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EXTENSION GROUPS FOR DG MODULES

SAEED NASSEH AND SEAN SATHER-WAGSTAFF

Abstract. Let $M$ and $N$ be differential graded (DG) modules over a positively graded commutative DG algebra $A$. We show that the Ext-groups $\text{Ext}^i_A(M, N)$ defined in terms of semi-projective resolutions are not in general isomorphic to the Yoneda Ext-groups $\text{YExt}^i_A(M, N)$ given in terms of equivalence classes of extensions. On the other hand, we show that these groups are isomorphic when the first DG module is semi-projective.

1. INTRODUCTION

Convention. In this paper, $R$ is a commutative ring with identity.

Given two $R$-modules $M$ and $N$, a classical result originating with work of Baer [4] states that $\text{Ext}^1_R(M, N)$, defined via projective/injective resolutions, is isomorphic to the abelian group $\text{YExt}^1_R(M, N)$ of equivalence classes of exact sequences of the form $0 \to N \to X \to M \to 0$. The purpose of this note is to discuss possible extensions of this result to the abelian category of differential graded (DG) modules over a positively graded commutative DG algebra $A$. See Section 2 for background information on this category.

Specifically, we show that Baer’s isomorphism fails in general in this context: Examples 3.1 and 3.2 exhibit DG $A$-modules $M, N$ with $\text{Ext}^1_A(M, N) \not\cong \text{YExt}^1_A(M, N)$. (See 2.4 and 2.6 below for definitions.) On the other hand, the following result shows that a reasonable hypothesis on the first module does yield such an isomorphism.

Theorem A. Let $A$ be a DG $R$-algebra, and let $N, Q$ be DG $A$-modules such that $Q$ is semi-projective. Then there is an isomorphism $\text{YExt}^i_A(Q, N) \cong \text{Ext}^i_A(Q, N)$ of abelian groups for all $i \geq 1$.

This is the main result of Section 3; see Proof 3.8. In the subsequent Section 4 we discuss some properties of $\text{YExt}$ with respect to truncations.

It is worth noting here that we apply results from this paper in our answer to a question of Vasconcelos in [12]. Specifically, in that paper, we investigate DG $A$-modules $C$ with $\text{Ext}^1_A(C, C) = 0$ using geometric techniques. These techniques yield an isomorphism between $\text{YExt}^1_A(C, C)$ and a certain quotient of tangent spaces; it is then important for us to know when the vanishing of $\text{Ext}^1_A(C, C)$ implies the vanishing of related $\text{YExt}^1$-modules; see Proposition 4.4 below.

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2. DG Modules

We assume that the reader is familiar with the category of $R$-complexes and the derived category $\mathcal{D}(R)$. Standard references for these topics are [6, 7, 8, 9, 10, 13, 14]. For clarity, we include some definitions and notation.

**Definition 2.1.** In this paper, complexes of $R$-modules ("$R$-complexes" for short) are indexed homologically: $M = \cdots \to M_{n+2} \xrightarrow{\partial_{n+2}} M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots$. The degree of an element $m \in M$ is denoted $|m|$. The infimum and supremum of $M$ are the infimum and supremum, respectively, of the set $\{n \in \mathbb{Z} \mid H_n(M) \neq 0\}$. The tensor product of two $R$-complexes $M, N$ is denoted $M \otimes_R N$, and the Hom complex is denoted $\text{Hom}_R(M, N)$. A chain map $M \to N$ is a cycle in $\text{Hom}_R(M, N)_0$.

Next we discuss DG algebras and DG modules, which are treated in, e.g., [1, 2, 3, 4, 5, 11]. We follow the notation and terminology from [2, 5]; given the slight differences in the literature, though, we include a summary next.

