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Vanishing of Ext and Tor Over Fiber Products

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ABSTRACT. Consider a non-trivial fiber product $R = S \times_k T$ of local rings S, T with common residue field k. Given two finitely generate R-modules M and N, we show that if $\operatorname{Tor}_i^R(M,N) = 0 = \operatorname{Tor}_{i+1}^R(M,N)$ for some $i \geqslant 5$, then $\operatorname{pd}_R(M) \leqslant 1$ or $\operatorname{pd}_R(N) \leqslant 1$. From this, we deduce several consequence, for instance, that R satisfies the Auslander-Reiten Conjecture.

1. Introduction

Throughout this paper, let $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ be commutative local (meaning local and noetherian) rings. Let $S \xrightarrow{\pi_S} k \xleftarrow{\pi_T} T$ denote the natural surjections onto the common residue field, and assume that $S \neq k \neq T$. Throughout, let R denote the fiber product (i.e., the pull-back) ring

$$R := S \times_k T = \{(s,t) \in S \times T \mid \pi_S(s) = \pi_T(t)\}.$$

It is also commutative and local with maximal ideal $\mathfrak{m}_R = \mathfrak{m}_S \oplus \mathfrak{m}_T$ and residue field k. And it is universal with respect to the next commutative diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & T \arrow[d, "\pi_T"] \\ S \arrow[r, "\pi_S"] & k \end{tikzcd}$$

Also, the subsets $\mathfrak{m}_S, \mathfrak{m}_T \subseteq R$ are ideals of R, and we have ring isomorphisms $S \cong R/\mathfrak{m}_T$ and $T \cong R/\mathfrak{m}_S$.

Many results about R state that its properties reflect those of the rings S and T. For instance, Dress and Krämer [12, Remark 3] show that for every finitely generated R-module N, the second syzygy of N decomposes as a direct sum $N' \cong N_1 \oplus N_2$ where N_1 is a finitely generated S-module and N_2 is a finitely generated T-module. Other examples of this include work of Moore [19, Theorem 1.8] which shows how, given a finitely generated S-module M_1, one can use the minimal S-free resolution of M_1 with the minimal free resolutions of k over S and T to obtain the minimal R-free resolution of M_1. (Fact 2.2 below describes part of this construction.) See, e.g., the papers of Kostrikin and Šafarevič [16] and Lescot [18] for more results in this theme. The point here is that the module category of R is deeply related to the module categories of S and T.

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However, there are glaring counterpoints to this theme. For instance, one consequence of Lescot's work is the equality

$$\operatorname{depth}(R) = \min\{\operatorname{depth}(S), \operatorname{depth}(T), 1\}. \tag{$*$}$$

(See, e.g., Christensen, Striuli, and Veliche [11, (3.2.1)].) Technically, this describes the depth of R in terms of the depths of S and T, but it shows, for instance, that R is almost never Cohen-Macaulay, even when S and T are so.

The results of this paper give further counterpoints to this theme. For instance, the next result is contained in Theorems 2.7 and 2.12. In the language of Celikbas and Sather-Wagstaff [7] this result says that every R-module of infinite projective dimension is a "pd-test module".

Theorem 1.1. *Let M and N be finitely generated modules over the fiber product R.*

(a) *If $\operatorname{depth}(S) = 0$ or $\operatorname{depth}(T) = 0$ and $\operatorname{Tor}_i^R(M,N) = 0$ for some $i \geqslant 5$, then M or N is free over R.*
(b) *In general, if $\operatorname{Tor}_i^R(M,N) = 0 = \operatorname{Tor}_{i+1}^R(M,N)$ for some $i \geqslant 5$, then $\operatorname{pd}_R(M) \leqslant 1$ or $\operatorname{pd}_R(N) \leqslant 1$.*

This result gives another counterpoint to the above theme because, even if S and T have Tor-independent modules of infinite projective dimension (e.g., if $S = k[U,V,X,Y]/((U,V)^2 + (X,Y)^2) \cong T$), the theorem shows that R will not have such modules. Section 2 is primarily devoted to the proof of this result. In the subsequent Section 3, we explore the consequences for "depth formulas" over R; some of these are expected, others are surprising to us.

The final Section 4 documents Ext-vanishing results that follow from Theorem 1.1. For instance, the next result, contained in Theorem 4.5, shows that the fiber product R satisfies the Auslander-Reiten Conjecture, regardless of whether we know it for S or T.

Theorem 1.2. *Let M be a finitely generated module over the fiber product R. If $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for $i \geqslant 1$, then M is R-free.*

2. Tor-Vanishing

In this section we prove Theorem 1.1 from the introduction, beginning with some preliminary facts. Recall that R is a non-trivial fiber product, as described above.

Fact 2.1 ([12, Remark 3])**.** For every finitely generated R-module N, the second syzygy of N decomposes as a direct sum $N' \cong N_1 \oplus N_2$ where N_1 is a finitely generated S-module and N_2 is a finitely generated T-module; in other words, N_1 is a finitely generated R-module annihilated by $\mathfrak{m}_T$, and similarly for N_2. Moreover, the proof of [12, Remark 3] shows the following. The syzygy N' is a submodule of a finite-rank free R-module R^n, as the image of an R-linear map $f \colon R^m \to R^n$, and we have $N_2 = \operatorname{Im}(f) \cap \mathfrak{m}_T R^n \subseteq \mathfrak{m}_T R^n \cong \mathfrak{m}_T T^n$, and similarly for N_1. In particular, N_2 is a first syzygy over T and N_1 is a first syzygy over S.

Fact 2.2 ([19, Theorem 1.8])**.** Let M_1 be a finitely generated S-module. We describe part of a minimal R-free resolution of M_1, in terms of the following:

(1) Let $S^{\beta_2} \xrightarrow{d_2} S^{\beta_1} \xrightarrow{d_1} S^{\beta_0}$ be the beginning of a minimal S-free resolution of M_1. (In particular, β_i is the ith Betti number $\beta_i^S(M_1)$ of M_1 over S.)
(2) Let $S^{b_1} \xrightarrow{f_1} S$ be a minimal S-free presentation of k where f_1 is a $1 \times b_1$ matrix whose entries minimally generate $\mathfrak{m}_S$. (In particular, $b_1 = \beta_1^S(k)$.)

