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ON GORENSTEIN FIBER PRODUCTS AND APPLICATIONS

SAEED NASSEH, RYO TAKAHASHI, AND KELLER VANDEBOGERT

Abstract. We show that a non-trivial fiber product $S \times_k T$ of commutative noetherian local rings $S, T$ with a common residue field $k$ is Gorenstein if and only if it is a hypersurface of dimension 1. In this case, both $S$ and $T$ are regular rings of dimension 1. We also give some applications of this result.

1. Introduction

Throughout this paper, $(S, m_S, k)$ and $(T, m_T, k)$ are commutative noetherian local rings with a common residue field $k$, and $S \times_k T$ denotes the fiber product of $S$ and $T$ over $k$. Note that, $S \times_k T$ is the pull-back of the natural surjections $S \xrightarrow{\pi_S} k \xleftarrow{\pi_T} T$ and

$$S \times_k T = \{(s, t) \in S \times T \mid \pi_S(s) = \pi_T(t)\}.$$ 

This ring is a commutative noetherian local ring with maximal ideal $m_{S \times_k T} = m_S \oplus m_T$ and residue field $k$. Also, $m_S$ and $m_T$ are ideals of $S \times_k T$ and there are ring isomorphisms $S \cong (S \times_k T)/m_T$ and $T \cong (S \times_k T)/m_S$. If $S \neq k \neq T$ (or equivalently, $m_S \neq 0 \neq m_T$), then we say that $S \times_k T$ is a non-trivial fiber product. (For more information about fiber products, in addition to the references introduced below, see [5, 7, 9, 10, 11, 12, 13, 15].)

In case that $S = T$, it is shown in [1, Theorem 1.8] that $S \times_k S$ is Gorenstein if and only if $S$ is a regular ring of dimension 1. (See also D’Anna [6] and Shapiro [17].) In this note we give the following generalization of [1, Theorem 1.8] which we prove in 2.5; compare this result with Ogoma [14, Theorem 4].

Main Theorem. Let $S \times_k T$ be a non-trivial fiber product. The ring $S \times_k T$ is Gorenstein if and only if it is a hypersurface of dimension 1, and then both $S$ and $T$ are regular rings of dimension 1.

Moreover, in this case, $S \times_k T$ is isomorphic to a fiber product $Q/(p) \times_k Q/(q)$, where $Q$, $Q/(p)$, and $Q/(q)$ are regular local rings with residue field $k$ and $p, q \in Q$ are prime elements.

As applications of this theorem, we give a stronger version of a result of Takahashi [18, Theorem A] and prove a generalization of [2, Proposition 3.10] due to Atkins and Vraciu; see Corollaries 3.6 and 3.8.

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2. Proof Of Main Theorem

For the rest of this paper, $(R, \mathfrak{m}_R, \ell)$ will be a commutative noetherian local ring, and recall that the rings $S$ and $T$ are introduced in the Introduction. The following result is proved in [1] Proposition 1.7 when $S$ and $T$ are artinian.

Proposition 2.1. Let $S \times_k T$ be non-trivial fiber product. Then $S \times_k T$ is Cohen-Macaulay if and only if $\dim S \times_k T \leq 1$ and $S$ and $T$ are Cohen-Macaulay with $\dim S = \dim S \times_k T = \dim T$.

Proof. The assertion follows from the equalities
\[
\dim S \times_k T = \max\{\dim S, \dim T\}
\]
\[
\depth S \times_k T = \min\{\depth S, \depth T, 1\}.
\]
(See Lescot [10], or Christensen, Striuli, and Veliche [5, Remark 3.1].) □

The following lemma will be used in the proof of Main Theorem.

Lemma 2.2. Assume that $R$ is a hypersurface of dimension 1, and let $I$ be a non-zero ideal of $R$ such that $R/I$ is a regular ring of dimension 1. Then $R/I \cong Q/(f)$, where $Q$ is a regular local ring and $f \in Q$ is a prime element.

Proof. Let $R \cong Q/(g)$, where $(Q, \mathfrak{m}_Q, \ell)$ is a 2-dimensional regular local ring and $g \in \mathfrak{m}_Q$. Since $I$ is a prime ideal of $R$, it corresponds to a prime ideal $q/(g)$ of $Q/(g)$, where $q \in \Spec(Q) \cap V((g))$. If $g = g_1 g_2 \cdots g_n$ is a prime factorization of $g$ in $Q$, then there exists an integer $1 \leq i \leq n$ such that $g_i \in q$. Let $f := g_i$, and note that $\ht_Q(q) = 1 = \ht_Q((f))$ because $\ht_R(I) = 0$. Hence, $q = (f)$. Therefore,
\[
\frac{R}{T} \cong \frac{Q/(g)}{q/(g)} = \frac{Q/(g)}{(f)/(g)} \cong \frac{Q}{(f)}
\]
as desired. □

Next we introduce some notations and discuss some results from [9] and [10] to use in the proof of our Main Theorem. (See also [5].)