**Definition 2.2.** A positively graded commutative differential graded $R$-algebra (DG $R$-algebra for short) is an $R$-complex $A$ equipped with a chain map $\mu^A : A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is associative, unital, and graded commutative such that $A_i = 0$ for $i < 0$. The map $\mu^A$ is the product on $A$. Given a DG $R$-algebra $A$, the underlying algebra is the graded commutative $R$-algebra $A^0 = \oplus_{i \geq 0} A_i$.

A differential graded module over a DG $R$-algebra $A$ (DG $A$-module for short) is an $R$-complex $M$ with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map $\mu^M$ is the scalar multiplication on $M$. The underlying $A^0$-module associated to $M$ is the $A^0$-module $M^0 = \oplus_{i \in \mathbb{Z}} M_i$.

The DG $A$-module $\text{Hom}_A(M, N)$ is the subcomplex of $\text{Hom}_R(M, N)$ of the $A$-linear homomorphisms. A morphism $M \to N$ of DG $A$-modules is a cycle in $\text{Hom}_A(M, N)_0$. Projective objects in the category of DG $A$-modules are called categorically projective. Quasiisomorphisms of DG $A$-modules are identified by the symbol $\simeq$, also used for the "quasiisomorphic" equivalence relation.

Two important DG $R$-algebras to keep in mind are $R$ itself and, more generally, the Koszul complex over $R$ (on a finite sequence of elements of $R$) with the exterior product. A DG $R$-module is just an $R$-complex, and a morphism of DG $R$-modules is simply a chain map.

**Remark 2.3.** Let $A$ be a DG $R$-algebra. The category of DG $A$-modules is an abelian category with enough projectives.

**Definition 2.4.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. For each $i \geq 0$ we have a well-defined Yoneda Ext group $\text{YExt}_A^i(M, N)$, defined in terms of a resolution of $M$ by categorically projective DG $A$-modules:

$$\cdots \to Q_1 \to Q_0 \to M \to 0.$$ 

A standard result shows that $\text{YExt}_A^1(M, N)$ is isomorphic to the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$ of DG $A$-modules under the Baer sum; see, e.g., [15] (3.4.6) and the proof of Theorem 3.3.

We now turn to the derived category $\mathcal{D}(A)$, and related notions.

**Definition 2.5.** Let $A$ be a DG $R$-algebra. A DG $A$-module $Q$ is graded-projective if $\text{Hom}_A(Q, -)$ preserves surjective morphisms, that is, if $Q^i$ is a projective graded
$R^2$-module. The DG module $Q$ is semi-projective if $\text{Hom}_A(Q, -)$ respects surjective quasiisomorphisms, that is, if $Q$ is graded-projective and respects quasiisomorphisms. A semi-projective resolution of $M$ is a quasiisomorphism $L \xrightarrow{\sim} M$ of DG $A$-modules such that $L$ is semi-projective.

**Fact 2.6.** Let $A$ be a DG $R$-algebra. Then every DG $A$-module has a semi-projective resolution.

**Definition 2.7.** Let $A$ be a DG $R$-algebra. The derived category $D(A)$ is formed from the category of DG $A$-modules by formally inverting the quasiisomorphisms; see [11]. Isomorphisms in $D(A)$ are identified by the symbol $\simeq$.

The derived functor $R\text{Hom}_A(M, N)$ is defined via a semi-projective resolution $P \xrightarrow{\sim} M$, as $R\text{Hom}_A(M, N) \simeq \text{Hom}_A(P, N)$. For each $i \in \mathbb{Z}$, set $\text{Ext}^i_A(M, N) := H_{-i}(R\text{Hom}_A(M, N))$.

### 3. DG Ext vs. Yoneda Ext

We begin this section with examples of DG $A$-modules $M$ and $N$ such that $\text{Ext}^1_A(M, N) \neq Y\text{Ext}^1_A(M, N)$. These present two facets of the distinctness of Ext and YExt, as the first example has $M$ and $N$ both bounded, while the second one (from personal communication with Avramov) has $M$ graded-projective.

