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Extension Groups for DG Modules

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Abstract. Let M and N be differential graded (DG) modules over a positively graded commutative DG algebra A. We show that the Ext-groups $\mathrm{Ext}^i_A(M,N)$ defined in terms of semi-projective resolutions are not in general isomorphic to the Yoneda Ext-groups $\mathrm{YExt}^i_A(M,N)$ given in terms of equivalence classes of extensions. On the other hand, we show that these groups are isomorphic when the first DG module is semi-projective.

1. Introduction

Convention. In this paper, R is a commutative ring with identity.

Given two R-modules M and N, a classical result originating with work of Baer [4] states that $\mathrm{Ext}^1_R(M,N)$, defined via projective/injective resolutions, is isomorphic to the abelian group $\mathrm{YExt}^1_R(M,N)$ of equivalence classes of exact sequences of the form $0 \to N \to X \to M \to 0$. The purpose of this note is to discuss possible extensions of this result to the abelian category of differential graded (DG) modules over a positively graded commutative DG algebra A. See Section 2 for background information on this category.

Specifically, we show that Baer's isomorphism fails in general in this context: Examples 3.1 and 3.2 exhibit DG A-modules M, N with $\mathrm{Ext}^1_A(M,N) \ncong \mathrm{YExt}^1_A(M,N)$. (See 2.4 and 2.6 below for definitions.) On the other hand, the following result shows that a reasonable hypothesis on the first module does yield such an isomorphism.

Theorem A. *Let A be a DG R-algebra, and let N, Q be DG A-modules such that Q is semi-projective. Then there is an isomorphism $\mathrm{YExt}^i_A(Q,N) \cong \mathrm{Ext}^i_A(Q,N)$ of abelian groups for all $i \geqslant 1$.*

This is the main result of Section 3; see Proof 3.8. In the subsequent Section 4, we discuss some properties of YExt with respect to truncations.

It is worth noting here that we apply results from this paper in our answer to a question of Vasconcelos in [12]. Specifically, in that paper, we investigate DG A-modules C with $\mathrm{Ext}^1_A(C,C) = 0$ using geometric techniques. These techniques yield an isomorphism between $\mathrm{YExt}^1_A(C,C)$ and a certain quotient of tangent spaces; it is then important for us to know when the vanishing of $\mathrm{Ext}^1_A(C,C)$ implies the vanishing of related YExt^1-modules; see Proposition 4.4 below.

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2. DG Modules

We assume that the reader is familiar with the category of R-complexes and the derived category $\mathcal{D}(R)$. Standard references for these topics are [6, 7, 9, 10, 13, 14]. For clarity, we include some definitions and notation.

Definition 2.1. In this paper, complexes of R-modules ("R-complexes" for short) are indexed homologically: $M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots$. The degree of an element $m \in M$ is denoted $|m|$. The *infimum* and *supremum* of M are the infimum and supremum, respectively, of the set $\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$. The *tensor product* of two R-complexes M, N is denoted $M \otimes_R N$, and the *Hom complex* is denoted $\operatorname{Hom}_R(M,N)$. A *chain map* $M \to N$ is a cycle in $\operatorname{Hom}_R(M,N)_0$.

Next we discuss DG algebras and DG modules, which are treated in, e.g., [1, 2, 3, 5, 8, 11]. We follow the notation and terminology from [2, 5]; given the slight differences in the literature, though, we include a summary next.

Definition 2.2. A *positively graded commutative differential graded R-algebra* (*DG R-algebra* for short) is an R-complex A equipped with a chain map $\mu^A : A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is associative, unital, and graded commutative such that $A_i = 0$ for $i < 0$. The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \oplus_{i \geqslant 0} A_i$.

A *differential graded module over* a DG R-algebra A (*DG A-module* for short) is an R-complex M with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is the *scalar multiplication* on M. The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \oplus_{i \in \mathbb{Z}} M_i$.

The DG A-module $\operatorname{Hom}_A(M,N)$ is the subcomplex of $\operatorname{Hom}_R(M,N)$ of the A-linear homomorphisms. A *morphism* $M \to N$ of DG A-modules is a cycle in $\operatorname{Hom}_A(M,N)_0$. Projective objects in the category of DG A-modules are called *categorically projective*. Quasiisomorphisms of DG A-modules are identified by the symbol $\simeq$, also used for the "quasiisomorphic" equivalence relation.

