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Group Theory and Particles

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GROUP THEORY AND PARTICLES

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by

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ABSTRACT

We begin by a brief overview of the notion of groups and Lie groups. We then explain what group representations are and give their main properties. Finally, we show how group representation form a natural framework to understand the Standard Model of physics.

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CHAPTER 1

INTRODUCTION

The standard model of particle is an attempt to describe the interactions between the elementary particles and three of the four fundamental forces: electromagnetism, weak interactions, and strong interactions. The standard model is not perfect; it fails to describe some symmetries, gravity, dark matter, and a few other known facts. However, the standard Model has successfully predicted the existence of previously undiscovered particles such as the Higgs Boson which exemplifies how important of a tool it is to modern physics.

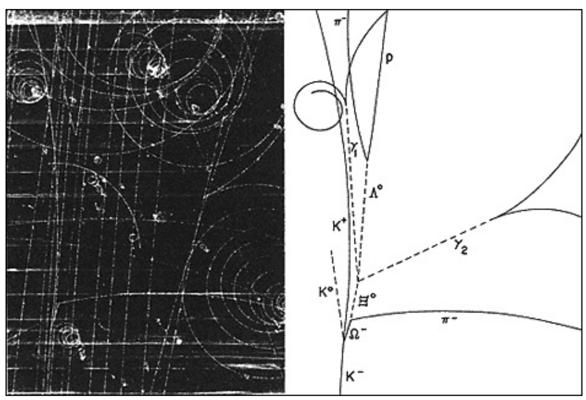

Figure 1.1: Interaction between particles capture in a bubble chamber.

The derivation of the standard model begins with the fermions then the leptons. Within the fermions, the nucleons and pions are the first the be represented. We will derive this representation of nucleons and pions since the beginning of the standard model. The entire standard model was derived over many years through the contributions of numerous physicists. The physics behind the derivation of the standard model is complicated to say the least. On the contrary The mathematics behind the standard model is straight forward. Of course some physics preliminaries are necessary such as how to model particles and describe forces mathematically. Given these, the standard model can be derived mathematically using representations of Lie groups, Lie algebras, and Hilbert spaces.

CHAPTER 2

GROUPS AND LIE GROUPS

2.1 Groups

Groups are one of the fundamental notions in mathematics. They are the structure that naturally encodes symmetries. Symmetries can be composed, reversed, etc. this is the essence of a group.

Definition 2.1. *A group is a pair $(G, *)$ consisting of a set G and of a binary operation $* : G \times G \to G$ satisfying the following axioms*

1. *$*$ is associative, that is $\forall g, h, i \in G$, $g * (h * i) = (g * h) * i$*

2. *there exists a neutral element e, i.e. there is an $e \in G$ such that $\forall g \in G, e * g = g * e = g$*

3. *each element has an inverse: $\forall g \in G$, $\exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$*

Some groups have the extra property that the operation $*$ is commutative, we call these groups commutative or Abelian.

Definition 2.2. *An Abelian group is a group $(G, *)$ such that*

1. *$*$ is commutative, i.e. $\forall g, h \in G$ we have $g * h = h * g$*

Some of the most common examples of groups are $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R}_{>}, \times)$ and $(\mathbb{R}^{\times} = \{x \in R, x \neq 0\}, \times)$. All these examples are infinite and Abelian. Some examples of finite groups are $(\{\pm 1\}, \times)$ and $(\{\pm 1, \pm i\}, \times)$ which are both Abelian. A first example of non-Abelian group is the group $GL(2, \mathbb{R})$ of 2×2 invertible real matrices under multiplication.

When studying the relation between groups, we need a notion of functions compatible with the structure inherent to groups.

Definition 2.3. *A group homomorphism between* $(G, *)$ *and* $(H, \circ)$ *is a function* $\phi : G \to H$ *such that*

$$\forall a, b \in G, \phi(a * b) = \phi(a) \circ \phi(b).$$

A typical example of group homomorphism is the determinant, $det : GL(n, \mathbb{R}) \to (\mathbb{R}^{\times}, \times) : A \mapsto \det A$. Notice that while ϕ carries some of the structure between the groups, these groups can still be very different; for example, the determinant map here above sends a non-Abelian group to an Abelian group. A homomorphism that preserves the whole group structure will be called an isomorphism:

Definition 2.4. *A group isomorphism is a bijective homomorphism.*

For example the exponential map, $x \mapsto e^x$, gives an isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}_{>}, \times)$.