(3) Let $T^{c_2} \xrightarrow{g_2} T^{c_1} \xrightarrow{g_1} T$ be the beginning of a minimal T-free resolution of k where g_1 is a $1 \times c_1$ matrix whose entries minimally generate $\mathfrak{m}_T$. (In particular, $c_1 = \beta_1^T(k)$.)

Then a minimal R-free resolution of M_1 begins as follows.

$$R^{\beta_2} \oplus R^{c_1\beta_1} \oplus R^{b_1c_1\beta_0} \oplus R^{c_2\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_2} & \widehat{g_1} & 0 & 0 \\ 0 & 0 & \widehat{f_1} & \overline{g_2} \end{smallmatrix}\right]} R^{\beta_1} \oplus R^{c_1\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_1} & \overline{g_1} \end{smallmatrix}\right]} R^{\beta_0} \tag{2.2.1}$$

Here each entry in each matrix is induced by the corresponding map from (1)–(3) above. For instance, $\widetilde{d_2}$ uses the same matrix as d_2, only considered over R; in particular, these entries are in $\mathfrak{m}_S$. And $\widehat{g_1}$ uses β_1-many copies of the matrix for g_1; in particular, these entries are in $\mathfrak{m}_T$.

Our first lemma is akin to [20, Lemma 3.2], though our proof is vastly different.

Lemma 2.3. *Let M_1, N_1 be finitely generated S-modules, and let M_2, N_2 be finitely generated T-modules. Then there are isomorphisms over the fiber product R.*

$$\begin{aligned}
\operatorname{Tor}_1^R(M_1, N_1) &\cong \operatorname{Tor}_1^S(M_1, N_1) \oplus \left(\frac{N_1}{\mathfrak{m}_S N_1}\right)^{\beta_1^T(k)\beta_0^S(M_1)} \\
&\cong \operatorname{Tor}_1^S(M_1, N_1) \oplus \left(\frac{M_1}{\mathfrak{m}_S M_1}\right)^{\beta_1^T(k)\beta_0^S(N_1)} \\
\operatorname{Tor}_1^R(M_2, N_2) &\cong \operatorname{Tor}_1^T(M_2, N_2) \oplus \left(\frac{N_2}{\mathfrak{m}_T N_2}\right)^{\beta_1^S(k)\beta_0^T(M_2)} \\
&\cong \operatorname{Tor}_1^T(M_2, N_2) \oplus \left(\frac{M_2}{\mathfrak{m}_T M_2}\right)^{\beta_1^S(k)\beta_0^T(N_2)} \\
\operatorname{Tor}_1^R(M_1, N_2) &\cong \operatorname{Tor}_1^T(k, N_2)^{\beta_0^S(M_1)} \oplus \left(\frac{N_2}{\mathfrak{m}_T N_2}\right)^{\beta_1^S(M_1)} \\
&\cong \operatorname{Tor}_1^S(M_1, k)^{\beta_0^T(N_2)} \oplus \left(\frac{M_1}{\mathfrak{m}_S M_1}\right)^{\beta_1^T(N_2)} \\
\operatorname{Tor}_1^R(M_2, N_1) &\cong \operatorname{Tor}_1^T(k, M_2)^{\beta_0^S(N_1)} \oplus \left(\frac{M_2}{\mathfrak{m}_T M_2}\right)^{\beta_1^S(N_1)} \\
&\cong \operatorname{Tor}_1^S(N_1, k)^{\beta_0^T(M_2)} \oplus \left(\frac{N_1}{\mathfrak{m}_S N_1}\right)^{\beta_1^T(M_2)} .
\end{aligned}$$

Proof. We verify the first isomorphism. The others are obtained similarly. Compute $\operatorname{Tor}_1^R(M_1, N_1)$ by tensoring the sequence (2.2.1) with N_1.

$$N_1^{\beta_2} \oplus N_1^{c_1\beta_1} \oplus N_1^{b_1c_1\beta_0} \oplus N_1^{c_2\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_2}' & \widehat{g_1}' & 0 & 0 \\ 0 & 0 & \widehat{f_1}' & \overline{g_2}' \end{smallmatrix}\right]} N_1^{\beta_1} \oplus N_1^{c_1\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_1}' & \overline{g_1}' \end{smallmatrix}\right]} N_1^{\beta_0} \tag{2.3.1}$$

Since N_1 is an S-module, it is annihilated by $\mathfrak{m}_T$, so the fact that the entries of $\widehat{g_1}$ are in $\mathfrak{m}_T$ implies that $\widehat{g_1}' = 0$, and similarly for $\overline{g_1}'$ and $\overline{g_2}'$. Thus, the complex (2.3.1) has the following form.

$$N_1^{\beta_2} \oplus N_1^{c_1\beta_1} \oplus N_1^{b_1c_1\beta_0} \oplus N_1^{c_2\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_2}' & 0 & 0 & 0 \\ 0 & 0 & \widehat{f_1}' & 0 \end{smallmatrix}\right]} N_1^{\beta_1} \oplus N_1^{c_1\beta_0} \xrightarrow{\left[\begin{smallmatrix} \widetilde{d_1}' & 0 \end{smallmatrix}\right]} N_1^{\beta_0}$$

From this, it follows that we have

$$\begin{aligned}\operatorname{Tor}_1^R(M_1,N_1) &\cong \mathrm{H}\left(N_1^{\beta_2} \xrightarrow{\widetilde{d_2}'} N_1^{\beta_1} \xrightarrow{\widetilde{d_1}'} N_1^{\beta_0}\right) \oplus \mathrm{H}\left(N_1^{b_1c_1\beta_0} \xrightarrow{\widehat{f_1}'} N_1^{c_1\beta_0} \xrightarrow{0} N_1^{\beta_0}\right)\\ &\cong \operatorname{Tor}_1^S(M_1,N_1) \oplus \left(\frac{N_1}{\mathfrak{m}_S N_1}\right)^{c_1\beta_0}\end{aligned}$$

as desired. □

The next two lemmas are essentially applications of the previous one, for use in our first main theorem.

