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EXISTENCE OF SOLUTIONS FOR A VARIABLE EXPONENT SYSTEM WITHOUT PS CONDITIONS

LI YIN, YUAN LIANG, QIHU ZHANG, CHUNSHAN ZHAO

Abstract. In this article, we study the existence of solution for the following elliptic system of variable exponents with perturbation terms

$$
\begin{gathered}
-\operatorname{div}|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda a(x)|u|^{\gamma(x)-2}u+F_u(x,u,v) \quad \text{in } \mathbb{R}^N,\\
-\operatorname{div}|\nabla v|^{q(x)-2}\nabla v)+|v|^{q(x)-2}v=\lambda b(x)|v|^{\delta(x)-2}v+F_v(x,u,v) \quad \text{in } \mathbb{R}^N,\\
u\in W^{1,p(\cdot)}(\mathbb{R}^N), v\in W^{1,q(\cdot)}(\mathbb{R}^N),
\end{gathered}
$$

where the corresponding functional does not satisfy PS conditions. We obtain a sufficient condition for the existence of solution and also present a result on asymptotic behavior of solutions at infinity.

1. Introduction

The study of differential equations and variational problems with variable exponent has attracted intense research interests in recent years. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity [1, 7, 19, 26]. The following variable exponent flow is an important model in image processing [7]:

$$
\begin{gathered}
u_t-\operatorname{div}|\nabla u|^{p(x)-2}\nabla u)+\lambda(u-u_0)=0, \quad \text{in } \Omega\times[0,T],\\
u(x,t)=g(x), \quad \text{on } \partial\Omega\times[0,T],\\
u(x,0)=u_0.
\end{gathered}
$$

The main benefit of this flow is the manner in which it accommodates the local image information. We refer to [14, 18, 24] for the existence of solution of variable exponent problems on bounded domain.

In this article, we consider the existence of solutions for the system

$$
\begin{gathered}
-\operatorname{div}|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda a(x)|u|^{\gamma(x)-2}u+F_u(x,u,v) \quad \text{in } \mathbb{R}^N,\\
-\operatorname{div}|\nabla v|^{q(x)-2}\nabla v)+|v|^{q(x)-2}v=\lambda b(x)|v|^{\delta(x)-2}v+F_v(x,u,v) \quad \text{in } \mathbb{R}^N, \quad (1.1)\\
u\in W^{1,p(\cdot)}(\mathbb{R}^N), \quad v\in W^{1,q(\cdot)}(\mathbb{R}^N),
\end{gathered}
$$

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where $p, q \in C(\mathbb{R}^N)$ are Lipschitz continuous and $p(\cdot), q(\cdot) >> 1$, the notation $h_1(\cdot) >> h_2(\cdot)$ means $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}(h_1(x) - h_2(x)) > 0$,

$$-\Delta_{p(x)}u := -\operatorname{div}|\nabla u|^{p(x)-2}\nabla u)$$

which is called the $p(x)$-Laplacian. When $p(\cdot) \equiv p$ (a constant), $p(x)$-Laplacian becomes the usual p-Laplacian. The terms $\lambda a(x)|u|^{\gamma(x)-2}u$ and $\lambda b(x)|v|^{\delta(x)-2}v$ are the perturbation terms. The $p(x)$ -Laplacian possesses more complicated nonlinearities than the p-Laplacian (see [15]). Many methods and results for p-Laplacian are invalid for $p(x)$-Laplacian.

The PS condition is very important in the study of the existence of solution via variational methods. According to [21, Theorem 2.8], if a $C^1(X, \mathbb{R})$ functional f satisfies the Mountain Pass Geometry, then it has a PS sequence $\{x_n\}$ which satisfies $f(x_n) \to c$ which is the mountain pass level and $f'(x_n) \to 0$. By [21, Theorem 2.9] it follows that if f also satisfies the PS condition, passing to a subsequence, then $x_n \to x_0$ in X, and then x_0 is a critical point of f, that is $f'(x_0) = 0$. In the study of this problems in the bounded domain, since we have the compact embedding from a Sobolev space to a Lebesgue space, so we have the PS condition when we study the case of subcritical growth condition. For the unbounded domain, we cannot get the compact embedding in general, so we do not have the PS condition.

It is well known that a main difficulty in the study of elliptic equations in $\mathbb{R}^N$ is the lack of compactness. Many methods have been used to overcome this difficulty. One type of methods is that under some additional conditions we can recover the required compact imbedding theorem, for example, the weighting method [16, 25], and the symmetry method [23]. If equations are periodic, the corresponding energy functionals are invariant under period-translation. We refer to [2]–[5] and references cited therein for the applications of this method to the p-Laplacian equations, the Schrödinger equations and the biharmonic equations etc.

Sometimes we can compare the original equation with its limiting equation at infinity. Especially, we can compare the corresponding critical values of the functionals for these two equations when the existence of the ground state solution for the limiting equation is known. Usually the limiting equations are homogeneous, but in [2]-[5] the limiting equations are periodic. We also refer to [13] for the existence of solution for $p(x)$-Laplacian equations with periodic conditions.

In this article we consider the existence and the asymptotic behavior of solutions near infinity for a variable exponent system with perturbations that does not satisfy periodic conditions, which implies the corresponding functional does not satisfy PS conditions on unbounded domain. We will also give a sufficient condition for the existence of solutions for the system (1.1). Our method is to compare the original equation with its limiting equation at infinity without perturbation. These results also partially generalize the results in [13] and [20].

In this article, we make the following assumptions.

(A0) $p(\cdot), q(\cdot)$ are Lipschitz continuous, $1 << p(\cdot), q(\cdot) << N$, $1 << \gamma(\cdot) << p(\cdot)$, $a(\cdot) \in L^{\frac{p(\cdot)}{p(\cdot)-\gamma(\cdot)}}(\mathbb{R}^N)$, $1 << \delta(\cdot) << q(\cdot)$, $b(\cdot) \in L^{\frac{q(\cdot)}{q(\cdot)-\delta(\cdot)}}(\mathbb{R}^N)$, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ satisfies

$$|F_u(x, u, v)| \le C(|u|^{p(x)-1} + |u|^{\alpha(x)-1} + |v|^{q(x)/\alpha^0(x)} + |v|^{\beta(x)/\alpha^0(x)}),$$
$$|F_v(x, u, v)| \le C(|v|^{q(x)-1} + |v|^{\beta(x)-1} + |u|^{p(x)/\beta^0(x)} + |u|^{\alpha(x)/\beta^0(x)}),$$

where $F_u = \frac{\partial}{\partial u}F$, $F_v = \frac{\partial}{\partial v}F$, $\alpha, \beta \in C(\mathbb{R}^N)$ satisfy

$$p(\cdot) << \alpha(\cdot) << p^*(\cdot), q(\cdot) << \beta(\cdot) << q^*(\cdot),$$

$h^0(\cdot)$ denotes the conjugate function of $h(\cdot)$, that is $\frac{1}{h(x)} + \frac{1}{h^0(x)} \equiv 1$, and

$$p^*(x) = \begin{cases} Np(x)/(N-p(x)), & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases}$$

(A1) There exist constants $\theta_1 > p^+$ and $\theta_2 > q^+$, such that F satisfies the following conditions

$$0 \leq sF_s(x,s,t), 0 \leq tF_t(x,s,t), \quad \forall (x,s,t) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R},$$

$$0 < F(x,s,t) \leq \frac{1}{\theta_1} sF_s(x,s,t) + \frac{1}{\theta_2} tF_t(x,s,t), \quad \forall x \in \mathbb{R}^N, \forall (s,t) \in \mathbb{R} \times \mathbb{R} \backslash \{(0,0)\}.$$

(A2) For $(s,t) \in \mathbb{R}^2$, the function $sF_s(x, \tau^{1/\theta_1}s, \tau^{1/\theta_2}t)/\tau^{\frac{\theta_1-1}{\theta_1}}$ and the function $tF_t(x, \tau^{1/\theta_1}s, \tau^{1/\theta_2}t)/\tau^{\frac{\theta_2-1}{\theta_2}}$ are increasing with respect to $\tau > 0$.

(A3) There is a measurable function $\widetilde{F}(s,t)$ such that

$$\lim_{|x| \to +\infty} F(x,s,t) = \widetilde{F}(s,t)$$

for bounded $|s| + |t|$ uniformly, and

$$|\widetilde{F}(s,t)| + |\widetilde{F}_s(s,t)s| + |\widetilde{F}_t(s,t)t| \leq C(|s|^{p^+} + |s|^{\alpha^-} + |t|^{q^+} + |t|^{\beta^-}), \quad \forall (s,t) \in \mathbb{R}^2,$$

and when $|x| \geq R$ the following inequalities hold

$$|F(x,s,t) - \widetilde{F}(s,t)| \leq \varepsilon(R)(|s|^{p(x)} + |s|^{p^*(x)} + |t|^{q(x)} + |t|^{q^*(x)}),$$

$$\begin{aligned} &|F_s(x,s,t) - \widetilde{F}_s(s,t)| \\ &\leq \varepsilon(R)(|s|^{p(x)-1} + |s|^{p^*(x)-1} + |t|^{q(x)(p^*(x)-1)/p^*(x)} + |t|^{q^*(x)(p^*(x)-1)/p^*(x)}), \\ &|F_t(x,s,t) - \widetilde{F}_t(s,t)| \\ &\leq \varepsilon(R)(|s|^{p(x)(q^*(x)-1)/q^*(x)} + |s|^{p^*(x)(q^*(x)-1)/q^*(x)} + |t|^{q(x)-1} + |t|^{q^*(x)-1}), \end{aligned}$$

where $\varepsilon(R)$ satisfies $\lim_{R\to+\infty} \varepsilon(R) = 0$.

