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Extension Groups for DG Modules

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EXTENSION GROUPS FOR DG MODULES

SAEED NASSEH AND SEAN SATHER-WAGSTAFF

Abstract. Let $M$ and $N$ be differential graded (DG) modules over a positively graded commutative DG algebra $A$. We show that the Ext-groups $\text{Ext}_A^i(M,N)$ defined in terms of semi-projective resolutions are not in general isomorphic to the Yoneda Ext-groups $\text{YExt}_A^i(M,N)$ given in terms of equivalence classes of extensions. On the other hand, we show that these groups are isomorphic when the first DG module is semi-projective.

1. Introduction

Convention. In this paper, $R$ is a commutative ring with identity.

Given two $R$-modules $M$ and $N$, a classical result originating with work of Baer [4] states that $\text{Ext}_R^1(M,N)$, defined via projective/injective resolutions, is isomorphic to the abelian group $\text{YExt}_R^1(M,N)$ of equivalence classes of exact sequences of the form $0 \to N \to X \to M \to 0$. The purpose of this note is to discuss possible extensions of this result to the abelian category of differential graded (DG) modules over a positively graded commutative DG algebra $A$. See Section 2 for background information on this category.

Specifically, we show that Baer’s isomorphism fails in general in this context: Examples 3.1 and 3.2 exhibit DG $A$-modules $M$, $N$ with $\text{Ext}_A^1(M,N) \not\cong \text{YExt}_A^1(M,N)$. (See 2.4 and 2.6 below for definitions.) On the other hand, the following result shows that a reasonable hypothesis on the first module does yield such an isomorphism.

Theorem A. Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is semi-projective. Then there is an isomorphism $\text{YExt}_A^i(Q,N) \cong \text{Ext}_A^i(Q,N)$ of abelian groups for all $i \geq 1$.

This is the main result of Section 3; see Proof 3.8. In the subsequent Section 4 we discuss some properties of $\text{YExt}$ with respect to truncations.

It is worth noting here that we apply results from this paper in our answer to a question of Vasconcelos in [12]. Specifically, in that paper, we investigate DG $A$-modules $C$ with $\text{Ext}_A^1(C,C) = 0$ using geometric techniques. These techniques yield an isomorphism between $\text{YExt}_A^1(C,C)$ and a certain quotient of tangent spaces; it is then important for us to know when the vanishing of $\text{Ext}_A^1(C,C)$ implies the vanishing of related $\text{YExt}^1$-modules; see Proposition 4.4 below.

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2. DG Modules

We assume that the reader is familiar with the category of R-complexes and the derived category D(R). Standard references for these topics are [6, 7, 10, 13, 14]. For clarity, we include some definitions and notation.

Definition 2.1. In this paper, complexes of R-modules (“R-complexes” for short) are indexed homologically: \( M = \cdots \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} \cdots \).

The degree of an element \( m \in M \) is denoted \(|m|\). The infimum and supremum of \( M \) are the infimum and supremum, respectively, of the set \( \{ n \in \mathbb{Z} \mid H_n(M) \neq 0 \} \). The tensor product of two R-modules \( M, N \) is denoted \( M \otimes_R N \), and the Hom complex is denoted \( \text{Hom}_R(M, N) \). A chain map \( M \rightarrow N \) is a cycle in \( \text{Hom}_R(M, N)_0 \).

We now discuss DG algebras and DG modules, which are treated in, e.g., [1, 2, 3, 4, 5, 11]. We follow the notation and terminology from [2, 5]; given the slight differences in the literature, though, we include a summary next.

Definition 2.2. A positively graded commutative differential graded R-algebra (DG R-algebra for short) is an R-complex \( A \) equipped with a chain map \( \mu^A: A \otimes_R A \rightarrow A \) with \( ab := \mu^A(a \otimes b) \) that is associative, unital, and graded commutative such that \( A_i = 0 \) for \( i < 0 \). The map \( \mu^A \) is the product on \( A \). Given a DG R-algebra \( A \), the underlying algebra is the graded commutative R-algebra \( A^0 = \oplus_{i \geq 0} A_i \).

