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On Nowhere Zero 4-Flows in Regular Matroids

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Abstract

Walton and Welsh proved that if a coloopless regular matroid M does not have a minor in $\{M(K_{3,3}), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. Lai, Li and Poon proved that if M does not have a minor in $\{M(K_5), M^*(K_5)\}$, then M admits a nowhere zero 4-flow. We prove that if a coloopless regular matroid M does not have a minor in $\{M((P_{10})_{\bar{3}}), M^*(K_5)\}$, then M admits a nowhere zero 4-flow where $(P_{10})_{\bar{3}}$ is the graph obtained from the Petersen graph P_{10} by contracting 3 edges of a perfect matching. As both $M(K_{3,3})$ and $M(K_5)$ are contractions of $M((P_{10})_{\bar{3}})$, our result extends the results of Walton and Welsh and Lai, Li and Poon.

Keywords: nowhere zero flows; regular matroids cycle covers; excluded-minors.

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1 Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [4] for graphs, and Oxley [13] or Welsh [26] for matroids.

Throughout this paper, $\mathbf{Z}, \mathbf{Z}^+$ and $\mathbf{Z}_n$ denote the additive groups of the integers, the set of all positive integers, and the cyclic group of order n, respectively, and $\mathcal{R}$ denotes the family of all regular matroids. To be consistent with the matroid terminology, a nontrivial 2-regular connected graph will be called a **circuit**, and a disjoint union of circuits a **cycle**. Thus the empty set $\emptyset$ is the only independent cycle. As in [13], the set of all circuits of a matroid M is denoted by $\mathcal{C}(M)$. We further denote the set of all cycles of a matroid M by $\mathcal{C}_0(M)$. Note that as we allow empty unions, the empty set is also a cycle (in both graphs and matroids). For matroids $N_1, N_2, \cdots, N_k$, let $EX(N_1, N_2, \cdots, N_k)$ denote the collection of matroids such that a matroid $M \in EX(N_1, N_2, \cdots, N_k)$ if and only if M does not have a minor isomorphic to any one in $\{N_1, N_2, \cdots, N_k\}$. The Fano matroid F_7 is the vector matroid over GF(2) of the following matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Flow was initially defined for graphs. For a discussion on flow and flow conjectures, see Jaeger [9] or Zhang [27]. The definition of flow has a natural extension to regular matroids. Let $M \in \mathcal{R}$, where $\mathcal{R}$ is the family of regular matroids, and D_M be its incidence matrix of circuits against elements. An **orientation** $(w(D_M), w(D_{M^*}))$ is an assignment of $+, -$ signs to the "1" entries of D_M and D_{M^*}, respectively, so that the resulting matrices $w(D_M)$ and $w(D_{M^*})$ satisfy

$$w(D_M)w(D_{M^*})^T = 0.$$

Let A be an abelian group. For an element $a \in A$, and for integers $+1, -1, 0$, we adopt the convention to write $(+1) \cdot a = a$, $(-1) \cdot a = -a$ and $0 \cdot a = 0$. Let $F^*(M, A) = \{f : E(M) \mapsto A \setminus \{0\}\}$ denote the set of all functions from $E(M)$ into $A \setminus \{0\}$. A map $f \in F^*(M, A)$

can be viewed as an $|E(M)|$-dimensional column vector. For a regular matroid M with an orientation $(w(D_M), w(D_{M^*}))$, a map $f \in F^*(M, A)$ satisfying

$$w(D_{M^*}) \cdot f = 0$$

is a **nowhere zero A-flow** (A-NZF for short) of M. When $A = \mathbf{Z}$, a $\mathbf{Z}$-NZF f of M is called a **nowhere zero k-flow** (k-NZF for short) of M if $\forall e \in E(M)$, $0 < |f(e)| < k$.

For positive integers k and m, an **m-cycle k-cover** of a matroid M is a family of cycles $C_1, C_2, \cdots, C_m$ of M such that every element of $E(M)$ lies in exactly k members of these C_i's. It has been observed that a graph G admits a 4-NZF if and only if G has a 3-cycle 2-cover (for example, see Zhang [27]). The following fact will be needed, a formal proof of it can be found in [11].

Proposition 1.1 (Proposition 1.1 in [11]). *Let M be a regular matroid. Then M admits a 4-NZF if and only if M has a 3-cycle 2-cover.*

Let P_{10} be the Petersen graph. Tutte proposed the famous 4-flow conjecture as follows.