This article is organized as follows. In Section 2, we introduce some basic properties of the Lebesgue-Sobolev spaces with variable exponents and $p(x)$-Laplacian. In Section 3, we give the main results and the proofs.

2. Notation and preliminary results

Throughout this paper, the letters c, c_i, C_i, $i = 1, 2, \dots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process. To discuss problem (1.1), we need some preparations on space $W^{1,p(\cdot)}(\Omega)$ which we call variable exponent Sobolev space, where $\Omega \subset \mathbb{R}^N$ is an open domain. Firstly, we state some basic properties of spaces $W^{1,p(\cdot)}(\Omega)$ which we will use later (for details, see [9, 11, 12, 14]). Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}), h(x) \geq 1 \text{ for } x \in \overline{\Omega}\},$$

$$h^+_\Omega = \operatorname{ess\,sup}_{x\in\Omega} h(x), h^-_\Omega = \operatorname{ess\,inf}_{x\in\Omega} h(x), \text{ for any } h \in L^\infty(\Omega),$$

$$h^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} h(x), h^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^N} h(x), \text{ for any } h \in L^\infty(\mathbb{R}^N),$$

$$S(\Omega) = \{u : u \text{ is a real-valued measurable function on } \Omega\},$$

$$L^{p(\cdot)}(\Omega) = \{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

In this section, $p(\cdot)$ and $p_i(\cdot)$ are Lipschitz continuous unless otherwise noted. We introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot),\Omega} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and $(L^{p(\cdot)}(\Omega),\ |\cdot|_{p(\cdot),\Omega})$ becomes a Banach space, we call it variable exponent Lebesgue space.

If $\Omega = \mathbb{R}^N$, we will simply denote by $|\cdot|_{p(\cdot)}$ the norm on $L^{p(\cdot)}(\mathbb{R}^N)$.

Proposition 2.1 ([9]). *(i) The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot),\Omega})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p^0(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^0(x)} \equiv 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^0(\cdot)}(\Omega)$, we have*

$$|\int_\Omega uv\,dx| \le (\frac{1}{p_\Omega^-} + \frac{1}{(p^0)_\Omega^-})|u|_{p(\cdot),\Omega}|v|_{p^0(\cdot),\Omega}.$$

(ii) If Ω is bounded, p_1, $p_2 \in C_+(\overline{\Omega})$, $p_1(\cdot) \le p_2(\cdot)$ for any $x \in \overline{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.

Proposition 2.2 ([9]). *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies*

$$|f(x,s)| \le h(x) + d|s|^{p_1(x)/p_2(x)} \quad \text{for any } x \in \Omega, s \in \mathbb{R},$$

where p_1, $p_2 \in C_+(\overline{\Omega})$, $h \in L^{p_2(\cdot)}(\Omega)$, $h(x) \ge 0$, $d \ge 0$, then the Nemytskii operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f u)(x) = f(x,u(x))$ is continuous and bounded.

Proposition 2.3 ([9]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(\cdot)}(\Omega),$$

then

(i) $|u|_{p(\cdot),\Omega} < 1\ (= 1; > 1) \iff \rho(u) < 1\ (= 1; > 1)$;
(ii) $|u|_{p(\cdot),\Omega} > 1 \implies |u|^{p^-}_{p(\cdot),\Omega} \le \rho(u) \le |u|^{p^+}_{p(\cdot),\Omega}$; $|u|_{p(\cdot),\Omega} < 1 \implies |u|^{p^-}_{p(\cdot),\Omega} \ge \rho(u) \ge |u|^{p^+}_{p(\cdot),\Omega}$;
(iii) $|u|_{p(\cdot),\Omega} \to 0 \iff \rho(u) \to 0$; $|u|_{p(\cdot),\Omega} \to \infty \iff \rho(u) \to \infty$.

Proposition 2.4 ([9]). *If u, $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1,2,\dots$, then the following statements are equivalent.*

(1) $\lim_{n\to\infty} |u_n - u|_{p(\cdot),\Omega} = 0$;
(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;
(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Denote $Y = \prod_{i=1}^k L^{p_i(\cdot)}(\Omega)$ with the norm

$$\|y\|_Y = \sum_{i=1}^k y^i|_{p_i(\cdot),\Omega}, \forall y = (y^1,\dots,y^k) \in Y,$$

where $p_i(\cdot) \in C_+(\overline{\Omega})$, $i = 1,\dots,m$, then Y is a Banach space.

With a proof similar to proof in [6], we have:

Proposition 2.5. *Suppose $f(x,y) : \Omega \times \mathbb{R}^k \to \mathbb{R}^m$ is a Caratheodory function; that is, f satisfies*

(i) *For a.e. $x \in \Omega$, $y \to f(x,y)$ is a continuous function from $\mathbb{R}^k$ to $\mathbb{R}^m$,*
(ii) *For any $y \in \mathbb{R}^k$, $x \to f(x,y)$ is measurable.*

If there exist $p_1(\cdot), \dots, p_k(\cdot) \in C_+(\overline{\Omega})$, $1 \le \beta(\cdot) \in C(\overline{\Omega})$, $\rho(\cdot) \in L^{\beta(\cdot)}(\Omega)$ and positive constant $c > 0$ such that

$$|f(x,y)| \le \rho(x) + c\sum_{i=1}^{k} |y_i|^{p_i(x)/\beta(x)} \quad \text{for any } x \in \Omega, y \in \mathbb{R}^k,$$

then the Nemytskii operator from Y to $(L^{\beta(\cdot)}(\Omega))^m$ defined by $(N_f u)(x) = f(x,u(x))$ is continuous and bounded.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \in (L^{p(\cdot)}(\Omega))^N\},$$

with the norm

$$\|u\|_{p(\cdot),\Omega} = |u|_{p(\cdot),\Omega} + |\nabla u|_{p(\cdot),\Omega}, \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

If $\Omega = \mathbb{R}^N$, we will denote the norm on $W^{1,p(\cdot)}(\mathbb{R}^N)$ as $\|u\|_{p(\cdot)}$.

Denote

$$\|u\|'_{p(\cdot),\Omega} = \inf\{\lambda > 0 : \int_\Omega |\frac{\nabla u}{\lambda}|^{p(x)}\,dx + \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$
$$\|v\|'_{q(\cdot),\Omega} = \inf\{\lambda > 0 : \int_\Omega |\frac{\nabla v}{\lambda}|^{q(x)}\,dx + \int_\Omega |\frac{v(x)}{\lambda}|^{q(x)}\,dx \le 1\}.$$

It is easy to see that the norm $\|\cdot\|'_{p(\cdot),\Omega}$ is equivalent to $\|\cdot\|_{p(\cdot),\Omega}$ on $W^{1,p(\cdot)}(\Omega)$, and $\|\cdot\|'_{q(\cdot),\Omega}$ is equivalent to $\|\cdot\|_{q(\cdot),\Omega}$ on $W^{1,q(\cdot)}(\Omega)$. In the following, we will use $\|\cdot\|'_{p(\cdot),\Omega}$ instead of $\|\cdot\|_{p(\cdot),\Omega}$ on $W^{1,p(\cdot)}(\Omega)$, and use $\|\cdot\|'_{q(\cdot),\Omega}$ instead of $\|\cdot\|_{q(\cdot),\Omega}$ on $W^{1,q(\cdot)}(\Omega)$. We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Proposition 2.6 ([8, 9, 11])**.** *(i) $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable reflexive Banach spaces;*

(ii) If $p(\cdot)$ is Lipschitz continuous, $\alpha(\cdot)$ is measurable, and satisfies $p(\cdot) \le \alpha(\cdot) \le p^*(\cdot)$ for any $x \in \Omega$, then the imbedding from $W^{1,p(\cdot)}(\mathbb{R}^N)$ to $L^{\alpha(\cdot)}(\mathbb{R}^N)$ is continuous;

(iii) If Ω is bounded, $\alpha \in C_+(\overline{\Omega})$ and $\alpha(\cdot) < p^*(\cdot)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{\alpha(\cdot)}(\Omega)$ is compact and continuous.

Proposition 2.7 ([12, Lemma 3.1])**.** *Assume that $p : \mathbb{R}^N \to \mathbb{R}$ is a uniformly continuous function, if $\{u_n\}$ is bounded in $W^{1,p(\cdot)}(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^{\rho(x)}\,dx \to 0, n \to +\infty,$$

for some $r > 0$ and some $\rho \in L_+^\infty(\mathbb{R}^N)$ satisfying

$$p(\cdot) \le \rho(\cdot) << p^*(\cdot),$$

then $u_n \to 0$ in $L^{\alpha(\cdot)}(\mathbb{R}^N)$ for any α satisfying $p(\cdot) << \alpha(\cdot) << p^*(\cdot)$, where $B(y,r)$ is an open ball with center y and radius r.