2.3. Let $M$ be a finitely generated $R$-module. Recall that the Poincaré series and the Bass series of $M$, denoted $P^R_M(t)$ and $I^R_M(t)$, respectively, are the formal Laurent series defined as follows:
\[
P^R_M(t) := \sum_{i \geq 0} \rank_R(\Ext^i_R(M, \ell))t^i
\]
\[
I^R_M(t) := \sum_{i \geq 0} \rank_R(\Ext^i_R(\ell, M))t^i.
\]
We simply denote $I^R_M(t)$ by $I_R(t)$. The coefficient of $t^{\depth R}$ in $I_R(t)$ is called type of $R$, and is denoted $\gamma_R$. Note that $\gamma_R \neq 0$ and all the coefficients of $t^i$ in $I_R(t)$ for $i < \depth R$ are zero. Also, note that the constant term in $P^R_M(t)$ is 1.

2.4. By Kostrikin and Šafarevič [9] we have the equality
\[
\frac{1}{P^k_{S \times_k T}(t)} = \frac{1}{P^k_S(t)} + \frac{1}{P^k_T(t)} - 1
\]
(2.4.1)
which gives a relation between Poincaré series of $k$ over $S \times_k T$ and over the rings $S$ and $T$. Also, by a result of Lescot [10] Theorem 3.1] we have the following formulas:
If $S$ and $T$ are singular, then
\[
\frac{I_{S \times T}(t)}{P^k_{S \times T}(t)} = t + \frac{I_S(t)}{P^k_S(t)} + \frac{I_T(t)}{P^k_T(t)}.
\] (2.4.2)

If $S$ is singular and $T$ is regular with $\dim T = n$, then
\[
\frac{I_{S \times T}(t)}{P^k_{S \times T}(t)} = t + \frac{I_S(t)}{P^k_S(t)} - \frac{t^{n+1}}{(1+t)^n}.
\] (2.4.3)

If $S$ and $T$ are regular with $\dim S = m$ and $\dim T = n$, then
\[
\frac{I_{S \times T}(t)}{P^k_{S \times T}(t)} = t - \frac{t^{m+1}}{(1+t)^m} - \frac{t^{n+1}}{(1+t)^n}.
\] (2.4.4)

We are now ready to prove the Main Theorem.

2.5 (Proof of Main Theorem). Assume that $A := S \times_k T$ is a Gorenstein ring. By Proposition 2.1, we have $\dim A \leq 1$ and $S$ and $T$ are Cohen-Macaulay with $\dim S = \dim A = \dim T$. We prove the theorem by considering the following three cases, and when using the Poincaré and Bass series, we simply write $I$ and $P$ instead of $I(t)$ and $P(t)$.

Case 1: Assume that $S$ and $T$ are singular. Then by (2.4.1) and (2.4.2), we have
\[
I_A \left( P^k_T + P^k_S - P^k_{S \times T} \right) = tP^k_T P^k_S + I_T P^k_S + I_T P^k_S.
\] (2.5.1)

If $\dim A = 0$, then both $S$ and $T$ are Cohen-Macaulay of dimension zero. Now by looking at the constant terms on the left and of (2.5.1) we obtain $1 = \gamma_A = \gamma_S + \gamma_T$. But this is impossible because $\gamma_S$ and $\gamma_T$ are positive integers.

If $\dim A = 1$, then $S$ and $T$ are Cohen-Macaulay of dimension one. Now by looking at the coefficient of $t$ on the left and right of (2.5.1) we obtain $1 = \gamma_A = 1 + \gamma_S + \gamma_T$. Hence, $\gamma_S + \gamma_T = 0$, which is again impossible. Therefore, both of $S$ and $T$ cannot be singular, and Case 1 does not hold.

Case 2: Assume that $S$ is singular and $T$ is regular with $\dim T = n$. Then it follows from (2.4.1) and (2.4.3) that
\[
I_A \left( P^k_T + P^k_S - P^k_{S \times T} \right)(1 + t)^n = (t(1+t)^n - t^{n+1}) P^k_T P^k_S + (1 + t)^n I_T P^k_S.
\] (2.5.2)

If $\dim A = 0$, then we have $n = 0$. Since $T$ is regular, we must have $T = k$, which is a contradiction with our assumption.

If $\dim A = 1$, then $n = 1$. Now by looking at the coefficient of $t$ on the left and right of (2.5.2), we obtain $1 = \gamma_A = 1 + \gamma_S$. This implies that $\gamma_S = 0$, which is impossible. Hence, Case 2 also does not hold.

Therefore, the only possibility is the following case.

Case 3. Both $S$ and $T$ are regular rings. If $\dim A = 0$, then both $S$ and $T$ have dimension zero, and hence, both are equal to $k$. This contradiction shows that we must have $\dim A = 1 = \dim S = \dim T$. Therefore, by [5] (3.2) Observation, the ring $A$ is a hypersurface of dimension one.

For the second part of the Main Theorem, note that $S \cong A/m_T$ and $T \cong A/m_S$. Hence the assertion follows from Lemma 2.2 and its proof.

