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EXTENSION GROUPS FOR DG MODULES

SAEED NASSEH AND SEAN SATHER-WAGSTAFF

ABSTRACT. Let $M$ and $N$ be differential graded (DG) modules over a positively graded commutative DG algebra $A$. We show that the Ext-groups $\text{Ext}_A^i(M, N)$ defined in terms of semi-projective resolutions are not in general isomorphic to the Yoneda Ext-groups $\text{YExt}_A^i(M, N)$ given in terms of equivalence classes of extensions. On the other hand, we show that these groups are isomorphic when the first DG module is semi-projective.

1. INTRODUCTION

Convention. In this paper, $R$ is a commutative ring with identity.

Given two $R$-modules $M$ and $N$, a classical result originating with work of Baer [4] states that $\text{Ext}_R^1(M, N)$, defined via projective/injective resolutions, is isomorphic to the abelian group $\text{YExt}_R^1(M, N)$ of equivalence classes of exact sequences of the form $0 \to N \to X \to M \to 0$. The purpose of this note is to discuss possible extensions of this result to the abelian category of differential graded (DG) modules over a positively graded commutative DG algebra $A$. See Section 2 for background information on this category.

Specifically, we show that Baer’s isomorphism fails in general in this context: Examples 3.1 and 3.2 exhibit DG $A$-modules $M, N$ with $\text{Ext}_A^1(M, N) \neq \text{YExt}_A^1(M, N)$ (See 2.4 and 2.6 below for definitions.) On the other hand, the following result shows that a reasonable hypothesis on the first module does yield such an isomorphism.

Theorem A. Let $A$ be a DG $R$-algebra, and let $N, Q$ be DG $A$-modules such that $Q$ is semi-projective. Then there is an isomorphism $\text{YExt}_A^i(Q, N) \cong \text{Ext}_A^i(Q, N)$ of abelian groups for all $i \geq 1$.

This is the main result of Section 3; see Proof 3.8. In the subsequent Section 4 we discuss some properties of YExt with respect to truncations.

It is worth noting here that we apply results from this paper in our answer to a question of Vasconcelos in [12]. Specifically, in that paper, we investigate DG $A$-modules $C$ with $\text{Ext}_A^1(C, C) = 0$ using geometric techniques. These techniques yield an isomorphism between $\text{YExt}_A^1(C, C)$ and a certain quotient of tangent spaces; it is then important for us to know when the vanishing of $\text{Ext}_A^1(C, C)$ implies the vanishing of related YExt$^1$-modules; see Proposition 4.4 below.

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2. DG Modules

We assume that the reader is familiar with the category of $R$-complexes and the derived category $\mathcal{D}(R)$. Standard references for these topics are [6, 7, 9, 10, 13, 14]. For clarity, we include some definitions and notation.

**Definition 2.1.** In this paper, complexes of $R$-modules ("$R$-complexes" for short) are indexed homologically: $M = \cdots \to \partial_{n+2}^M M_{n+1} \to \partial_{n+1}^M M_n \to \partial_n^M M_{n-1} \to \cdots$. The degree of an element $m \in M$ is denoted $|m|$. The infimum and supremum of $M$ are the infimum and supremum, respectively, of the set $\{ n \in \mathbb{Z} \mid H_n(M) \neq 0 \}$. The tensor product of two $R$-complexes $M, N$ is denoted $M \otimes R N$, and the Hom complex is denoted $\text{Hom}_R(M, N)$. A chain map $M \to N$ is a cycle in $\text{Hom}_R(M, N)_0$.

Next we discuss DG algebras and DG modules, which are treated in, e.g., [1, 2, 3, 4, 5, 11]. We follow the notation and terminology from [2, 5]; given the slight differences in the literature, though, we include a summary next.

**Definition 2.2.** A positively graded commutative differential graded $R$-algebra (DG $R$-algebra for short) is an $R$-complex $A$ equipped with a chain map $\mu^A : A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is associative, unital, and graded commutative such that $A_i = 0$ for $i < 0$. The map $\mu^A$ is the product on $A$. Given a DG $R$-algebra $A$, the underlying algebra is the graded commutative $R$-algebra $A^0 = \oplus_{i \geq 0} A_i$.