Lemma 2.4. *Let M_1, N_1 be finitely generated S-modules. If $\operatorname{Tor}_1^R(M_1,N_1)=0$, then $M_1=0$ or $N_1=0$.*

Proof. Assume that $M_1 \neq 0$. Hence, $\beta_0^S(M_1)\neq 0$. Also, the assumption $T\neq k$ implies that $\beta_1^T(k)\neq 0$. From Lemma 2.3, we have

$$0=\operatorname{Tor}_1^R(M_1,N_1)\cong \operatorname{Tor}_1^S(M_1,N_1)\oplus\left(\frac{N_1}{\mathfrak{m}_S N_1}\right)^{\beta_1^T(k)\beta_0^S(M_1)}$$

so $N_1=\mathfrak{m}_S N_1$. Now by Nakayama's Lemma we conclude that $N_1=0$. □

Lemma 2.5. *Let M_1, N_1 be first syzygies over S of finitely generated S-modules, and let M_2, N_2 be first syzygies over T of finitely generated T-modules. Assume that $\operatorname{depth}(S)=0$ or $\operatorname{depth}(T)=0$, and set $M:=M_1\oplus M_2$ and $N:=N_1\oplus N_2$. Assume that $\operatorname{Tor}_1^R(M,N)=0$; then $M=0$ or $N=0$.*

Proof. Assume that $M_1\oplus M_2=M\neq 0$. We need to show that $N=0$. By symmetry, assume further that $M_1\neq 0$. The assumption $\operatorname{Tor}_1^R(M,N)=0$ implies that $\operatorname{Tor}_1^R(M_1,N_1)=0$, so Lemma 2.4 implies that $N_1=0$.

Suppose by way of contradiction that $N_2\neq 0$. Another application of the Tor-vanishing assumption, using Lemma 2.3, implies that

$$\begin{aligned}0&=\operatorname{Tor}_1^R(M_1,N_2)\\ &\cong \operatorname{Tor}_1^T(k,N_2)^{\beta_0^S(M_1)}\oplus\left(\frac{N_2}{\mathfrak{m}_T N_2}\right)^{\beta_1^S(M_1)}\\ &\cong \operatorname{Tor}_1^S(M_1,k)^{\beta_0^T(N_2)}\oplus\left(\frac{M_1}{\mathfrak{m}_S M_1}\right)^{\beta_1^T(N_2)}.\end{aligned}$$

Since $\beta_0^S(M_1)\neq 0\neq\beta_0^T(N_2)$, we conclude that $\operatorname{Tor}_1^T(k,N_2)=0=\operatorname{Tor}_1^S(M_1,k)$. Hence, $M_1\neq 0$ is free over S and $N_2\neq 0$ is free over T, so S is a summand of M_1 and T is a summand of N_2. Since M_1 is a syzygy over S, we have $S\subseteq M_1\subseteq \mathfrak{m}_S S^m$ for some $m\geqslant 1$. If $\operatorname{depth}(S)=0$, this is impossible. Indeed, this implies that there is an element $t\in\mathfrak{m}_S S^m$ such that the map $S\to\mathfrak{m}_S S^m$ given by $s\mapsto st$ is a monomorphism; but the socle $\operatorname{Soc}(S)\neq 0$ is contained in the kernel of this map.

On the other hand, the fact that N_2 is a syzygy over T yields a contradiction in the case $\operatorname{depth}(T)=0$. □

Remark 2.6. The following computation

$$\operatorname{Tor}_1^R(S,T)=\operatorname{Tor}_1^R(R/\mathfrak{m}_T,R/\mathfrak{m}_S)\cong(\mathfrak{m}_S\cap\mathfrak{m}_T)/(\mathfrak{m}_S\mathfrak{m}_T)=0$$

shows the necessity of the syzygy assumptions in the previous result, since $S,T\neq 0$. The isomorphism here is standard.

The following result contains Theorem 1.1(a) from the introduction.

Theorem 2.7. *Let M, N be finitely generated modules over the fiber product R. If* $\operatorname{depth}(S) = 0$ *or* $\operatorname{depth}(T) = 0$*, then the following conditions are equivalent.*

(i) $\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \gg 0$
(ii) $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i \geqslant 1$
(iii) $\operatorname{Tor}_i^R(M, N) = 0$ *for some* $i \geqslant 5$
(iv) $\operatorname{pd}_R(M) < \infty$ *or* $\operatorname{pd}_R(N) < \infty$
(v) M *or* N *is* R*-free*

Proof. Assume for this paragraph that there exists an integer $i \geqslant 5$ such that $\operatorname{Tor}_i^R(M, N) = 0$; we prove that M or N is R-free. Let M' be the second syzygy of M, and let N' be the $(i-3)$rd syzygy of N. Dimension-shift to conclude that $\operatorname{Tor}_1^R(M', N') \cong \operatorname{Tor}_3^R(M, N') \cong \operatorname{Tor}_i^R(M, N) = 0$. The assumption $i \geqslant 5$ implies $i - 3 \geqslant 2$. Thus, Fact 2.1 implies that $M' = M_1' \oplus M_2'$ and $N' = N_1' \oplus N_2'$ where M_1', N_1' are (finitely generated) first syzygies over S, and M_2', N_2' are (finitely generated) first syzygies over T. Thus, Lemma 2.5 implies that $M' = 0$ or $N' = 0$. In light of the construction of M' and N', it follows that $\operatorname{pd}_R(M) \leqslant 1 < \infty$ or $\operatorname{pd}_R(N) \leqslant i-4 < \infty$. By equation $(*)$ from the introduction, we have $\operatorname{depth}(R) = 0$. Thus, by the Auslander-Buchsbaum formula, we have $\operatorname{pd}_R(M) = 0$ or $\operatorname{pd}_R(N) = 0$.