2.2 Lie Groups

Some groups also have an additional geometric structure which is compatible with its algebraic structure, we call these groups Lie groups.

We must diverge for a moment to discuss manifolds, for it is necessary to understand Lie groups.

A n-dimensional manifold M is a topological space such that for all $p \in M$ there a neighborhood of p which is homeomorphic to n-dimensional Euclidean space. If the Euclidean space is $\mathbb{R}^{\ltimes}$ then its a real manifold and if $\mathbf{C^n}$ then its a complex manifold.

In practice, we require M to be covered by a collection of open sets U^α equipped with charts, i.e. homeomorphisms $x_\alpha : U^\alpha \to x_\alpha(U^\alpha) \subset \mathbb{R}^n, \mathbb{C}^n$.

A n-dimensional, real, smooth manifold M is real manifold such that all transition maps $x_\beta \circ x_\alpha^{-1} : x_\alpha\left(U^\alpha \cap U^\beta\right) \to x_\beta\left(U^\alpha \cap U^\beta\right)$, when $U^\alpha \cap U^\beta \neq \emptyset$, are smooth functions on $\mathbb{R}^{\ltimes}$.

Definition 2.5. *A Lie group is a group $(G, *)$ such that*

1. *G is a manifold*
2. *The map $G \times G \to G : (g, h) \mapsto g * h$ is smooth.*
3. *The map $G \to G : g \mapsto g^{-1}$ is smooth.*

The simplest example of Lie Group is $(\mathbb{R}, +)$. A more interesting example is the group of rotations in the plane around the origin, $SO(2) = \{R_\theta\}$, which can be identified with the unit circle $S^1 = \{(x, y), x^2 + y^2 = 1\} = \{e^{i\theta}\}$ via the map $R_\theta \mapsto e^{i\theta}$.

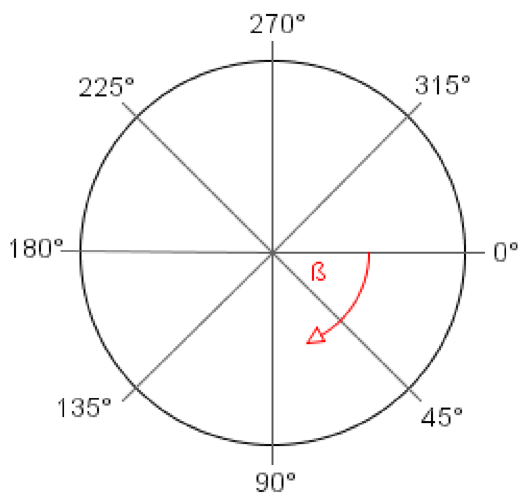

Figure 2.1: The trig circle or $S^1 = SO(2)$.

The composition of rotations, $R_\theta \circ R_\psi = R_{\theta+\psi}$ corresponds to the complex multiplication $e^{i\theta} e^{i\psi} = e^{i(\theta+\psi)}$, and the inversion of rotation $R_\theta \mapsto R_{-\theta}$ corresponds to the complex inverstion $e^{i\theta} \mapsto e^{-i\theta}$ and both these maps are smooth. Another important example of Lie group is the group of three dimensional representations $SO(3)$. Note that $SO(3)$, contrary to $SO(2)$ is not Abelian: tilting an object forward and then to the left is not the same as first tilting it to the left and then forward.

CHAPTER 3

REPRESENTATION THEORY

3.1 Group Representations

In nature, groups often don't appear in their full form bus as *representations*, i.e. they appear as families of transformations, or symmetries, whose composition is compatible with the group operation. The most common form of representation is that of linear representation, i.e. transformations of vector spaces, or transformations which can be put in matrix form (in the finite dimensional case). Let V be a vector space, we denote by $Aut(V)$ the group of linear isomorphisms from V to itself.