Two important DG R-algebras to keep in mind are R itself and, more generally, the Koszul complex over R (on a finite sequence of elements of R) with the exterior product. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Remark 2.3. Let A be a DG R-algebra. The category of DG A-modules is an abelian category with enough projectives.

Definition 2.4. Let A be a DG R-algebra, and let M, N be DG A-modules. For each $i \geqslant 0$ we have a well-defined *Yoneda Ext group* $\operatorname{YExt}^i_A(M,N)$, defined in terms of a resolution of M by categorically projective DG A-modules:

$$\cdots \to Q_1 \to Q_0 \to M \to 0.$$

A standard result shows that $\operatorname{YExt}^1_A(M,N)$ is isomorphic to the set of equivalence classes of exact sequences $0 \to N \to X \to M \to 0$ of DG A-modules under the Baer sum; see, e.g., [15, (3.4.6)] and the proof of Theorem 3.5.

We now turn to the derived category $\mathcal{D}(A)$, and related notions.

Definition 2.5. Let A be a DG R-algebra. A DG A-module Q is *graded-projective* if $\operatorname{Hom}_A(Q,-)$ preserves surjective morphisms, that is, if $Q^\natural$ is a projective graded

$R^\natural$-module. The DG module Q is *semi-projective* if $\mathrm{Hom}_A(Q,-)$ respects surjective quasiisomorphisms, that is, if Q is graded-projective and respects quasiisomorphisms. A *semi-projective resolution* of M is a quasiisomorphism $L \xrightarrow{\simeq} M$ of DG A-modules such that L is semi-projective.

Fact 2.6. Let A be a DG R-algebra. Then every DG A-module has a semi-projective resolution.

Definition 2.7. Let A be a DG R-algebra. The derived category $\mathcal{D}(A)$ is formed from the category of DG A-modules by formally inverting the quasiisomorphisms; see [11]. Isomorphisms in $\mathcal{D}(A)$ are identified by the symbol $\simeq$.

The derived functor $\mathbf{R}\mathrm{Hom}_A(M,N)$ is defined via a semi-projective resolution $P \xrightarrow{\simeq} M$, as $\mathbf{R}\mathrm{Hom}_A(M,N) \simeq \mathrm{Hom}_A(P,N)$. For each $i \in \mathbb{Z}$, set $\mathrm{Ext}^i_A(M,N) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_A(M,N))$.

3. DG Ext vs. Yoneda Ext

We begin this section with examples of DG A-modules M and N such that $\mathrm{Ext}^1_A(M,N) \ncong \mathrm{YExt}^1_A(M,N)$. These present two facets of the distinctness of Ext and YExt, as the first example has M and N both bounded, while the second one (from personal communication with Avramov) has M graded-projective.

Example 3.1. Let $R = k[\![X]\!]$, and consider the following exact sequence of DG R-modules, i.e., exact sequence of R-complexes:

$$0 \longrightarrow \underline{R} \longrightarrow \underline{R} \longrightarrow \underline{k} \longrightarrow 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "X"] \arrow[d, "1"'] & R \arrow[r] \arrow[d, "1"'] & k \arrow[r] \arrow[d, "1"'] & 0 \\
0 \arrow[r] & R \arrow[r, "X"] \arrow[d] & R \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

This sequence does not split over R (it is not even degree-wise split) so it gives a non-trivial class in $\mathrm{YExt}^1_R(\underline{k},\underline{R})$, and we conclude that $\mathrm{YExt}^1_R(\underline{k},\underline{R}) \neq 0$. On the other hand, $\underline{k}$ is homologically trivial, so we have $\mathrm{Ext}^1_R(\underline{k},\underline{R}) = 0$ since 0 is a semi-free resolution of $\underline{k}$.

Example 3.2. Let $R = k[X]/(X^2)$ and consider the following exact graded-projective DG R-module $M = \cdots \xrightarrow{X} R \xrightarrow{X} R \xrightarrow{X} \cdots$. Since M is exact, we have $\mathrm{Ext}^i_R(M,M) = 0$ for all i. We claim, however, that $\mathrm{YExt}^1_R(M,M) \neq 0$. To see this, first note that M is isomorphic to the suspension ΣM and that M is not contractible. Thus, the mapping cone sequence for the identity morphism id_M is isomorphic to one of the form $0 \to M \to X \to M \to 0$ and is not split.