A differential graded module over a DG $R$-algebra $A$ (DG $A$-module for short) is an $R$-complex $M$ with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map $\mu^M$ is the scalar multiplication on $M$. The underlying $A^0$-module associated to $M$ is the $A^0$-module $M^0 = \oplus_{n \in \mathbb{Z}} M_n$.

The DG $A$-module $\text{Hom}_A(M, N)$ is the subcomplex of $\text{Hom}_R(M, N)$ of the $A$-linear homomorphisms. A morphism $M \to N$ of DG $A$-modules is a cycle in $\text{Hom}_A(M, N)_0$. Projective objects in the category of DG $A$-modules are called categorically projective. Quasiisomorphisms of DG $A$-modules are identified by the symbol $\simeq$, also used for the "quasiisomorphic" equivalence relation.

Two important DG $R$-algebras to keep in mind are $R$ itself and, more generally, the Koszul complex over $R$ (on a finite sequence of elements of $R$) with the exterior product. A DG $R$-module is just an $R$-complex, and a morphism of DG $R$-modules is simply a chain map.

**Remark 2.3.** Let $A$ be a DG $R$-algebra. The category of DG $A$-modules is an abelian category with enough projectives.

**Definition 2.4.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. For each $i \geq 0$ we have a well-defined Yoneda Ext group $\text{YExt}^i_A(M, N)$, defined in terms of a resolution of $M$ by categorically projective DG $A$-modules:

$$\cdots \to Q_1 \to Q_0 \to M \to 0.$$  

A standard result shows that $\text{YExt}^1_A(M, N)$ is isomorphic to the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$ of DG $A$-modules under the Baer sum; see, e.g., [15, (3.4.6)] and the proof of Theorem 3.3.

We now turn to the derived category $\mathcal{D}(A)$, and related notions.

**Definition 2.5.** Let $A$ be a DG $R$-algebra. A DG $A$-module $Q$ is graded-projective if $\text{Hom}_A(Q, -)$ preserves surjective morphisms, that is, if $Q^i$ is a projective graded...
The DG module $Q$ is \textit{semi-projective} if $\text{Hom}_A(Q, -)$ respects surjective quasiisomorphisms, that is, if $Q$ is graded-projective and respects quasiisomorphisms. A \textit{semi-projective resolution} of $M$ is a quasiisomorphism $L \xrightarrow{\sim} M$ of DG $A$-modules such that $L$ is semi-projective.

**Fact 2.6.** Let $A$ be a DG $R$-algebra. Then every DG $A$-module has a semi-projective resolution.

**Definition 2.7.** Let $A$ be a DG $R$-algebra. The derived category $\mathcal{D}(A)$ is formed from the category of DG $A$-modules by formally inverting the quasiisomorphisms; see [11]. Isomorphisms in $\mathcal{D}(A)$ are identified by the symbol $\cong$.

The derived functor $R\text{Hom}_A(M, N)$ is defined via a semi-projective resolution $P \xrightarrow{\sim} M$, as $R\text{Hom}_A(M, N) \cong \text{Hom}_A(P, N)$. For each $i \in \mathbb{Z}$, set $\text{Ext}^i_A(M, N) := H^{-i}(R\text{Hom}_A(M, N))$.

3. **DG Ext vs. Yoneda Ext**

We begin this section with examples of DG $A$-modules $M$ and $N$ such that $\text{Ext}^1_A(M, N) \neq \text{YExt}^1_A(M, N)$. These present two facets of the distinctness of Ext and YExt, as the first example has $M$ and $N$ both bounded, while the second one (from personal communication with Avramov) has $M$ graded-projective.

