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On Gorenstein Fiber Products and Applications

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ON GORENSTEIN FIBER PRODUCTS AND APPLICATIONS

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Abstract. We show that a non-trivial fiber product $S \times_k T$ of commutative noetherian local rings S, T with a common residue field k is Gorenstein if and only if it is a hypersurface of dimension 1. In this case, both S and T are regular rings of dimension 1. We also give some applications of this result.

1. Introduction

Throughout this paper, $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ are commutative noetherian local rings with a common residue field k, and $S \times_k T$ denotes the fiber product of S and T over k. Note that, $S \times_k T$ is the pull-back of the natural surjections $S \xrightarrow{\pi_S} k \xleftarrow{\pi_T} T$ and

$$S \times_k T = \{(s,t) \in S \times T \mid \pi_S(s) = \pi_T(t)\}.$$

This ring is a commutative noetherian local ring with maximal ideal $\mathfrak{m}_{S\times_k T} = \mathfrak{m}_S \oplus \mathfrak{m}_T$ and residue field k. Also, $\mathfrak{m}_S$ and $\mathfrak{m}_T$ are ideals of $S \times_k T$ and there are ring isomorphisms $S \cong (S \times_k T)/\mathfrak{m}_T$ and $T \cong (S \times_k T)/\mathfrak{m}_S$. If $S \neq k \neq T$ (or equivalently, $\mathfrak{m}_S \neq 0 \neq \mathfrak{m}_T$), then we say that $S \times_k T$ is a non-trivial fiber product. (For more information about fiber products, in addition to the references introduced below, see [5, 7, 9, 10, 11, 12, 13, 15].)

In case that $S = T$, it is shown in [1, Theorem 1.8] that $S \times_k S$ is Gorenstein if and only if S is a regular ring of dimension 1. (See also D'Anna [6] and Shapiro [17].) In this note we give the following generalization of [1, Theorem 1.8] which we prove in 2.5; compare this result with Ogoma [14, Theorem 4].

Main Theorem. *Let $S \times_k T$ be a non-trivial fiber product. The ring $S \times_k T$ is Gorenstein if and only if it is a hypersurface of dimension 1, and then both S and T are regular rings of dimension 1.*

Moreover, in this case, $S \times_k T$ is isomorphic to a fiber product $Q/(p) \times_k Q/(q)$, where Q, $Q/(p)$, and $Q/(q)$ are regular local rings with residue field k and $p, q \in Q$ are prime elements.

As applications of this theorem, we give a stronger version of a result of Takahashi [18, Theorem A] and prove a generalization of [2, Proposition 3.10] due to Atkins and Vraciu; see Corollaries 3.6 and 3.8.

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2. Proof Of Main Theorem

For the rest of this paper, $(R, \mathfrak{m}_R, \ell)$ will be a commutative noetherian local ring, and recall that the rings S and T are introduced in the Introduction. The following result is proved in [1, Proposition 1.7] when S and T are artinian.

Proposition 2.1. *Let $S \times_k T$ be non-trivial fiber product. Then $S \times_k T$ is Cohen-Macaulay if and only if $\dim S \times_k T \leqslant 1$ and S and T are Cohen-Macaulay with $\dim S = \dim S \times_k T = \dim T$.*

Proof. The assertion follows from the equalities

$$\dim S \times_k T = \max\{\dim S, \dim T\}$$
$$\operatorname{depth} S \times_k T = \min\{\operatorname{depth} S, \operatorname{depth} T, 1\}.$$

(See Lescot [10], or Christensen, Striuli, and Veliche [5, Remark 3.1].) □

The following lemma will be used in the proof of Main Theorem.

Lemma 2.2. *Assume that R is a hypersurface of dimension 1, and let I be a non-zero ideal of R such that R/I is a regular ring of dimension 1. Then $R/I \cong Q/(f)$, where Q is a regular local ring and $f \in Q$ is a prime element.*

Proof. Let $R \cong Q/(g)$, where $(Q, \mathfrak{m}_Q, \ell)$ is a 2-dimensional regular local ring and $g \in \mathfrak{m}_Q$. Since I is a prime ideal of R, it corresponds to a prime ideal $\mathfrak{q}/(g)$ of $Q/(g)$, where $\mathfrak{q} \in \operatorname{Spec}(Q) \cap V((g))$. If $g = g_1 g_2 \cdots g_n$ is a prime factorization of g in Q, then there exists an integer $1 \leqslant i \leqslant n$ such that $g_i \in \mathfrak{q}$. Let $f := g_i$, and note that $\operatorname{ht}_Q(\mathfrak{q}) = 1 = \operatorname{ht}_Q((f))$ because $\operatorname{ht}_R(I) = 0$. Hence, $\mathfrak{q} = (f)$. Therefore,

$$\frac{R}{I} \cong \frac{Q/(g)}{\mathfrak{q}/(g)} = \frac{Q/(g)}{(f)/(g)} \cong \frac{Q}{(f)}$$

as desired. □

Next we introduce some notations and discuss some results from [9] and [10] to use in the proof of our Main Theorem. (See also [5].)