Definition 3.1. *A linear representation of a group* $(G,*)$ *is a homomorphism* $\sigma : G \to Aut(V) : g \mapsto \lambda_g$

Concretely, a linear representation on V means a family of isomorphisms parametrized by G, $\{\lambda_g\}_{g\in G}$ and whose composition is compatible with the group operation:

$$\lambda_g \circ \lambda_h = \lambda_{h*g}.$$

Let us study two examples of representations in detail :

Example 3.2. *Consider the family of matrices* $\Lambda_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R}$. *It is easy to check that* $\Lambda_x \circ \Lambda_y = \Lambda_{x+y}$. *In other words, the map* $\sigma : x \mapsto \Lambda_x$ *is a representation of* $(\mathbb{R}, +)$ *on the vector space* $V = \mathbb{R}^2$. *Since the map* σ *is injective, i.e. distinct elements of* $\mathbb{R}$ *are represented by different matrices, we talk about a faithfull representation.*

Example 3.3. *The determinant map* $\det : GL(2,\mathbb{R}) \to \mathbb{R}^\times$ *can also be thought of as a one dimensional representation as* $Aut(\mathbb{R}) = \mathbb{R}^\times$, *i.e.*

$$\det : GL(2,\mathbb{R}) \to Aut(\mathbb{R}) : A \mapsto \det A$$

where $\det A : \mathbb{R} \to \mathbb{R} : x \mapsto \det Ax$.

Example 3.4. *Consider now the group of rotations by multiples of* 60^o, $C_6 = \{R_\theta\}_{\theta=k\frac{\pi}{3},k=0,\ldots,5}$. *One can check that the map*

$$\sigma : R_{\theta=k\frac{\pi}{3}} \to \begin{pmatrix} 1 & 0 \\ 0 & (-1)^k \end{pmatrix}$$

is a representation.

Heuristically speaking, one could think of representations as being the 'shadow' of groups. They preserve some of their structure but might not always encompass the full story.

In what follows, we will try to get a better understanding of how representations can be constructed and deconstructed.

Besides the concrete examples above, there are three essential types of representations of a group which are ubiquitous : trivial, regular, and permutation.

The trivial representation is a representation of a finite group G such that $\rho(s) = 1 \forall s \in G$. This is a representation of degree 1. Hence the basis of V is $\{1\}$ and $dim(V) = 1$.

The regular representation of a finite group G on a finite dimensional vector space V with basis $(e_t)_{t\in G}$ is such that $\rho_s : V \to V : e_t \mapsto e_{st}$. The degree of this representation obviously equals the dimension of G given the basis of V is indexed by the elements of G.

The permutation representation of a finite group G is like the previous but G acts on a finite set X such that for each $s \in G$ there exists a permutation $x \mapsto sx$ satisfying $1x = x$ and $s(tx) = (st)x$ for $s, t \in G$ and $x \in X$. We then define a vector space V having basis $(e_x)_{x\in X}$ and hence $\rho_s : e_x \to e_{sx}$ for $s \in G$. The degree of this representation is the order (number of elements) of the group.

3.2 Comparing Representations

A group G can have multiple representations which means we must be able to compare two representations. Two representations ρ on V and ρ' on V' are isomorphic if there exists $\tau : V \to V'$ such that $\tau \circ \rho = \rho' \circ \tau$ or equivalently $TR = R'T$ if ρ and ρ' are given in matrix form R and R'.

3.3 Building new Representations

Using different basic representations and machinery from (multi)linear algebra, we can construct new representations.

3.3.1 Direct Sum

Let $\rho_s^1 : G \to GL(V_1)$ and $\rho_s^2 : G \to GL(V_2)$ be two representations of G. Then the direct sum of the two representations is defined as $(\rho^1 \oplus \rho^2)(s) = \rho^1(s) \oplus \rho^2(s)$. If we represent $\rho^1 \oplus \rho^2$ as a matrix, then it is the matrix whose diagonal blocks are those representing ρ^1 and ρ^2.

3.3.2 Tensor Product

An important operation between representations is the the tensor product.

Let $\rho_s^1 : G \to GL(V_1)$ and $\rho_s^2 : G \to GL(V_2)$ be two representations of G. Then the tensor product of the two representations is defined as $\rho^1(s_1) \otimes \rho^2(s_2) = \rho(s_1 \cdot s_2) = \rho^1(s_1) \cdot \rho^2(s_2)$ for $s_i \in G$ and $\rho_s \in GL(V_1 \otimes V_2)$.