**Example 3.1.** Let $R = k[X]$, and consider the following exact sequence of DG $R$-modules, i.e., exact sequence of $R$-complexes:

$$
0 \rightarrow R \rightarrow R \rightarrow k \rightarrow 0
$$

This sequence does not split over $R$ (it is not even degree-wise split) so it gives a non-trivial class in $\text{YExt}^1_R(k, R)$, and we conclude that $\text{YExt}^1_R(k, R) \neq 0$. On the other hand, $k$ is homologically trivial, so we have $\text{Ext}^1_R(k, R) = 0$ since 0 is a semi-free resolution of $k$.

**Example 3.2.** Let $R = k[X]/(X^2)$ and consider the following exact graded-projective DG $R$-module $M = \cdots \xrightarrow{X} R \xrightarrow{X} R \xrightarrow{X} \cdots$. Since $M$ is exact, we have $\text{Ext}^i_R(M, M) = 0$ for all $i$. We claim, however, that $\text{YExt}^1_R(M, M) \neq 0$. To see this, first note that $M$ is isomorphic to the suspension $\Sigma M$ and that $M$ is not contractible. Thus, the mapping cone sequence for the identity morphism $\text{id}_M$ is isomorphic to one of the form $0 \rightarrow M \rightarrow X \rightarrow M \rightarrow 0$ and is not split.
The definition of the isomorphism \( Y\text{Ext}^i_A(Q, N) \to \text{Ext}^i_A(Q, N) \) for \( i = 1 \) in Theorem A is contained in the following construction. The subsequent lemma and theorem show that \( \Psi \) is a well-defined isomorphism.

**Construction 3.3.** Let \( A \) be a DG \( R \)-algebra, and let \( N, Q \) be DG \( A \)-modules such that \( Q \) is graded-projective. Define \( \Psi : Y\text{Ext}^1_A(Q, N) \to \text{Hom}_A(Q, N) \) as follows. Note that if \( Q \) is semi-projective, then \( \text{Ext}^1_A(Q, N) \cong \text{H}_{-1}(\text{Hom}_A(Q, N)) \), which fits with what we have in Theorem A.

Let \( \zeta \in Y\text{Ext}^1_A(Q, N) \) be represented by the sequence

\[
0 \to N \xrightarrow{f} X \xrightarrow{g} Q \to 0. \tag{3.3.1}
\]

Since \( Q \) is graded-projective, this sequence is graded-split, that is there are elements \( h \in \text{Hom}_A(X, N)_0 \) and \( k \in \text{Hom}_A(Q, X)_0 \) with \( hf = \text{id}_N \), \( gk = \text{id}_Q \), \( hk = 0 \), and \( fh + kg = \text{id}_X \).

Thus, the sequence (3.3.1) is isomorphic to one of the form

\[
\begin{array}{ccccccc}
0 & \to & N_i & \xrightarrow{\partial^N_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \to & 0 \\
\downarrow{\epsilon_i} & & \downarrow{\partial^X_i} & & \downarrow{\partial^Q_i} & & & \\
0 & \to & N_{i-1} & \xrightarrow{\partial^N_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \to & 0 \\
\downarrow{\epsilon_{i-1}} & & \downarrow{\partial^X_{i-1}} & & \downarrow{\partial^Q_{i-1}} & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \tag{3.3.2}
\]

where \( \epsilon_j \) is the natural inclusion and \( \pi_j \) is the natural surjection for each \( j \). Since this diagram comes from a graded-splitting of (3.3.1), the scalar multiplication on the middle column of (3.3.2) is the natural one \( a \left[ \frac{p}{q} \right] = \left[ \frac{ap}{aq} \right] \). (We write elements of \( N_i \oplus Q_i \) as column vectors.)