Conjecture 1.1 (Tutte [21] and [22], Matthews [12]). *Let G be a 2-edge-connected graph. If G does not have a P_{10}-minor, then G admits a 4-NZF.*

One matroid version of the conjecture can be stated as follows.

Conjecture 1.2. *If M is a coloopless regular matroid such that $M \in EX(M(P_{10}), M^*(K_5))$, then M admits a 4-NZF.*

The Four-Color theorem can be stated in terms of nowhere zero flows. Applying the Four-Color theorem, and the duality between colorings and nowhere zero flows, the following results are implied.

Theorem 1.1 (Appel and Haken [1], Appel, Haken and Hoch [2], Robertson, Sanders, Seymour and Thomas [14]). *Every 2-edge-connected planar graph admits a 4-NZF.*

Theorem 1.2 (Walton and Welsh [24]). *If $M \in EX(M(K_{3,3}), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

In the proof of a conjecture proposed by Jensen and Toft [10], Lai, Li and Poon applied the Four-Color Theorem to prove the following Theorem 1.3, which is an approach to Conjecture 1.2.

Theorem 1.3 (Lai, Li and Poon, [11]). *If $M \in EX(M(K_5), M^*(K_5)) \cap \mathcal{R}$ is a coloopless matroid, then M admits a 4-NZF.*

Let N be a perfect matching of P_{10}. For a positive integer $\mu \leq 5$, let $(P_{10})_{\bar{\mu}}$ be the graph obtained from P_{10} by contracting the edges in the edge set F, where $F \subseteq N$ and $|F| = \mu$.

Remark. It is not hard to see that for two perfect matchings N and N' of P_{10}, if $F \subseteq N$, $F' \subseteq N'$ are two edge sets and $|F| = |F'|$, then $P_{10}/F \cong P_{10}/F'$. Hence $(P_{10})_{\bar{\mu}}$ is well defined.

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Theorem 1.4 (Wang, Zhang and Zhang [25]). *Every bridgeless graph without a minor isomorphic to $(P_{10})_{\bar{3}}$ admits a 4-NZF.*

The main objective of this paper is to prove the following theorem, which generalizes Theorem 1.3, and is also an approach to Conjecture 1.2.

Theorem 1.5. *If M is a coloopless matroid such that $M \in EX(M((P_{10})_{\bar{3}}), M^*(K_5)) \cap \mathcal{R}$, then M admits a 4-NZF.*

The definition of flow has no natural extension to binary matroids whereas cycle cover is defined for general matroids. In view of Proposition 1.1 and the excluded-minor characterization of regular matroids, Theorem 1.5 is equivalent to saying that if a coloopless binary matroid $M \in EX(F_7, F_7^*, M((P_{10})_{\bar{3}}), M^*(K_5))$, then M has a 3-cycle 2-cover. In Section 3 we will show that this result can be extended in the following form.

Corollary 1.6. *Let M be a coloopless binary matroid. If $M \in EX(F_7^*, M((P_{10})_{\bar{3}}), M^*(K_5))$, M has a 3-cycle 2-cover.*

As the matroid F_7^* does not have a 3-cycle 2-cover (to be shown in Section 3), Corollary 1.6 does not hold if F_7^* is not excluded.

In Section 2, we extract a decomposition theorem for regular matroids without $M(K_5)$ or $M^*(K_5)$ minors from the well known decomposition theorems of Seymour [16] and Wagner [23]. In Section 3, this theorem will be employed to prove Theorem 1.5 and Corollary 1.6.

2 Decomposition of Regular Matroids

In this paper, we use $\triangle$ to denote both a set operator and a matroid operator. Given two sets X and Y, the symmetric difference of X and Y is defined as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Definition 2.1. Suppose that M_1, M_2 are binary matroids on E_1 and E_2, respectively. We follow Seymour [16] and define a new binary matroid $M_1 \triangle M_2$ to be the matroid with ground set equal to $E_1 \triangle E_2$ and with its set of cycles equal to

$$\{C_1 \triangle C_2 \subseteq E_1 \triangle E_2 : C_i \text{ is a cycle of } M_i,\ i = 1, 2\}. \tag{1}$$

Definition 2.2. Three special cases of this operation are introduced by Seymour ([16] and [17]) as follows.

(i) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **1-sum** of M_1 and M_2.

(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{z\}$, say, and z is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **2-sum** of M_1 and M_2.

(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, say, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **3-sum** of M_1 and M_2.