3.3.3 Symmetric Square and Alternating Square

There are two other important types of representations which appear naturally when understanding the tensor product.

Let $\sigma : V \otimes V \to V \otimes V$ be the linear involution defined on the basis $(e_i \cdot e_j)$ of $V \otimes V$ as $\sigma(e_i \cdot e_j) = e_j \cdot e_i$.

Then the tensor product can be decomposed as follows $V \otimes V = Sym^2(V) \oplus Alt^2(V)$ where the symmetric square Sym^2 and Alternating square Alt^2 are the eigenspaces of σ associated to ± 1.

More precisely,

$$\overset{2}{Sym}(V) = \{z \in V \otimes V : \sigma(z) = z\}$$

with basis $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$ and with dimension $\frac{n(n+1)}{2}$ and

$$\overset{2}{Alt}(V) = \{z \in V \otimes V : \sigma(z) = -z\}$$

with basis $(e_i \cdot e_j - e_j \cdot e_i)_{i \leq j}$ and with dimension $\frac{n(n-1)}{2}$.

This decomposition follows from the fact that $\sigma(e_ie_j + e_je_i) = e_je_i + e_ie_j = z$ and $\sigma(e_ie_j - e_je_i) = e_je_i - e_ie_j = -z$.

One can then show that a representation of G on V induces a representation on the square, the Symmetric square and the Alternating square.

Example 3.5. *Consider the representation of $(\mathbb{Z}/2\mathbb{Z} = \{0, 1\}, +)$ on $V = \mathbb{C}$ defined by $\rho(0) : z \mapsto z$ and $\rho(1) : z \mapsto -z$. The direct sum representation on $\mathbb{C} \oplus \mathbb{C}$ is defined by $(\rho \oplus \rho)(1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The tensor product representation on $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ is the trivial representation as $(\rho \otimes \rho)(-1) = (-1) \otimes (-1) = 1$. In this example, the Symmetric square $Sym^2\,\mathbb{C} = \mathbb{C}$ with the trivial representation as well and Alt^2 is trivial.*

3.4 Reducible and Irreducible Representations

Given a representation on a vector space V, it is natural to ask what happens at the level of subspaces of V.

Definition 3.6. *A sub-representation of a representation of G on V is the data of W, a subspace of V invariant under the action of G and the restriction of each element of $Aut(V)$ in the image of the representation to $Aut(W)$.*

This means that if $\rho : G \to Aut(V)$ is a representation on V, for all $x \in W$, $\rho_s(x) \in W$ as soon as $s \in W$. We will denote ρ_s restricted to a subspace W as ρ_s^W or more simply ρ^W. Notice the sub-representation $\rho^W : G \to GL(W)$ is an automorphism which makes the subscript redundant.

A lot of properties regarding sub-representation descend directly from properties of vector subspaces from linear algebra.

An example of sub-representation comes form decomposable representations, i.e. representations V which can be constructed as the direct sum of two other non-trivial representations $V = V_1 \oplus V_2$. In that case, both V_1 and V_2 are sub-representations of V.

Given a sub-representations then raises the question whether a representation can be decomposed into 'smaller pieces'.

Definition 3.7. *A representation is irreducible (also denoted by irrep) if there does not exist a proper non trivial vector subspaces W of V invariant under the action of G.*

A reducible representation is a representation which is not irreducible.

One would now be tempted to reconstruct a reducible representation from its sub-representations. However this is not always possible.

Proposition 3.8. *A decomposable representation is reducible.*

... But a reducible representation is not always decomposable.

Example 3.9. *The representation of* $(\mathbb{R}, +)$ *given in Example 3.2 contains* $\mathbb{R} = \{(x, 0)^t\}$ *as an invariant subspace (it is reducible) but it is not hard to show that it is not decomposable!*

However, not all is lost, when working with finite groups and vector spaces over fields of characteristic 0 (or not dividing the order of the group, then Maschke's theorem tells us that reducible and decomposable are the same. In its simplest form, we have:

Theorem 3.10. *(Maschke) Let* G *be a finite group, and let* V *be a vector field over* $\mathbb{R}$ *or* $\mathbb{C}$*. If* U *is a sub-representation of* V*, then there is a subrepresentation* W *of* V *such that* $V = U \oplus W$*.*

3.5 Character Theory

Character theory is important tool for comparing representations. Let $a : V \to V$ be a linear map. With respect to the basis (e_i) of V, a can represented by the matrix (a_{ij}).