The fact that (3.3.2) commutes implies that \( \partial^X_i \) has a specific form:

\[
\partial^X_i = \begin{bmatrix} \partial^N_i & \lambda_i \\ 0 & \partial^Q_i \end{bmatrix}. \tag{3.3.3}
\]

Here, we have \( \lambda_i : Q_i \to N_{i-1} \), that is, \( \lambda = \{ \lambda_i \} \in \text{Hom}_R(Q, N)_{-1} \). Since the horizontal maps in the sequence (3.3.2) are morphisms of DG \( A \)-modules, it follows that \( \lambda \) is a cycle in \( \text{Hom}_A(Q, N)_{-1} \). Thus, \( \lambda \) represents a homology class in \( H_{-1}(\text{Hom}_A(Q, N)) \), and we define \( \Psi : Y\text{Ext}^1_A(Q, N) \to H_{-1}(\text{Hom}_A(Q, N)) \) by setting \( \Psi(\zeta) \) equal to \( [\lambda] \) the homology class of \( \lambda \) in \( H_{-1}(\text{Hom}_A(Q, N)) \).

**Lemma 3.4.** Let \( A \) be a DG \( R \)-algebra, and let \( N, Q \) be DG \( A \)-modules such that \( Q \) is graded-projective. Then the map \( \Psi : Y\text{Ext}^1_A(Q, N) \to H_{-1}(\text{Hom}_A(Q, N)) \) from Construction 3.3 is well-defined.
Proof. Let $\zeta \in \text{YExt}_A^1(Q, N)$ be represented by the sequence (3.3.2), and let $\zeta$ be represented by another exact sequence

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & N_{i-1} & \xrightarrow{\epsilon_{i-1}} & N_{i-1} \oplus Q_{i-1} & \xrightarrow{\pi_{i-1}} & Q_{i-1} & \rightarrow & 0 \\
\end{array}
\]

(3.4.1)

where

\[
\partial_i^{X'} = \begin{bmatrix} \partial_i^N & \lambda_i' \\ 0 & \partial_i^Q \end{bmatrix}.
\]

(3.4.2)

We need to show that $\lambda - \lambda' \in \text{Im}(\partial_0^{\text{Hom}_A(Q, N)})$. The sequences (3.3.2) and (3.4.1) are equivalent in $\text{YExt}_A^1(Q, N)$, so for each $i$ there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & N_i & \xrightarrow{\epsilon_i} & N_i \oplus Q_i & \xrightarrow{\pi_i} & Q_i & \rightarrow & 0 \\
\end{array}
\]

(3.4.3)

where the middle vertical arrow describes a DG $A$-module isomorphism, and such that the following diagram commutes for all $i$

\[
\begin{array}{ccccccccc}
N_i \oplus Q_i & \xrightarrow{\left[ \begin{array}{cc} \partial_i^N & \lambda_i \\ 0 & \partial_i^Q \end{array} \right]} & N_i \oplus Q_i \\
N_{i-1} \oplus Q_{i-1} & \xrightarrow{\left[ \begin{array}{cc} \partial_{i-1}^N & \lambda_{i-1}' \\ 0 & \partial_{i-1}^Q \end{array} \right]} & N_{i-1} \oplus Q_{i-1} \\
\end{array}
\]

(3.4.4)

The fact that diagram (3.4.4) commutes implies that $u_i = \text{id}_{N_i}$, $x_i = \text{id}_{Q_i}$, and $w_i = 0$. Also, the fact that the middle vertical arrow in diagram (3.4.3) describes a DG $A$-module morphism implies that the sequence $\nu_i: Q_i \rightarrow N_i$ respects scalar multiplication, i.e., we have $v \in \text{Hom}_A(Q, N)$. The fact that diagram (3.4.4) commutes implies that $\lambda_i - \lambda_i' = \partial_i^N v_i - v_{i-1} \partial_i^Q$. We conclude that $\lambda - \lambda' = \partial_0^{\text{Hom}_A(Q, N)}(v) \in \text{Im}(\partial_0^{\text{Hom}_A(Q, N)})$, so $\Psi$ is well-defined.

The next result contains the case $i = 1$ of Theorem A from the introduction, because if $Q$ is semi-projective, then $\text{Ext}_A^1(Q, N) \cong H_{-1}(\text{Hom}_A(Q, N))$.

