C\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{K_1,K_2}K_1^{1/2}K_2^{1/2}\sum_{\Delta\le 1}\Delta^{-3/2}\sum_{K_3\sim\Delta K_0^{2/3}}K_3^{1/2} \\
&\int u^*_{K_0}v^*_K|u_{K_1}||u_{K_2}|\sum_{L\le CK_3^3}|h_{K_0,jK_3,L}|dxdt\,.
\end{aligned}
\tag{11.41}
$$

Applying Hölder inequality with L^4 norms for first two functions and L^6 for the last three functions, and then using (11.37), we get

$$
\begin{aligned}
&C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{K_1,K_2}K_1^{1/2}K_2^{1/2}\sum_{\Delta\le 1}\Delta^{-3/2}\sum_{K_3\sim\Delta K_0^{2/3}}K_3^{1/2} \\
&\|u_{K_0}\|_4\|v^*_K\|_4\|u_{K_1}\|_6\|u_{K_2}\|_6\sum_{L\le CK_3^3}\frac{L}{K_0^2}\|u_{jK_3,L}\|_6\,,
\end{aligned}
\tag{11.42}
$$

which is bounded by

$$
\begin{aligned}
&C\mathfrak{M}_1\|\phi\|_{H^s}\|u\|_{Y_s}\sum_{K_0}\sum_{K\le K_0}K^s\sum_{\Delta\le 1}\Delta^{-3/2}\sum_{L\le C\Delta^3K_0^2}\frac{L}{K_0^2} \\
&\|u_{K_0}\|_4\|v^*_K\|_4\sum_{K_1}K_1^{1/2}\|u_{K_1}\|_{0+,\frac12}\sum_{K_2}K_2^{1/2}\|u_{K_2}\|_{0+,\frac12}\sum_{K_3}K_3^{1/2}\|u_{jK_3,L}\|_{0+,\frac12} \\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s}\sum_{\Delta\le1}\Delta^{3/2}\sum_{K_0}\sum_{K\le CK_0}K^s\|u_{K_0}\|_4\|v_K\|_4 \\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^3_{Y_s}\left(\sum_{K_0}K_0^{2s}\|u_{K_0}\|^2_{0,\frac13}\right)^{1/2}\left(\sum_{K}\|v_K\|^2_{0,\frac13}\right)^{1/2} \\
&\le C\mathfrak{M}_1\|\phi\|^2_{H^s}\|u\|^4_{Y_s}\,.
\end{aligned}
\tag{11.43}
$$

This completes the case (11.34).

For the contribution of (11.35), (8.10) is bounded by

$$
\begin{aligned}
&\sum_{K_1}\|u_{K_1}\|_\infty\sum_{K_2}\|u_{K_2}\|_\infty\sum_{K_3}\|u_{K_3}\|_\infty\int\sum_{K_0}\sum_{K\le CK_0}K^sv^*_KK_0u^*_{K_0} \\
&\sum_{2^{10}K_3^3<L\le 2^{-5}K_0^2}|h_{K_0,jK_3,L}|dxdt\,,
\end{aligned}
\tag{11.44}
$$

which is dominated by

$$
C\|u\|^2_{Y_s}\sum_{K_3}\|u_{K_3}\|_\infty\sum_{\substack{\Delta\le 2^{-5}\\ \Delta\text{ dyadic}}}\sum_{K_0}\sum_{K\le CK_0}K^s\int K_0u^*_{K_0}v^*_K\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\le\Delta K_0^2}}|h_{K_0,jK_3,L}|dxdt\,.
\tag{11.45}
$$

Using Cauchy-Schwarz inequality, we estimate (11.45) further by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-5}\\ \Delta \text{ dyadic}}} \Delta^{-\frac{1}{2}} \sum_{K_0} \sum_{K\leq CK_0} K^s \int u_{K_0}^* v_K^* \left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq \Delta K_0^2}} L|h_{K_0,jK_3,L}|^2 \right)^{1/2} dxdt\,. \tag{11.46}$$

Employing Hölder inequality with L^4 norms for the first two functions and L^2 for the last one, we bound (11.46) by

$$C\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-5}\\ \Delta \text{ dyadic}}} \Delta^{-\frac{1}{2}} \sum_{K_0} \sum_{K\leq CK_0} K^s \|u_{K_0}\|_4 \|v_K\|_4 \cdot \left\| \left(\sum_{\substack{2^{10}K_3^3<L\\ \frac{\Delta}{2}K_0^2<L\leq \Delta K_0^2}} L|h_{K_0,jK_3,L}|^2 \right)^{1/2} \right\|_2 . \tag{11.47}$$

From (10.55), (11.47) is majorized by

$$\begin{aligned} & C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{\substack{\Delta\leq 2^{-5}\\ \Delta \text{ dyadic}}} \Delta^{\frac{1}{2}} \sum_{K_0} \sum_{K\leq CK_0} K^s \|u_{K_0}\|_4 \|v_K\|_4 \\ & \leq C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^3 \left(\sum_{K_0} K_0^{2s}\|u_{K_0}\|_{0,\frac{1}{3}}^2\right)^{1/2} \left(\sum_{K} \|v_K\|_{0,\frac{1}{3}}^2\right)^{1/2} \\ & \leq C\mathfrak{M}_1\|\phi\|_{H^s}^2\|u\|_{Y_s}^4 . \end{aligned} \tag{11.48}$$

This finishes the proof for the case (11.35).

For the contribution of (11.36), we estimate (8.10) by

$$\sum_{K_1,K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \int \sum_{K_0} K_0 u_{K_0}^* \sum_{L>2^{-5}K_0^2} |h_{K_0,jK_3,L}| \sum_{K\leq CK_0} K^s v_K^* dxdt\,. \tag{11.49}$$

By Cauchy-Schwarz inequality, (11.49) is bounded by

$$\sum_{K_1,K_2} \|u_{K_1}\|_\infty \|u_{K_2}\|_\infty \sum_{K_3} \|u_{K_3}\|_\infty \sum_{K_0} \sum_{K\leq CK_0} K^s \int v_K^* u_{K_0}^* \left(\sum_{L>2^{-10}K_0^2} L|h_{K_0,jK_3,L}|^2 \right)^{1/2} dxdt\,. \tag{11.50}$$

Employing Hölder inequality with L^4 norms for the first two functions and L^2 norm for the last one, we dominate (11.50) by

$$\begin{aligned} & C\mathfrak{M}_1\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{K_0} \sum_{K\le CK_0} K^s \|u_{K_0}\|_4 \|v_K\|_4 \left\| \left(\sum_{L>2^{-5}K_0^2} L|u_{jK_3,L}|^2 \right)^{1/2} \right\|_2 \\ & \le C\mathfrak{M}_1\|u\|_{Y_s}^2 \sum_{K_3} \|u_{K_3}\|_\infty \sum_{K_0} \sum_{K\le CK_0} K^s \|u_{K_0}\|_{0,\frac{1}{3}} \|v_K\|_{0,\frac{1}{3}} \|u\|_{0,\frac{1}{2}} \\ & \le C\mathfrak{M}_1 \|\phi\|_{H^s} \|u\|_{Y_s}^4 . \end{aligned} \tag{11.51}$$

Hence we complete the case of (11.36).

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