Definition 3.11. *The character* χ *of* a *equals* $Tr\,(a) = \sum_i a_{ii}$.

This definiton is independent of the choice of basis the character of a is simply the sum of its eigenvalues.

We can therefore define the character of a representation $\rho : G \to GL\,(V)$ as the scalar function

$$\chi_\rho\,(s) = Tr\,(\rho_s)\,.$$

A few notable properties of χ:

1. $\chi(1) = n$
2. $\chi(s^{-1}) = \chi(s)^*$ such that $*$ denotes conjugate.
3. $\chi(tst^{-1}) = \chi(s)$ (characters are constant along conjugacy classes)

Proof. 1. $\rho(1) = 1$ and $Tr(1) = n$.

2. This follows from by the property of eigenvalues that $\lambda^* = \lambda^{-1}$.
3. Follows from the fact that $Tr(uv) = Tr(vu)$.

□

Now let χ_1 and χ_2 be the characters of representation ρ^1 and ρ^2. Then the character for the direct sum of representation is $\chi = \chi_1 + \chi_2$ and the character for tensor product is $\chi = \chi_1\chi_2$.

An important lemma for character theory is Schur's Lemma. Schur's Lemma states that two irreps of a group can only be trivially related.

Theorem 3.12. *(Schur) Given two irreps of G $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ and a linear map $f : V_1 \to V_2$ st $\rho^1 \circ f = f \circ \rho^2$. If ρ^1 and ρ^2 are not isomorphic then there no G equivariant maps between V_1 and V_2.*
If ρ^1 and ρ^2 are isomorphic then the only G equivariant maps between V_1 and V_2 are the trivial map and homotheties (a scalar multiple of identity).

Proof. (sketch) The former is trivial in that the two representation are not isomorphic which means there does not exists $f : V_1 \to V_2$ so f must be 0.

The latter follows from the fact that if $V_1 = V_2$ then $f : V_1 \to V_2$ must equal one of its eigenvalue.

Now let $f^0 = \frac{1}{g}\sum_{t\in G}\left(\rho_t^2\right)^{-1} f\rho_t^1$. f^0 has properties defined by Schurs Lemma with a homothety ratio of $\frac{1}{n}Tr(f)$ with $n = dim(V_1)$. This ratio is found by the property that f^0 equals an eigenvalue. $Tr(\lambda) = n\lambda \Rightarrow \lambda = \frac{1}{n}Tr(f)$. The book denotes f^0 as h^0 and proving the previous simply requires showing $\rho^1 \circ h^0 = h^0 \circ \rho^2$ $\square$

Now in order to calculate the characters of a representation we need the following property of orthogonality of characters. Orthogonality of characters will help us decompose representations into irreps.

Define

$$(\chi_1 \| \chi_2) = < \chi_1, \chi_2 > = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g).$$

Then if χ is the character of an irreducible representation $(\chi \| \chi) = 1$. Furthermore if two representations are not isomorphic with characters χ and χ' then $(\chi \| \chi') = \delta_{ij}/n = 0$ given $i \neq j$.

One of consequences of this propertyis that two representations with the same character are isomorphic. The basic idea behind this is a character $\chi = a_1\chi_1 + \ldots + a_n\chi_n$ where χ_i is the character of the irreducible representation W_i and $(\chi \| \chi) = \sum_{i=1}^{i=n} a_i^2$.

The most important consequence is the irreducibility criteria. Assume $(\chi \| \chi) > 0$ then if $(\chi \| \chi) = 1$ then it is clear from what was previously stated that V is isomorphic to an irreducible representation W_i. This seemingly trivial observation is actually essential and makes characters the major tool in understanding the structure of representations.

Example 3.13. *Let us work out the character table of the symmetric group S_3.*

S_3 has 3 conjugacy classes indexed by the partitions of 3*:* $1+1+1 = 1+2 = 3$.