Theorem 3.5. Let $A$ be a DG $R$-algebra, and let $N$, $Q$ be DG $A$-modules such that $Q$ is graded-projective. Then the map $\Psi: \text{YExt}_A^1(Q, N) \rightarrow H_{-1}(\text{Hom}_A(Q, N))$ from Construction 3.3 is a group isomorphism.

Proof. We break the proof into three claims.
Claim 1. $\Psi$ is additive. Let $\zeta, \zeta' \in \text{YExt}_A^1(Q, N)$ be represented by exact sequences
$$0 \to N \overset{i}{\to} X \overset{p}{\to} Q \to 0 \text{ and } 0 \to N \overset{i'}{\to} X' \overset{q}{\to} Q \to 0$$
respectively, where $X_i = N_i \oplus Q_i = X'_i$ and the differentials $\partial^X$ and $\partial^{X'}$ are described as in (3.3.3) and (3.4.2), respectively. We need to show that the Baer sum $\zeta + \zeta'$ is represented by an exact sequence $0 \to N \overset{\pi}{\to} \tilde{X} \overset{\tau}{\to} Q \to 0$, where $\tilde{X}_i = N_i \oplus Q_i$ and $\partial^{\tilde{X}} = \left[ \begin{array}{c} \partial^X \lambda_i + \lambda'_i \\ 0 & \partial^Q \end{array} \right]$, with scalar multiplication $a \left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} ap \\ aq \end{array} \right]$. Note that it is straightforward to show that the sequence $\tilde{X}$ defined in this way is a DG $A$-module, and the natural maps $N \overset{i}{\to} \tilde{X} \overset{\pi}{\to} Q$ are $A$-linear, using the analogous properties for $X$ and $X'$.

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram

$$
\begin{array}{ccc}
X'' & \overset{\pi}{\to} & X' \\
\downarrow & & \downarrow \\
X & \overset{\pi'}{\to} & Q.
\end{array}
$$

The DG module $X''$ is a submodule of the direct sum $X \oplus X'$, so each $X''_i$ is the submodule of

$$(X \oplus X')_1 = X_1 \oplus X'_1 \cong N_1 \oplus Q_1 \oplus N_1 \oplus Q_1$$

consisting of all vectors $[x, x']$ such that $\pi'(x') = \pi_1(x)$, that is, all vectors of the form $[p \ q \ p' \ q']$ such that $q = q'$. In other words, we have

$$N_1 \oplus Q_1 \oplus N_1 \xrightarrow{\pi''} X''_1 \quad (3.5.1)$$

where the isomorphism is given by $[p \ q \ p' \ q'] \mapsto [p \ q \ p']^T$. The differential on $X \oplus X'$ is the natural diagonal map. So, under the isomorphism (3.5.1), the differential on $X''$ has the form

$$\partial^{X''}_1 = \left[ \begin{array}{c} \partial^X \lambda_i \\ 0 \\ 0 \\ \partial^Q \end{array} \right] \mapsto N_i \oplus Q_i \oplus N_i \to N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \cong X''_{i-1}.$$  

The next step in the construction of $\zeta + \zeta'$ is to build $\tilde{X}$, which is the cokernel of the morphism $\gamma : N \to X''$ given by $p \mapsto \left[ \begin{array}{c} \partial \lambda_i \\ 0 \end{array} \right]$. That is, since $\gamma$ is injective, the complex $\tilde{X}$ is determined by the exact sequence $0 \to N \overset{i}{\to} X'' \overset{\pi}{\to} \tilde{X} \to 0$. It is straightforward to show that this sequence has the following form

$$
\begin{array}{cccc}
0 & \to & N_i & \xrightarrow{\partial^X} & N_i \oplus Q_i \oplus N_i & \xrightarrow{\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]} & N_i \oplus Q_i & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \quad & \downarrow & & \downarrow \\
0 & \to & N_{i-1} & \xrightarrow{\partial^X} & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} & \xrightarrow{\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]} & N_{i-1} \oplus Q_{i-1} & \to & 0.
\end{array}
$$