The definition of the isomorphism $\mathrm{YExt}^i_A(Q,N) \to \mathrm{Ext}^i_A(Q,N)$ for $i=1$ in Theorem A is contained in the following construction. The subsequent lemma and theorem show that Ψ is a well-defined isomorphism.

Construction 3.3. Let A be a DG R-algebra, and let N, Q be DG A-modules such that Q is graded-projective. Define $\Psi\colon \mathrm{YExt}^1_A(Q,N) \to \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$ as follows. Note that if Q is semi-projective, then $\mathrm{Ext}^1_A(Q,N) \cong \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$, which fits with what we have in Theorem A.

Let $\zeta \in \mathrm{YExt}^1_A(Q,N)$ be represented by the sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} Q \to 0. \tag{3.3.1}$$

Since Q is graded-projective, this sequence is *graded-split*, that is there are elements $h \in \mathrm{Hom}_A(X,N)_0$ and $k \in \mathrm{Hom}_A(Q,X)_0$ with

$$hf = \mathrm{id}_N \qquad gk = \mathrm{id}_Q \qquad hk = 0 \qquad fh + kg = \mathrm{id}_X\,.$$

Thus, the sequence (3.3.1) is isomorphic to one of the form

$$\begin{tikzcd}
& \vdots \arrow[d, "\partial^N_{i+1}"'] & \vdots \arrow[d, "\partial^X_{i+1}"'] & \vdots \arrow[d, "\partial^Q_{i+1}"'] & \\
0 \arrow[r] & N_i \arrow[r, "\epsilon_i"] \arrow[d, "\partial^N_i"'] & N_i \oplus Q_i \arrow[r, "\pi_i"] \arrow[d, "\partial^X_i"'] & Q_i \arrow[r] \arrow[d, "\partial^Q_i"'] & 0 \\
0 \arrow[r] & N_{i-1} \arrow[r, "\epsilon_{i-1}"] \arrow[d, "\partial^N_{i-1}"'] & N_{i-1} \oplus Q_{i-1} \arrow[r, "\pi_{i-1}"] \arrow[d, "\partial^X_{i-1}"'] & Q_{i-1} \arrow[r] \arrow[d, "\partial^Q_{i-1}"'] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd} \tag{3.3.2}$$

where ϵ_j is the natural inclusion and π_j is the natural surjection for each j. Since this diagram comes from a graded-splitting of (3.3.1), the scalar multiplication on the middle column of (3.3.2) is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$. (We write elements of $N_i \oplus Q_i$ as column vectors.)

The fact that (3.3.2) commutes implies that ∂^X_i has a specific form:

$$\partial^X_i = \begin{bmatrix} \partial^N_i & \lambda_i \\ 0 & \partial^Q_i \end{bmatrix}. \tag{3.3.3}$$

Here, we have $\lambda_i\colon Q_i \to N_{i-1}$, that is, $\lambda = \{\lambda_i\} \in \mathrm{Hom}_R(Q,N)_{-1}$. Since the horizontal maps in the sequence (3.3.2) are morphisms of DG A-modules, it follows that λ is a cycle in $\mathrm{Hom}_A(Q,N)_{-1}$. Thus, λ represents a homology class in $\mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$, and we define $\Psi\colon \mathrm{YExt}^1_A(Q,N) \to \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$ by setting $\Psi(\zeta)$ equal to $[\lambda]$ the homology class of λ in $\mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$.

Lemma 3.4. *Let A be a DG R-algebra, and let N, Q be DG A-modules such that Q is graded-projective. Then the map $\Psi\colon \mathrm{YExt}^1_A(Q,N) \to \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$ from Construction 3.3 is well-defined.*

Proof. Let $\zeta \in \mathrm{YExt}^1_A(Q,N)$ be represented by the sequence (3.3.2), and let ζ be represented by another exact sequence