Using the previous paragraph, for $n = \mathrm{i}, \ldots, \mathrm{iv}$ we have $(n) \implies (\mathrm{v})$. The converses of these implications are routine. $\square$

Theorem 2.7 gives the following slightly weaker version of a result of Nasseh and Yoshino [20, Theorem 3.1]; see also [5, Proposition 5.2]. This result uses the "trivial extension" $S \ltimes k$; as an additive abelian group, this is $S \oplus k$, and multiplication is given by the formula $(s, x)(t, y) := (st, sy + tx)$. One shows readily that there is an isomorphism $S \ltimes k \cong S \times_k (k[x]/(x^2))$. Our result is slightly weaker than Nasseh and Yoshino's result because that result only requires $i \geqslant 3$; note that they also give an example showing that this range of i-values is optimal in their setting.

Corollary 2.8. *Assume that M and N are non-zero finitely generated $S \ltimes k$-modules such that* $\operatorname{Tor}_i^{S \ltimes k}(M, N) = 0$ *for some* $i \geqslant 5$*; then M or N is free over $S \ltimes k$.*

Here is an example showing that in the general setting of Theorem 2.7, vanishing of $\operatorname{Tor}_3^R(M, N)$ is not enough to guarantee $\operatorname{pd}_R(M) < \infty$ or $\operatorname{pd}_R(N) < \infty$, in contrast to the special case of [20, Theorem 3.1]. Neither are two sequential vanishings $\operatorname{Tor}_2^R(M, N) = 0 = \operatorname{Tor}_3^R(M, N)$ enough. (See also Theorem 2.12 below.) In the example, we use the following straightforward fact: If $0 \neq s \in \mathfrak{m}_S$ and $0 \neq t \in \mathfrak{m}_T$, then $\operatorname{Ann}_R(s + t) = \operatorname{Ann}_S(s) \oplus \operatorname{Ann}_T(t)$.

Example 2.9. Consider the local artinian rings $S := k[U, V]/(U^2, V^2)$ and $T := k[X, Y]/(X^2, Y^2)$, and set $R := S \times_k T$ as usual. Use lower-case letters u, v, x, y to represent the residues of the variables U, V, X, Y in R and in the respective rings S and T. With $M = R/(u+x)R$ and $N = R/(v+y)R$, we claim that $\operatorname{Tor}_i^R(M, N) = 0$ if and only if $i = 2$ or 3.

Note that M and N are not free over R, since they have non-trivial annihilators. Since $\operatorname{depth}(R) = 0$, the Auslander-Buchsbaum formula implies that M and N have infinite projective dimension. By Theorem 2.7, it follows that $\operatorname{Tor}_i^R(M, N) \neq 0$ for all $i \geqslant 5$. In addition, we have $\operatorname{Tor}_0^R(M, N) \cong M \otimes_R N \cong R/(u + x, y + v)R \neq 0$.

Thus, it remains to show that $\operatorname{Tor}_1^R(M,N) \neq 0 = \operatorname{Tor}_2^R(M,N) = \operatorname{Tor}_3^R(M,N) \neq \operatorname{Tor}_4^R(M,N)$. To this end, we note the following straightforward computation.

$$(uS \oplus xT) \cap (vS \oplus yT) = uvS \oplus xyT = (uS \oplus xT) \cdot (vS \oplus yT) \tag{2.9.1}$$

Next, we explain the first equality in the following display

$$(u+x)R \cap (v+y)R = uvS \oplus xyT \neq (uv+xy)R = (u+x)R \cdot (v+y)R. \tag{2.9.2}$$

Here, the containment $(u+x)R \cap (v+y)R \subseteq uvS \oplus xyT$ follows from (2.9.1) since we have $(u+x)R \subseteq (uS \oplus xT)$ and $(v+y)R \subseteq (vS \oplus yT)$. The reverse containment follows from the equalities $v(u+x) = uv = u(v+y)$ and $x(v+y) = xy = y(u+x)$. (The other steps in (2.9.2) are even more straightforward.) Similarly, we have

$$(uS \oplus xT) \cap (v+y)R = (uS \oplus xT) \cdot (v+y)R. \tag{2.9.3}$$

Now we show how this helps us to compute $\operatorname{Tor}_i^R(M,N)$. We make repeated use of the formula $\operatorname{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ$, first for $i = 1$:

$$\operatorname{Tor}_1^R(M,N) = \operatorname{Tor}_1^R(R/(u+x)R, R/(v+y)R) \cong \frac{(u+x)R \cap (v+y)R}{(u+x)R \cdot (v+y)R} \neq 0$$

where the non-vanishing is from (2.9.2). Similarly, as the paragraph preceding this example shows $\operatorname{Ann}_R(u+x) = uS \oplus xT$, the next computation follows by dimension-shifting and using (2.9.3).

$$\operatorname{Tor}_2^R(M,N) = \operatorname{Tor}_1^R(R/(uS \oplus xT), R/(v+y)R) \cong \frac{(uS \oplus xT) \cap (v+y)R}{(uS \oplus xT) \cdot (v+y)R} = 0$$

Similarly, from (2.9.1) we have $\operatorname{Tor}_3^R(M,N) = 0$. Thus, it remains to show that $\operatorname{Tor}_4^R(M,N) \neq 0$. To this end, again by dimension-shifting we have

$$\begin{aligned}
\operatorname{Tor}_4^R(M,N) &\cong \operatorname{Tor}_2^R(R/(uS \oplus xT), R/(vS \oplus yT)) \\
&\cong \operatorname{Tor}_1^R(uS \oplus xT, R/(vS \oplus yT)) \\
&\cong \operatorname{Tor}_1^R(uS, R/(vS \oplus yT)) \oplus \operatorname{Tor}_1^R(xT, R/(vS \oplus yT)).
\end{aligned}$$

So, it suffices to show that $\operatorname{Tor}_1^R(uS, R/(vS \oplus yT)) \neq 0$, which we show next.

$$\begin{aligned}
\operatorname{Tor}_1^R(uS, R/(vS \oplus yT)) &\cong \operatorname{Tor}_1^R(R/\operatorname{Ann}_R(u), R/(vS \oplus yT)) \\
&\cong \operatorname{Tor}_1^R(R/(uS \oplus \mathfrak{m}_T), R/(vS \oplus yT)) \\
&\cong \frac{(uS \oplus \mathfrak{m}_T) \cap (vS \oplus yT)}{(uS \oplus \mathfrak{m}_T) \cdot (vS \oplus yT)} \\
&= \frac{uvS \oplus yT}{uvS \oplus y\mathfrak{m}_T} \\
&\neq 0
\end{aligned}$$

This completes the example.