**Example 3.1.** Let $R = k[X]$, and consider the following exact sequence of DG $R$-modules, i.e., exact sequence of $R$-complexes:

\[
0 \rightarrow R \xrightarrow{X} R \xrightarrow{1} k \rightarrow 0
\]

This sequence does not split over $R$ (it is not even degree-wise split) so it gives a non-trivial class in $Y\text{Ext}^1_R(k, R)$, and we conclude that $Y\text{Ext}^1_R(k, R) \neq 0$. On the other hand, $k$ is homologically trivial, so we have $\text{Ext}^1_R(k, k) = 0$ since 0 is a semi-free resolution of $k$.

**Example 3.2.** Let $R = k[X]/(X^2)$ and consider the following exact graded-projective DG $R$-module $M = \cdots \xrightarrow{X} R \xrightarrow{X} R \xrightarrow{X} \cdots$. Since $M$ is exact, we have $\text{Ext}^i_R(M, M) = 0$ for all $i$. We claim, however, that $Y\text{Ext}^1_R(M, M) \neq 0$. To see this, first note that $M$ is isomorphic to the suspension $\Sigma M$ and that $M$ is not contractible. Thus, the mapping cone sequence for the identity morphism $\text{id}_M$ is isomorphic to one of the form $0 \rightarrow M \rightarrow X \rightarrow M \rightarrow 0$ and is not split.
The definition of the isomorphism $\text{YExt}^i_A(Q, N) \to \text{Ext}^i_A(Q, N)$ for $i = 1$ in Theorem A is contained in the following construction. The subsequent lemma and theorem show that $\Psi$ is a well-defined isomorphism.

**Construction 3.3.** Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Define $\Psi : \text{YExt}^1_A(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ as follows. Note that if $Q$ is semi-projective, then $\text{Ext}^1_A(Q, N) \cong H_{-1}(\text{Hom}_A(Q, N))$, which fits with what we have in Theorem A.

Let $\zeta \in \text{YExt}^1_A(Q, N)$ be represented by the sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} Q \to 0.$$  \hfill (3.3.1)

Since $Q$ is graded-projective, this sequence is graded-split, that is there are elements $h \in \text{Hom}_A(X, N)_0$ and $k \in \text{Hom}_A(Q, X)_0$ with

$$hf = \text{id}_N \quad gk = \text{id}_Q \quad hk = 0 \quad fh + kg = \text{id}_X.$$

Thus, the sequence (3.3.1) is isomorphic to one of the form

$$\begin{array}{ccccccc}
0 & \to & N & \xrightarrow{\epsilon_i} & N \oplus Q & \xrightarrow{\pi_i} & Q & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N_{i-1} & \xrightarrow{\epsilon_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$  \hfill (3.3.2)

where $\epsilon_j$ is the natural inclusion and $\pi_j$ is the natural surjection for each $j$. Since this diagram comes from a graded-splitting of (3.3.1), the scalar multiplication on the middle column of (3.3.2) is the natural one $a [p] = [ap]$. (We write elements of $N_i \oplus Q_i$ as column vectors.)

The fact that (3.3.2) commutes implies that $\partial^X_i$ has a specific form:

$$\partial^X_i = \begin{bmatrix} \phi_i^N & \lambda_i \\ 0 & \phi_i^Q \end{bmatrix}.$$  \hfill (3.3.3)

Here, we have $\lambda_i : Q_i \to N_{i-1}$, that is, $\lambda = \{\lambda_i\} \in \text{Hom}_R(Q, N)_{-1}$. Since the horizontal maps in the sequence (3.3.2) are morphisms of DG $A$-modules, it follows that $\lambda$ is a cycle in $\text{Hom}_A(Q, N)_{-1}$. Thus, $\lambda$ represents a homology class in $H_{-1}(\text{Hom}_A(Q, N))$, and we define $\Psi : \text{YExt}^1_A(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ by setting $\Psi(\zeta)$ equal to $[\lambda]$ the homology class of $\lambda$ in $H_{-1}(\text{Hom}_A(Q, N))$.