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^N_{i+1}"'] & \vdots \arrow[d, "\partial^{X'}_{i+1}"'] & \vdots \arrow[d, "\partial^Q_{i+1}"'] & \\
0 \arrow[r] & N_i \arrow[r, "\epsilon_i"] \arrow[d, "\partial^N_i"'] & N_i \oplus Q_i \arrow[r, "\pi_i"] \arrow[d, "\partial^{X'}_i"'] & Q_i \arrow[r] \arrow[d, "\partial^Q_i"'] & 0 \\
0 \arrow[r] & N_{i-1} \arrow[r, "\epsilon_{i-1}"] \arrow[d, "\partial^N_{i-1}"'] & N_{i-1} \oplus Q_{i-1} \arrow[r, "\pi_{i-1}"] \arrow[d, "\partial^{X'}_{i-1}"'] & Q_{i-1} \arrow[r] \arrow[d, "\partial^Q_{i-1}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{3.4.1}$$

where

$$\partial^{X'}_i = \begin{bmatrix} \partial^N_i & \lambda'_i \\ 0 & \partial^Q_i \end{bmatrix}. \tag{3.4.2}$$

We need to show that $\lambda - \lambda' \in \mathrm{Im}(\partial_0^{\mathrm{Hom}_A(Q,N)})$. The sequences (3.3.2) and (3.4.1) are equivalent in $\mathrm{YExt}^1_R(Q,N)$, so for each i there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N_i \arrow[r, "\epsilon_i"] \arrow[d, "="'] & N_i \oplus Q_i \arrow[r, "\pi_i"] \arrow[d, "\left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]"', "\cong"] & Q_i \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & N_i \arrow[r, "\epsilon_i"] & N_i \oplus Q_i \arrow[r, "\pi_i"] & Q_i \arrow[r] & 0
\end{tikzcd} \tag{3.4.3}$$

where the middle vertical arrow describes a DG A-module isomorphism, and such that the following diagram commutes for all i

$$\begin{tikzcd}
N_i \oplus Q_i \arrow[r, "\left[\begin{smallmatrix} u_i & v_i \\ w_i & x_i \end{smallmatrix}\right]", "\cong"'] \arrow[d, "\left[\begin{smallmatrix} \partial^N_i & \lambda_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]"'] & N_i \oplus Q_i \arrow[d, "\left[\begin{smallmatrix} \partial^N_i & \lambda'_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]"] \\
N_{i-1} \oplus Q_{i-1} \arrow[r, "\left[\begin{smallmatrix} u_{i-1} & v_{i-1} \\ w_{i-1} & x_{i-1} \end{smallmatrix}\right]", "\cong"'] & N_{i-1} \oplus Q_{i-1}.
\end{tikzcd} \tag{3.4.4}$$

The fact that diagram (3.4.3) commutes implies that $u_i = \mathrm{id}_{N_i}$, $x_i = \mathrm{id}_{Q_i}$, and $w_i = 0$. Also, the fact that the middle vertical arrow in diagram (3.4.3) describes a DG A-module morphism implies that the sequence $v_i \colon Q_i \to N_i$ respects scalar multiplication, i.e., we have $v \in \mathrm{Hom}_A(Q,N)_0$. The fact that diagram (3.4.4) commutes implies that $\lambda_i - \lambda'_i = \partial^N_i v_i - v_{i-1}\partial^Q_i$. We conclude that $\lambda - \lambda' = \partial_0^{\mathrm{Hom}_A(Q,N)}(v) \in \mathrm{Im}(\partial_0^{\mathrm{Hom}_A(Q,N)})$, so Ψ is well-defined. $\square$

The next result contains the case $i = 1$ of Theorem A from the introduction, because if Q is semi-projective, then $\mathrm{Ext}^1_A(Q,N) \cong \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$.

Theorem 3.5. *Let A be a DG R-algebra, and let N, Q be DG A-modules such that Q is graded-projective. Then the map $\Psi \colon \mathrm{YExt}^1_A(Q,N) \to \mathrm{H}_{-1}(\mathrm{Hom}_A(Q,N))$ from Construction 3.3 is a group isomorphism.*

Proof. We break the proof into three claims.