Denote $X_1 = W^{1,p(\cdot)}(\mathbb{R}^N)$, $X_2 = W^{1,q(\cdot)}(\mathbb{R}^N)$, $X = X_1 \times X_2$. Let us endow the norm $\|\cdot\|$ on X as

$$\|(u,v)\| = \max\{\|u\|_{p(\cdot)}, \|v\|_{q(\cdot)}\}.$$

The dual space of X will be denoted by X^*, then for any $\Theta \in X^*$, there exist $f \in (W^{1,p(\cdot)}(\mathbb{R}^N))^*$ and $g \in (W^{1,q(\cdot)}(\mathbb{R}^N))^*$ such that $\Theta(u,v) = f(u) + g(v)$. We denote $\|\cdot\|_*$, $\|\cdot\|_{*,p(\cdot)}$ and $\|\cdot\|_{*,q(\cdot)}$ the norms of $X^*, (W^{1,p(\cdot)}(\mathbb{R}^N))^*$ and $(W^{1,q(\cdot)}(\mathbb{R}^N))^*$, respectively. Obviously $X^* = (W^{1,p(\cdot)}(\mathbb{R}^N))^* \times (W^{1,q(\cdot)}(\mathbb{R}^N))^*$ and

$$\|\Theta\|_* = \|f\|_{*,p(\cdot)} + \|g\|_{*,q(\cdot)}, \quad \forall \Theta \in X.$$

For every (u,v) and (φ,ψ) in X, set

$$\begin{gathered}
\Phi_1(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)}\,dx, \\
\Phi_2(v) = \int_{\mathbb{R}^N} \frac{1}{q(x)} |\nabla v|^{q(x)}\,dx + \int_{\mathbb{R}^N} \frac{1}{q(x)} |v|^{q(x)}\,dx, \\
\Phi(u,v) = \Phi_1(u) + \Phi_2(v), \\
\Psi(u,v) = \int_{\mathbb{R}^N} \{\lambda[\frac{a(x)}{\gamma(x)}|u|^{\gamma(x)} + \frac{b(x)}{\delta(x)}|v|^{\delta(x)}] + F(x,u,v)\}\,dx, \\
\Psi_1(u,v) = \int_{\mathbb{R}^N} \lambda \frac{a(x)}{\gamma(x)} |u|^{\gamma(x)}\,dx + \int_{\mathbb{R}^N} \lambda \frac{b(x)}{\delta(x)} |v|^{\delta(x)}\,dx.
\end{gathered}$$

It follows from Proposition 2.5 that $\Phi \in C^1(X,\mathbb{R})$, then

$$\begin{gathered}
\Phi'(u,v)(\varphi,\psi) = D_1\Phi(u,v)(\varphi) + D_2\Phi(u,v)(\psi), \forall(\varphi,\psi) \in X, \\
\Psi'(u,v)(\varphi,\psi) = D_1\Psi(u,v)(\varphi) + D_2\Psi(u,v)(\psi), \forall(\varphi,\psi) \in X,
\end{gathered}$$

where

$$\begin{gathered}
D_1\Phi(u,v)(\varphi) = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx + \int_{\mathbb{R}^N} |u|^{p(x)-2}u\varphi\,dx = \Phi_1'(u)(\varphi), \\
\forall \varphi \in X_1, \\
D_2\Phi(u,v)(\psi) = \int_{\mathbb{R}^N} |\nabla v|^{p(x)-2}\nabla v \nabla\psi\,dx + \int_{\mathbb{R}^N} |v|^{p(x)-2}v\psi\,dx = \Phi_2'(v)(\psi), \\
\forall \psi \in X_2, \\
D_1\Psi(u,v)(\varphi) = \int_{\mathbb{R}^N} [\lambda a(x)|u|^{\gamma(x)-2}u + \frac{\partial}{\partial u}F(x,u,v)]\varphi\,dx, \quad \forall \varphi \in X_1, \\
D_2\Psi(u,v)(\psi) = \int_{\mathbb{R}^N} [\lambda b(x)|v|^{\delta(x)-2}v + \frac{\partial}{\partial v}F(x,u,v)]\psi\,dx, \quad \forall \psi \in X_2.
\end{gathered}$$

The integral functional associated with the problem (1.1) is

$$J(u,v) = \Phi(u,v) - \Psi(u,v).$$

Without loss of generality, we may assume that $F(x,0,0) = 0$, then we have

$$F(x,u,v) = \int_0^1 [u\partial_2 F(x,tu,tv) + v\partial_3 F(x,tu,tv)]dt,$$

where ∂_j denotes the partial derivative of F with respect to its j-th variable. The condition (A0) holds

$$|F(x,u,v)| \le c(|u|^{p(x)} + |u|^{\alpha(x)} + |v|^{q(x)} + |v|^{\beta(x)}). \tag{2.1}$$

From Proposition 2.5 and condition (A0), it is easy to see that $J \in C^1(X,\mathbb{R})$ and satisfies

$$J'(u,v)(\varphi,\psi) = D_1J(u,v)(\varphi) + D_2J(u,v)(\psi), \quad \forall(\varphi,\psi) \in X,$$

where

$$\begin{aligned} D_1J(u,v)(\varphi) &= D_1\Phi(u,v)(\varphi) - D_1\Psi(u,v)(\varphi), \quad \forall\varphi \in X_1,\\ D_2J(u,v)(\psi) &= D_2\Phi(u,v)(\psi) - D_2\Psi(u,v)(\psi), \quad \forall\psi \in X_2. \end{aligned}$$

Obviously,

$$\|J'(u,v)\|_* = \|D_1J(u,v)\|_{*,p(\cdot)} + \|D_2J(u,v)\|_{*,q(\cdot)}.$$

We say $(u,v) \in X$ is a critical point of J if

$$J'(u,v)(\varphi,\psi) = 0, \quad \forall(\varphi,\psi) \in X.$$

Proposition 2.8 ([22]). *(i) Φ is a convex functional;*

(ii) Φ' is strictly monotone, that is, for any (u_1,v_1) , $(u_2,v_2) \in X$ with $(u_1,v_1) \neq (u_2,v_2)$, we have

$$(\Phi'(u_1,v_1) - \Phi'(u_2,v_2))(u_1-u_2,v_1-v_2) > 0,$$

(iii) Φ' is a mapping of type (S_+), that is if $(u_n,v_n) \rightharpoonup (u,v)$ in X and

$$\limsup_{n\to\infty}[\Phi'(u_n,v_n) - \Phi'(u,v)](u_n-u,v_n-v) \leq 0,$$

then $(u_n,v_n) \to (u,v)$ in X.

(iv) $\Phi' : X \to X^*$ is a bounded homeomorphism.

Theorem 2.9. *$\Psi_1 \in C^1(X,\mathbb{R})$ and Ψ_1,Ψ_1' are weakly-strongly continuous, that is, $(u_n,v_n) \rightharpoonup (u,v)$ implies $\Psi_1(u_n,v_n) \to \Psi_1(u,v)$ and $\Psi_1'(u_n,v_n) \to \Psi_1'(u,v)$.*

The proof is similar to the proof of [25, Theorem 3.2], we omit it here.

3. Main results and their proofs

In this section, we state the main results at first, and using the critical point theory, we prove the existence of solutions for problem (1.1), and the asymptotic behavior of solutions near infinity.

We say that $(u,v) \in X$ is a weak solution for (1.1), if

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx + \int_{\mathbb{R}^N} |u|^{p(x)-2}u\cdot\varphi\,dx\\ &= \int_{\mathbb{R}^N} \{\lambda a(x)|u|^{\gamma(x)-2}u + F_u(x,u,v)\}\varphi\,dx, \quad \forall\varphi \in X_1,\\ &\int_{\mathbb{R}^N} |\nabla v|^{q(x)-2}\nabla v\cdot\nabla\psi\,dx + \int_{\mathbb{R}^N} |v|^{q(x)-2}v\cdot\psi\,dx\\ &= \int_{\mathbb{R}^N} \{\lambda b(x)|v|^{\delta(x)-2}v + F_v(x,u,v)\}\psi\,dx, \quad \forall\psi \in X_2. \end{aligned}$$

It is easy to see that the critical point of J is a solution for (1.1).

Similar to the proof of [18, Theorem 5], from (A1) we have

$$F(x,\tau^{1/\theta_1}s,\tau^{1/\theta_2}t) \geq \tau F(x,s,t), \quad \forall(x,s,t) \in \mathbb{R}^N\times\mathbb{R}^2,\ \tau \geq 1, \tag{3.1}$$

$$F(x,\tau^{1/\theta_1}s,\tau^{1/\theta_2}t) \leq \tau F(x,s,t), \quad \forall(x,s,t) \in \mathbb{R}^N\times\mathbb{R}^2,\ 0 \leq \tau \leq 1. \tag{3.2}$$

In fact, from (A0) and (A1) we have

(A0') $0 \le F(x,s,t) \le \sigma |s|^{p(x)} + C(\sigma)|s|^{\alpha(x)} + \sigma |t|^{q(x)} + C(\sigma)|t|^{\beta(x)}$, where σ is a small enough positive constant.

Denote

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = (0,0), \gamma(1) = (u^*, v^*)\},$$
$$c = \inf_{\gamma \in \Gamma} \max_{(u,v)\in\gamma} J(u,v),$$

where $(u^*, v^*) \in X$ satisfies $J(u^*, v^*) < 0$.

