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# On Gorenstein Fiber Products and Applications

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## ON GORENSTEIN FIBER PRODUCTS AND APPLICATIONS

SAEED NASSEH, RYO TAKAHASHI, AND KELLER VANDEBOGERT

ABSTRACT. We show that a non-trivial fiber product  $S \times_k T$  of commutative noetherian local rings  $S, T$  with a common residue field  $k$  is Gorenstein if and only if it is a hypersurface of dimension 1. In this case, both  $S$  and  $T$  are regular rings of dimension 1. We also give some applications of this result.

## 1. INTRODUCTION

Throughout this paper,  $(S, \mathfrak{m}_S, k)$  and  $(T, \mathfrak{m}_T, k)$  are commutative noetherian local rings with a common residue field  $k$ , and  $S \times_k T$  denotes the fiber product of  $S$  and  $T$  over  $k$ . Note that,  $S \times_k T$  is the pull-back of the natural surjections  $S \xrightarrow{\pi_S} k \xleftarrow{\pi_T} T$  and

$$S \times_k T = \{(s, t) \in S \times T \mid \pi_S(s) = \pi_T(t)\}.$$

This ring is a commutative noetherian local ring with maximal ideal  $\mathfrak{m}_{S \times_k T} = \mathfrak{m}_S \oplus \mathfrak{m}_T$  and residue field  $k$ . Also,  $\mathfrak{m}_S$  and  $\mathfrak{m}_T$  are ideals of  $S \times_k T$  and there are ring isomorphisms  $S \cong (S \times_k T)/\mathfrak{m}_T$  and  $T \cong (S \times_k T)/\mathfrak{m}_S$ . If  $S \neq k \neq T$  (or equivalently,  $\mathfrak{m}_S \neq 0 \neq \mathfrak{m}_T$ ), then we say that  $S \times_k T$  is a non-trivial fiber product. (For more information about fiber products, in addition to the references introduced below, see [5, 7, 9, 10, 11, 12, 13, 15].)

In case that  $S = T$ , it is shown in [1, Theorem 1.8] that  $S \times_k S$  is Gorenstein if and only if  $S$  is a regular ring of dimension 1. (See also D'Anna [6] and Shapiro [17].) In this note we give the following generalization of [1, Theorem 1.8] which we prove in 2.5; compare this result with Ogoma [14, Theorem 4].

**Main Theorem.** *Let  $S \times_k T$  be a non-trivial fiber product. The ring  $S \times_k T$  is Gorenstein if and only if it is a hypersurface of dimension 1, and then both  $S$  and  $T$  are regular rings of dimension 1.*

*Moreover, in this case,  $S \times_k T$  is isomorphic to a fiber product  $Q/(p) \times_k Q/(q)$ , where  $Q$ ,  $Q/(p)$ , and  $Q/(q)$  are regular local rings with residue field  $k$  and  $p, q \in Q$  are prime elements.*

As applications of this theorem, we give a stronger version of a result of Takahashi [18, Theorem A] and prove a generalization of [2, Proposition 3.10] due to Atkins and Vraciu; see Corollaries 3.6 and 3.8.

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## 2. PROOF OF MAIN THEOREM

For the rest of this paper,  $(R, \mathfrak{m}_R, \ell)$  will be a commutative noetherian local ring, and recall that the rings  $S$  and  $T$  are introduced in the Introduction. The following result is proved in [1, Proposition 1.7] when  $S$  and  $T$  are artinian.

**Proposition 2.1.** *Let  $S \times_k T$  be non-trivial fiber product. Then  $S \times_k T$  is Cohen-Macaulay if and only if  $\dim S \times_k T \leq 1$  and  $S$  and  $T$  are Cohen-Macaulay with  $\dim S = \dim S \times_k T = \dim T$ .*

*Proof.* The assertion follows from the equalities

$$\begin{aligned} \dim S \times_k T &= \max\{\dim S, \dim T\} \\ \text{depth } S \times_k T &= \min\{\text{depth } S, \text{depth } T, 1\}. \end{aligned}$$

(See Lescot [10], or Christensen, Striuli, and Veliche [5, Remark 3.1].) □

The following lemma will be used in the proof of Main Theorem.