Claim 1. Ψ is additive. Let $\zeta, \zeta' \in \mathrm{YExt}^1_A(Q, N)$ be represented by exact sequences $0 \to N \xrightarrow{\epsilon} X \xrightarrow{\pi} Q \to 0$ and $0 \to N \xrightarrow{\epsilon'} X' \xrightarrow{\pi'} Q \to 0$ respectively, where $X_i = N_i \oplus Q_i = X'_i$ and the differentials ∂^X and $\partial^{X'}$ are described as in (3.3.3) and (3.4.2), respectively. We need to show that the Baer sum $\zeta + \zeta'$ is represented by an exact sequence $0 \to N \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q \to 0$, where $\widetilde{X}_i = N_i \oplus Q_i$ and $\partial^{\widetilde{X}}_i = \left[\begin{smallmatrix} \partial^N_i & \lambda_i + \lambda'_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]$, with scalar multiplication $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$. Note that it is straightforward to show that the sequence $\widetilde{X}$ defined in this way is a DG A-module, and the natural maps $N \xrightarrow{\widetilde{\epsilon}} \widetilde{X} \xrightarrow{\widetilde{\pi}} Q$ are A-linear, using the analogous properties for X and X'.

We construct the Baer sum in two steps. The first step is to construct the pull-back diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d, "\pi'"] \\ X \arrow[r, "\pi"] & Q. \end{tikzcd}$$

The DG module X'' is a submodule of the direct sum $X \oplus X'$, so each X''_i is the submodule of

$$(X \oplus X')_i = X_i \oplus X'_i \cong N_i \oplus Q_i \oplus N_i \oplus Q_i$$

consisting of all vectors $\left[\begin{smallmatrix} x \\ x' \end{smallmatrix}\right]$ such that $\pi'_i(x') = \pi_i(x)$, that is, all vectors of the form $[p \quad q \quad p' \quad q']^T$ such that $q = q'$. In other words, we have

$$N_i \oplus Q_i \oplus N_i \xrightarrow{\cong} X''_i \tag{3.5.1}$$

where the isomorphism is given by $[p \quad q \quad p']^T \mapsto [p \quad q \quad p' \quad q]^T$. The differential on $X \oplus X'$ is the natural diagonal map. So, under the isomorphism (3.5.1), the differential on X'' has the form

$$X''_i \cong N_i \oplus Q_i \oplus N_i \xrightarrow{\partial^{X''}_i = \left[\begin{smallmatrix} \partial^N_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^N_i \end{smallmatrix}\right]} N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \cong X''_{i-1}.$$

The next step in the construction of $\zeta + \zeta'$ is to build $\widetilde{X}$, which is the cokernel of the morphism $\gamma \colon N \to X''$ given by $p \mapsto \left[\begin{smallmatrix} -p \\ 0 \\ p \end{smallmatrix}\right]$. That is, since γ is injective, the complex $\widetilde{X}$ is determined by the exact sequence $0 \to N \xrightarrow{\gamma} X'' \xrightarrow{\tau} \widetilde{X} \to 0$. It is straightforward to show that this sequence has the following form

$$\begin{tikzcd} 0 \arrow[r] & N_i \arrow[r, "\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]"] \arrow[d, "\partial^N_i"'] & N_i \oplus Q_i \oplus N_i \arrow[r, "\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]"] \arrow[d, "\left[\begin{smallmatrix} \partial^N_i & \lambda_i & 0 \\ 0 & \partial^Q_i & 0 \\ 0 & \lambda'_i & \partial^N_i \end{smallmatrix}\right]"] & N_i \oplus Q_i \arrow[r] \arrow[d, "\left[\begin{smallmatrix} \partial^N_i & \lambda_i + \lambda'_i \\ 0 & \partial^Q_i \end{smallmatrix}\right]"] & 0 \\ 0 \arrow[r] & N_{i-1} \arrow[r, "\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]"] & N_{i-1} \oplus Q_{i-1} \oplus N_{i-1} \arrow[r, "\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right]"] & N_{i-1} \oplus Q_{i-1} \arrow[r] & 0. \end{tikzcd}$$

By inspecting the right-most column of this diagram, we see that $\widetilde{X}$ has the desired form. Furthermore, checking the module structures at each step of the construction, we see that the scalar multiplication on $\widetilde{X}$ is the natural one $a\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = \left[\begin{smallmatrix} ap \\ aq \end{smallmatrix}\right]$. This concludes the proof of Claim 1.