Denote

$$\begin{aligned}\widehat{J}(u,v) = &\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \\ &+ \int_{\mathbb{R}^N} \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx - \int_{\mathbb{R}^N} \widetilde{F}(u,v)\,dx,\end{aligned}$$
$$\mathcal{N} = \{(u,v) \in X : \widehat{J}'(u,v)(\frac{1}{\theta_1}u, \frac{1}{\theta_2}v) = 0, (u,v) \neq 0\},$$
$$J^\infty = \inf_{(u,v)\in\mathcal{N}} \widehat{J}(u,v).$$

Now our results can be stated as follows.

Theorem 3.1. *If F satisfies* (A0)–(A3)*, the positive parameter λ is small enough and $c < J^\infty$, then* (1.1) *possesses a nontrivial solution.*

Next, we give an application of Theorem 3.1, that is, a sufficient condition for $c < J^\infty$.

We say $h(x)$ is periodic and its period is $A = \{a_1, a_2, \dots, a_N\}$ where $a_i \ge 0$, $1 \le i \le N$, if

$$h(x) = h(x + n_i a_i e_i), \quad \forall x \in \mathbb{R}^N,$$

where n_i are integers and $\langle e_1, \dots, e_N\rangle$ is the standard basis of $\mathbb{R}^N$.

Denote

$$Q(x_o, A) = \{x \in \mathbb{R}^N : (x - x_o)e_i \in [0, a_i]\}.$$

Theorem 3.2. *If $p(\cdot), q(\cdot)$ are periodic and their periods is A, $F(x,s,t)$ satisfies* (A0)–(A3)*, and there exist $\tau, \delta > 0$ and $p(\cdot) << \alpha_o(\cdot) << p^*(\cdot)$, $q(\cdot) << \beta_o(\cdot) << q^*(\cdot)$ such that*

$$\begin{gathered} F(x,s,t) \ge \widetilde{F}(s,t), \quad \forall (x,s,t) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+, \\ F(x,s,t) \ge \widetilde{F}(s,t) + \tau s^{\alpha_o(x)-1} + \tau t^{\beta_o(x)-1}, \\ \forall (x,s,t) \in B(Q(x_o, A), \delta) \times \mathbb{R}^+ \times \mathbb{R}^+, \end{gathered} \tag{3.3}$$

then (1.1) *possesses at least one nontrivial solution when λ is small enough.*

Next we give the behavior of solutions near infinity.

Theorem 3.3. *Suppose* (A0)–(A3) *hold, $a, b \in L^\infty(\mathbb{R}^N)$. If u is a weak solution for problem* (1.1)*, then $u, v \in C^{1,\alpha}(\mathbb{R}^N)$, $u(x) \to 0, |\nabla u(x)| \to 0, v(x) \to 0$ and $|\nabla v(x)| \to 0$ as $|x| \to \infty$.*

3.1. **Proof of Theorem 3.1.** For the proof, we need to do some preparations.

Lemma 3.4. *If F satisfies* (A0)–(A2)*, and the parameter λ is small enough, then J satisfies the Mountain Pass Geometry, that is,*

- (i) *There exist positive numbers ρ and α such that $J(u,v) \geq \alpha$ for any $(u,v) \in X$ with $\|(u,v)\| = \rho$;*
- (ii) *$J(0,0) = 0$, and there exists $(u,v) \in X$ with $\|(u,v)\| > \rho$ such that $J(u,v) < 0$.*

Proof. (i) Recall that (A0)–(A1) imply (A0'). Then from (A0') we have

$$|F(x,u,v)| \leq \varepsilon(|u|^{p^+} + |v|^{q^+}) + C(\varepsilon)(|u|^{\alpha(x)} + |v|^{\beta(x)}).$$

Suppose ε and λ are small enough. We have

$$\begin{aligned}
J(u,v) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_{\mathbb{R}^N} \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx \\
&\quad - \int_{\mathbb{R}^N} \lambda[\frac{a(x)}{\gamma(x)}|u|^{\gamma(x)} + \frac{b(x)}{\delta(x)}|v|^{\delta(x)}]\,dx - \int_{\mathbb{R}^N} F(x,u,v)\,dx \\
&\geq \Phi(u,v) - \lambda \int_{\mathbb{R}^N} (|u|^{p(x)} + |v|^{q(x)})\,dx - \lambda C_1 \\
&\quad - \varepsilon \int_{\mathbb{R}^N} [|u|^{p^+} + |v|^{q^+}]\,dx - C(\varepsilon)\int_{\mathbb{R}^N} (|u|^{\alpha(x)} + |v|^{\beta(x)})\,dx \\
&\geq \frac{1}{2}\Phi(u,v) - C(\varepsilon)\int_{\mathbb{R}^N} (|u|^{\alpha(x)} + |v|^{\beta(x)})\,dx.
\end{aligned}$$

Since

$$p(\cdot) << \alpha(\cdot) << p^*(\cdot), q(\cdot) << \beta(\cdot) << q^*(\cdot),$$

there exists a positive constant ε_0 such that

$$\alpha(\cdot) - p(\cdot) \geq 2\varepsilon_0 \quad \text{and} \quad \beta(\cdot) - q(\cdot) \geq 2\varepsilon_0.$$

Since p is Liptchitz cntinuous, we can divide $\mathbb{R}^N$ into countable disjoint cube $\Omega_n, n = 1, 2, \dots$, each one has the same side length, such that $\cup_{n=1}^\infty \Omega_n = \mathbb{R}^N$ and for any $n = 1, 2, \dots$, the following inequalities hold

$$\inf_{x\in\Omega_n} \alpha(x) - \sup_{x\in\Omega_n} p(x) \geq \varepsilon_0 \quad \text{and} \quad \inf_{x\in\Omega_n} \beta(x) - \sup_{x\in\Omega_n} q(x) \geq \varepsilon_0.$$

Denote $\alpha^-_{\Omega_n} = \inf_{x\in\Omega_n} \alpha(x), p^+_{\Omega_n} = \sup_{x\in\Omega_n} p(x)$. Suppose the positive number $\rho < 1$ is small enough and $\|(u,v)\| = \rho$, from Propositions 2.3 and 2.6, it follows that

$$\begin{aligned}
C(\varepsilon)\int_{\Omega_n} |u|^{\alpha(x)} \leq C(\varepsilon)|u|^{\alpha^-_{\Omega_n}}_{\alpha(\cdot),\Omega_n} &\leq \frac{1}{8p^+}\|u\|^{p^+_{\Omega_n}}_{p(\cdot),\Omega_n} \\
&\leq \frac{1}{8}\int_{\Omega_n} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx.
\end{aligned}$$

Similarly, we have

$$C(\varepsilon)\int_{\Omega_n} |v|^{\beta(x)}\,dx \leq \frac{1}{8}\int_{\Omega_n} \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx.$$

Thus, we have

$$J(u,v) \geq \frac{1}{2}\Phi(u,v) - C(\varepsilon)\int_{\mathbb{R}^N} (|u|^{\alpha(x)} + |v|^{\beta(x)})\,dx$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{\infty}\int_{\Omega_n}\frac{1}{p(x)}(|\nabla u|^{p(x)}\,dx + |u|^{p(x)})\,dx \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}\int_{\Omega_n}\frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx \\
&\quad - \sum_{n=1}^{\infty}C(\varepsilon)\int_{\Omega_n}(|u|^{\alpha(x)} + |v|^{\beta(x)})\,dx \\
&\ge \frac{1}{8}\Phi(u,v).
\end{aligned}
$$

It means that assertion (i) holds.

(ii) Obviously, $J(0,0)=0$. When $t\ge 1$, by (3.1), we have

$$
\begin{aligned}
&J(t^{1/\theta_1}u, t^{1/\theta_2}v) \\
&= \int_{\mathbb{R}^N}\frac{1}{p(x)}t^{\frac{p(x)}{\theta_1}}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_{\mathbb{R}^N}\frac{1}{q(x)}t^{\frac{q(x)}{\theta_2}}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx \\
&\quad - \int_{\mathbb{R}^N}\lambda[\frac{a(x)}{\gamma(x)}t^{\frac{\gamma(x)}{\theta_1}}|u|^{\gamma(x)} + \frac{b(x)}{\delta(x)}t^{\frac{\delta(x)}{\theta_2}}|v|^{\delta(x)}]\,dx - \int_{\mathbb{R}^N}F(x,t^{1/\theta_1}u,t^{1/\theta_2}v)\,dx \\
&\le \int_{\mathbb{R}^N}\frac{1}{p(x)}t^{\frac{p(x)}{\theta_1}}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_{\mathbb{R}^N}\frac{1}{q(x)}t^{\frac{q(x)}{\theta_2}}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx \\
&\quad - \int_{\mathbb{R}^N}\lambda[\frac{a(x)}{\gamma(x)}t^{\frac{\gamma(x)}{\theta_1}}|u|^{\gamma(x)} + \frac{b(x)}{\delta(x)}t^{\frac{\delta(x)}{\theta_2}}|v|^{\delta(x)}]\,dx - \int_{\mathbb{R}^N}tF(x,u,v)\,dx.
\end{aligned}
$$

Note that $\gamma(x) << p(x)$, $\delta(x) << q(x)$, $\theta_1 > p^+$ and $\theta_2 > q^+$, then for any nontrivial $(u,v)\in X$, it is not hard to check

$$
J(t^{1/\theta_1}u, t^{1/\theta_2}v) \to -\infty \quad \text{as } t\to+\infty.
$$

□

We remark that it is easy to see that $J^\infty > 0$.