A differential graded module over a DG R-algebra \( A \) (DG A-module for short) is an R-complex \( M \) with a chain map \( \mu^M: A \otimes_R M \rightarrow M \) such that the rule \( am := \mu^M(a \otimes m) \) is associative and unital. The map \( \mu^M \) is the scalar multiplication on \( M \). The underlying \( A^0 \)-module associated to \( M \) is the \( A^0 \)-module \( M^0 = \oplus_{i \in \mathbb{Z}} M_i \).

The DG A-module \( \text{Hom}_A(M, N) \) is the subcomplex of \( \text{Hom}_R(M, N) \) of the \( A \)-linear homomorphisms. A morphism \( M \rightarrow N \) of DG A-modules is a cycle in \( \text{Hom}_A(M, N)_0 \). Projective objects in the category of DG A-modules are called categorically projective. Quasiisomorphisms of DG A-modules are identified by the symbol \( \simeq \), also used for the “quasiisomorphic” equivalence relation.

Two important DG R-algebras to keep in mind are \( R \) itself and, more generally, the Koszul complex over \( R \) (on a finite sequence of elements of \( R \)) with the exterior product. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Remark 2.3. Let \( A \) be a DG R-algebra. The category of DG A-modules is an abelian category with enough projectives.

Definition 2.4. Let \( A \) be a DG R-algebra, and let \( M, N \) be DG A-modules. For each \( i \geq 0 \) we have a well-defined Yoneda Ext group \( \text{YExt}_A^i(M, N) \), defined in terms of a resolution of \( M \) by categorically projective DG A-modules:

\[ \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0. \]

A standard result shows that \( \text{YExt}_A^i(M, N) \) is isomorphic to the set of equivalence classes of exact sequences \( 0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0 \) of DG A-modules under the Baer sum; see, e.g., [15 (3.4.6)] and the proof of Theorem 3.3.

We now turn to the derived category \( D(A) \), and related notions.

Definition 2.5. Let \( A \) be a DG R-algebra. A DG A-module \( Q \) is graded-projective if \( \text{Hom}_A(Q, -) \) preserves surjective morphisms, that is, if \( Q^i \) is a projective graded
$R^2$-module. The DG module $Q$ is semi-projective if $\text{Hom}_A(Q, -)$ respects surjective quasiisomorphisms, that is, if $Q$ is graded-projective and respects quasiisomorphisms. A semi-projective resolution of $M$ is a quasiisomorphism $L \xrightarrow{\cong} M$ of DG $A$-modules such that $L$ is semi-projective.

**Fact 2.6.** Let $A$ be a DG $R$-algebra. Then every DG $A$-module has a semi-projective resolution.

**Definition 2.7.** Let $A$ be a DG $R$-algebra. The derived category $D(A)$ is formed from the category of DG $A$-modules by formally inverting the quasiisomorphisms; see [11]. Isomorphisms in $D(A)$ are identified by the symbol $\cong$.

The derived functor $R\text{Hom}_A(M, N)$ is defined via a semi-projective resolution $P \xrightarrow{\cong} M$, as $R\text{Hom}_A(M, N) \cong \text{Hom}_A(P, N)$. For each $i \in \mathbb{Z}$, set $\text{Ext}^i_A(M, N) := H^{-i}(R\text{Hom}_A(M, N))$.

**3. DG Ext vs. Yoneda Ext**

We begin this section with examples of DG $A$-modules $M$ and $N$ such that $\text{Ext}^1_A(M, N) \neq \text{YExt}^1_A(M, N)$. These present two facets of the distinctness of Ext and YExt, as the first example has $M$ and $N$ both bounded, while the second one (from personal communication with Avramov) has $M$ graded-projective.

**Example 3.1.** Let $R = k[\langle X \rangle]$, and consider the following exact sequence of DG $R$-modules, i.e., exact sequence of $R$-complexes:

$$
0 \rightarrow R \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{0} k \rightarrow 0.
$$

This sequence does not split over $R$ (it is not even degree-wise split) so it gives a non-trivial class in $\text{YExt}^1_R(k, R)$, and we conclude that $\text{YExt}^1_R(k, R) \neq 0$. On the other hand, $k$ is homologically trivial, so we have $\text{Ext}^1_R(k, R) = 0$ since $0$ is a semi-free resolution of $k$.