Remark 2.10. As in the preceding example, one can show that the ring $R = k[\![U,V]\!]/(UV) \times_k k[\![X,Y]\!]/(XY)$ with the modules $M = R/(u+x)R$ and $N = R/(v+y)R$ satisfy $\operatorname{Tor}_i^R(M,N) = 0$ if and only if $i = 1$ or 3. (Note that this ring fits in the context of Theorem 2.12 below.)

Next, we work on the case where S and T both have positive depth.

Lemma 2.11. *Let M_1, N_1 be first syzygies over S of finitely generated S-modules, and let M_2, N_2 be first syzygies over T of finitely generated T-modules. Set $M := M_1 \oplus M_2$ and $N := N_1 \oplus N_2$. Assume that $\operatorname{Tor}_1^R(M,N) = 0 = \operatorname{Tor}_2^R(M,N)$; then $M = 0$ or $N = 0$.*

Proof. As in the beginning of the proof of Lemma 2.5, we assume that $M_1 \neq 0$, and conclude that $N_1 = 0$. And we suppose by way of contradiction that $N_2 \neq 0$. An application of (a symmetric version of) Lemma 2.4 implies that $M_2 = 0$.

By Fact 2.2, a minimal R-free presentation of $M_1 = M$ has the following form:

$$R^{\beta_1} \oplus R^{c_1\beta_0} \xrightarrow{[\widetilde{d_1} \;\; \overline{g_1}]} R^{\beta_0}.$$

In particular, we have

$$\operatorname{Im}(\overline{g_1}) = \operatorname{Im}(g_1)R^{\beta_0} = \mathfrak{m}_T R^{\beta_0} \neq 0 \tag{2.11.1}$$

where the non-vanishing follows from the assumption $T \neq k$.

Next, set $M' = \operatorname{Im}[\widetilde{d_1} \;\; \overline{g_1}] \subseteq R^{\beta_0}$, i.e., M' is the first syzygy of M in the above minimal presentation. Since M is a first syzygy, the module M' is a second syzygy, and Fact 2.1 implies that $M' \cong M_1' \oplus M_2'$ where $\mathfrak{m}_T M_1' = 0 = \mathfrak{m}_S M_2'$. Moreover, this Fact explains the first equality in the next display.

$$M_2' = \operatorname{Im}[\widetilde{d_1} \;\; \overline{g_1}] \cap (\mathfrak{m}_T R^{\beta_0}) \supseteq \operatorname{Im}(\overline{g_1}) \cap (\mathfrak{m}_T R^{\beta_0}) = \mathfrak{m}_T R^{\beta_0} \neq 0 \tag{2.11.2}$$

The containment here is straightforward; the second equality and the non-vanishing are from (2.11.1).

Dimension-shifting gives $0 = \operatorname{Tor}_2^R(M,N) \cong \operatorname{Tor}_1^R(M',N)$. The condition $M_2' \neq 0$ from (2.11.2), with Lemma 2.4 implies that $N_2 = 0$, contradicting the supposition in the first paragraph of this proof. In other words, we must have $N_2 = 0$. With the already established $N_1 = 0$, we conclude that $N = 0$, as desired. □

Next, we establish Theorem 1.1(b) from the introduction.

Theorem 2.12. *Let M and N be finitely generated modules over the fiber product R. Then the following conditions are equivalent.*

(i) $\operatorname{Tor}_i^R(M,N) = 0$ *for* $i \gg 0$
(ii) $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \geqslant 2$
(iii) $\operatorname{Tor}_i^R(M,N) = 0 = \operatorname{Tor}_{i+1}^R(M,N)$ *for some* $i \geqslant 5$
(iv) $\operatorname{pd}_R(M) < \infty$ *or* $\operatorname{pd}_R(N) < \infty$
(v) $\operatorname{pd}_R(M) \leqslant 1$ *or* $\operatorname{pd}_R(N) \leqslant 1$

Proof. Argue as in the proof of Theorem 2.7, using Lemma 2.11 in lieu of 2.5. □

In the preceding result, the next remark shows that even if one assumes that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geqslant 1$, one cannot conclude that M or N is free (unless, of course, one is in the setting of Theorem 2.7).

Remark 2.13. Indeed, assume that S and T have positive depth, so we have $\operatorname{depth}(R) = 1$ by equation (3.1.1). Let $f \in \mathfrak{m}_R$ be R-regular, and set $M := R/fR$. Then $\operatorname{pd}_R(M) = 1$, so $\operatorname{Tor}_i^R(M,-) = 0$ for all $i \geqslant 2$. By dimension-shifting, if N is a syzygy over R of infinite projective dimension, e.g., if N is a non-principal ideal of R, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geqslant 1$, even though M and N are not free.

In light of the preceding theorems and examples, we pose the following.

Question 2.14. Let M, N be finitely generated modules over the fiber product R. If $\mathrm{Tor}_4^R(M,N) = 0$, must one of the modules M, N have finite projective dimension?

3. Auslander's Depth Formula

In this section, we document some consequences of the above results for Auslander's "depth formula" from [2, Theorem 1.2]. This subject has received considerable attention recently; see, e.g., work of Araya and Yoshino [1], Christensen and Jorgensen [10], and Foxby [13].