**Lemma 2.2.** *Assume that  $R$  is a hypersurface of dimension 1, and let  $I$  be a non-zero ideal of  $R$  such that  $R/I$  is a regular ring of dimension 1. Then  $R/I \cong Q/(f)$ , where  $Q$  is a regular local ring and  $f \in Q$  is a prime element.*

*Proof.* Let  $R \cong Q/(g)$ , where  $(Q, \mathfrak{m}_Q, \ell)$  is a 2-dimensional regular local ring and  $g \in \mathfrak{m}_Q$ . Since  $I$  is a prime ideal of  $R$ , it corresponds to a prime ideal  $\mathfrak{q}/(g)$  of  $Q/(g)$ , where  $\mathfrak{q} \in \text{Spec}(Q) \cap V((g))$ . If  $g = g_1 g_2 \cdots g_n$  is a prime factorization of  $g$  in  $Q$ , then there exists an integer  $1 \leq i \leq n$  such that  $g_i \in \mathfrak{q}$ . Let  $f := g_i$ , and note that  $\text{ht}_Q(\mathfrak{q}) = 1 = \text{ht}_Q((f))$  because  $\text{ht}_R(I) = 0$ . Hence,  $\mathfrak{q} = (f)$ . Therefore,

$$\frac{R}{I} \cong \frac{Q/(g)}{\mathfrak{q}/(g)} = \frac{Q/(g)}{(f)/(g)} \cong \frac{Q}{(f)}$$

as desired. □

Next we introduce some notations and discuss some results from [9] and [10] to use in the proof of our Main Theorem. (See also [5].)

**2.3.** Let  $M$  be a finitely generated  $R$ -module. Recall that the *Poincaré series* and the *Bass series* of  $M$ , denoted  $P_R^M(t)$  and  $I_M^R(t)$ , respectively, are the formal Laurent series defined as follows:

$$\begin{aligned} P_R^M(t) &:= \sum_{i \geq 0} \text{rank}_\ell(\text{Ext}_R^i(M, \ell)) t^i \\ I_M^R(t) &:= \sum_{i \geq 0} \text{rank}_\ell(\text{Ext}_R^i(\ell, M)) t^i. \end{aligned}$$

We simply denote  $I_M^R(t)$  by  $I_R(t)$ . The coefficient of  $t^{\text{depth } R}$  in  $I_R(t)$  is called *type* of  $R$ , and is denoted  $\gamma_R$ . Note that  $\gamma_R \neq 0$  and all the coefficients of  $t^i$  in  $I_R(t)$  for  $i < \text{depth } R$  are zero. Also, note that the constant term in  $P_R^\ell(t)$  is 1.

**2.4.** By Kostrikin and Šafarevič [9] we have the equality

$$\frac{1}{P_{S \times_k T}^k(t)} = \frac{1}{P_S^k(t)} + \frac{1}{P_T^k(t)} - 1 \tag{2.4.1}$$

which gives a relation between Poincaré series of  $k$  over  $S \times_k T$  and over the rings  $S$  and  $T$ . Also, by a result of Lescot [10, Theorem 3.1] we have the following formulas:

If  $S$  and  $T$  are singular, then

$$\frac{I_{S \times_k T}(t)}{P_{S \times_k T}^k(t)} = t + \frac{I_S(t)}{P_S^k(t)} + \frac{I_T(t)}{P_T^k(t)}. \quad (2.4.2)$$

If  $S$  is singular and  $T$  is regular with  $\dim T = n$ , then

$$\frac{I_{S \times_k T}(t)}{P_{S \times_k T}^k(t)} = t + \frac{I_S(t)}{P_S^k(t)} - \frac{t^{n+1}}{(1+t)^n}. \quad (2.4.3)$$

If  $S$  and  $T$  are regular with  $\dim S = m$  and  $\dim T = n$ , then

$$\frac{I_{S \times_k T}(t)}{P_{S \times_k T}^k(t)} = t - \frac{t^{m+1}}{(1+t)^m} - \frac{t^{n+1}}{(1+t)^n}. \quad (2.4.4)$$

We are now ready to prove the Main Theorem.

**2.5 (Proof of Main Theorem).** Assume that  $A := S \times_k T$  is a Gorenstein ring. By Proposition 2.1, we have  $\dim A \leq 1$  and  $S$  and  $T$  are Cohen-Macaulay with  $\dim S = \dim A = \dim T$ . We prove the theorem by considering the following three cases, and when using the Poincaré and Bass series, we simply write  $I$  and  $P$  instead of  $I(t)$  and  $P(t)$ .