**Lemma 3.4.** Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Then the map $\Psi : \text{YExt}^1_A(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ from Construction 3.3 is well-defined.
Proof. Let \( \zeta \in \text{YExt}^1_A(Q,N) \) be represented by the sequence (3.3.2), and let \( \zeta \) be represented by another exact sequence

\[
\begin{array}{ccccccccc}
0 & \to & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \to & 0 \\
\downarrow{\partial^N_i} & & \downarrow{\partial^{X'}_i} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_i} & & \\
0 & \to & N_{i-1} & \xrightarrow{\epsilon_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \to & 0 \\
\end{array}
\]

(3.4.1)

where

\[
\partial^X_i = \begin{bmatrix} \partial^N_i & \lambda'_i \\ 0 & \partial^Q_i \end{bmatrix}.
\]

(3.4.2)

We need to show that \( \lambda - \lambda' \in \text{Im}(\partial^0_{\text{Hom}_A(Q,N)}) \). The sequences (3.3.2) and (3.4.1) are equivalent in \( \text{YExt}^1_R(Q,N) \), so for each \( i \) there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \to & 0 \\
\downarrow{\partial^N_i} & & \downarrow{[u_i \ v_i]} & & \downarrow{\pi_i} & & \downarrow{\epsilon_i} & & \\
0 & \to & N_{i-1} & \xrightarrow{\epsilon_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \to & 0 \\
\end{array}
\]

(3.4.3)

where the middle vertical arrow describes a DG \( A \)-module isomorphism, and such that the following diagram commutes for all \( i \)

\[
\begin{array}{ccccccccc}
N_i \oplus Q_i & \xrightarrow{[u_i \ v_i]} & N_i \oplus Q_i \\
\downarrow{\partial^N_i \lambda_i} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_i} \\
N_{i-1} \oplus Q_{i-1} & \xrightarrow{[u_{i-1} \ v_{i-1}]} & N_{i-1} \oplus Q_{i-1} \\
\end{array}
\]

(3.4.4)

The fact that diagram (3.4.3) commutes implies that \( u_i = \text{id}_{N_i}, \ v_i = \text{id}_{Q_i} \), and \( w_i = 0 \). Also, the fact that the middle vertical arrow in diagram (3.4.3) describes a DG \( A \)-module morphism implies that the sequence \( \epsilon_i : Q_i \to N_i \) respects scalar multiplication, i.e., we have \( v \in \text{Hom}_A(Q,N) \). The fact that diagram (3.4.4) commutes implies that \( \lambda_i - \lambda'_i = \partial^N_i v_i - v_{i-1} \partial^Q_i \). We conclude that \( \lambda - \lambda' = \partial^0_{\text{Hom}_A(Q,N)}(v) \in \text{Im}(\partial^0_{\text{Hom}_A(Q,N)}) \), so \( \Psi \) is well-defined.

The next result contains the case \( i = 1 \) of Theorem A from the introduction, because if \( Q \) is semi-projective, then \( \text{Ext}^1_A(Q,N) \cong \text{H}_{-1}(\text{Hom}_A(Q,N)) \).

**Theorem 3.5.** Let \( A \) be a DG \( R \)-algebra, and let \( N, Q \) be DG \( A \)-modules such that \( Q \) is graded-projective. Then the map \( \Psi : \text{YExt}^1_A(Q,N) \to \text{H}_{-1}(\text{Hom}_A(Q,N)) \) from Construction 3.3 is a group isomorphism.