Lemma 3.5. *If F satisfies* (A0)–(A2), *$\{(u_n,v_n)\}$ is a PS sequence of J, that is $J(u_n,v_n)\to c$ which is the mountain pass level, and $J'(u_n,v_n)\to 0$, then $\{(u_n,v_n)\}$ is bounded.*

Proof. Since $1 << \gamma(\cdot) << p(\cdot)$, $a(\cdot)\in L^{\frac{p(\cdot)}{p(\cdot)-\gamma(\cdot)}}(\mathbb{R}^N)$, $1 << \delta(\cdot) << q(\cdot)$, $b(\cdot)\in L^{\frac{q(\cdot)}{q(\cdot)-\delta(\cdot)}}(\mathbb{R}^N)$, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^N}\lambda[\frac{a(x)}{\gamma(x)}|u|^{\gamma(x)} + \frac{b(x)}{\delta(x)}|v|^{\delta(x)}]\,dx| \\
&\le \int_{\mathbb{R}^N}[|\lambda a(x)||u|^{\gamma(x)} + |\lambda b(x)||v|^{\delta(x)}]\,dx \\
&\le \int_{\mathbb{R}^N}[\frac{\gamma(x)}{p(x)}(\varepsilon_1)^{\frac{p(x)}{\gamma(x)}}|u|^{p(x)} + \frac{p(x)-\gamma(x)}{p(x)}|\frac{1}{\varepsilon_1}\lambda a(x)|^{\frac{p(x)}{p(x)-\gamma(x)}}]\,dx \\
&\quad + \int_{\mathbb{R}^N}[\frac{\delta(x)}{q(x)}(\varepsilon_1)^{\frac{q(x)}{\delta(x)}}|v|^{q(x)} + \frac{q(x)-\delta(x)}{q(x)}|\frac{1}{\varepsilon_1}\lambda b(x)|^{\frac{q(x)}{q(x)-\delta(x)}}]\,dx \\
&\le \varepsilon_1\int_{\mathbb{R}^N}[|u|^{p(x)} + |v|^{q(x)}]\,dx + C(\varepsilon_1),
\end{aligned}
$$

where ε_1 is a positive small enough constant.

By (A1), we have for large values of n

$$
\begin{aligned}
&c+1\\
&\geq J(u_n,v_n)\\
&=\int_{\mathbb{R}^N}\frac{1}{p(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx+\int_{\mathbb{R}^N}\frac{1}{q(x)}(|\nabla v_n|^{q(x)}+|v_n|^{q(x)})\,dx\\
&\quad-\int_{\mathbb{R}^N}\lambda[\frac{a(x)}{\gamma(x)}|u_n|^{\gamma(x)}+\frac{b(x)}{\delta(x)}|v_n|^{\delta(x)}]\,dx-\int_{\mathbb{R}^N}F(x,u_n,v_n)\,dx\\
&\geq\int_{\mathbb{R}^N}\frac{1}{p(x)}\{|\nabla u_n|^{p(x)}+|u_n|^{p(x)}-\frac{u_n}{\theta_1}F_u(x,u_n,v_n)\}\,dx\\
&\quad+\int_{\mathbb{R}^N}\frac{1}{q(x)}\{|\nabla v_n|^{q(x)}+|v_n|^{q(x)}-\frac{v_n}{\theta_2}F_v(x,u_n,v_n)\}\,dx\\
&\quad-\int_{\mathbb{R}^N}\lambda[\frac{a(x)}{\gamma(x)}|u_n|^{\gamma(x)}+\frac{b(x)}{\delta(x)}|v_n|^{\delta(x)}]\,dx,\\
&=\int_{\mathbb{R}^N}(\frac{1}{p(x)}-\frac{1}{\theta_1})(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx+J'(u_n,v_n)(\frac{1}{\theta_1}u_n,\frac{1}{\theta_2}v_n)\\
&\quad+\int_{\mathbb{R}^N}(\frac{1}{q(x)}-\frac{1}{\theta_2})(|\nabla v_n|^{q(x)}+|v_n|^{q(x)})\,dx-\frac{l}{2}\int_{\mathbb{R}^N}[|u_n|^{p(x)}+|v_n|^{q(x)}]\,dx-C\\
&\geq\frac{l}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx+\frac{l}{2}\int_{\mathbb{R}^N}(|\nabla v_n|^{q(x)}+|v_n|^{q(x)})\,dx\\
&\quad-\frac{1}{\theta_1}\|D_1J(u_n,v_n)\|_{*,p(\cdot)}\|u_n\|_{p(\cdot)}-\frac{1}{\theta_2}\|D_2J(u_n,v_n)\|_{*,q(\cdot)}\|v_n\|_{q(\cdot)}-C,
\end{aligned}
$$

where $l=\min\{(\frac{1}{p^+}-\frac{1}{\theta_1}),(\frac{1}{q^+}-\frac{1}{\theta_2})\}$.

Without loss of generality, we assume that $\|v_n\|_{q(\cdot)}\leq\|u_n\|_{p(\cdot)}\to\infty$, $n=1,2,\dots$. Therefore for large enough n, we have

$$
c+1\geq\frac{l}{2}\|u_n\|_{p(\cdot)}^{p^-}-(\frac{1}{\theta_1}\|D_1J(u_n,v_n)\|_{*,p(\cdot)}+\frac{1}{\theta_2}\|D_2J(u_n,v_n)\|_{*,q(\cdot)})\|u_n\|_{p(\cdot)}-C.
$$

This is a contradiction. Thus $\{\|u_n\|_{p(\cdot)}\}$ and $\{\|v_n\|_{q(\cdot)}\}$ are bounded. □

Lemma 3.6. . *Suppose F satisfies* (A0)–(A3), *$\{(u_n,v_n)\}$ satisfy $J(u_n,v_n)\to c>0$, where c is the mountain pass level, $J'(u_n,v_n)\to 0$, λ is small enough, passing to a subsequence still labeled by n, we have*

(i) $\{(u_n,v_n)\}$ has a nontrivial weak limit $(u,v)\in X$ or

$$
\int_{\mathbb{R}^N}\widetilde{F}_u(u_n,v_n)u_n\,dx+\int_{\mathbb{R}^N}\widetilde{F}_v(u_n,v_n)v_n\,dx\geq\delta>0;
$$

(ii) If $c<J^\infty$, then $\{(u_n,v_n)\}$ has a nontrivial weak limit.

Proof. (i) It follows from Lemma 3.5 that $\{(u_n,v_n)\}$ is bounded in X. Without loss of generality, we may assume that $(u_n,v_n)\rightharpoonup(u,v)$ in X. If $(u,v)=(0,0)$, then Proposition 2.6 implies

$$
\begin{aligned}
&u_n\to 0\quad\text{in } L_{\rm loc}^{\alpha(\cdot)}(\mathbb{R}^N),p(\cdot)\leq\alpha(\cdot)<p^*(\cdot),\\
&v_n\to 0\quad\text{in } L_{\rm loc}^{\beta(\cdot)}(\mathbb{R}^N),q(\cdot)\leq\beta(\cdot)<q^*(\cdot).
\end{aligned}\tag{3.4}
$$

Since $(u_n, v_n) \rightharpoonup (0,0)$, then Theorem 2.9 implies

$$\int_{\mathbb{R}^N} \lambda \frac{a(x)}{\gamma(x)} |u_n|^{\gamma(x)}\, dx = o(1) = \int_{\mathbb{R}^N} \lambda \frac{b(x)}{\delta(x)} |v_n|^{\delta(x)}\, dx. \tag{3.5}$$

Recall that (A0)–(A1) imply (A0'). Then from (A0'), (A3) and (3.4), it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (F_u(x,u_n,v_n) - \widetilde{F}_u(u_n,v_n))u_n\, dx\Big| \\
&\le \int_{|x|\ge R} |F_u(x,u_n,v_n) - \widetilde{F}_u(u_n,v_n)||u_n|\, dx \\
&\quad + C\int_{|x|\le R} (|u_n|^{p(x)} + |u_n|^{\alpha(x)} + |v_n|^{q(x)} + |v_n|^{\beta(x)})\, dx \\
&\le \varepsilon(R)\int_{|x|\ge R} (|u_n|^{p(x)} + |u_n|^{\alpha(x)} + |v_n|^{q(x)} + |v_n|^{\beta(x)})\, dx \\
&\quad + C\int_{|x|\le R} (|u_n|^{p(x)} + |u_n|^{\alpha(x)} + |v_n|^{q(x)} + |v_n|^{\beta(x)})\, dx,
\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} F_u(x,u_n,v_n)u_n\, dx = \int_{\mathbb{R}^N} \widetilde{F}_u(u_n,v_n)u_n\, dx + o(1) \quad \text{as } n\to+\infty. \tag{3.6}$$