2.3. Let M be a finitely generated R-module. Recall that the *Poincaré series* and the *Bass series* of M, denoted $P_R^M(t)$ and $I_M^R(t)$, respectively, are the formal Laurent series defined as follows:

$$P_R^M(t) := \sum_{i \geqslant 0} \operatorname{rank}_\ell(\operatorname{Ext}_R^i(M, \ell))t^i$$
$$I_M^R(t) := \sum_{i \geqslant 0} \operatorname{rank}_\ell(\operatorname{Ext}_R^i(\ell, M))t^i.$$

We simply denote $I_R^R(t)$ by $I_R(t)$. The coefficient of $t^{\operatorname{depth} R}$ in $I_R(t)$ is called *type of* R, and is denoted γ_R. Note that $\gamma_R \neq 0$ and all the coefficients of t^i in $I_R(t)$ for $i < \operatorname{depth} R$ are zero. Also, note that the constant term in $P_R^\ell(t)$ is 1.

2.4. By Kostrikin and Šafarevič [9] we have the equality

$$\frac{1}{P_{S \times_k T}^k(t)} = \frac{1}{P_S^k(t)} + \frac{1}{P_T^k(t)} - 1 \tag{2.4.1}$$

which gives a relation between Poincaré series of k over $S \times_k T$ and over the rings S and T. Also, by a result of Lescot [10, Theorem 3.1] we have the following formulas:

If S and T are singular, then

$$\frac{I_{S\times_k T}(t)}{P^k_{S\times_k T}(t)} = t + \frac{I_S(t)}{P^k_S(t)} + \frac{I_T(t)}{P^k_T(t)}. \tag{2.4.2}$$

If S is singular and T is regular with $\dim T = n$, then

$$\frac{I_{S\times_k T}(t)}{P^k_{S\times_k T}(t)} = t + \frac{I_S(t)}{P^k_S(t)} - \frac{t^{n+1}}{(1+t)^n}. \tag{2.4.3}$$

If S and T are regular with $\dim S = m$ and $\dim T = n$, then

$$\frac{I_{S\times_k T}(t)}{P^k_{S\times_k T}(t)} = t - \frac{t^{m+1}}{(1+t)^m} - \frac{t^{n+1}}{(1+t)^n}. \tag{2.4.4}$$

We are now ready to prove the Main Theorem.

2.5 (*Proof of Main Theorem*)**.** Assume that $A := S \times_k T$ is a Gorenstein ring. By Proposition 2.1, we have $\dim A \leqslant 1$ and S and T are Cohen-Macaulay with $\dim S = \dim A = \dim T$. We prove the theorem by considering the following three cases, and when using the Poincaré and Bass series, we simply write I and P instead of $I(t)$ and $P(t)$.

Case 1: Assume that S and T are singular. Then by (2.4.1) and (2.4.2) we have

$$I_A\left(P^k_T + P^k_S - P^k_T P^k_S\right) = tP^k_T P^k_S + I_S P^k_T + I_T P^k_S. \tag{2.5.1}$$

If $\dim A = 0$, then both S and T are Cohen-Macaulay of dimension zero. Now by looking at the constant terms on the left and right of (2.5.1) we obtain $1 = \gamma_A = \gamma_S + \gamma_T$. But this is impossible because γ_S and γ_T are positive integers.

If $\dim A = 1$, then S and T are Cohen-Macaulay of dimension one. Now by looking at the coefficient of t on the left and right of (2.5.1) we obtain $1 = \gamma_A = 1 + \gamma_S + \gamma_T$. Hence, $\gamma_S + \gamma_T = 0$, which is again impossible. Therefore, both of S and T cannot be singular, and Case 1 does not hold.

Case 2: Assume that S is singular and T is regular with $\dim T = n$. Then it follows from (2.4.1) and (2.4.3) that

$$I_A\left(P^k_T + P^k_S - P^k_T P^k_S\right)(1+t)^n = \left(t(1+t)^n - t^{n+1}\right)P^k_T P^k_S + (1+t)^n I_S P^k_T. \tag{2.5.2}$$

If $\dim A = 0$, then we have $n = 0$. Since T is regular, we must have $T = k$, which is a contradiction with our assumption.

If $\dim A = 1$, then $n = 1$. Now by looking at the coefficient of t on the left and right of (2.5.2) we obtain $1 = \gamma_A = 1 + \gamma_S$. This implies that $\gamma_S = 0$, which is impossible. Hence, Case 2 also does not hold.

Therefore, the only possibility is the following case.

Case 3. Both S and T are regular rings. If $\dim A = 0$, then both S and T have dimension zero, and hence, both are equal to k. This contradiction shows that we must have $\dim A = 1 = \dim S = \dim T$. Therefore, by [5, (3.2) Observation], the ring A is a hypersurface of dimension one.

For the second part of the Main Theorem, note that $S \cong A/\mathfrak{m}_T$ and $T \cong A/\mathfrak{m}_S$. Hence the assertion follows from Lemma 2.2 and its proof. $\square$

We conclude this section with the following result that will be used later.