Case 1: Assume that  $S$  and  $T$  are singular. Then by (2.4.1) and (2.4.2) we have

$$I_A (P_T^k + P_S^k - P_T^k P_S^k) = t P_T^k P_S^k + I_S P_T^k + I_T P_S^k. \quad (2.5.1)$$

If  $\dim A = 0$ , then both  $S$  and  $T$  are Cohen-Macaulay of dimension zero. Now by looking at the constant terms on the left and right of (2.5.1) we obtain  $1 = \gamma_A = \gamma_S + \gamma_T$ . But this is impossible because  $\gamma_S$  and  $\gamma_T$  are positive integers.

If  $\dim A = 1$ , then  $S$  and  $T$  are Cohen-Macaulay of dimension one. Now by looking at the coefficient of  $t$  on the left and right of (2.5.1) we obtain  $1 = \gamma_A = 1 + \gamma_S + \gamma_T$ . Hence,  $\gamma_S + \gamma_T = 0$ , which is again impossible. Therefore, both of  $S$  and  $T$  cannot be singular, and Case 1 does not hold.

Case 2: Assume that  $S$  is singular and  $T$  is regular with  $\dim T = n$ . Then it follows from (2.4.1) and (2.4.3) that

$$I_A (P_T^k + P_S^k - P_T^k P_S^k) (1+t)^n = (t(1+t)^n - t^{n+1}) P_T^k P_S^k + (1+t)^n I_S P_T^k. \quad (2.5.2)$$

If  $\dim A = 0$ , then we have  $n = 0$ . Since  $T$  is regular, we must have  $T = k$ , which is a contradiction with our assumption.

If  $\dim A = 1$ , then  $n = 1$ . Now by looking at the coefficient of  $t$  on the left and right of (2.5.2) we obtain  $1 = \gamma_A = 1 + \gamma_S$ . This implies that  $\gamma_S = 0$ , which is impossible. Hence, Case 2 also does not hold.

Therefore, the only possibility is the following case.

Case 3. Both  $S$  and  $T$  are regular rings. If  $\dim A = 0$ , then both  $S$  and  $T$  have dimension zero, and hence, both are equal to  $k$ . This contradiction shows that we must have  $\dim A = 1 = \dim S = \dim T$ . Therefore, by [5, (3.2) Observation], the ring  $A$  is a hypersurface of dimension one.

For the second part of the Main Theorem, note that  $S \cong A/\mathfrak{m}_T$  and  $T \cong A/\mathfrak{m}_S$ . Hence the assertion follows from Lemma 2.2 and its proof.  $\square$

We conclude this section with the following result that will be used later.

**Proposition 2.6.** *A non-trivial fiber product  $A := S \times_k T$  is not regular.*

*Proof.* If  $A$  is a regular ring, then by Proposition 2.1 we have  $\dim A \leq 1$ . Now by the Auslander-Buchsbaum formula we have  $\text{pd}_A(A/\mathfrak{m}_T) \leq 1$ . This implies that  $\text{pd}_A(\mathfrak{m}_T) = 0$ , and hence  $\mathfrak{m}_T$  is a free  $A$ -module. But this cannot happen because  $\mathfrak{m}_S \mathfrak{m}_T = 0$ , and  $\mathfrak{m}_S \neq 0$ . Therefore,  $A$  is not a regular ring.  $\square$

### 3. APPLICATIONS

This section contains some applications of the Main Theorem. In particular, we give a stronger version of a result of Takahashi and prove a generalization of a result of Atkins and Vraciu; see Corollaries 3.6 and 3.8 below.

We start with a result of Ogoma [13, Lemma 3.1] that plays an essential role in this section.

**3.1.** Let  $\mathfrak{a} \subseteq R$  be an ideal of  $R$  that has a decomposition  $\mathfrak{a} = I \oplus J$ , where  $I$  and  $J$  are non-zero ideals of  $R$ . Then there is an isomorphism  $R \cong (R/I) \times_{R/\mathfrak{a}} (R/J)$ . This isomorphism is naturally defined by  $r \mapsto (r + I, r + J)$  for  $r \in R$ .

As an immediate observation of this discussion we record the following result.