**Example 3.2.** Let $R = k[X]/(X^2)$ and consider the following exact graded-projective DG $R$-module $M = \cdots \xrightarrow{X} R \xrightarrow{X} R \xrightarrow{X} \cdots$. Since $M$ is exact, we have $\text{Ext}^i_R(M, M) = 0$ for all $i$. We claim, however, that $\text{YExt}^1_R(M, M) \neq 0$. To see this, first note that $M$ is isomorphic to the suspension $\Sigma M$ and that $M$ is not contractible. Thus, the mapping cone sequence for the identity morphism $\text{id}_M$ is isomorphic to one of the form $0 \rightarrow M \rightarrow X \rightarrow M \rightarrow 0$ and is not split.
The definition of the isomorphism $\text{YExt}_A^i(Q, N) \to \text{Ext}_A^i(Q, N)$ for $i = 1$ in Theorem A is contained in the following construction. The subsequent lemma and theorem show that $\Psi$ is a well-defined isomorphism.

**Construction 3.3.** Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Define $\Psi: \text{YExt}_A^1(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ as follows. Note that if $Q$ is semi-projective, then $\text{Ext}_A^1(Q, N) \cong H_{-1}(\text{Hom}_A(Q, N))$, which fits with what we have in Theorem A.

Let $\zeta \in \text{YExt}_A^1(Q, N)$ be represented by the sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} Q \to 0. \quad (3.3.1)$$

Since $Q$ is graded-projective, this sequence is graded-split, that is there are elements $h \in \text{Hom}_A(X, N)_0$ and $k \in \text{Hom}_A(Q, X)_0$ with

$$hf = \text{id}_N \quad gk = \text{id}_Q \quad hk = 0 \quad fh + kg = \text{id}_X.$$ 

Thus, the sequence (3.3.1) is isomorphic to one of the form

$$
\begin{array}{ccccccccc}
0 & & N_i & & N_i \oplus Q_i & & Q_i & & 0 \\
\downarrow{\partial^N_i} & & \downarrow{\epsilon_i} & & \downarrow{\pi_i} & & \downarrow{\partial^Q_i} & & \\
0 & & N_{i-1} & & N_{i-1} \oplus Q_{i-1} & & Q_{i-1} & & 0 \\
\downarrow{\partial^N_{i-1}} & & \downarrow{\epsilon_{i-1}} & & \downarrow{\pi_{i-1}} & & \downarrow{\partial^Q_{i-1}} & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\end{array} \quad (3.3.2)
$$

where $\epsilon_j$ is the natural inclusion and $\pi_j$ is the natural surjection for each $j$. Since this diagram comes from a graded-splitting of (3.3.1), the scalar multiplication on the middle column of (3.3.2) is the natural one $a[p^q] = [ap^q]$. (We write elements of $N_i \oplus Q_i$ as column vectors.)

The fact that (3.3.2) commutes implies that $\partial^X_i$ has a specific form:

$$\partial^X_i = \begin{bmatrix} \partial^N_i & \lambda_i \\ 0 & \partial^Q_i \end{bmatrix}. \quad (3.3.3)$$

Here, we have $\lambda_i: Q_i \to N_{i-1}$, that is, $\lambda = \{\lambda_i\} \in \text{Hom}_R(Q, N)_{-1}$. Since the horizontal maps in the sequence (3.3.2) are morphisms of DG $A$-modules, it follows that $\lambda$ is a cycle in $\text{Hom}_A(Q, N)_{-1}$. Thus, $\lambda$ represents a homology class in $H_{-1}(\text{Hom}_A(Q, N))$, and we define $\Psi: \text{YExt}_A^1(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ by setting $\Psi(\zeta)$ equal to $[\lambda]$ the homology class of $\lambda$ in $H_{-1}(\text{Hom}_A(Q, N))$.