Claim 2. Ψ is injective. Suppose that $\zeta \in \operatorname{Ker}(\Psi)$ is represented by the displays (3.3.1)–(3.3.3). The condition $\Psi(\zeta) = 0$ says that $\lambda \in \operatorname{Im}(\partial_0^{\operatorname{Hom}_A(Q,N)})$, so there is an element $s \in \operatorname{Hom}_A(Q,N)_0$ such that $\lambda = \partial_0^{\operatorname{Hom}_A(Q,N)}(s)$. Thus, for each i we have $\lambda_i = \partial_i^N s_i - s_{i-1}\partial_i^Q$. From this, it is straightforward to show that the following diagram commutes:

$$\begin{tikzcd}
N_i \oplus Q_i \arrow[r, "\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]", "\cong"'] \arrow[d, "\left[\begin{smallmatrix} \partial_i^N & \lambda_i \\ 0 & \partial_i^Q \end{smallmatrix}\right]"'] & N_i \oplus Q_i \arrow[d, "\left[\begin{smallmatrix} \partial_i^N & 0 \\ 0 & \partial_i^Q \end{smallmatrix}\right]"] \\
N_{i-1} \oplus Q_{i-1} \arrow[r, "\left[\begin{smallmatrix} 1 & s_{i-1} \\ 0 & 1 \end{smallmatrix}\right]", "\cong"'] & N_{i-1} \oplus Q_{i-1}.
\end{tikzcd}$$

From the fact that s is A-linear, it follows that the maps $\left[\begin{smallmatrix} 1 & s_i \\ 0 & 1 \end{smallmatrix}\right]$ describe an A-linear isomorphism $X \xrightarrow{\cong} N \oplus Q$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "="'] & X \arrow[r, "\pi"] \arrow[d, "\cong"'] & Q \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & N \arrow[r, "\epsilon"] & N \oplus Q \arrow[r, "\pi"] & Q \arrow[r] & 0.
\end{tikzcd}$$

In other words, the sequence (3.3.1) splits, so we have $\zeta = 0$, and Ψ is injective. This concludes the proof of Claim 2.

Claim 3. Ψ is surjective. For this, let $\xi \in \mathrm{H}_{-1}(\operatorname{Hom}_A(Q,N))$ be represented by $\lambda \in \operatorname{Ker}(\partial_{-1}^{\operatorname{Hom}_A(Q,N)})$. Using the fact that λ is A-linear such that $\partial_{-1}^{\operatorname{Hom}_A(Q,N)}(\lambda) = 0$, one checks directly that the displays (3.3.2)–(3.3.3) describe an exact sequence of DG A-module homomorphisms of the form (3.3.1) whose image under Ψ is ξ. This concludes the proof of Claim 3 and the proof of the theorem. □

Remark 3.6. After the results of this paper were announced, Avramov, et al. [2] established the following generalization of Theorem 3.5.

Proposition 3.7. *Let A be a DG R-algebra, and let M and N be DG A-modules. There is a monomorphism of abelian groups*

$$\kappa\colon\ \mathrm{H}_0(\operatorname{Hom}_A(\Sigma^{-1}M,N)) \to \operatorname{YExt}^1_U(M,N)$$

with image equal to the set of equivalence classes of graded-split exact sequences of the form $0 \to N \to X \to M \to 0$.

To see how this generalizes Theorem 3.5, first note that if M is graded-projective, then the map κ is bijective, as in this case every element of $\operatorname{YExt}^1_U(M,N)$ is graded-split; thus, we have $\mathrm{H}_{-1}(\operatorname{Hom}_A(M,N)) \cong \mathrm{H}_0(\operatorname{Hom}_A(\Sigma^{-1}M,N)) \cong \operatorname{YExt}^1_U(M,N)$.

Proof 3.8 (Proof of Theorem A)**.** Using Theorem 3.5, we need only justify the isomorphism $\operatorname{YExt}^i_A(Q,N) \cong \operatorname{Ext}^i_A(Q,N)$ for $i \geqslant 2$. Let

$$L_\bullet^+ = \cdots \xrightarrow{\partial_2^L} L_1 \xrightarrow{\partial_1^L} L_0 \xrightarrow{\pi} Q \to 0$$

be a resolution of Q by categorically projective DG A-modules. Since each L_j is categorically projective, we have $\operatorname{YExt}^i_A(L_j,-) = 0$ for all $i \geqslant 1$ and $L_j \simeq 0$ for each j, so we have $\operatorname{Ext}^i_A(L_j,-) = 0$ for all i. Set $Q_i := \operatorname{Im}\partial_i^L$ for each $i \geqslant 1$. Each

L_i is graded-projective, so the fact that Q is graded-projective implies that each Q_i is graded-projective.