By inspecting the right-most column of this diagram, we see that $\tilde{X}$ has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on $\tilde{X}$ is the natural one $a \left[ \begin{array}{c} p \\ q \end{array} \right] = \left[ \begin{array}{c} ap \\ aq \end{array} \right]$. This concludes the proof of Claim 1.
Claim 2. Ψ is injective. Suppose that ζ ∈ Ker(Ψ) is represented by the displays (3.3.1) - (3.3.3). The condition Ψ(ζ) = 0 says that λ ∈ Im(∂_0^{Hom_A(Q,N)}), so there is an element s ∈ Hom_A(Q,N) such that λ = ∂_0^{Hom_A(Q,N)}(s). Thus, for each i we have λ_i = ∂_i^N s_i - s_{i-1} ∂_i^Q. From this, it is straightforward to show that the following diagram commutes:

\[
\begin{array}{c}
N_i \oplus Q_i \\
\downarrow \quad \quad \downarrow \\
N_{i-1} \oplus Q_{i-1}
\end{array}
\begin{array}{c}
\left[ \begin{array}{c}
\partial_i^N \\
0
\end{array} \right] \\
\left[ \begin{array}{c}
\lambda_i \\
0
\end{array} \right]
\right] \\
\left[ \begin{array}{c}
\partial_i^Q
\end{array} \right]
\end{array}
\begin{array}{c}
N_i \oplus Q_i \\
\downarrow \quad \quad \downarrow \\
N_{i-1} \oplus Q_{i-1}
\end{array}
\begin{array}{c}
\left[ \begin{array}{c}
\partial_i^N \\
0
\end{array} \right] \\
\left[ \begin{array}{c}
0 \\
1
\end{array} \right]
\right] \\
\left[ \begin{array}{c}
\partial_i^Q
\end{array} \right]
\end{array}
\]

From the fact that s is A-linear, it follows that the maps \([\begin{array}{c}1 \\ s_i \end{array}]\) describe an A-linear isomorphism \(X \xrightarrow{\cong} N \oplus Q\) making the following diagram commute:

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{\epsilon} X \xrightarrow{\pi} Q \rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{\epsilon} N \oplus Q \xrightarrow{\pi} Q \rightarrow 0
\end{array}
\]

In other words, the sequence (3.3.1) splits, so we have ζ = 0, and Ψ is injective. This concludes the proof of Claim 2.

Claim 3. Ψ is surjective. For this, let ξ ∈ H_{−1}(Hom_A(Q,N)) be represented by a λ ∈ Ker(∂_1^{Hom_A(Q,N)}). Using the fact that λ is A-linear such that ∂_1^{Hom_A(Q,N)}(λ) = 0, one checks directly that the displays (3.3.2) - (3.3.3) describe an exact sequence of DG A-module homomorphisms of the form (3.3.1) whose image under Ψ is ξ. This concludes the proof of Claim 3 and the proof of the theorem.

Remark 3.6. After the results of this paper were announced, Avramov, et al. established the following generalization of Theorem 3.5.

Proposition 3.7. Let A be a DG R-algebra, and let M and N be DG A-modules. There is a monomorphism of abelian groups

\[
\kappa: H_0(Hom_A(\Sigma^{-1} M, N)) \rightarrow YExt^1_U(M, N)
\]

with image equal to the set of equivalence classes of graded-split exact sequences of the form 0 → N → X → M → 0.

To see how this generalizes Theorem 3.3, first note that if M is graded-projective, then the map κ is bijective, as in this case every element of YExt^1_U(M, N) is graded-split; thus, we have H_{−1}(Hom_A(M, N)) ≅ H_0(Hom_A(\Sigma^{-1} M, N)) ≅ YExt^1_U(M, N).