**Lemma 3.4.** Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Then the map $\Psi: \text{YExt}_A^1(Q, N) \to H_{-1}(\text{Hom}_A(Q, N))$ from Construction 3.3 is well-defined.
Proof. Let $\zeta \in \text{YExt}^1_A(Q,N)$ be represented by the sequence (3.3.2), and let $\zeta$ be represented by another exact sequence

$$
\begin{array}{ccccccccc}
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0 \\
\downarrow{\partial^N_i} & & \downarrow{\partial^N_i} & & \downarrow{\partial^Q_{i+1}} & & \downarrow{\partial^Q_{i+1}} & & \downarrow{\partial^Q_{i+1}} \\
0 & \rightarrow & N_{i-1} & \xrightarrow{\epsilon_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \rightarrow & 0 \\
\downarrow{\partial^N_{i-1}} & & \downarrow{\partial^N_{i-1}} & & \downarrow{\partial^Q_i} & & \downarrow{\partial^Q_{i-1}} & & \downarrow{\partial^Q_{i-1}} \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
$$

(3.4.1)

where

$$\partial^X_i = \begin{bmatrix} \partial^N_i & \lambda_i' \\ 0 & \partial^Q_i \end{bmatrix}.$$  

(3.4.2)

We need to show that $\lambda - \lambda' \in \text{Im}(\partial^0_0\text{Hom}_A(Q,N))$. The sequences (3.3.2) and (3.4.1) are equivalent in $\text{YExt}^1_R(Q,N)$, so for each $i$ there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0 \\
\downarrow{\text{id}} & & \downarrow{\text{id}} & & \downarrow{\text{id}} & & \downarrow{\text{id}} & & \downarrow{\text{id}} \\
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0
\end{array}
$$

(3.4.3)

where the middle vertical arrow describes a DG $A$-module isomorphism, and such that the following diagram commutes for all $i$

$$
\begin{array}{cccccc}
N_i \oplus Q_i & \xrightarrow{[u_i v_i]} & N_i \oplus Q_i & \xrightarrow{[\partial^N_i \lambda_i] \oplus [0 \partial^Q_i]} & N_{i-1} \oplus Q_{i-1} \\
N_i \oplus Q_i & \xrightarrow{[u_i v_i]} & N_i \oplus Q_i & \xrightarrow{[\partial^N_i \lambda_i'] \oplus [0 \partial^Q_i]} & N_{i-1} \oplus Q_{i-1}
\end{array}
$$

(3.4.4)

The fact that diagram (3.4.3) commutes implies that $u_i = \text{id}_{N_i}$, $x_i = \text{id}_{Q_i}$, and $w_i = 0$. Also, the fact that the middle vertical arrow in diagram (3.4.3) describes a DG $A$-module morphism implies that the sequence $\lambda_i: Q_i \rightarrow N_i$ respects scalar multiplication, i.e., we have $v \in \text{Hom}_A(Q,N)_0$. The fact that diagram (3.4.4) commutes implies that $\lambda_i - \lambda_i' = \partial^N_1 v_i - v_{i-1} \partial^Q_i$. We conclude that $\lambda - \lambda' = \partial^0_0\text{Hom}_A(Q,N)(v) \in \text{Im}(\partial^0_0\text{Hom}_A(Q,N))$, so $\Psi$ is well-defined. $\square$

The next result contains the case $i = 1$ of Theorem A from the introduction, because if $Q$ is semi-projective, then $\text{Ext}^i_A(Q,N) \cong H_{-1}(\text{Hom}_A(Q,N))$.

**Theorem 3.5.** Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Then the map $\Psi: \text{YExt}^1_A(Q,N) \rightarrow H_{-1}(\text{Hom}_A(Q,N))$ from Construction 3.3 is a group isomorphism.