For part of the proof of the first result of this section, we work in the derived category $\mathcal{D}(R)$ with objects equal to the R-complexes (i.e., the chain complexes of R-modules) indexed homologically. References for this include Christensen, Foxby, and Holm [17], Hartshorne and Grothendieck [15], and Verdier [22, 23]. We say that an R-complex X is *homologically finite* if the total homology module $\mathrm{H}(X) = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(X)$ is finitely generated. We set $\mathrm{Ext}^i_R(X,Y) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,Y))$ and $\mathrm{depth}_R(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_R(k,X) \neq 0\}$.

Theorem 3.1. *Let M, N be non-zero finitely generated modules over the fiber product R such that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \gg 0$. Set*

$$q := \max\{i \geqslant 0 \mid \mathrm{Tor}_i^R(M,N) \neq 0\}.$$

Assume that at least one of the following conditions holds.

(1) $q = 0$
(2) $\mathrm{depth}_R(\mathrm{Tor}_q^R(M,N)) \leqslant 1$
(3) $\mathrm{depth}(S) = 0$ *or* $\mathrm{depth}(T) = 0$
(4) *one of the modules M, N is a syzygy of a finitely generated R-module*
(5) *one of the modules M, N is S or T*
(6) $\mathrm{depth}(N) = 0$ *and* $\mathrm{pd}_R(N) = \infty$

Then we have $q \leqslant 1$ and

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(N) = \mathrm{depth}(R) + \mathrm{depth}_R(\mathrm{Tor}_q^R(M,N)) - q. \tag{3.1.1}$$

Proof. Theorem 2.12 implies that $q \leqslant 1$ and, say, $\mathrm{pd}_R(M) \leqslant 1$. Thus it remains to establish equation (3.1.1) under any of the conditions (1)–(6). In cases (1)–(2) this is from [2, Theorem 1.2]. Theorem 2.7 shows that (3) $\Longrightarrow$ (1). In particular, we assume for the remainder of this proof that $\mathrm{depth}(S), \mathrm{depth}(T) \geqslant 1$.

(4) In the case where N is a syzygy of some finitely generated R-module L, for $i \geqslant 1$ we have

$$\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i+1}^R(M,L) = 0$$

since $\mathrm{pd}_R(M) \leqslant 1$, by dimension-shifting. On the other hand, if M is a syzygy of L, we have $\mathrm{pd}_R(L) \leqslant \mathrm{depth}(R) = 1$ by Auslander-Buchsbaum, so M is free. In each case, we conclude that $q = 0$, so we are done by case (1).

(5) The assumption $\mathrm{depth}(S) \geqslant 1$ yields an S-regular element $f \in \mathfrak{m}_S$. It follows readily that $fR \cong fS \cong S$ so S is a syzygy over R, namely, the first syzygy of R/fR. Symmetrically, we see that T is an R-syzygy, so this case follows from the preceding one.

(6) Assume that $\mathrm{depth}(N) = 0$ and $\mathrm{pd}_R(N) = \infty$. Assume further that $q = 1 = \mathrm{pd}_R(M)$ and $\mathrm{depth}(S), \mathrm{depth}(T) \geqslant 1$, otherwise we are in the situation of case (1)

or (3). Thus, we have $\operatorname{depth}(R) = 1$ by equation $(*)$ from the introduction, and $\operatorname{depth}_R(M) = 0$ by the Auslander-Buchsbaum formula. We need to show that

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) - 1.$$

In light of our assumptions, this reduces to showing that $\operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) = 0$. That is, we need to show that $\mathfrak{m}_R \in \operatorname{Ass}_R(\operatorname{Tor}_1^R(M,N))$.

From [13, Lemma 2.1], we have

$$\begin{aligned}\operatorname{depth}_R(M \otimes_R^{\mathbf{L}} N) &= \operatorname{depth}_R(M) + \operatorname{depth}_R(N) - \operatorname{depth}(R)\\ &= -1\\ &= -\sup\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(M,N) \neq 0\}.\end{aligned}$$

Thus, according to [8, (1.6.6)], we have $\mathfrak{m}_R \in \operatorname{Ass}_R(\operatorname{Tor}_1^R(M,N))$, as desired. $\square$

Because of the strength of Theorems 2.7 and 2.12, we were surprised to find the next examples which show that the depth formula (3.1.1) fails over the fiber product R in the general case $q = 1$ when $\operatorname{depth}(S), \operatorname{depth}(T) \geqslant 1$.

Example 3.2. Set $S := k[\![U,V,W]\!]$ and $T := k[\![X,Y,Z]\!]$ with prime ideals $\mathfrak{p} := US$ and $\mathfrak{q} := (Y,Z)T$. We consider the fiber product R and the R-modules $M = R/(U+X)R$ and $N = S/\mathfrak{p} \oplus T/\mathfrak{q}$. Since $U + X$ is R-regular, we have $\operatorname{pd}_R(M) = 1$. Using the minimal free resolution $0 \to R \xrightarrow{U+X} R \to M \to 0$ we have the first isomorphism in the next display

$$\begin{aligned}\operatorname{Tor}_1^R(M,N) &\cong \operatorname{Ker}\left(N \xrightarrow{U+X} N\right)\\ &\cong \operatorname{Ker}\left(S/\mathfrak{p} \xrightarrow{U+X} S/\mathfrak{p}\right) \oplus \operatorname{Ker}\left(T/\mathfrak{q} \xrightarrow{U+X} T/\mathfrak{q}\right)\\ &\cong \operatorname{Ker}\left(S/\mathfrak{p} \xrightarrow{U} S/\mathfrak{p}\right) \oplus \operatorname{Ker}\left(T/\mathfrak{q} \xrightarrow{X} T/\mathfrak{q}\right)\\ &\cong S/\mathfrak{p}.\end{aligned}$$

The second isomorphism is from the definition of N, and the third one follows from the vanishings $X(S/\mathfrak{p}) = 0 = U(T/\mathfrak{q})$. For the last isomorphism here, note that X is $(T/\mathfrak{q})$-regular and that $U(S/\mathfrak{p}) = 0$.