Proposition 2.6. *A non-trivial fiber product $A := S \times_k T$ is not regular.*

Proof. If A is a regular ring, then by Proposition 2.1 we have $\dim A \leqslant 1$. Now by the Auslander-Buchsbaum formula we have $\mathrm{pd}_A(A/\mathfrak{m}_T) \leqslant 1$. This implies that $\mathrm{pd}_A(\mathfrak{m}_T) = 0$, and hence $\mathfrak{m}_T$ is a free A-module. But this cannot happen because $\mathfrak{m}_S\mathfrak{m}_T = 0$, and $\mathfrak{m}_S \neq 0$. Therefore, A is not a regular ring. □

3. Applications

This section contains some applications of the Main Theorem. In particular, we give a stronger version of a result of Takahashi and prove a generalization of a result of Atkins and Vraciu; see Corollaries 3.6 and 3.8 below.

We start with a result of Ogoma [13, Lemma 3.1] that plays an essential role in this section.

3.1. Let $\mathfrak{a} \subseteq R$ be an ideal of R that has a decomposition $\mathfrak{a} = I \oplus J$, where I and J are non-zero ideals of R. Then there is an isomorphism $R \cong (R/I) \times_{R/\mathfrak{a}} (R/J)$. This isomorphism is naturally defined by $r \mapsto (r + I, r + J)$ for $r \in R$.

As an immediate observation of this discussion we record the following result.

Proposition 3.2. *A local ring is a non-trivial fiber product of the form $S \times_k T$ if and only if its maximal ideal is decomposable.*

From Proposition 2.6 we obtain the following result.

Corollary 3.3. *If $\mathfrak{m}_R$ is decomposable, then R is not regular.*

The next result follows directly from [12, Corollary 4.2].

Corollary 3.4. *Assume that $\mathfrak{m}_R$ is decomposable. For finitely generated R-modules M and N if $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, then $\mathrm{pd}_R(M) \leqslant 1$ or $\mathrm{id}_R(N) \leqslant 1$.*

Remark 3.5. Corollary 3.4 shows in particular that if $\mathfrak{m}_R$ is decomposable, then R satisfies the Auslander-Reiten Conjecture, that is, if for a finitely generated R-module M we have $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then M is a free R-module. (See [4] for details about this conjecture.)

The following is a stronger version of a result of Takahashi [18, Theorem A]. Note that in this corollary we do not assume that R is complete; see Remark 3.7.

Corollary 3.6. *If $\mathfrak{m}_R$ is decomposable, then the following are equivalent.*

(i) *There is a finitely generated R-module E of finite injective dimension such that $\mathrm{Ext}^i_R(E,R) = 0$ for all $i \gg 0$;*
(ii) *R is Gorenstein;*
(iii) *R is a hypersurface of dimension 1. In this case, R is isomorphic to a fiber product $Q/(p) \times_\ell Q/(q)$, where Q, $Q/(p)$, and $Q/(q)$ are regular local rings with residue field ℓ and $p, q \in Q$ are prime elements;*
(iv) *There is a finitely generated R-module M with infinite projective dimension such that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \gg 0$.*

Proof. By Proposition 3.2, the ring R is a non-trivial fiber product. Let us assume that $R = S \times_k T$. (In particular, $\ell = k$ in this case.)

(i) $\Longrightarrow$ (ii). It follows from our vanishing assumption and Corollary 3.4 that R is Gorenstein or $\mathrm{pd}_R(E) < \infty$. In the latter case, R is also Gorenstein by Foxby [8].

(ii) $\Longrightarrow$ (iii) follows directly from the Main Theorem.

(iii) $\Longrightarrow$ (iv). By Corollary 3.3, the ring R is not regular. Thus, by Auslander-Buchsbaum and Serre [3, 16] we have $\operatorname{pd}_R(k) = \infty$. Since R is Gorenstein we also have $\operatorname{Ext}^i_R(k, R) = 0$ for all $i \gg 0$.

(iv) $\Longrightarrow$ (i). Since M has infinite projective dimension, our vanishing assumption and Corollary 3.4 imply that R is a Gorenstein ring. This completes the proof. □

Remark 3.7. In Corollary 3.6, if we further assume that R is a quotient of a regular ring, then by [18, Theorem 3.2.4] the ring R is isomorphic to a quotient $A/(xy)$ of a regular local ring A of dimension 2, where x, y is a regular system of parameters for A.

The following is a generalization of [2, Proposition 3.10]. Recall that a finitely generated R-module X is *totally reflexive* if $\operatorname{Hom}_R(\operatorname{Hom}_R(X,R),R) \cong X$ and $\operatorname{Ext}^i_R(X,R) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(X,R),R)$ for all $i > 0$.

Corollary 3.8. *Assume that $\mathfrak{m}_R$ is decomposable. If R is artinian, then R has no non-free finitely generated module M such that $\operatorname{Ext}^i_R(M,R) = 0$ for all $i \gg 0$. In particular, R has no non-free totally reflexive modules.*

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