Proof. We break the proof into three claims.
Claim 1. \( \Psi \) is additive. Let \( \zeta, \zeta' \in \text{YExt}_A^1(Q, N) \) be represented by exact sequences
\[
0 \to N \xrightarrow{\xi} X \xrightarrow{\pi} Q \to 0 \quad \text{and} \quad 0 \to N \xrightarrow{\xi'} X' \xrightarrow{\pi'} Q \to 0
\]
respectively, where \( X_i = N_i \oplus Q_i = X'_i \) and the differentials \( \partial^X_i \) and \( \partial^X'_i \) are described as in (3.3.3) and (3.4.2), respectively. We need to show that the Baer sum \( \zeta + \zeta' \) is represented by an exact sequence
\[
0 \to N \xrightarrow{\xi} \tilde{X} \xrightarrow{\pi} Q \to 0,
\]
where \( \tilde{X} = N_i \oplus Q_i \) and \( \partial^X_i = \begin{bmatrix} \partial^N_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^N_i \end{bmatrix} \)
with scalar multiplication \( a \left[ \begin{smallmatrix} p \\ q \\ p' \\ q' \end{smallmatrix} \right] = \left[ \begin{smallmatrix} a_p \\ a_q \\ a_{p'} \\ a_{q'} \end{smallmatrix} \right] \). Note that it is straightforward to show that the sequence \( \tilde{X} \) defined in this way is a DG \( A \)-module, and the natural maps \( N \xrightarrow{\xi} \tilde{X} \xrightarrow{\pi} Q \) are \( A \)-linear, using the analogous properties for \( X \) and \( X' \).

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram
\[
\begin{array}{ccc}
X'' & \xrightarrow{\pi''} & X' \\
\downarrow & & \downarrow \pi'
\end{array}
\]
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Q
\end{array}
\]

The DG module \( X'' \) is a submodule of the direct sum \( X \oplus X' \), so each \( X''_i \) is the submodule of
\[
(X \oplus X')_i = X_i \oplus X'_i \cong N_i \oplus Q_i \oplus N_i \oplus Q_i
\]
consisting of all vectors \( \begin{bmatrix} x \\ x' \end{bmatrix} \) such that \( \pi'_i(x') = \pi_i(x) \), that is, all vectors of the form \( \begin{bmatrix} p \\ q \\ p' \\ q' \end{bmatrix} \) such that \( q = q' \). In other words, we have
\[
N_i \oplus Q_i \oplus N_i \xrightarrow{\cong} X''_i (3.5.1)
\]
where the isomorphism is given by \( \begin{bmatrix} p \\ q \\ p' \\ q' \end{bmatrix} \mapsto \begin{bmatrix} p \\ q \end{bmatrix} \). The differential on \( X \oplus X' \) is the natural diagonal map. So, under the isomorphism (3.5.1), the differential on \( X'' \) has the form
\[
\partial^X''_i = \begin{bmatrix} \partial^N_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^N_i \end{bmatrix}
\]

\[
X'' \cong N_i \oplus Q_i \oplus N_i \xrightarrow{\pi''} N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \cong X''_{i-1}.
\]

The next step in the construction of \( \zeta + \zeta' \) is to build \( \tilde{X} \), which is the cokernel of the morphism \( \gamma: N \to X'' \) given by \( p \mapsto \begin{bmatrix} 0 & p \end{bmatrix} \). That is, since \( \gamma \) is injective, the complex \( \tilde{X} \) is determined by the exact sequence
\[
0 \to N \xrightarrow{\xi} X'' \xrightarrow{\pi''} \tilde{X} \to 0.
\]
It is straightforward to show that this sequence has the following form
\[
\begin{array}{ccc}
0 & \xrightarrow{\partial_{i-1}^N} & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_i \to 0
\end{array}
\]
\[
\begin{array}{ccc}
0 & \xrightarrow{\partial_{i-1}^N} & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_i \to 0
\end{array}
\]
\[
\begin{array}{ccc}
0 & \xrightarrow{\partial_{i-1}^N} & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \oplus Q_i \to 0
\end{array}
\]
By inspecting the right-most column of this diagram, we see that \( \tilde{X} \) has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on \( \tilde{X} \) is the natural one \( a \left[ \begin{smallmatrix} p \\ q \\ p' \\ q' \end{smallmatrix} \right] = \left[ \begin{smallmatrix} a_p \\ a_q \\ a_{p'} \\ a_{q'} \end{smallmatrix} \right] \). This concludes the proof of Claim 1.
Claim 2. $\Psi$ is injective. Suppose that $\zeta \in \text{Ker}(\Psi)$ is represented by the displays (3.3.1)–(3.3.3). The condition $\Psi(\zeta) = 0$ says that $\lambda \in \text{Im}(\partial^0_{\text{Hom}_A(Q,N)})$, so there is an element $s \in \text{Hom}_A(Q,N)_0$ such that $\lambda = \partial^0_{\text{Hom}_A(Q,N)}(s)$. Thus, for each $i$ we have $\lambda_i = \partial^N_i s_i - s_{i-1} \partial^Q_i$. From this, it is straightforward to show that the following diagram commutes:

$$
\begin{array}{ccc}
N_i \oplus Q_i & \xrightarrow{[1 \ s_i \ \ 0 \ 1]} & N_i \oplus Q_i \\
\partial^N_i \lambda_i & \downarrow \cong & \partial^Q_i \\
N_{i-1} \oplus Q_{i-1} & \xrightarrow{\begin{bmatrix} 1 & s_{i-1} \\ 0 & 1 \end{bmatrix} \ 
\cong} & N_{i-1} \oplus Q_{i-1}.
\end{array}
$$

From the fact that $s$ is $A$-linear, it follows that the maps $[1 \ s_i \ \ 0 \ 1]$ describe an $A$-linear isomorphism $X \cong N \oplus Q$ making the following diagram commute:

$$
\begin{array}{ccc}
0 & \xrightarrow{\epsilon} & N & \xrightarrow{\pi} & Q & \xrightarrow{\epsilon} & 0 \\
\cong & & \cong & & \cong & & \cong
\end{array}
$$

In other words, the sequence (3.3.1) splits, so we have $\zeta = 0$, and $\Psi$ is injective. This concludes the proof of Claim 2.

Claim 3. $\Psi$ is surjective. For this, let $\xi \in H_{-1}(\text{Hom}_A(Q,N))$ be represented by $\lambda \in \text{Ker}(\partial^{-1}_{\text{Hom}_A(Q,N)})$. Using the fact that $\lambda$ is $A$-linear such that $\partial^{-1}_{\text{Hom}_A(Q,N)}(\lambda) = 0$, one checks directly that the displays (3.3.2)–(3.3.3) describe an exact sequence of DG $A$-module homomorphisms of the form (3.3.1) whose image under $\Psi$ is $\xi$. This concludes the proof of Claim 3 and the proof of the theorem. \(\square\)

Remark 3.6. After the results of this paper were announced, Avramov, et al. \(\cite{2}\) established the following generalization of Theorem 3.5.

Proposition 3.7. Let $A$ be a DG $R$-algebra, and let $M$ and $N$ be DG $A$-modules. There is a monomorphism of abelian groups

$$
\kappa: H_0(\text{Hom}_A(\Sigma^{-1}M,N)) \to \text{YExt}_U^1(M,N)
$$

with image equal to the set of equivalence classes of graded-split exact sequences of the form $0 \to N \to X \to M \to 0$.

To see how this generalizes Theorem 3.5, first note that if $M$ is graded-projective, then the map $\kappa$ is bijective, as in this case every element of $\text{YExt}_U^1(M,N)$ is graded-split; thus, we have $H_{-1}(\text{Hom}_A(M,N)) \cong H_0(\text{Hom}_A(\Sigma^{-1}M,N)) \cong \text{YExt}_U^1(M,N)$.

Proof 3.8 (Proof of Theorem A). Using Theorem 3.5, we need only justify the isomorphism $\text{YExt}_A^i(Q,N) \cong \text{Ext}_A^i(Q,N)$ for $i \geq 2$. Let

$$
L_+^i = \cdots \xrightarrow{\partial^i_0} L_1 \xrightarrow{\partial^i_1} L_0 \xrightarrow{\pi} Q \to 0
$$

be a resolution of $Q$ by categorically projective DG $A$-modules. Since each $L_j$ is categorically projective, we have $\text{YExt}_A^i(L_j,-) = 0$ for all $i \geq 1$ and $L_j \cong 0$ for each $j$, so we have $\text{Ext}_A^i(L_j,-) = 0$ for all $i$. Set $Q_i := \text{Im} \partial^i_0$ for each $i \geq 1$. Each
Let $L_i$ be graded-projective, so the fact that $Q$ is graded-projective implies that each $Q_i$ is graded-projective.