**Proposition 3.2.** *A local ring is a non-trivial fiber product of the form  $S \times_k T$  if and only if its maximal ideal is decomposable.*

From Proposition 2.6 we obtain the following result.

**Corollary 3.3.** *If  $\mathfrak{m}_R$  is decomposable, then  $R$  is not regular.*

The next result follows directly from [12, Corollary 4.2].

**Corollary 3.4.** *Assume that  $\mathfrak{m}_R$  is decomposable. For finitely generated  $R$ -modules  $M$  and  $N$  if  $\text{Ext}_R^i(M, N) = 0$  for all  $i \gg 0$ , then  $\text{pd}_R(M) \leq 1$  or  $\text{id}_R(N) \leq 1$ .*

**Remark 3.5.** Corollary 3.4 shows in particular that if  $\mathfrak{m}_R$  is decomposable, then  $R$  satisfies the Auslander-Reiten Conjecture, that is, if for a finitely generated  $R$ -module  $M$  we have  $\text{Ext}_R^i(M, M \oplus R) = 0$  for all  $i > 0$ , then  $M$  is a free  $R$ -module. (See [4] for details about this conjecture.)

The following is a stronger version of a result of Takahashi [18, Theorem A]. Note that in this corollary we do not assume that  $R$  is complete; see Remark 3.7.

**Corollary 3.6.** *If  $\mathfrak{m}_R$  is decomposable, then the following are equivalent.*

- (i) *There is a finitely generated  $R$ -module  $E$  of finite injective dimension such that  $\text{Ext}_R^i(E, R) = 0$  for all  $i \gg 0$ ;*
- (ii)  *$R$  is Gorenstein;*
- (iii)  *$R$  is a hypersurface of dimension 1. In this case,  $R$  is isomorphic to a fiber product  $Q/(p) \times_\ell Q/(q)$ , where  $Q$ ,  $Q/(p)$ , and  $Q/(q)$  are regular local rings with residue field  $\ell$  and  $p, q \in Q$  are prime elements;*
- (iv) *There is a finitely generated  $R$ -module  $M$  with infinite projective dimension such that  $\text{Ext}_R^i(M, R) = 0$  for all  $i \gg 0$ .*

*Proof.* By Proposition 3.2, the ring  $R$  is a non-trivial fiber product. Let us assume that  $R = S \times_k T$ . (In particular,  $\ell = k$  in this case.)

(i)  $\implies$  (ii). It follows from our vanishing assumption and Corollary 3.4 that  $R$  is Gorenstein or  $\text{pd}_R(E) < \infty$ . In the latter case,  $R$  is also Gorenstein by Foxby [8].

(ii)  $\implies$  (iii) follows directly from the Main Theorem.

(iii)  $\implies$  (iv). By Corollary 3.3, the ring  $R$  is not regular. Thus, by Auslander-Buchsbaum and Serre [3, 16] we have  $\text{pd}_R(k) = \infty$ . Since  $R$  is Gorenstein we also have  $\text{Ext}_R^i(k, R) = 0$  for all  $i \gg 0$ .

(iv)  $\implies$  (i). Since  $M$  has infinite projective dimension, our vanishing assumption and Corollary 3.4 imply that  $R$  is a Gorenstein ring. This completes the proof.  $\square$

**Remark 3.7.** In Corollary 3.6, if we further assume that  $R$  is a quotient of a regular ring, then by [18, Theorem 3.2.4] the ring  $R$  is isomorphic to a quotient  $A/(xy)$  of a regular local ring  $A$  of dimension 2, where  $x, y$  is a regular system of parameters for  $A$ .

The following is a generalization of [2, Proposition 3.10]. Recall that a finitely generated  $R$ -module  $X$  is *totally reflexive* if  $\text{Hom}_R(\text{Hom}_R(X, R), R) \cong X$  and  $\text{Ext}_R^i(X, R) = 0 = \text{Ext}_R^i(\text{Hom}_R(X, R), R)$  for all  $i > 0$ .

**Corollary 3.8.** *Assume that  $\mathfrak{m}_R$  is decomposable. If  $R$  is artinian, then  $R$  has no non-free finitely generated module  $M$  such that  $\text{Ext}_R^i(M, R) = 0$  for all  $i \gg 0$ . In particular,  $R$  has no non-free totally reflexive modules.*

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