**Proof.** We break the proof into three claims.
Claim 1. \( \Psi \) is additive. Let \( \zeta, \zeta' \in \text{YExt}^1_{\mathcal{A}}(Q, N) \) be represented by exact sequences \( 0 \to N \xrightarrow{\zeta} X \xrightarrow{\pi} Q \to 0 \) and \( 0 \to N \xrightarrow{\zeta'} X' \xrightarrow{\pi'} Q \to 0 \) respectively, where \( X_i = N_i \oplus Q_i = X'_i \) and the differentials \( \partial^X \) and \( \partial^{X'} \) are described as in (3.3.3) and (3.4.2), respectively. We need to show that the Baer sum \( \zeta + \zeta' \) is represented by an exact sequence \( 0 \to N \xrightarrow{\zeta} \tilde{X} \xrightarrow{\tilde{\pi}} Q \to 0 \), where \( \tilde{X}_i = N_i \oplus Q_i \) and \( \partial^\tilde{X} = \begin{bmatrix} \partial^X \lambda_i + \lambda'_i \\ 0 \\ \partial^Q \end{bmatrix} \), with scalar multiplication \( a \begin{bmatrix} p \\ q \\ p' \end{bmatrix} = \begin{bmatrix} ap \\ aq \\ ap' \end{bmatrix} \). Note that it is straightforward to show that the sequence \( \tilde{X} \) defined in this way is a DG \( \mathcal{A} \)-module, and the natural maps \( N \xrightarrow{\zeta} \tilde{X} \xrightarrow{\tilde{\pi}} Q \) are \( \mathcal{A} \)-linear, using the analogous properties for \( X \) and \( X' \).

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{\gamma} & X' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & Q.
\end{array}
\]

The DG module \( X'' \) is a submodule of the direct sum \( X \oplus X' \), so each \( X''_i \) is the submodule of

\[
(X \oplus X')_i = X_i \oplus X'_i \cong N_i \oplus Q_i \oplus N_i \oplus Q_i,
\]

consisting of all vectors \( \begin{bmatrix} x \\ x' \end{bmatrix} \) such that \( \pi'_i(x') = \pi_i(x) \), that is, all vectors of the form \( \begin{bmatrix} p \\ q \\ p' \\ q' \end{bmatrix} \) such that \( q = q' \). In other words, we have

\[
N_i \oplus Q_i \oplus N_i \xrightarrow{\sim} X''_i
\]

where the isomorphism is given by \( \begin{bmatrix} p \\ q \\ p' \end{bmatrix} \mapsto \begin{bmatrix} p \\ q \end{bmatrix}^T \). The differential on \( X \oplus X' \) is the natural diagonal map. So, under the isomorphism (3.5.1), the differential on \( X'' \) has the form

\[
\partial^\tilde{X} = \begin{bmatrix} \partial^X \lambda_i \\ 0 \\ \partial^Q \end{bmatrix}
\]

The next step in the construction of \( \zeta + \zeta' \) is to build \( \tilde{X} \), which is the cokernel of the morphism \( \gamma: N \to X'' \) given by \( p \mapsto \begin{bmatrix} -p \\ p \end{bmatrix} \). That is, since \( \gamma \) is injective, the complex \( \tilde{X} \) is determined by the exact sequence \( 0 \to N \xrightarrow{\zeta} X'' \xrightarrow{\tilde{\pi}} \tilde{X} \to 0 \). It is straightforward to show that this sequence has the following form

\[
\begin{array}{ccccccc}
0 & \to & N_i & \xrightarrow{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} & N_i \oplus Q_i \oplus N_i & \xrightarrow{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} & N_i \oplus Q_i & \to & 0 \\
\partial^\tilde{X} & & & & & & & & \\
0 & \to & N_{i-1} & \xrightarrow{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} & \xrightarrow{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} & N_{i-1} \oplus Q_{i-1} & \to & 0.
\end{array}
\]