For $i = 1, 2, 3$, an i-sum of M_1, M_2 is denoted as $M_1 \oplus_i M_2$. The 1-sum $M_1 \oplus_1 M_2$ is also written as $M_1 \oplus M_2$. Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is known that R_{10}^* is isomorphic to R_{10}. Based on the notion of matroid sums, Seymour proved the following decomposition theorem for regular matroids.

Theorem 2.1 (Seymour [16]). *Let M be a regular matroid. One of the following must hold.*
(i) M is graphic.
(ii) M is cographic.
(iii) $M \cong R_{10}$.
(iv) For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

If a matroid M is isomorphic to the cycle matroid of a planar graph, then M is called a **planar matroid**. Thus a matroid M is planar if and only if M^* is planar. Let H_8 denote the graph depicted in the figure below.

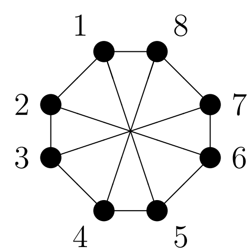

Figure 1: The graph H_8

Wagner's original statement of his decomposition theorem is in pure graph theory terms. A matroidal version is given as follows (see Seymour [16] and [17]).

Theorem 2.2 (Wagner [23]). *Let M be a graphic matroid that does not contain a minor isomorphic to $M(K_5)$. One of the following must hold.*
(i) M is a planar matroid.
(ii) $M \cong M(H_8)$.
(iii) $M \cong M(K_{3,3})$.
(iv) For some $i \in \{1, 2, 3\}$, $M = M_1 \oplus_i M_2$ is the i-sum of two matroids M_1 and M_2, such that both M_1 and M_2 are proper minors of M.

Proposition 2.1 (Propositions 4.2.11, 8.3.1 and 12.4.16, in [13]). *Each of the following holds:*

(i) The matroid M is not 2-connected, if and only if for some proper non-empty subset T of $E(M)$, $M = (M|T) \oplus (M|(E \setminus T))$. Note that $M|T$ and $M|(E \setminus T)$ are both proper minors of M.

(ii) The matroid M is 2-connected but not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

(iii) If M is a 3-connected binary matroid and a 3-sum of M_1 and M_2, then M_1 and M_2 are isomorphic to proper minors of M.

3 The Proofs of Theorem 1.5 and Corollary 1.6

In view of Proposition 1.1, we will prove Theorem 1.5 by showing that M has a 3-cycle 2-cover given the assumption of the theorem. We first establish some lemmas.

Proposition 3.1. *Each of the following holds.*

(i) Each of $M(H_8)$, $M^*(H_8)$, $M(K_{3,3})$, $M^*(K_{3,3})$, R_{10}, F_7 has a 3-cycle 2-cover.

(ii) F_7^* does not have a 3-cycle 2-cover.

These results follow from known facts about tangential 2-block. See for example the discussion on Tutte's tangential 2-block conjecture in [5]. The results can also be verified directly in a straightforward way.

Lemma 3.1. *Suppose that M_1, M_2 are binary matroids and each of M_1 and M_2 has a 3-cycle 2-cover. Then for a binary matroid M, the following holds.*

(i) If $M = M_1 \oplus M_2$ is a 1-sum of M_1 and M_2, then M also has a 3-cycle 2-cover.

(ii) If $M = M_1 \oplus_2 M_2$ is a 2-sum of M_1 and M_2, then M also has a 3-cycle 2-cover.

Proof. We omit the proof here as it is similar to the Lemma 3.3 of [11]. □

Definition 3.1. Suppose that M_1, M_2 are binary matroids and $Z = \{e_1, e_2, e_3\} = E(M_1) \cap E(M_2)$ is a circuit in both M_1 and M_2. Let $L = M(K_4)$ with $E(L) = \{e_1, e_2, e_3, f_1, f_2, f_3\}$ such that $Z = \{e_1, e_2, e_3\}$ is a circuit of L and $Z' = \{f_1, f_2, f_3\}$ is a cocircuit of L, and such that $\{e_j, f_j\}$ is a perfect matching of K_4, for each $j \in \{1, 2, 3\}$. Define $N_i = M_i \oplus_3 L$, for $i \in \{1, 2\}$.

With the same notation in Definition 3.1, we observe that for each $i \in \{1, 2\}$, if $Z = E(M_i) \cap E(L)$, then $E(N_i) \cap E(L) = Z'$. Moreover, for each $i \in \{1, 2\}$,

$$M_i \oplus_3 L = N_i \text{ and } N_i \Delta L = M_i. \tag{2}$$

By Definition 3.1, if M_1 and M_2 are coloopless, then N_1 and N_2 are also coloopless. We have the following known result.