Proof 3.8 (Proof of Theorem [A]). Using Theorem 3.3, we need only justify the isomorphism YExt^i_A(Q, N) ≅ Ext^i_A(Q, N) for i ≥ 2. Let

\[
L_j^+ = \cdots \xrightarrow{\partial_j^L} L_1 \xrightarrow{\partial_0^L} L_0 \xrightarrow{\pi} Q \rightarrow 0
\]

be a resolution of Q by categorically projective DG A-modules. Since each L_j is categorically projective, we have YExt^i_A(L_j, −) = 0 for all i ≥ 1 and L_j ≃ 0 for each j, so we have Ext^i_A(L_j, −) = 0 for all i. Set Q_i := Im ∂_i^L for each i ≥ 1. Each
$L_i$ is graded-projective, so the fact that $Q$ is graded-projective implies that each $Q_i$ is graded-projective.

Now, a straightforward dimension-shifting argument explains the first and third isomorphisms in the following display for $i \geq 2$:

$$\text{YExt}^i_A(Q, N) \cong \text{YExt}^i_A(Q_{i-1}, N) \cong \text{Ext}^i_A(Q_{i-1}, N) \cong \text{Ext}^i_A(Q, N).$$

The second isomorphism is from Theorem 3.5 since each $Q_i$ is graded-projective. □

The next example shows that one can have $\text{YExt}^0_A(Q, N) \not\cong \text{Ext}^0_A(Q, N)$, even when $Q$ is semi-free.

**Example 3.9.** Continue with the assumptions and notation of Example 3.1, and set $Q = N = \mathbb{R}$. It is straightforward to show that the morphisms $\mathbb{R} \to \mathbb{R}$ are precisely given by multiplication by fixed elements of $\mathbb{R}$, so we have the first step in the next display:

$$\text{YExt}^0_A(\mathbb{R}, \mathbb{R}) \cong \mathbb{R} \not\cong 0 = \text{Ext}^0_A(\mathbb{R}, \mathbb{R}).$$

The third step follows from the condition $\mathbb{R} \simeq 0$.

**Remark 3.10.** It is perhaps worth noting that our proofs can also be used to give the isomorphisms from Theorem $\square$ when $Q$ is not necessarily semi-projective, but $N$ is “semi-injective”.

4. YExt$^1$ AND TRUNCATIONS

For our work in [12], we need to know how YExt respects the following notion.

**Definition 4.1.** Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. Given an integer $n$, the $n$th soft left truncation of $M$ is the complex

$$\tau(M)_{(\leq n)} := \cdots \to M_n/\text{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by $\partial^M$. In other words, $\tau(M)_{(\leq n)}$ is the quotient DG $A$-module $M/M'$ where $M'$ is the following DG submodule of $M$:

$$M' = \cdots \to M_{n+2} \to M_{n+1} \to \text{Im}(\partial^M_{n+1}) \to 0.$$  

Note that we have $M' \simeq 0$ if and only if $n \geq \text{sup}(M)$, so the natural morphism $\rho: M \to \tau(M)_{(\leq n)}$ of DG $A$-modules yields an isomorphism in $\mathcal{D}(A)$ if and only if $n \geq \text{sup}(M)$.

**Proposition 4.2.** Let $A$ be a DG $R$-algebra, and let $M$ and $N$ be DG $A$-modules. Assume that $n$ is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $\text{YExt}^1_A(\tau(M)_{(\leq n)}, N) \to \text{YExt}^1_A(M, N)$ induced by the morphism $\rho: M \to \tau(M)_{(\leq n)}$ from Definition 4.1 is a monomorphism.

**Proof.** Let $\Upsilon$ denote the map $\text{YExt}^1_A(\tau(M)_{(\leq n)}, N) \to \text{YExt}^1_A(M, N)$ induced by $\rho$. Let $\alpha \in \text{Ker}(\Upsilon) \subseteq \text{YExt}^1_A(\tau(M)_{(\leq n)}, N)$ be represented by the exact sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} \tau(M)_{(\leq n)} \to 0. \quad (4.2.1)$$

Note that, since $N_i = 0 = (\tau(M)_{(\leq n)})_i$ for all $i > n$, we have $X_i = 0$ for all $i > n$. Our assumptions imply that $0 = \Upsilon([\alpha]) = [\beta]$ where $\beta$ comes from the following
The middle row \( \beta \) of this diagram is split exact since \([\beta] = 0\), so there is a morphism \( F : \tilde{X} \to N \) of DG \( A \)-modules such that \( F \circ \tilde{f} = \text{id}_N \). Note that \( K \) has the form
\[
K = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} \text{Im}(\partial^M_{n+1}) \to 0
\] because of the right-most column of the diagram.