By inspecting the right-most column of this diagram, we see that \( \tilde{X} \) has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on \( \tilde{X} \) is the natural one \( a \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} ap \\ aq \end{bmatrix} \). This concludes the proof of Claim 1.
Claim 2. Ψ is injective. Suppose that ζ ∈ Ker(Ψ) is represented by the displays (3.3.1)–(3.3.3). The condition Ψ(ζ) = 0 says that λ ∈ Im(∂^0_{Hom_A(Q,N)}), so there is an element s ∈ Hom_A(Q, N)_0 such that λ = ∂^0_{Hom_A(Q,N)}(s). Thus, for each i we have λ_i = ∂^N_i s_i - s_{i-1} ∂^Q_i. From this, it is straightforward to show that the following diagram commutes:

\[
\begin{array}{c}
N_i ⊕ Q_i \xrightarrow{\begin{bmatrix} 1 & s_i \\ 0 & 1 \end{bmatrix}} N_i ⊕ Q_i \\
N_{i-1} ⊕ Q_{i-1} \xrightarrow{\begin{bmatrix} 1 & s_{i-1} \\ 0 & 1 \end{bmatrix}} N_{i-1} ⊕ Q_{i-1}.
\end{array}
\]

From the fact that s is A-linear, it follows that the maps \( \begin{bmatrix} 1 & s_i \\ 0 & 1 \end{bmatrix} \) describe an A-linear isomorphism \( X \xrightarrow{\cong} N ⊕ Q \) making the following diagram commute:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\epsilon} & X & \xrightarrow{\pi} & Q & \longrightarrow & 0 \\
 & & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & \\
0 & \longrightarrow & N & \xrightarrow{\epsilon} & N ⊕ Q & \xrightarrow{\pi} & Q & \longrightarrow & 0.
\end{array}
\]

In other words, the sequence (3.3.1) splits, so we have ζ = 0, and Ψ is injective. This concludes the proof of Claim 2.

Claim 3. Ψ is surjective. For this, let ξ ∈ H_{-1}(Hom_A(Q, N)) be represented by λ ∈ Ker(∂^0_{Hom_A(Q,N)}). Using the fact that λ is A-linear such that ∂^0_{Hom_A(Q,N)}(λ) = 0, one checks directly that the displays (3.3.2)–(3.3.3) describe an exact sequence of DG A-module homomorphisms of the form (3.3.1) whose image under Ψ is ξ. This concludes the proof of Claim 3 and the proof of the theorem.

□

Remark 3.6. After the results of this paper were announced, Avramov, et al. 2 established the following generalization of Theorem 3.5.

Proposition 3.7. Let A be a DG R-algebra, and let M and N be DG A-modules. There is a monomorphism of abelian groups

\[ \kappa : H_0(Hom_A(Σ^{-1}M, N)) \rightarrow Y\text{Ext}^i_U(M, N) \]

with image equal to the set of equivalence classes of graded-split exact sequences of the form \( 0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0 \).

To see how this generalizes Theorem 3.5, first note that if M is graded-projective, then the map \( \kappa \) is bijective, as in this case every element of \( Y\text{Ext}^i_U(M, N) \) is graded-split; thus, we have \( H_{-1}(Hom_A(M, N)) \cong H_0(Hom_A(Σ^{-1}M, N)) \cong Y\text{Ext}^i_U(M, N) \).

Proof 3.8 (Proof of Theorem A). Using Theorem 3.5, we need only justify the isomorphism \( Y\text{Ext}^i_A(Q, N) \cong \text{Ext}^i_A(Q, N) \) for \( i \geq 2 \). Let

\[ L_i^+ = \cdots \xrightarrow{\partial^L_i} L_{i-1} \xrightarrow{\partial^L_i} L_0 \xrightarrow{\pi} Q \rightarrow 0 \]

be a resolution of Q by categorically projective DG A-modules. Since each \( L_j \) is categorically projective, we have \( Y\text{Ext}^i_A(L_j, -) = 0 \) for all \( i \geq 1 \) and \( L_j \cong 0 \) for each \( j \), so we have \( \text{Ext}^i_A(L_j, -) = 0 \) for all \( i \). Set \( Q_i := \text{Im} \partial^L_i \) for each \( i \geq 1 \). Each
$L_i$ is graded-projective, so the fact that $Q$ is graded-projective implies that each $Q_i$ is graded-projective.