These conjugacy classes are represented by the identity (e) *(1 element), the transposition* (12) *(3 elements), and the class of the cyclic rotations* (123) *(2 elements).*

To compute χ_1:

$< \chi_1, \chi_1 > = \frac{1}{6}(\chi_1^2(e) + 3\chi_1^2(12) + 2\chi_1^2(123)) = 1$

$\rightarrow \chi_1^2(e) + 3\chi_1^2(12) + 2\chi_1^2(123) = 6$

$\rightarrow \chi_1(e) = \chi_1(12) = \chi_1(123) = 1$

To compute χ_2*:*

$< \chi_1, \chi_2 >= \frac{1}{6}(\chi_1(e)\chi_2(e) + 3\chi_1(12)\chi_2(12) + 2\chi_1(123)\chi_2(123)) = 0$

$\rightarrow \chi_2(e) + 3\chi_2(12) + 2\chi_2(123) = 0$

and

$< \chi_2, \chi_2 >= \frac{1}{6}(\chi_2^2(e) + 3\chi_2^2(12) + 2\chi_2^2(123)) = 1$

$\rightarrow \chi_2^2(e) + 3\chi_2^2(12) + 2\chi_2^2(123) = 6$

$\rightarrow \chi_2(e), \chi_2(12), \chi_2(123) = \pm 1$ *so* $\chi_2(e) = \chi_2(123) = 1$ *and* $\chi_2(12) = -1.$

To compute χ_3*:*

$< \chi_1, \chi_3 >= \frac{1}{6}(\chi_1(e)\chi_3(e) + 3\chi_1(12)\chi_3(12) + 2\chi_1(123)\chi_3(123)) = 0$

$\Rightarrow \chi_3(e) + 3\chi_3(12) + 2\chi_3(123) = 0 \rightarrow \chi_3(e) + 2\chi_3(123) = -3\chi_3(12)$

and

$< \chi_2, \chi_3 >= \frac{1}{6}(\chi_2(e)\chi_3(e) + 3\chi_2(12)\chi_3(12) + 2\chi_2(123)\chi_3(123)) = 0$

$\rightarrow \chi_3(e) - 3\chi_3(12) + 2\chi_3(123) = 0 \rightarrow \chi_3(e) + 2\chi_3(123) = 3\chi_3(12)$

$\rightarrow \chi_3(12) = 0$ *so* $\chi_3(e) + 2\chi_3(123) = 0.$

Finally,

$< \chi_3, \chi_3 >= \frac{1}{6}(\chi_3^2(e) + 3\chi_3^2(12) + 2\chi_3^2(123)) = 1$ *and*

$\rightarrow \chi_3^2(e) + 2\chi_3^2(123) = 6$ *so* $\chi_3(e) = 2$ *and* $\chi_3(123) = -1.$

3.6 Compact Groups and Unitary Representations

Our discussion above focused mainly on finite groups whence the summation appearing everywhere. If we allow G to be a compact Lie group, a lot of the theory can simply be transferred. If G is a compact group we can can associate a G-invariant (in the words of Hurwitz) or Haar measure. This measure plays the role of the summation and allows a.o. to compute the average of group elements, mimicking the finite case.

A Hilbert space H is a vector space with inner product $< f, g >$ such that H is a complete metric space under the norm. (This means that every Cauchy sequence in the metric space converges.) Given a complex Hilbert space, we can look at a special class of operators, unitary operators. These are surjective bounded operator on a Hilbert space **preserving** the inner product. They are the complex analogue of orthogonal operators. For a complex Hilbert space V, we denote by $U(V)$ the group of unitary operators. A crucial type of representations for physicist is that of unitary representation. Namely,

Definition 3.14. *A unitary representation of a Lie group G on V is a group homomorphism*

$$\rho : G \to U(V)$$

such that that for each fixed $v \in V$ the map $g \mapsto \rho(g)v$ is continuous.

Example 3.15. *Let $V = L^2(\mathbb{R})$ be the space of square integrable functions on the real line. Take G to be the additive group of real numbers. There is a unitary representation of the real numbers on V through shifting :*

$$(a \in \mathbb{R}, f \in V) \mapsto f(x - a)$$

Examples of unitary representations are ubiquitous in quantum mechanics and quantum field theory where they encode the symmetries, G, to which a set of states are constrained.