We conclude this section with the following result that will be used later.

Proposition 2.6. A non-trivial fiber product $A := S \times_k T$ is not regular.
If $A$ is a regular ring, then by Proposition 2.1 we have $\dim A \leq 1$. Now by the Auslander-Buchsbaum formula we have $\text{pd}_A(A/m_T) = 0$, and hence $m_T$ is a free $A$-module. But this cannot happen because $m_Sm_T = 0$, and $m_S \neq 0$. Therefore, $A$ is not a regular ring.

3. Applications

This section contains some applications of the Main Theorem. In particular, we give a stronger version of a result of Takahashi and prove a generalization of a result of Atkins and Vraciu; see Corollaries 3.6 and 3.8 below.

We start with a result of Ogoma [13, Lemma 3.1] that plays an essential role in this section.

3.1. Let $a \subseteq R$ be an ideal of $R$ that has a decomposition $a = I \oplus J$, where $I$ and $J$ are non-zero ideals of $R$. Then there is an isomorphism $R \cong (R/I) \times R/a(R/J)$. This isomorphism is naturally defined by $r \mapsto (r + I, r + J)$ for $r \in R$.

As an immediate observation of this discussion we record the following result.

Proposition 3.2. A local ring is a non-trivial fiber product of the form $S \times_k T$ if and only if its maximal ideal is decomposable.

From Proposition 2.6 we obtain the following result.

Corollary 3.3. If $m_R$ is decomposable, then $R$ is not regular.

The next result follows directly from [12, Corollary 4.2].

Corollary 3.4. Assume that $m_R$ is decomposable. For finitely generated $R$-modules $M$ and $N$ if $\text{Ext}_R^i(M, N) = 0$ for all $i \gg 0$, then $\text{pd}_R(M) \leq 1$ or $\text{id}_R(N) \leq 1$.

Remark 3.5. Corollary 3.4 shows in particular that if $m_R$ is decomposable, then $R$ satisfies the Auslander-Reiten Conjecture, that is, if for a finitely generated $R$-module $M$ we have $\text{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$, then $M$ is a free $R$-module. (See [4] for details about this conjecture.)

The following is a stronger version of a result of Takahashi [18, Theorem A]. Note that in this corollary we do not assume that $R$ is complete; see Remark 3.7.

Corollary 3.6. If $m_R$ is decomposable, then the following are equivalent.

(i) There is a finitely generated $R$-module $E$ of finite injective dimension such that $\text{Ext}_R^i(E, R) = 0$ for all $i \gg 0$;
(ii) $R$ is Gorenstein;
(iii) $R$ is a hypersurface of dimension 1. In this case, $R$ is isomorphic to a fiber product $Q/(p) \times_\ell Q/(q)$, where $Q$, $Q/(p)$, and $Q/(q)$ are regular local rings with residue field $\ell$ and $p, q \in Q$ are prime elements;
(iv) There is a finitely generated $R$-module $M$ with infinite projective dimension such that $\text{Ext}_R^i(M, R) = 0$ for all $i \gg 0$.

Proof. By Proposition 3.2, the ring $R$ is a non-trivial fiber product. Let us assume that $R = S \times_k T$. (In particular, $\ell = k$ in this case.)

(i) $\implies$ (ii). It follows from our vanishing assumption and Corollary 3.4 that $R$ is Gorenstein or $\text{pd}_R(E) < \infty$. In the latter case, $R$ is also Gorenstein by Foxby [8].

(ii) $\implies$ (iii) follows directly from the Main Theorem.
(iii) \implies (iv). By Corollary 3.3 the ring $R$ is not regular. Thus, by Auslander-Buchsbaum and Serre [3, 16] we have $pd_R(k) = \infty$. Since $R$ is Gorenstein we also have $\text{Ext}_R^i(k, R) = 0$ for all $i \gg 0$.

(iv) \implies (i). Since $M$ has infinite projective dimension, our vanishing assumption and Corollary 3.4 imply that $R$ is a Gorenstein ring. This completes the proof. □

Remark 3.7. In Corollary 3.6, if we further assume that $R$ is a quotient of a regular ring, then by [18, Theorem 3.2.4] the ring $R$ is isomorphic to a quotient $A/(xy)$ of a regular local ring $A$ of dimension 2, where $x, y$ is a regular system of parameters for $A$.

The following is a generalization of [2, Proposition 3.10]. Recall that a finitely generated $R$-module $X$ is \textit{totally reflexive} if $\text{Hom}_R(\text{Hom}_R(X, R), R) \cong X$ and $\text{Ext}_R^i(X, R) = 0 = \text{Ext}_R^i(\text{Hom}_R(X, R), R)$ for all $i > 0$.

Corollary 3.8. Assume that $m_R$ is decomposable. If $R$ is artinian, then $R$ has no non-free finitely generated module $M$ such that $\text{Ext}_R^i(M, R) = 0$ for all $i \gg 0$. In particular, $R$ has no non-free totally reflexive modules.

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