Now, a straightforward dimension-shifting argument explains the first and third isomorphisms in the following display for $i \geq 2$:

$$Y\text{Ext}^i_A(Q, N) \cong Y\text{Ext}^i_A(Q_{i-1}, N) \cong \text{Ext}^i_A(Q_{i-1}, N) \cong \text{Ext}^i_A(Q, N).$$

The second isomorphism is from Theorem 3.5 since each $Q_i$ is graded-projective. □

The next example shows that one can have $Y\text{Ext}_A^0(Q, N) \not\cong \text{Ext}_A^0(Q, N)$, even when $Q$ is semi-free.

**Example 3.9.** Continue with the assumptions and notation of Example 3.1, and set $Q = N = A$. It is straightforward to show that the morphisms $R \to R$ are precisely given by multiplication by fixed elements of $R$, so we have the first step in the next display:

$$Y\text{Ext}_A^0(R, R) \cong R \not= 0 = \text{Ext}_A^0(R, R).$$

The third step follows from the condition $R \simeq 0$.

**Remark 3.10.** It is perhaps worth noting that our proofs can also be used to give the isomorphisms from Theorem 3.4 when $Q$ is not necessarily semi-projective, but $N$ is “semi-injective”.

4. **$Y\text{Ext}^1$ and Truncations**

For our work in [12], we need to know how $Y\text{Ext}$ respects the following notion.

**Definition 4.1.** Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. Given an integer $n$, the $n$th soft left truncation of $M$ is the complex

$$\tau(M)_{(\leq n)} := \cdots \to 0 \to M_n / \text{Im}(\partial_{n+1}^M) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by $\partial^M$. In other words, $\tau(M)_{(\leq n)}$ is the quotient DG $A$-module $M/M'$ where $M'$ is the following DG submodule of $M$:

$$M' = \cdots \to M_{n+2} \to M_{n+1} \to \text{Im}(\partial_{n+1}^M) \to 0.$$ 

Note that we have $M' \simeq 0$ if and only if $n \geq \text{sup}(M)$, so the natural morphism $\rho: M \to \tau(M)_{(\leq n)}$ of DG $A$-modules yields an isomorphism in $\mathcal{D}(A)$ if and only if $n \geq \text{sup}(M)$.

**Proposition 4.2.** Let $A$ be a DG $R$-algebra, and let $M$ and $N$ be DG $A$-modules. Assume that $n$ is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $Y\text{Ext}_A^1(\tau(M)_{(\leq n)}), N) \to Y\text{Ext}_A^1(M, N)$ induced by the morphism $\rho: M \to \tau(M)_{(\leq n)}$ from Definition 4.1 is a monomorphism.

**Proof.** Let $\Upsilon$ denote the map $Y\text{Ext}_A^1(\tau(M)_{(\leq n)}), N) \to Y\text{Ext}_A^1(M, N)$ induced by $\rho$. Let $\alpha \in \text{Ker}(\Upsilon) \subseteq Y\text{Ext}_A^1(\tau(M)_{(\leq n)}), N)$ be represented by the exact sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} \tau(M)_{(\leq n)} \to 0. \quad (4.2.1)$$

Note that, since $N_i = 0 = (\tau(M)_{(\leq n)})_i$, for all $i > n$, we have $X_i = 0$ for all $i > n$. Our assumptions imply that $0 = \Upsilon([\alpha]) = [\beta]$ where $\beta$ comes from the following
The middle row $\beta$ of this diagram is split exact since $[\beta] = 0$, so there is a morphism $F : \tilde{X} \to N$ of DG $A$-modules such that $F \circ \tilde{f} = \text{id}_N$. Note that $K$ has the form

$$K = \cdots \to \partial^M_{n+2} M_{n+1} \to \partial^M_{n+1} \text{Im}(\partial^M_{n+1}) \to 0$$

because of the right-most column of the diagram.