Now, a straightforward dimension-shifting argument explains the first and third isomorphisms in the following display for $i \geq 2$:

$$\text{YExt}_A^1(Q, N) \cong \text{YExt}_A^1(Q_{i-1}, N) \cong \text{Ext}_A^1(Q_{i-1}, N) \cong \text{Ext}_A^1(Q, N).$$

The second isomorphism is from Theorem 3.5 since each $Q_i$ is graded-projective. □

The next example shows that one can have $\text{YExt}_A^0(Q, N) \ncong \text{Ext}_A^0(Q, N)$, even when $Q$ is semi-free.

**Example 3.9.** Continue with the assumptions and notation of Example 3.1, and set $Q = N = \mathbb{R}$. It is straightforward to show that the morphisms $\mathbb{R} \to \mathbb{R}$ are precisely given by multiplication by fixed elements of $R$, so we have the first step in the next display:

$$\text{YExt}_A^0(\mathbb{R}, \mathbb{R}) \cong \mathbb{R} \neq 0 = \text{Ext}_A^0(\mathbb{R}, \mathbb{R}).$$

The third step follows from the condition $\mathbb{R} \simeq 0$.

**Remark 3.10.** It is perhaps worth noting that our proofs can also be used to give the isomorphisms from Theorem A when $Q$ is not necessarily semi-projective, but $N$ is “semi-injective”.

4. **YExt$^1$ and Truncations**

For our work in [12], we need to know how YExt respects the following notion.

**Definition 4.1.** Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. Given an integer $n$, the $n$th soft left truncation of $M$ is the complex

$$\tau(M)_{\leq n} := \cdots \to 0 \to M_n \to \text{Im}(\partial_{n+1}^M) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by $\partial^M$. In other words, $\tau(M)_{\leq n}$ is the quotient DG $A$-module $M/M'$ where $M'$ is the following DG submodule of $M$:

$$M' = \cdots \to M_{n+2} \to M_{n+1} \to \text{Im}(\partial_{n+1}^M) \to 0.$$ 

Note that we have $M' \simeq 0$ if and only if $n \geq \text{sup}(M)$, so the natural morphism $\rho: M \to \tau(M)_{\leq n}$ of DG $A$-modules yields an isomorphism in $\mathcal{D}(A)$ if and only if $n \geq \text{sup}(M)$.

**Proposition 4.2.** Let $A$ be a DG $R$-algebra, and let $M$ and $N$ be DG $A$-modules. Assume that $n$ is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $\text{YExt}_A^1(\tau(M)_{\leq n}, N) \to \text{YExt}_A^1(M, N)$ induced by the morphism $\rho: M \to \tau(M)_{\leq n}$ from Definition 4.1 is a monomorphism.

**Proof.** Let $\Upsilon$ denote the map $\text{YExt}_A^1(\tau(M)_{\leq n}, N) \to \text{YExt}_A^1(M, N)$ induced by $\rho$. Let $\alpha \in \text{Ker}(\Upsilon) \subseteq \text{YExt}_A^1(\tau(M)_{\leq n}, N)$ be represented by the exact sequence

$$0 \to N \overset{f}{\to} X \overset{g}{\to} \tau(M)_{\leq n} \to 0. \tag{4.2.1}$$

Note that, since $N_i = 0 = (\tau(M)_{\leq n})_i$, for all $i > n$, we have $X_i = 0$ for all $i > n$. Our assumptions imply that $0 = \Upsilon([\alpha]) = [\beta]$ where $\beta$ comes from the following...
The middle row $\beta$ of this diagram is split exact since $[\beta] = 0$, so there is a morphism $F: \tilde{X} \to N$ of DG $A$-modules such that $F \circ \tilde{f} = \text{id}_N$. Note that $K$ has the form

$$K = \cdots \frac{\partial^M_{n+2}}{M_{n+1}} \xrightarrow{\partial^M_{n+1}} \text{Im}(\partial^M_{n+1}) \to 0$$

because of the right-most column of the diagram.