From this, we have

$$\operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) = \operatorname{depth}_R(S/\mathfrak{p}) = 2$$

and the next display is by construction

$$\operatorname{depth}_R(N) = \min\{\operatorname{depth}_R(S/\mathfrak{p}), \operatorname{depth}_R(T/\mathfrak{q})\} = \min\{2,1\} = 1.$$

As we have $\operatorname{depth}(R) = 1$ by equation $(*)$ from the introduction, the fact that $U + X$ is R-regular implies $\operatorname{depth}_R(M) = 0$. From these facts, we find that

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = 1 < 2 = \operatorname{depth}(R) + \operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) - 1$$

so the depth formula (3.1.1) fails here.

Example 3.3. Set $S = k[\![u,x,y,z,a]\!]$ and $T = k[\![v]\!]$. In [1, Example 2.9], the authors construct an ideal $I \subseteq S$ such that the module $N := S/I$ satisfies $\operatorname{depth}_S(N) = 3$ and $\operatorname{depth}_S(\operatorname{Tor}_1^S(S/aS,N)) = 2$. The module S/aS has projective dimension 1 over S, and one checks easily that

$$\operatorname{depth}_S(S/aS) + \operatorname{depth}_S(N) = 7 > 6 = \operatorname{depth}(S) + \operatorname{depth}_S(\operatorname{Tor}_1^S(S/aS,N)) - 1.$$

Let R be the fiber product, as usual, and set $M = R/(a+v)R$. By equation $(*)$ from the introduction, we have $\operatorname{depth}(R) = 1$. The element $a+v$ is R-regular, so $\operatorname{pd}_R(M) = 1$. Also, one has $\operatorname{depth}_R(N) = \operatorname{depth}_S(N) = 3$. In the next display, the isomorphisms are routine, and the equality is from the condition $vN = 0$.

$$\operatorname{Tor}_1^R(M,N) \cong \operatorname{Ker}(N \xrightarrow{a+v} N) = \operatorname{Ker}(N \xrightarrow{a} N) \cong \operatorname{Tor}_1^S(S/aS, N)$$

We conclude that $\operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) = \operatorname{depth}_S(\operatorname{Tor}_1^S(S/aS,N)) = 2$, so

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = 3 > 2 = \operatorname{depth}(R) + \operatorname{depth}_R(\operatorname{Tor}_1^R(M,N)) - 1$$

so the depth formula (3.1.1) fails here, in the opposite way from Example 3.2.

The next result shows that the previous example is, in a sense, minimal with respect to the particular failure of the depth formula (3.1.1).

Proposition 3.4. *Let $(A, \mathfrak{m}_A)$ be a commutative local noetherian ring, and let M, N be finitely generated A-modules such that $\operatorname{pd}_A(M) = 1$. If $\operatorname{depth}_A(N) \leqslant 2$, then*

$$\operatorname{depth}_A(M) + \operatorname{depth}_A(N) \leqslant \operatorname{depth}(A) + \operatorname{depth}_A(\operatorname{Tor}_1^A(M,N)) - 1.$$

Proof. In the case $\operatorname{depth}_R(N) = 0$, the desired inequality follows by the Auslander-Buchsbaum formula for M. So, we assume for the rest of the proof that $\operatorname{depth}_A(N)$ is 1 or 2. Consider a minimal free resolution

$$0 \to A^m \to A^n \to M \to 0$$

wherein $m, n \neq 0$ since $\operatorname{pd}_A(M) = 1$. It follows that there is an exact sequence

$$0 \to \operatorname{Tor}_1^A(M,N) \to N^m \to N^n \to M \otimes_A N \to 0.$$

Break this into short exact sequences

$$0 \to \operatorname{Tor}_1^A(M,N) \to N^m \to V \to 0 \qquad\qquad 0 \to V \to N^n \to M \otimes_A N \to 0$$

and apply the Depth Lemma; the assumption $1 \leqslant \operatorname{depth}_A(N) \leqslant 2$ implies that

$$\operatorname{depth}_A(V) \geqslant \min\{\operatorname{depth}_A(N), \operatorname{depth}_A(M \otimes_A N) + 1\} \geqslant 1$$
$$\operatorname{depth}_A(\operatorname{Tor}_1^A(M,N)) \geqslant \min\{\operatorname{depth}_A(N), \operatorname{depth}_A(V) + 1\} \geqslant \operatorname{depth}_A(N).$$

In light of the Auslander-Buchsbaum formula for M, which has projective dimension 1, the preceding display yields the desired inequality. □

4. Ext-Vanishing

The results of this section, like those of Section 2, give counterpoints to the theme discussed in Section 1: the conclusions here hold over the fiber product R, whether or not they hold over S or T.

It is worth noting here that much of the machinery we invoke in the proof of the next result is developed in significant generality in the forthcoming [6]. Thus, we only sketch the proof.

Proposition 4.1. *Let M and N be homologically finite complexes over the fiber product R such that $\operatorname{Ext}_R^i(M,N) = 0$ for $i \gg 0$, then $\operatorname{pd}_R(M) < \infty$ or $\operatorname{id}_R(N) < \infty$.*

Proof. First, note that if X, Y are homologically finite R-complexes such that $\operatorname{Tor}_i^R(X,Y)=0$ for $i\gg 0$, then $\operatorname{pd}_R(X)<\infty$ or $\operatorname{pd}_R(Y)<\infty$. Indeed, take sufficiently high syzygies in minimal R-free resolutions of X and Y. These are finitely generated and Tor-independent, so the conclusion follows from Theorem 2.12.

Let K denote the Koszul complex over R on a finite generating sequence for $\mathfrak{m}_R$. It follows that the complex $K\otimes_R^{\mathbf{L}}N=K\otimes_R N$ is homologically finite and, furthermore, has homology annihilated by $\mathfrak{m}_R$. Thus, it has finite length total homology module. Moreover, we have $\operatorname{id}_R(K\otimes_R^{\mathbf{L}}N)<\infty$ if and only if $\operatorname{id}_R(N)<\infty$.