Similar to the proof of (3.6), we can verify

$$\int_{\mathbb{R}^N} F_v(x,u_n,v_n)v_n\, dx = \int_{\mathbb{R}^N} \widetilde{F}_v(u_n,v_n)v_n\, dx + o(1) \quad \text{when } n\to+\infty, \tag{3.7}$$

$$\int_{\mathbb{R}^N} F(x,u_n,v_n)\, dx = \int_{\mathbb{R}^N} \widetilde{F}(u_n,v_n)\, dx + o(1) \quad \text{as } n\to+\infty. \tag{3.8}$$

Since $F(x,u,v)\ge 0$ and $J(u_n,v_n)\to c>0$, we have

$$\Phi(u_n,v_n) - \Psi_1(u_n,v_n) \ge J(u_n,v_n) \ge C_1 > 0, \quad \text{for } n \backsimeq \infty, \tag{3.9}$$

which together with (3.5)-(3.8) and $J'(u_n,v_n)\to 0$ implies

$$\int_{\mathbb{R}^N} \widetilde{F}_u(u_n,v_n)u_n\, dx + \int_{\mathbb{R}^N} \widetilde{F}_v(u_n,v_n)v_n\, dx \ge \delta > 0. \tag{3.10}$$

(ii) By (A0), (3.1) and (3.2), there exist $t_n>0$ such that $(t_n^{1/\theta_1}u_n, t_n^{1/\theta_2}v_n)\in\mathcal{N}$; that is,

$$\begin{aligned}
&\frac{1}{\theta_1}\int_{\mathbb{R}^N} \Big(|\nabla t_n^{1/\theta_1}u_n|^{p(x)} + |t_n^{1/\theta_1}u_n|^{p(x)}\Big)\, dx \\
&+ \frac{1}{\theta_2}\int_{\mathbb{R}^N} \Big(|\nabla t_n^{1/\theta_2}v_n|^{q(x)} + |t_n^{1/\theta_2}v_n|^{q(x)}\Big)\, dx \\
&= \frac{1}{\theta_1}\int_{\mathbb{R}^N} \widetilde{F}_u(t_n^{1/\theta_1}u_n, t_n^{\frac{1}{\theta_2}}v_n)t_n^{1/\theta_1}u_n\, dx \\
&\quad + \frac{1}{\theta_2}\int_{\mathbb{R}^N} \widetilde{F}_v(t_n^{1/\theta_1}u_n, t_n^{\frac{1}{\theta_2}}v_n)t_n^{1/\theta_2}v_n\, dx.
\end{aligned} \tag{3.11}$$

Suppose (u,v) is trivial, then (3.10) is valid. Noting that $\{(u_n,v_n)\}$ is bounded in X. Obviously, there exist positive constants c_1 and c_2 such that

$$c_1 \le t_n \le c_2. \tag{3.12}$$

From (3.5) and (3.12), we have

$$\int_{\mathbb{R}^N} \lambda \frac{a(x)}{\gamma(x)} |t_n^{1/\theta_1} u_n|^{\gamma(x)}\,dx = o(1) = \int_{\mathbb{R}^N} \lambda \frac{b(x)}{\delta(x)} |t_n^{1/\theta_2} v_n|^{\delta(x)}\,dx. \tag{3.13}$$

Since $J'(u_n, v_n) \to 0$ and $\{(u_n, v_n)\}$ is bounded in X, it follows from (3.6) and (3.7) that

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx \\ &= \int_{\mathbb{R}^N} F_u(x, u_n, v_n) u_n\,dx + o(1) = \int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n) u_n\,dx + o(1), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla v_n|^{q(x)} + |v_n|^{q(x)})\,dx \\ &= \int_{\mathbb{R}^N} F_v(x, u_n, v_n) v_n\,dx + o(1) = \int_{\mathbb{R}^N} \widetilde{F}_v(u_n, v_n) v_n\,dx + o(1). \end{aligned} \tag{3.15}$$

Obviously, there exist $\xi_n, \eta_n \in \mathbb{R}^N$ such that

$$\begin{gathered} \int_{\mathbb{R}^N} \Big(|\nabla t_n^{1/\theta_1} u_n|^{p(x)} + |t_n^{1/\theta_1} u_n|^{p(x)}\Big)\,dx = t_n^{\frac{p(\xi_n)}{\theta_1}} \int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big)\,dx, \\ \int_{\mathbb{R}^N} \Big(|\nabla t_n^{1/\theta_2} v_n|^{q(x)} + |t_n^{1/\theta_2} v_n|^{q(x)}\Big)\,dx = t_n^{\frac{q(\eta_n)}{\theta_2}} \int_{\mathbb{R}^N} (|\nabla v_n|^{q(x)} + |v_n|^{q(x)})\,dx, \end{gathered}$$

which together with (3.11), (3.14) and (3.15) implies

$$\begin{aligned} &\frac{1}{\theta_1} t_n^{\frac{p(\xi_n)}{\theta_1}} [\int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n) u_n\,dx + o(1)] + \frac{1}{\theta_2} t_n^{\frac{q(\eta_n)}{\theta_2}} [\int_{\mathbb{R}^N} \widetilde{F}_v(u_n, v_n) v_n\,dx + o(1)] \\ &= \frac{1}{\theta_1} \int_{\mathbb{R}^N} \widetilde{F}_u(t_n^{1/\theta_1} u_n, t_n^{\frac{1}{\theta_2}} v_n) t_n^{1/\theta_1} u_n\,dx + \frac{1}{\theta_2} \int_{\mathbb{R}^N} \widetilde{F}_v(t_n^{1/\theta_1} u_n, t_n^{\frac{1}{\theta_2}} v_n) t_n^{1/\theta_2} v_n\,dx. \end{aligned}$$

Thus

$$\begin{aligned} &\frac{1}{\theta_1} t_n^{\frac{p(\xi_n)}{\theta_1}} \{\int_{\mathbb{R}^N} [\widetilde{F}_u(t_n^{1/\theta_1} u_n, t_n^{\frac{1}{\theta_2}} v_n) t_n^{\frac{1-p(\xi_n)}{\theta_1}} u_n - \widetilde{F}_u(u_n, v_n) u_n]\,dx + o(1)\} \\ &+ \frac{1}{\theta_2} t_n^{\frac{q(\eta_n)}{\theta_2}} \{\int_{\mathbb{R}^N} [\widetilde{F}_v(t_n^{1/\theta_1} u_n, t_n^{\frac{1}{\theta_2}} v_n) t_n^{\frac{1-q(\xi_n)}{\theta_2}} v_n - \widetilde{F}_v(u_n, v_n) v_n]\,dx + o(1)\} \\ &= 0. \end{aligned} \tag{3.16}$$

From (A2), it is easy to see that

$$\partial_2 F(x, \tau^{1/\theta_1} s, \tau^{1/\theta_2} t) s / |\tau|^{\frac{\theta_1 - 1}{\theta_1}} \quad \text{and} \quad \partial_3 F(x, \tau^{1/\theta_1} s, \tau^{1/\theta_2} t) t / |\tau|^{\frac{\theta_2 - 1}{\theta_2}}$$

are increasing about τ when $\tau > 0$; obviously, $\partial_1 \widetilde{F}(\tau^{1/\theta_1} s, \tau^{1/\theta_2} t) s / |\tau|^{\frac{\theta_1 - 1}{\theta_1}}$ and $\partial_2 \widetilde{F}(\tau^{1/\theta_1} s, \tau^{1/\theta_2} t) t / |\tau|^{\frac{\theta_2 - 1}{\theta_2}}$ are increasing when $\tau > 0$. By (A1) and (3.16), we have

(1) If $t_n \geq 1$, then

$$\begin{aligned} 0 \leq &\frac{1}{\theta_1} t_n^{\frac{p(\xi_n)}{\theta_1}} (t_n^{\frac{\theta_1 - p(\xi_n)}{\theta_1}} - 1) \int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n) u_n\,dx \\ &+ \frac{1}{\theta_2} t_n^{\frac{q(\eta_n)}{\theta_2}} (t_n^{\frac{\theta_2 - q(\xi_n)}{\theta_2}} - 1) \int_{\mathbb{R}^N} \widetilde{F}_v(u_n, v_n) v_n\,dx \leq o(1); \end{aligned} \tag{3.17}$$

(2) If $t_n < 1$, then

$$\begin{aligned} 0 \le & \frac{1}{\theta_1} t_n^{\frac{p(\xi_n)}{\theta_1}} (1 - t_n^{\frac{\theta_1 - p(\xi_n)}{\theta_1}}) \int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n) u_n \, dx \\ & + \frac{1}{\theta_2} t_n^{\frac{q(\eta_n)}{\theta_2}} (1 - t_n^{\frac{\theta_2 - q(\xi_n)}{\theta_2}}) \int_{\mathbb{R}^N} \widetilde{F}_v(u_n, v_n) v_n \, dx \le o(1). \end{aligned} \tag{3.18}$$

From (3.10), (3.12), (3.17) and (3.18), it follows that

$$\lim_{n\to\infty} t_n = 1. \tag{3.19}$$

Together with (3.5), (3.8) and the definition of (u_n, v_n), we have

$$c = J(u_n, v_n) + o(1) = \widehat{J}(u_n, v_n) + o(1). \tag{3.20}$$

From the bounded continuity of Nemytskii operator, we can see

$$\widehat{J}(u_n, v_n) = \widehat{J}(t_n^{1/\theta_1} u_n, t_n^{1/\theta_2} v_n) + o(1). \tag{3.21}$$