Now, a straightforward dimension-shifting argument explains the first and third isomorphisms in the following display for $i \geqslant 2$:

$$\mathrm{YExt}^i_A(Q,N) \cong \mathrm{YExt}^1_A(Q_{i-1},N) \cong \mathrm{Ext}^1_A(Q_{i-1},N) \cong \mathrm{Ext}^i_A(Q,N).$$

The second isomorphism is from Theorem 3.5 since each Q_i is graded-projective. $\square$

The next example shows that one can have $\mathrm{YExt}^0_A(Q,N) \ncong \mathrm{Ext}^0_A(Q,N)$, even when Q is semi-free.

Example 3.9. Continue with the assumptions and notation of Example 3.1, and set $Q = N = \underline{R}$. It is straightforward to show that the morphisms $\underline{R} \to \underline{R}$ are precisely given by multiplication by fixed elements of R, so we have the first step in the next display:

$$\mathrm{YExt}^0_A(\underline{R},\underline{R}) \cong R \neq 0 = \mathrm{Ext}^0_A(\underline{R},\underline{R}).$$

The third step follows from the condition $\underline{R} \simeq 0$.

Remark 3.10. It is perhaps worth noting that our proofs can also be used to give the isomorphisms from Theorem A when Q is not necessarily semi-projective, but N is "semi-injective".

4. YExt^1 and Truncations

For our work in [12], we need to know how YExt respects the following notion.

Definition 4.1. Let A be a DG R-algebra, and let M be a DG A-module. Given an integer n, the *nth soft left truncation of M* is the complex

$$\tau(M)_{(\leqslant n)} := \cdots \to 0 \to M_n/\operatorname{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M. In other words, $\tau(M)_{(\leqslant n)}$ is the quotient DG A-module M/M' where M' is the following DG submodule of M:

$$M' = \cdots \to M_{n+2} \to M_{n+1} \to \operatorname{Im}(\partial^M_{n+1}) \to 0.$$

Note that we have $M' \simeq 0$ if and only if $n \geqslant \sup(M)$, so the natural morphism $\rho\colon M \to \tau(M)_{(\leqslant n)}$ of DG A-modules yields an isomorphism in $\mathcal{D}(A)$ if and only if $n \geqslant \sup(M)$.

Proposition 4.2. *Let A be a DG R-algebra, and let M and N be DG A-modules. Assume that n is an integer such that $N_i = 0$ for all $i > n$. Then the natural map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N) \to \mathrm{YExt}^1_A(M,N)$ induced by the morphism $\rho\colon M \to \tau(M)_{(\leqslant n)}$ from Definition 4.1 is a monomorphism.*

Proof. Let Υ denote the map $\mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N) \to \mathrm{YExt}^1_A(M,N)$ induced by ρ. Let $\alpha \in \operatorname{Ker}(\Upsilon) \subseteq \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)},N)$ be represented by the exact sequence

$$0 \to N \xrightarrow{f} X \xrightarrow{g} \tau(M)_{(\leqslant n)} \to 0. \tag{4.2.1}$$

Note that, since $N_i = 0 = (\tau(M)_{(\leqslant n)})_i$ for all $i > n$, we have $X_i = 0$ for all $i > n$. Our assumptions imply that $0 = \Upsilon([\alpha]) = [\beta]$ where β comes from the following pull-back diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & K \arrow[r, "="] \arrow[d, "\widetilde{h}"'] & K \arrow[r] \arrow[d, "h"'] & 0 \\
\beta : & 0 \arrow[r] & N \arrow[r, "\widetilde{f}"] \arrow[d, "="'] & \widetilde{X} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{\rho}"'] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[r] \arrow[d, "\rho"'] & 0 \\
\alpha : & 0 \arrow[r] & N \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] \arrow[d] & \tau(M)_{(\leqslant n)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0. &
\end{tikzcd}
\tag{4.2.2}
$$

The middle row β of this diagram is split exact since $[\beta] = 0$, so there is a morphism $F\colon \widetilde{X} \to N$ of DG A-modules such that $F \circ \widetilde{f} = \mathrm{id}_N$. Note that K has the form

$$
K = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} \mathrm{Im}(\partial^M_{n+1}) \to 0 \tag{4.2.3}
$$

because of the right-most column of the diagram.