Lemma 3.2. *Let N be a connected binary matroid with $r(N) \geq 4$, and let $Z = \{e_1, e_2, e_3\}$ be a 3-circuit of N. Then for some disjoint subsets $T_1, T_2 \subseteq E(N) - Z$, $(N - T_1)/T_2 \cong K_4$, where Z is a 3-circuit of $(N - T_1)/T_2$.*

Lemma 3.3. *$M_1 \oplus_3 M_2 = N_1 \Delta N_2$.*

Proof. We shall show that both sides have the same set of cycles. By Definition 2.1, for any $C \in \mathcal{C}(M_1 \oplus_3 M_2)$, $C = C_1 \Delta C_2$ with $C_1 \in \mathcal{C}_0(M_1)$, $C_2 \in \mathcal{C}_0(M_2)$ and $C_1 \cap Z = C_2 \cap Z = W$. If $W = \emptyset$, then $C \in \mathcal{C}_0(N_1 \Delta N_2)$, by (1) in Definition 2.1. Similarly, if $W = Z$, then $C_1 \Delta C_2 = (C_1 \Delta Z) \Delta (C_2 \Delta Z) \in \mathcal{C}_0(N_1 \Delta N_2)$. Thus we assume that $2 \geq |W| \geq 1$.

If $|W| = 1$, then without loss of generality, we assume that $W = \{e_1\}$. Thus $C' = \{e_1, f_2, f_3\}$ is a circuit of $L = M(K_4)$, and so by (1), $C_1' = C_1 \Delta C' \in \mathcal{C}_0(M_1 \oplus_3 L) = \mathcal{C}_0(N_1)$, Similarly, $C_2' = C_2 \Delta C' \in \mathcal{C}_0(N_2)$. It follows by (1) that $C_1' \Delta C_2' \in \mathcal{C}_0(N_1 \Delta N_2)$. Since

$$C_1' \Delta C_2' = C_1 \Delta C' \Delta C_2 \Delta C' = C_1 \Delta C_2,$$

it follows that $C_1 \Delta C_2 \in \mathcal{C}_0(N_1 \Delta N_2)$.

Now suppose $|W| = 2$. Then without loss of generality, we assume that $W = \{e_1, e_2\}$. Thus $C'' = \{e_1, e_2, f_2, f_3\}$ is a circuit of $L = M(K_4)$, and so by (1), $C_1'' = C_1 \Delta C'' \in \mathcal{C}_0(M_1 \oplus_3 L) = \mathcal{C}_0(N_1)$, Similarly, $C_2'' = C_2 \Delta C'' \in \mathcal{C}_0(N_2)$. It follows by (1) that $C_1'' \Delta C_2'' \in \mathcal{C}_0(N_1 \Delta N_2)$. Since

$$C_1'' \Delta C_2'' = C_1 \Delta C' \Delta C_2 \Delta C' = C_1 \Delta C_2,$$

it follows that $C_1 \Delta C_2 \in \mathcal{C}_0(N_1 \Delta N_2)$. This proves that $\mathcal{C}_0(M_1 \oplus_3 M_2) \subseteq \mathcal{C}_0(N_1 \Delta N_2)$.

Conversely, pick an arbitrary $D = D_1 \Delta D_2 \in \mathcal{C}_0(N_1 \Delta N_2)$, with $D_i \in \mathcal{C}_0(N_i)$, $i \in \{1, 2\}$. Then $D_1 \cap Z' = D_2 \cap Z' = W'$. Since D_i is a circuit and Z' is a cocircuit, and since N_i is binary, $|W'| \in \{0, 2\}$. If $W' = \emptyset$, then for each i, $D_i \in \mathcal{C}_0(M_i - Z)$ and so by (1), $D_i \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. As $D_1 \Delta D_2$ is a disjoint union of D_1 and D_2, $D_1 \Delta D_2 \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. Thus we assume that $|W'| = 2$. Without loss of generality, we may assume that $W' = \{f_1, f_2\}$. Let $D' = \{f_1, f_2, e_3\}$. Then for $i \in \{1, 2\}$, D' is a circuit in L such that $D'\mathcal{Z}' = \mathcal{W}' = \mathcal{D}_\rangle\mathcal{Z}'$. It follows by (2) that $C_i = D' \Delta D_i$ is a cycle of M_i. Moreover, as

$$C_1 \Delta C_2 = (D' \Delta D_1) \Delta (D' \Delta D_2) = D_1 \Delta D_2,$$

we conclude that $D_1 \Delta D_2 \in \mathcal{C}_0(M_1 \oplus_3 M_2)$. This proves that $\mathcal{C}_0(N_1 \Delta N_2) \subseteq \mathcal{C}_0(M_1 \oplus_3 M_2)$, and so it completes the proof for this lemma. □