Note that $(t_n^{1/\theta_1} u_n, t_n^{1/\theta_2} v_n) \in \mathcal{N}$. It follows from (3.19), (3.20) and (3.21) that

$$c = \widehat{J}(t_n^{1/\theta_1} u_n, t_n^{1/\theta_2} v_n) + o(1) \ge J^\infty + o(1) \to J^\infty > c.$$

This is a contradiction. □

Proof of Theorem 3.1. From Lemmas 3.4 and 3.5, we know that there exist a bounded PS sequence $\{(u_n, v_n)\} \subset X$ such that

$$J(u_n, v_n) \to c > 0, \qquad J'(u_n, v_n) \to 0,$$

where c is the mountain pass level of J. Moreover, from Proposition 2.6 we have

$$\begin{aligned} u_n \to u \quad & \text{in } L^{\alpha(\cdot)}_{\rm loc}(\mathbb{R}^N), \quad p(\cdot) \le \alpha(\cdot) << p^*(\cdot), \\ v_n \to v \quad & \text{in } L^{\beta(\cdot)}_{\rm loc}(\mathbb{R}^N), \quad q(\cdot) \le \beta(\cdot) << q^*(\cdot), \end{aligned} \tag{3.22}$$

then

$$u_n \to u \text{ a.e. in } \mathbb{R}^N, \text{ and } v_n \to v \text{ a.e. in } \mathbb{R}^N. \tag{3.23}$$

Since $c < J^\infty$, Lemma 3.6 implies that (u, v) is nontrivial. It only remains to prove that (u, v) is a solution for (1.1). Since $J'(u_n, v_n) \to 0$ as $n \to \infty$, for any $(\varphi, \psi) \in X$, we have

$$\begin{gathered} \int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi + |u_n|^{p(x)-2} u_n \varphi) \, dx \\ - \int_{\mathbb{R}^N} \{\lambda a(x) |u_n|^{\gamma(x)-2} u_n + F_u(x, u_n, v_n)\} \varphi \, dx \to 0 \\ \int_{\mathbb{R}^N} (|\nabla v_n|^{q(x)-2} \nabla v_n \nabla \psi + |v_n|^{q(x)-2} v_n \psi) \, dx \\ - \int_{\mathbb{R}^N} \{\lambda b(x) |v_n|^{\delta(x)-2} v_n + F_v(x, u_n, v_n)\} \psi \, dx \to 0. \end{gathered}$$

Since $\{\|u_n\|_{p(\cdot)}\}$ and $\{\|v_n\|_{q(\cdot)}\}$ are bounded, for any $(\varphi, \psi) \in X$, it is easy to see that the following two groups are uniformly integrable in $\mathbb{R}^N$,

$$\begin{gathered} \{(|u_n|^{p(x)-1} + |\lambda a(x)| |u_n|^{\gamma(x)-1} + |F_u(x, u_n, v_n)|) \cdot |\varphi|\}, \\ \{(|v_n|^{q(x)-1} + |\lambda b(x)| |v_n|^{\delta(x)-1} + |F_v(x, u_n, v_n)|) \cdot |\psi|\}. \end{gathered}$$

Combining this, (3.23) and Vitali convergent theorem implies

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{p(x)-2}u_n\varphi\,dx - \int_{\mathbb{R}^N} \{\lambda a(x)|u_n|^{\gamma(x)-2}u_n + F_u(x,u_n,v_n)\}\varphi\,dx \qquad (3.24)\\
&\to \int_{\mathbb{R}^N} |u|^{p(x)-2}u\varphi\,dx - \int_{\mathbb{R}^N} \{\lambda a(x)|u|^{\gamma(x)-2}u + F_u(x,u,v)\}\varphi\,dx \quad \text{as } n\to\infty,\\
&\int_{\mathbb{R}^N} |v_n|^{q(x)-2}v_n\psi\,dx - \int_{\mathbb{R}^N} \{\lambda b(x)|v_n|^{\delta(x)-2}v_n + F_v(x,u_n,v_n)\}\psi\,dx \qquad (3.25)\\
&\to \int_{\mathbb{R}^N} |v|^{q(x)-2}v\psi\,dx - \int_{\mathbb{R}^N} \{\lambda b(x)|v|^{\delta(x)-2}v + F_v(x,u,v)\}\psi\,dx \quad \text{as } n\to\infty.
\end{aligned}$$

Thus, to prove that (u,v) is a weak solution of (1.1), we only need to prove that for any $(\varphi,\psi)\in X$ there holds

$$\begin{gathered}
\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi\,dx \to \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx, \quad n\to+\infty,\\
\int_{\mathbb{R}^N} |\nabla v_n|^{q(x)-2}\nabla v_n\nabla\psi\,dx \to \int_{\mathbb{R}^N} |\nabla v|^{q(x)-2}\nabla v\nabla\psi\,dx, \quad n\to+\infty.
\end{gathered} \tag{3.26}$$

Choose $\phi\in C_0^\infty(\mathbb{R}^N)$ with $0\le\phi\le 1$, we have

$$\int_{\mathbb{R}^N} \phi(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla w|^{p(x)-2}\nabla w)(\nabla u_n-\nabla w)\,dx \ge 0, \forall w\in X_1. \tag{3.27}$$

Since $\{(u_n,v_n)\}$ is bounded in X and $J'(u_n,v_n)\to 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} [|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(\phi(u_n-w)) + |u_n|^{p(x)-2}u_n\phi(u_n-w)]\,dx\\
&= \int_{\mathbb{R}^N} [\lambda a(x)|u_n|^{\gamma(x)-2}u_n + F_u(x,u_n,v_n)]\phi(u_n-w)\,dx + o(1),
\end{aligned} \tag{3.28}$$

for all $w\in X_1$. It follows from (3.27) and (3.28) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \{[\lambda a(x)|u_n|^{\gamma(x)-2}u_n + F_u(x,u_n,v_n)] - |u_n|^{p(x)-2}u_n\}\phi(u_n-w)\,dx\\
&- \int_{\mathbb{R}^N} (u_n-w)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi\,dx\\
&- \int_{\mathbb{R}^N} \phi|\nabla w|^{p(x)-2}\nabla w(\nabla u_n-\nabla w)\,dx + o(1)\ge 0,\ \forall w\in X_1.
\end{aligned} \tag{3.29}$$

Note that (u_n,v_n) is bounded in X, we may assume

$$\begin{gathered}
(u_n,v_n)\rightharpoonup(u,v) \quad \text{in } X,\\
\nabla u_n\rightharpoonup\nabla u \quad \text{in } (L^{p(\cdot)}(\mathbb{R}^N))^N, \qquad (3.30)\\
\nabla v_n\rightharpoonup\nabla v \quad \text{in } (L^{q(\cdot)}(\mathbb{R}^N))^N\\
|\nabla u_n|^{p(x)-2}\nabla u_n\rightharpoonup T \quad \text{in } (L^{p^0(\cdot)}(\mathbb{R}^N))^N, \qquad (3.31)\\
|\nabla v_n|^{q(x)-2}\nabla v_n\rightharpoonup S \quad \text{in } (L^{q^0(\cdot)}(\mathbb{R}^N))^N.
\end{gathered}$$

Note that ϕ has compact support, letting $n \to +\infty$, according to (3.24), (3.29), (3.30) and (3.31), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \{[\lambda a(x)|u|^{\gamma(x)-2}u + F_u(x,u,v)] - |u|^{p(x)-2}u\}\phi(u-w)\,dx \\ &- \int_{\mathbb{R}^N} (u-w)T\nabla\phi\,dx - \int_{\mathbb{R}^N} \phi|\nabla w|^{p(x)-2}\nabla w(\nabla u - \nabla w)\,dx \ge 0, \end{aligned} \tag{3.32}$$

for all $w \in X_1$. On the other hand $|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup T$ $J'(u_n,v_n) \to 0$, which implies that

$$\int_{\mathbb{R}^N} (T\nabla\varphi + |u|^{p(x)-2}u\varphi)\,dx = \int_{\mathbb{R}^N} [\lambda a(x)|u|^{\gamma(x)-2}u + F_u(x,u,v)]\varphi\,dx. \tag{3.33}$$

Set $\varphi = \phi(u-w)$, it follows from (3.32) and (3.33) that

$$\int_{\mathbb{R}^N} \phi(T - |\nabla w|^{p(x)-2}\nabla w)(\nabla u - \nabla w)\,dx \ge 0, \quad \forall w \in X_1. \tag{3.34}$$

Set $w = u - \varepsilon\xi$, where $\xi \in X_1$, $\varepsilon > 0$. From (3.34) we have

$$\int_{\mathbb{R}^N} \phi(T - |\nabla u|^{p(x)-2}\nabla u)\nabla\xi\,dx \ge 0, \quad \forall \xi \in X_1,$$

then

$$\int_{\mathbb{R}^N} \phi(T - |\nabla u|^{p(x)-2}\nabla u)\nabla\xi\,dx = 0, \quad \forall \xi \in X_1,$$

it is easy to see that

$$\int_{\mathbb{R}^N} (T - |\nabla u|^{p(x)-2}\nabla u)\nabla\xi\,dx = 0, \forall \xi \in X_1.$$

Thus (3.26) is valid. Therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + |u|^{p(x)-2}u\varphi)\,dx \\ &= \int_{\mathbb{R}^N} [\lambda a(x)|u|^{\gamma(x)-2}u + F_u(x,u,v)]\varphi\,dx, \forall \varphi \in X_1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla v|^{q(x)-2}\nabla v\nabla\psi + |v|^{q(x)-2}v\psi)\,dx \\ &= \int_{\mathbb{R}^N} [\lambda b(x)|v|^{\delta(x)-2}v + F_v(x,u,v)]\psi\,dx, \forall \psi \in X_2. \end{aligned}$$

Thus (u,v) is a solution of (1.1). $\square$

3.2. **Proof of Theorem 3.2.** Motivated by the property of translation invariant for p-Laplacian, we get a sufficient condition for $c < J^\infty$. To prove Theorem 3.2, we need the following Lemma.