CHAPTER 4

MODELING PARTICLES

4.1 Physics Preliminaries

Since the darkest ages, humans have tried to give a meaning to the world around them. Myths and legends are nothing but an attempt to give a sense to our surroundings. As civilization and science developed, mankind was able to offer deeper and deeper insight on why our world is the way it is. In particular, over the last centuries, we have observed that there are several essential forces holding everything together. The first one, gravity, is the force guiding planets through the sky and apples to the ground. It affects massive objects and sends them one to another. Since Newton we have had the chance to get a good local understanding of gravity, but it is thanks to Einstein's theory of relativity that we now have a pretty comprehensive model of how our world is shaped by gravity. The second force is the electro-magnetic force. Known first to us in the form or lightning, or to classical civilizations as the capacity of wool fibers to stick to amber (ἤλεκτρον, elektron(!) in greek). The observation by Danish scholar Hans Christian Ørsted in 1820 that electric currents affect compasses led nineteenth century physicist searching for a theory of electro-magnetism.

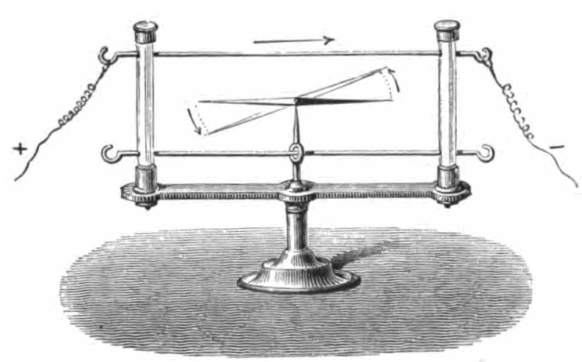

Figure 4.1: School apparatus (1876) used to replicate Ørsted's experiment.

Their quest proved successful and culminated in the notorious Maxwell equations.

The problem with these two forces is that they could not explain how protons stuck together in the nucleus of atoms. This apparent paradox led physicist to hypothesize the existence of a third force, the strong force, which would be reason why protons stay together –at close ranges– in the nucleus. Eventually, physicist also had to deduce that there was a fourth force, the weak force, that would affect sub-atomic particles and cause radioactive decay. While these four forces seem sufficient to explain (most of) our world, physicists were trying to understand if it would be possible to find a unique canvas in which all these forces would fit. This quest for a unified theory has been the grail of theoretical physicists for the last hundred years.

So far, the quest has been pretty succesfull. Electricity and Magnetism were combined in the nineteenth century into electromagnetism. Electromagnetism and the weak force were then united in the electro weak force. The electro weak and the strong force at higher energies combine then into Grand Unified Theories.

However, the search is not over as there is yet to be a model model combining the above forces with gravity.

In the quest to unify forces, physicists have discovered that the atoms that make our universe and were supposed to be fundamental (atom comes from the greek ἄτομος , which cannot be cut) were actually made of smaller particles.

This particles which carry matter and forces are the basic building blocks of the theories described above and form the Standard Model of particles.

In this chapter, we will mainly focus on the first three forces and show how representation theory offers us a canvas to understand and model the different forces and particles appearing in our world.

While we will not be able to offer a detailed view of the standard model, we will explain the basic ideas behind

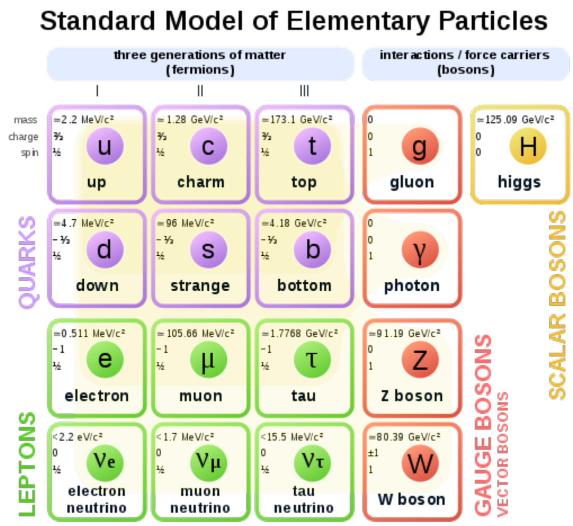

Figure 4.2: The Standard Model of elementary particles with bosons and fermions.