Let $E:=E_R(k)$ be the injective hull of k over R. Since $\operatorname{H}(K\otimes_R^{\mathbf{L}}N)$ has finite length, so does $\operatorname{H}(\mathbf{R}\operatorname{Hom}_R(K\otimes_R^{\mathbf{L}}N,E))$; in particular, $\mathbf{R}\operatorname{Hom}_R(K\otimes_R^{\mathbf{L}}N,E)$ is homologically finite. With the "Hom-evaluation" isomorphism of Avramov and Foxby [4, Lemma 4.4(I)]

$$\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,K\otimes_R^{\mathbf{L}}N),E)\simeq M\otimes_R^{\mathbf{L}}\mathbf{R}\operatorname{Hom}_R(K\otimes_R^{\mathbf{L}}N,E)$$

the preceding paragraph implies that $\operatorname{pd}_R(M)<\infty$ or $\operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(K\otimes_R^{\mathbf{L}}N,E))<\infty$. It follows readily that $\operatorname{pd}_R(M)<\infty$ or $\operatorname{id}_R(K\otimes_R^{\mathbf{L}}N)<\infty$, i.e., $\operatorname{pd}_R(M)<\infty$ or $\operatorname{id}_R(N)<\infty$, as desired. □

Next, we document some consequences of the preceding proposition.

Corollary 4.2. *Let M, N be finitely generated modules over the fiber product R. If $\operatorname{Ext}_R^i(M,N)=0$ for $i\gg 0$, then $\operatorname{pd}_R(M)\leqslant 1$ or $\operatorname{id}_R(N)\leqslant 1$.*

Proof. This follows from Proposition 4.1, via the Auslander-Buchsbaum and Bass formulas, since $\operatorname{depth}(R)\leqslant 1$ by equation $(*)$. □

Corollary 4.3. *Assume that $\operatorname{depth}(S)=0$ and that S or T is not artinian. Let M, N be non-zero finitely generated modules over the fiber product R. If $\operatorname{Ext}_R^i(M,N)=0$ for $i\gg 0$, then M is R-free.*

Proof. Equation $(*)$ shows that $\operatorname{depth}(R)=0$. Thus, the preceding result (with the Auslander-Buchsbaum and Bass formulas) shows that M is free or N is injective. Since R surjects onto S and T, one of which is not artinian, we know that R is not artinian, so it does not have a finitely generated injective module. In particular, N is not injective, so M is free. □

The next result is immediate from the previous one. It says that the fiber product R has "Ext-index" at most $\operatorname{depth}(R)\leqslant 1$. See equation $(*)$.

Corollary 4.4. *Let M, N be finitely generated modules over the fiber product R such that $\operatorname{Ext}_R^i(M,N)=0$ for $i\gg 0$. Then $\operatorname{Ext}_R^i(M,N)=0$ for all $i>\operatorname{depth}(R)$.*

The next result says in particular that the fiber product R satisfies the Auslander-Reiten Conjecture. It contains Theorem 1.2 from the introduction, and it follows, e.g., from the previous corollary and [9, Theorem 2.3].

Theorem 4.5. *Let M be a finitely generated module over the fiber product R.*
(a) *If $\operatorname{Ext}_R^i(M,M\oplus R)=0$ for $i\gg 0$, then $\operatorname{pd}_R(M)\leqslant 1$.*
(b) *If $\operatorname{Ext}_R^i(M,M\oplus R)=0$ for $i\geqslant 1$, then M is R-free.*

The next result provides yet another cointerpoint the the theme from Section 1, as one can easily construct examples where S and T have non-trivial semidualizing complexes (even modules); recall that a homologically finite R-complex C is *semidualizing* if $\mathbf{R}\operatorname{Hom}_R(C,C)\simeq R$ in $\mathcal{D}(R)$.

Corollary 4.6. *The fiber product R has at most two semidualizing complexes, up to shift-isomorphism, namely R and a dualizing R-complex (if R has one).*

Proof. By definition, if C is a semidualizing R-complex, then it is homologically finite such that $\operatorname{Ext}^i_R(C,C)=0$ for all $i \geqslant 1$. Thus, Proposition 4.1 implies that $\operatorname{pd}_R(C)<\infty$ or $\operatorname{id}_R(C)<\infty$, that is, $C \simeq \Sigma^n R$ for some $n \in \mathbb{Z}$ (by a result of Christensen [8, Theorem 8.1]) or C is dualizing (by definition). □

It is not clear to us when the fiber product R has a dualizing complex. Of course, if it does, then so do the homomorphic images S and T. On the other hand, if S and T are complete, then so is R by a result of Grothendieck [14, (19.3.2.1)], so R has a dualizing complex in this case. Hence, we pose the following.

Question 4.7. If S and T admit dualizing complexes, must R also admit one?

Next, we recall Auslander and Bridger's G-dimension [3]. A finitely generated R-module G is *totally reflexive* if $\operatorname{Hom}_R(\operatorname{Hom}_R(G,R),R)\cong G$ and $\operatorname{Ext}^i_R(G,R)=0=\operatorname{Ext}^i_R(\operatorname{Hom}_R(G,R),R)$ for all $i \geqslant 1$. Every finitely generated free R-module is totally reflexive, so every finitely generated R-module M has a resolution by totally reflexive modules. If M has a bounded resolution by totally reflexive modules, then the G-dimension of M is the length of the shortest such resolution. The following result says, in the language of Takahashi [21], that our ring R is "G-regular".

Corollary 4.8. *Assume that the fiber product R is not Gorenstein.*[1] *Let M be a finitely generated R-module. Then one has $\operatorname{pd}_R(M)=\operatorname{G-dim}_R(M)$. In particular, M is totally reflexive if and only if it is free.*

Proof. By [3, 4.13], it suffices to assume that $\operatorname{G-dim}_R(M)<\infty$, and prove that $\operatorname{pd}_R(M)<\infty$. This assumption implies that we have $\operatorname{Ext}^i_R(M,R)=0$ for $i \gg 0$, so Corollary 4.2 implies that $\operatorname{pd}_R(M)<\infty$ or $\operatorname{id}_R(R)<\infty$. Since R is not Gorenstein, it follows that $\operatorname{pd}_R(M)<\infty$, as desired. □

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[1] See [11, Section 3] for a discussion of this assumption.

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