We claim that $F \circ \widetilde{h} = 0$. It suffices to check this degree-wise. When $i > n$, we have $N_i = 0$, so $F_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. When $i < n$, the display (4.2.3) shows that $K_i = 0$, so $\widetilde{h}_i = 0$, and $F_i \circ \widetilde{h}_i = 0$. For $i = n$, we first note that the display (4.2.3) shows that ∂^K_{n+1} is surjective. In the following diagram, the faces with solid arrows commute because $\widetilde{h}$ and F are morphisms:

$$
\begin{tikzcd}
0 \arrow[dr, dotted] \arrow[dd, dotted] & & K_{n+1} \arrow[ll, dotted] \arrow[dr, "\widetilde{h}_{n+1}"] \arrow[dd, "\partial^K_{n+1}"'] & \\
 & 0 \arrow[dd] & & \widetilde{X}_{n+1} \arrow[ll, "F_{n+1}"] \arrow[dd, "\partial^{\widetilde{X}}_{n+1}"] \\
0 \arrow[dr, dotted] & & K_n \arrow[ll, dotted] \arrow[dr, "\widetilde{h}_n"] & \\
 & N_n & & \widetilde{X}_n \arrow[ll, "F_n"]
\end{tikzcd}
$$

Since ∂^K_{n+1} is surjective, a simple diagram chase shows that $F_n \circ \widetilde{h}_n = 0$. This establishes the claim.

To conclude the proof, note that the previous claim shows that the map $K \to 0$ is a left-splitting of the top row of diagram (4.2.2) that is compatible with the left-splitting F of the middle row. It is now straightforward to show that F induces a morphism $\overline{F}\colon X \to N$ of DG A-modules that left-splits the bottom row of diagram (4.2.2). Since this row represents $\alpha \in \mathrm{YExt}^1_A(\tau(M)_{(\leqslant n)}, N)$, we conclude that $[\alpha] = 0$, so Υ is a monomorphism. $\square$

The next example shows that the monomorphism from Proposition 4.2 may not be an isomorphism.

Example 4.3. Continue with the assumptions and notation of Example 3.1. The following diagram describes a non-zero element of $\mathrm{YExt}^1_R(M,N)$:

$$0 \longrightarrow N \longrightarrow \underline{R} \longrightarrow M \longrightarrow 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "1"] \arrow[d, "1"'] & R \arrow[r] \arrow[d, "\pi"'] & 0 \\
0 \arrow[r] & R \arrow[r, "X"] \arrow[d] & R \arrow[r, "\pi"] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

It is straightforward to show that $\tau(M)_{(\leqslant 0)}=0$, so we have

$$0=\mathrm{YExt}^1_A(\tau(M)_{(\leqslant 0)},N)\hookrightarrow \mathrm{YExt}^1_A(M,N)\neq 0$$

thus this map is not an isomorphism.

Proposition 4.4. *Let A be a DG R-algebra, and let C be a semi-projective DG A-module such that $\mathrm{Ext}^1_R(C,C)=0$. For $n\geqslant \sup(C)$, one has*

$$\mathrm{YExt}^1_A(C,C)=0=\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)},\tau(C)_{(\leqslant n)}).$$

Proof. From Theorem 3.5, we have $\mathrm{YExt}^1_A(C,C)\cong \mathrm{Ext}^1_A(C,C)=0$. For the remainder of the proof, assume without loss of generality that $\sup(C)<\infty$. Another application of Theorem 3.5 explains the first step in the next display:

$$\mathrm{YExt}^1_A(C,\tau(C)_{(\leqslant n)})\cong \mathrm{Ext}^1_A(C,\tau(C)_{(\leqslant n)})\cong \mathrm{Ext}^1_A(C,C)=0.$$

The second step comes from the assumption $n\geqslant \sup(C)$ which guarantees that the natural morphism $C\to\tau(C)_{(\leqslant n)}$ represents an isomorphism in $\mathcal{D}(A)$. Proposition 4.2 implies that $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)},\tau(C)_{(\leqslant n)})$ is isomorphic to a subgroup of $\mathrm{YExt}^1_A(C,\tau(C)_{(\leqslant n)})=0$, so $\mathrm{YExt}^1_A(\tau(C)_{(\leqslant n)},\tau(C)_{(\leqslant n)})=0$, as desired. □

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