Lemma 3.4. *Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid. With the same notation in Definition 3.1, for each $i \in \{1, 2\}$, N_i is a minor of M.*

Proof. By symmetry, it suffices to show that N_1 is a minor of M. Since $M = M_1 \oplus_3 M_2$, and since M is 3-connected, M_2 is also 3-connected. By Lemma 3.4, M_2 has a minor $L \cong M(K_4)$ such that $Z = E(M_1) \cap E(M_2)$ is a 3-circuit of L. It follows that $N_1 = M_1 \oplus_3 L$ is a minor of $M = M_1 \oplus_3 M_2$. □

Lemma 3.5. *Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid. With the same notation in Definition 3.1, if each of N_1 and N_2 has a 3-cycle 2-cover, then M also has a 3-cycle 2-cover.*

Proof. For any $C \in \mathcal{C}(N_1)$, $C = C_1 \Delta C_0$ and $C_1 \cap Z = C_0 \cap Z$ where $C_1 \in \mathcal{C}(M_1)$ and $C_0 \in \mathcal{C}(L)$. Therefore, $C \cap Z' = C_0 \cap Z'$. Since Z' is a cocircuit of L, $C_0 \cap Z' \equiv 0 (\text{mod } 2)$, and so $C \cap Z' \equiv 0 (\text{mod } 2)$. This implies Z' is a cocircuit of N_1. Similarly, Z' is a cocircuit of N_2. $Z' \in \mathcal{C}(N_1^*) \cap \mathcal{C}(N_2^*)$.

Let $\{C_j^i : j = 1, 2, 3\}$ be a 3-cycle 2-cover of N_i, for $i = 1, 2$. Since $Z' \in \mathcal{C}(N_i^*)$, $|C_i^j \cap Z'| = 2$. Without loss of generality, we can assume that $f_i \notin C_i^j$, then $\{C_i^1 \Delta C_i^2 : i = 1, 2, 3\}$ is a 3-cycle 2-cover of M. □

Proof of Theorem 1.5. By the way of contradiction, assume Theorem 1.5 is not true, then there is a matroid M such that $M \in EX(M((P_{10})_{\bar{3}}), M^*(K_5)) \cap \mathcal{R}$ which does not admit a 4-NZF. Choose M so that $|E(M)|$ is minimum.

By Theorem 2.1, M is one of the following: graphic, cographic, R_{10} or $M = M_1 \oplus_i M_2$ for some $i \in \{1, 2, 3\}$.

(i) M is not graphic. Otherwise since M is $(P_{10})_{\bar{3}}$-minor free, by theorem 1.4, M admits a 4-NZF, a contradiction.

(ii) M is not cographic. Otherwise if M is cographic, then $M \in EX(M(K_5))$, and by Theorem 1.3, M has a 3-cycle 2-cover, a contradiction.

(iii) $M \ncong R_{10}$ by Proposition 3.1.

Therefore, $M = M_1 \oplus_i M_2$ for $i \in \{1, 2, 3\}$.

Claim 3.1. $i \notin \{1, 2\}$.

Otherwise, by the minimality of M, both M_1 and M_2 have 3-cycle 2-covers. By Lemma 3.1, M also has a 3-cycle 2-cover, a contradiction.

By Claim 3.1, $M = M_1 \oplus_3 M_2$

(iv) Therefore, M has to be a 3-sum.

Now suppose $M = M_1 \oplus_3 M_2$. By Definition 3.1 and Lemma 3.4, N_i is a minor of M for $i = 1, 2$ and so $N_i \in EX(M((P_{10})_{\bar{3}}), M^*(K_5))$. By the minimality of M, both N_1 and N_2 have 3-cycle 2-covers. By Lemma 3.5, M has a 3-cycle 2-cover, a contradiction.

□

For binary matroids without F_7^* minor, Seymour has established the following decomposition theorem.

Theorem 3.6 ((6.4) in Seymour [17])**.** *Every binary matroid without F_7^* minor may be obtained by means of proper 1-sums or 2-sums from regular matroids and copies of F_7.*

Corollary 1.6 follows directly from Proposition 3.1, Lemma 3.1, Theorem 3.6 and Theorem 1.5.

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