4.2 Representations and Particles

Theories studied by physicists have "internal symmetry groups" known as gauge group, which usually takes the form of a compact Lie group G. In terms of representation theory, elementary particles are considered as elements of a Hilbert space endowed with a unitary representation of G. More precisely, fundamental particles are basis elements of irreducible representations of G.

In this framework, different theories (electromagnetism,...) correspond to different Hilbert Spaces and gauge groups and unifying forces means finding a larger group containing the previous groups as gauge groups.

Ideally, an ultimate theory should be based on a simple group, i.e. which cannot be decomposed.

4.3 Hilbert Spaces in Physics

In quantum mechanics, the state of any physical system by a unit vector in a complex Hilbert Space. For example, the state of a particle on a live is described by an element

$$\psi \in L^2(\mathbb{R})$$

. When studying the atom, Heisenberg hypothesized that the neutron and proton would be two facets of a same particle which was named nucleon.

As a nucleon is a proton or a neutron, we can assume that it is an element of $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ where we take that both neutron and proton live in $\mathbb{C}$.

With respect to this decomposition, the basis vectors of $\mathbb{C}^2$ then correspond to the proton and neutron:

$$\text{proton} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{neutron} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For a given Hilbert space, the probability that a system in state $\sigma \in H$ will be observed in state $\varphi \in H$ is $| < \sigma, \varphi > |^2$. In our case, the proton and neutron are orthogonal hence cannot exist at the same time. However, nothing prevents a generalized nucleon to take the form

$$\alpha p + \beta n$$

where $\alpha^2 + \beta^2 = 1$.

While Heisenberg's assumption proved to be too simplistic, the nucleon model was a good start to further understand how to mathematically understand particles.

The next step to model particles was made when Cassen and Condon supposed that the difference between proton and neutron might be similar to the difference in spin measured between particles. They called this new invariant isospin.

In practice, there is an action of $SU(2)$ on $\mathbb{C}^2$ and therefore an induced action of its Lie Algebra $\mathfrak{su}(2)$.

They expected that at certain level of energies, forces should be invariant under the action, i.e. that our particles are part of a representation and not only a self-standing Hilbert space.

Now, for forces to be invariant under a symmetry, it means that physical processes are linear maps compatible with the group action : they are intertwining operators.

An intertwining operator $F : V \to V$ is a linear operator such that $F(g\sigma) = gF(\sigma)$ for every $\sigma \in V$ and $g \in G$ where G is a group with unitary representation on V.

Now it is know that any intertwining operator respects the action of the lie algebra, in this case $su(2)$. Hence for any $T \in su(2)$ and $c \in \mathbb{C}^2$ we have that $F(Tc) = TF(c)$.

Now consider the eigen vectors $T_j \in \mathbb{C}^2$ of $su(2)$; we know there exist an eigen value for each T_j. So for $g \in su(2)$, $T_j g = (i\lambda) g$ for some eigen value λ

Thus we have that $F(T_j g) = F(i\lambda g) = i\lambda F(g)$. So $F(g)$ is also an eigen vector of $T_i \in \mathbb{C}^2$ with the same eigen value.

We know that the eigenvectors of $su(2)$ are simply the basis vectors, hence our particles.

At this stage, one sees that up to a change of variable, the isospin invariant is nothing but the eigenvalue of Lie algebra operator associated to the sub-representation in which the particle lives.

Isopin distinguishes the proton and neutron states of a nucleon; the proton is the isopin up state and the neutron is the the isopin down state.

While so far we have only talked about the 'static' particles, it is essential to also understand phenomena that allow a particle to transit from one stage to another (e.g. from neutron to proton). To do this we would need to look at larger groups than $SU(2)$ and allow for new particles as the pion. This extension is a topic that I plan to explore further in the future.

CHAPTER 5
CONCLUSION

Thus we have shown how operators and representations of groups can serve to model particles. By studying the representation of nucleons in $\mathbb{C}^2$ we see how it is possible to build representations that model the relations between fundamental particles. While this introduction is very pedestrian, it is important to remember that finding these representations and operators required the tremendous efforts of numerous physicist; so while in hindsight the mathematical process seems straight forward, creating such a straight forward model was not simple. We encourage the interested reader in further examining the standard models intricacies discovering the beauties behind this carefully crafted model.

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