We claim that \( F \circ \tilde{h} = 0 \). It suffices to check this degree-wise. When \( i > n \), we have \( N_i = 0 \), so \( F_i = 0 \), and \( F_i \circ \tilde{h}_i = 0 \). When \( i < n \), the display (4.2.3) shows that \( K_i = 0 \), so \( \tilde{h}_i = 0 \), and \( F_i \circ \tilde{h}_i = 0 \). For \( i = n \), we first note that the display (4.2.3) shows that \( \partial^K_{n+1} \) is surjective. In the following diagram, the faces with solid arrows commute because \( \tilde{h} \) and \( F \) are morphisms:

Since \( \partial^K_{n+1} \) is surjective, a simple diagram chase shows that \( F_n \circ \tilde{h}_n = 0 \). This establishes the claim.

To conclude the proof, note that the previous claim shows that the map \( K \to 0 \) is a left-splitting of the top row of diagram (4.2.2) that is compatible with the left-splitting \( F \) of the middle row. It is now straightforward to show that \( F \) induces a morphism \( \overline{F} : X \to N \) of DG \( A \)-modules that left-splits the bottom row of diagram (4.2.2). Since this row represents \( \alpha \in \text{YExt}_A^1(\tau(M)(\leq n), N) \), we conclude that \([\alpha] = 0\), so \( \Upsilon \) is a monomorphism. \( \square \)
The next example shows that the monomorphism from Proposition 4.2 may not be an isomorphism.

**Example 4.3.** Continue with the assumptions and notation of Example 3.1. The following diagram describes a non-zero element of $\text{YExt}_R^1(M, N)$:

$$
\begin{array}{ccccccc}
0 & \rightarrow & N & \rightarrow & R & \rightarrow & M & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & R & \rightarrow & R & \rightarrow & 0 \\
0 & \rightarrow & R & \rightarrow & R & \rightarrow & k & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0,
\end{array}
$$

It is straightforward to show that $\tau(M)_{\leq 0} = 0$, so we have

$$0 = \text{YExt}_A^1(\tau(M)_{\leq 0}, N) \not\sim \text{YExt}_A^1(M, N) \neq 0$$

thus this map is not an isomorphism.

**Proposition 4.4.** Let $A$ be a DG $R$-algebra, and let $C$ be a semi-projective DG $A$-module such that $\text{Ext}_R^1(C, C) = 0$. For $n \geq \text{sup}(C)$, one has

$$\text{YExt}_A^1(C, C) = 0 = \text{YExt}_A^1(\tau(C)_{\leq n}, \tau(C)_{\leq n}).$$

**Proof.** From Theorem 3.5, we have $\text{YExt}_A^1(C, C) \cong \text{Ext}_A^1(C, C) = 0$. For the remainder of the proof, assume without loss of generality that $\text{sup}(C) < \infty$. Another application of Theorem 3.5 explains the first step in the next display:

$$\text{YExt}_A^1(C, \tau(C)_{\leq n}) \cong \text{Ext}_A^1(C, \tau(C)_{\leq n}) \cong \text{Ext}_A^1(C, C) = 0.$$

The second step comes from the assumption $n \geq \text{sup}(C)$ which guarantees that the natural morphism $C \rightarrow \tau(C)_{\leq n}$ represents an isomorphism in $\text{D}(A)$. Proposition 4.2 implies that $\text{YExt}_A^1(\tau(C)_{\leq n}, \tau(C)_{\leq n})$ is isomorphic to a subgroup of $\text{YExt}_A^1(C, \tau(C)_{\leq n}) = 0$, so $\text{YExt}_A^1(\tau(C)_{\leq n}, \tau(C)_{\leq n}) = 0$, as desired. □

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