We claim that $F \circ \tilde{h} = 0$. It suffices to check this degree-wise. When $i > n$, we have $N_i = 0$, so $F_i = 0$, and $F_i \circ \tilde{h}_i = 0$. When $i < n$, the display (4.2.3) shows that $K_i = 0$, so $\tilde{h}_i = 0$, and $F_i \circ \tilde{h}_i = 0$. For $i = n$, we first note that the display (4.2.3) shows that $\partial^K_{n+1}$ is surjective. In the following diagram, the faces with solid arrows commute because $\tilde{h}$ and $F$ are morphisms:

Since $\partial^K_{n+1}$ is surjective, a simple diagram chase shows that $F_n \circ \tilde{h}_n = 0$. This establishes the claim.

To conclude the proof, note that the previous claim shows that the map $K \to 0$ is a left-splitting of the top row of diagram (4.2.2) that is compatible with the left-splitting $F$ of the middle row. It is now straightforward to show that $F$ induces a morphism $\overline{F} : X \to N$ of DG $A$-modules that left-splits the bottom row of diagram (4.2.2). Since this row represents $\alpha \in \text{YExt}_A^1(\tau(M)(\leq n), N)$, we conclude that $[\alpha] = 0$, so $\Upsilon$ is a monomorphism. \qed
The next example shows that the monomorphism from Proposition 4.2 may not be an isomorphism.

**Example 4.3.** Continue with the assumptions and notation of Example 3.1. The following diagram describes a non-zero element of \( \text{YExt}_1^A(M, N) \):

\[
\begin{array}{ccccccc}
0 & \rightarrow & N & \rightarrow & R & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & R & \rightarrow & R & \rightarrow & 0 \\
\downarrow & & \downarrow & & 1 & & \pi & & \\
0 & \rightarrow & R & \rightarrow & R & \rightarrow & \pi & \rightarrow & k & \rightarrow & 0 \\
\end{array}
\]

It is straightforward to show that \( \tau(M)_{(\leq 0)} = 0 \), so we have

\[
0 = \text{YExt}_1^A(\tau(M)_{(\leq 0)}, N) \rightarrow \text{YExt}_1^A(M, N) \neq 0
\]

thus this map is not an isomorphism.

**Proposition 4.4.** Let \( A \) be a DG \( R \)-algebra, and let \( C \) be a semi-projective DG \( A \)-module such that \( \text{Ext}_1^A(C, C) = 0 \). For \( n \geq \sup(C) \), one has

\[
\text{YExt}_1^A(C, C) = 0 = \text{YExt}_1^A(\tau(C)_{(\leq n)}, \tau(C)_{(\leq n)}).
\]

**Proof.** From Theorem 3.5, we have \( \text{YExt}_1^A(C, C) \cong \text{Ext}_1^A(C, C) = 0 \). For the remainder of the proof, assume without loss of generality that \( \sup(C) < \infty \). Another application of Theorem 3.5 explains the first step in the next display:

\[
\text{YExt}_1^A(C, \tau(C)_{(\leq n)}) \cong \text{Ext}_1^A(C, \tau(C)_{(\leq n)}) \cong \text{Ext}_1^A(C, C) = 0.
\]

The second step comes from the assumption \( n \geq \sup(C) \) which guarantees that the natural morphism \( C \rightarrow \tau(C)_{(\leq n)} \) represents an isomorphism in \( D(A) \). Proposition 4.2 implies that \( \text{YExt}_1^A(\tau(C)_{(\leq n)}, \tau(C)_{(\leq n)}) \) is isomorphic to a subgroup of \( \text{YExt}_1^A(C, \tau(C)_{(\leq n)}) = 0 \), so \( \text{YExt}_1^A(\tau(C)_{(\leq n)}, \tau(C)_{(\leq n)}) = 0 \), as desired. \( \square \)

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