We claim that $F \circ \tilde{h} = 0$. It suffices to check this degree-wise. When $i > n$, we have $N_i = 0$, so $F_i = 0$, and $F_i \circ \tilde{h}_i = 0$. When $i < n$, the display (4.2.3) shows that $K_i = 0$, so $\tilde{h}_i = 0$, and $F_i \circ \tilde{h}_i = 0$. For $i = n$, we first note that the display (4.2.3) shows that $\partial^K_{n+1}$ is surjective. In the following diagram, the faces with solid arrows commute because $\tilde{h}$ and $F$ are morphisms:

Since $\partial^K_{n+1}$ is surjective, a simple diagram chase shows that $F_n \circ \tilde{h}_n = 0$. This establishes the claim.

To conclude the proof, note that the previous claim shows that the map $K \to 0$ is a left-splitting of the top row of diagram (4.2.2) that is compatible with the left-splitting $F$ of the middle row. It is now straightforward to show that $F$ induces a morphism $\overline{F}: X \to N$ of DG $A$-modules that left-splits the bottom row of diagram (4.2.2). Since this row represents $\alpha \in \text{YExt}^1_A(\tau(M)(\leq n), N)$, we conclude that $[\alpha] = 0$, so $\Upsilon$ is a monomorphism. $\square$
The next example shows that the monomorphism from Proposition 4.2 may not be an isomorphism.

**Example 4.3.** Continue with the assumptions and notation of Example 3.1. The following diagram describes a non-zero element of $\text{YExt}^1_R(M, N)$:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N & \rightarrow & \mathbb{R} & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & R & \rightarrow & \pi & \rightarrow & 0 \\
\downarrow & & \downarrow & & 1 & & \downarrow & & \\
0 & \rightarrow & R & \rightarrow & R & \rightarrow & \pi & \rightarrow & 0 \\
\downarrow & & \downarrow & & 1 & & \downarrow & & \\
0 & \rightarrow & k & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0.
\end{array}
\]

It is straightforward to show that $\tau(M)_{\leq 0} = 0$, so we have

$0 = \text{YExt}^1_A(\tau(M)_{\leq 0}, N) \nrightarrow \text{YExt}^1_A(M, N) \neq 0$

thus this map is not an isomorphism.

**Proposition 4.4.** Let $A$ be a DG $R$-algebra, and let $C$ be a semi-projective DG $A$-module such that $\text{Ext}^1_R(C, C) = 0$. For $n \geq \text{sup}(C)$, one has

$\text{YExt}^1_A(C, C) = 0 = \text{YExt}^1_A(\tau(C)_{\leq n}, \tau(C)_{\leq n})$.

**Proof.** From Theorem 3.5, we have $\text{YExt}^1_A(C, C) \cong \text{Ext}^1_A(C, C) = 0$. For the remainder of the proof, assume without loss of generality that $\text{sup}(C) < \infty$. Another application of Theorem 3.5 explains the first step in the next display:

$\text{YExt}^1_A(C, \tau(C)_{\leq n}) \cong \text{Ext}^1_A(C, \tau(C)_{\leq n}) \cong \text{Ext}^1_A(C, C) = 0$.

The second step comes from the assumption $n \geq \text{sup}(C)$ which guarantees that the natural morphism $C \rightarrow \tau(C)_{\leq n}$ represents an isomorphism in $D(A)$. Proposition 4.2 implies that $\text{YExt}^1_A(\tau(C)_{\leq n}, \tau(C)_{\leq n})$ is isomorphic to a subgroup of $\text{YExt}^1_A(C, \tau(C)_{\leq n}) = 0$, so $\text{YExt}^1_A(\tau(C)_{\leq n}, \tau(C)_{\leq n}) = 0$, as desired. \qed

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**References**


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