Lemma 3.7. *If F satisfies* (A0)–(A2)*, then for any $(u,v) \in X\backslash\{(0,0)\}$, there exists a unique $t(u,v) > 0$ such that*

(1) $\widehat{J}((t(u,v))^{1/\theta_1}u, (t(u,v))^{\frac{1}{\theta_2}}v) = \max_{s\in[0,+\infty)} \widehat{J}(s^{\frac{1}{\theta_1}}u, s^{1/\theta_2}v)$,

(2) $((t(u,v))^{1/\theta_1}u, (t(u,v))^{1/\theta_2}v) \in \mathcal{N}$,

(3) *The operator* $(u,v)\mapsto t(u,v)$ *is continuous from* $X\backslash\{(0,0)\}$ *to* $(0,+\infty)$, *and the operator* $(u,v)\mapsto ((t(u,v))^{\frac{1}{\theta_1}}u,(t(u,v))^{1/\theta_2}v)$ *is a homeomorphism from the unit sphere in* X *to* $\mathcal{N}$.

Proof. For any $(u,v)\in X\backslash\{(0,0)\}$, define

$$g(t)=\widehat{J}(t^{1/\theta_1}u,t^{1/\theta_2}v),\quad \forall t\in[0,+\infty).$$

(1) Similar to the proof of (3.11), we have $g(t)>0$ as $t>0$ is small enough, and $g(t)<0$ as $t\to+\infty$. Obviously, g is continuous, then g attains it's maximum in $(0,+\infty)$.

(2) From (A2), it is not hard to check that $(t^{1/\theta_1}u,t^{1/\theta_2}v)\in\mathcal{N}$ if and only if $tg'(t)=0$; that is,

$$\begin{aligned}
&\int_{\mathbb{R}^N}[|\nabla u|^{p(x)}\frac{1}{\theta_1}t^{\frac{p(x)}{\theta_1}}\,dx+|u|^{p(x)}\frac{1}{\theta_1}t^{\frac{p(x)}{\theta_1}}]\,dx\\
&+\int_{\mathbb{R}^N}[|\nabla v|^{q(x)}\frac{1}{\theta_2}t^{\frac{q(x)}{\theta_2}}\,dx+|v|^{q(x)}\frac{1}{\theta_2}t^{\frac{q(x)}{\theta_1}}]\,dx\\
&=\int_\Omega\widetilde{F}_1(t^{1/\theta_1}u,t^{1/\theta_2}v)\frac{1}{\theta_1}t^{1/\theta_1}u\,dx+\int_\Omega\widetilde{F}_2(t^{1/\theta_1}u,t^{1/\theta_2}v)\frac{1}{\theta_2}t^{1/\theta_2}v\,dx,
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{1}{\theta_1}[|\nabla u|^{p(x)}t^{\frac{p(x)}{\theta_1}-1}\,dx+|u|^{p(x)}t^{\frac{p(x)}{\theta_1}-1}]\,dx\\
&+\int_{\mathbb{R}^N}\frac{1}{\theta_2}[|\nabla v|^{q(x)}t^{\frac{q(x)}{\theta_2}-1}\,dx+|v|^{q(x)}t^{\frac{q(x)}{\theta_1}-1}]\,dx\\
&=\int_\Omega\frac{1}{\theta_1}\frac{\widetilde{F}_1(t^{1/\theta_1}u,t^{1/\theta_2}v)}{t^{\frac{\theta_1-1}{\theta_1}}}u\,dx+\int_\Omega\frac{1}{\theta_2}\frac{\widetilde{F}_2(t^{1/\theta_1}u,t^{1/\theta_2}v)}{t^{\frac{\theta_2-1}{\theta_2}}}v\,dx.
\end{aligned}$$

It follows from (3.1) and (3.2) that the left hand is strictly decreasing with respect to t, while the right hand is increasing. Thus $g'(t)=0$ has a unique solution $t(u,v)$ such that $((t(u,v))^{1/\theta_1}u,(t(u,v))^{1/\theta_2}v)\in\mathcal{N}$.

We claim that $g(t)$ is increasing on $[0,t(u,v)]$, and decreasing on $[t(u,v),+\infty)$. Denote $(u_*,v_*)=((t(u,v))^{1/\theta_1}u,(t(u,v))^{1/\theta_2}v)$. Define $\rho(t)=\widehat{J}(t^{1/\theta_1}u_*,t^{1/\theta_2}v_*)$. We only need to prove that $\rho(t)$ is increasing on $[0,1]$, and $\rho(t)$ is decreasing on $[1,+\infty)$. From (1), it is easy to see that there exists $t_\#>0$ such that

$$\rho(t_\#)=\max_{t\geq0}\widehat{J}(t^{1/\theta_1}u_*,t^{1/\theta_2}v_*),$$

therefore $\rho'(t_\#)=0$.

Suppose $t>1$. By (A2), we have

$$\begin{aligned}
&\rho'(t)\\
&=\int_{\mathbb{R}^N}\frac{1}{\theta_1}t^{\frac{p(x)}{\theta_1}-1}(|\nabla u_*|^{p(x)}+|u_*|^{p(x)})\,dx+\int_{\mathbb{R}^N}\frac{1}{\theta_2}t^{\frac{q(x)}{\theta_2}-1}(|\nabla v_*|^{q(x)}+|v_*|^{q(x)})\,dx\\
&\quad-\int_{\mathbb{R}^N}\widetilde{F}_1(t^{1/\theta_1}u_*,t^{1/\theta_2}v_*)\frac{1}{\theta_1}t^{\frac{1}{\theta_1}-1}u_*\,dx-\int_{\mathbb{R}^N}\widetilde{F}_2(t^{1/\theta_1}u_*,t^{1/\theta_2}v_*)\frac{1}{\theta_2}t^{\frac{1}{\theta_2}-1}v_*\,dx\\
&<\int_{\mathbb{R}^N}\frac{1}{\theta_1}(|\nabla u_*|^{p(x)}+|u_*|^{p(x)})\,dx+\int_{\mathbb{R}^N}\frac{1}{\theta_2}(|\nabla v_*|^{q(x)}+|v_*|^{q(x)})\,dx\\
&\quad-\int_{\mathbb{R}^N}\widetilde{F}_1(u_*,v_*)\frac{1}{\theta_1}u_*\,dx-\int_{\mathbb{R}^N}\widetilde{F}_2(u_*,v_*)\frac{1}{\theta_2}v_*\,dx
\end{aligned}$$

$$= \widehat{J}'(u_*, v_*)(\frac{1}{\theta_1}u_*, \frac{1}{\theta_2}v_*) = 0.$$

Thus $\rho(t)$ is strictly decreasing when $t > 1$.

Suppose $t < 1$. Similarly, we have

$$\rho'(t) > \widehat{J}'(u_*, v_*)(\frac{1}{\theta_1}u_*, \frac{1}{\theta_2}v_*) = 0.$$

Thus $\rho(t)$ is strictly increasing when $t < 1$. Therefore $g(t)$ is increasing on $[0, t(u,v)]$ and decreasing on $[t(u,v), +\infty)$.

(3) We only need to proof that $t(\cdot,\cdot)$ is continuously. Let $(u_m, v_m) \to (u, v)$ in X, then $\widehat{J}(t^{1/\theta_1}u_m, t^{1/\theta_2}v_m) \to \widehat{J}(t^{\frac{1}{\theta_1}}u, t^{1/\theta_2}v)$. We choose a constant t_0 large enough such that $\widehat{J}(t_0^{\frac{1}{\theta_1}}u, t_0^{1/\theta_2}v) < 0$, then there exists a $M > 0$ such that

$$\widehat{J}(t_0^{1/\theta_1}u_m, t_0^{1/\theta_2}v_m) < 0$$

for any $m > M$. Therefore $t(u_m, v_m) < t_0$ when $m > M$, then $\{t(u_m, v_m)\}$ has a convergent subsequence $\{t(u_{m_j}, v_{m_j})\}$ satisfying $t(u_{m_j}, v_{m_j}) \to t_*$. Thus

$$\widehat{J}((t(u_{m_j}, v_{m_j}))^{1/\theta_1}u_{m_j}, (t(u_{m_j}, v_{m_j}))^{1/\theta_2}v_{m_j}) \to \widehat{J}(t_*^{1/\theta_1}u, t_*^{1/\theta_2}v).$$

From (1) we know that

$$\begin{aligned}
&\widehat{J}((t(u_{m_j}, v_{m_j}))^{\frac{1}{\theta_1}}u_{m_j}, (t(u_{m_j}, v_{m_j}))^{1/\theta_2}v_{m_j}) \\
&\geq \widehat{J}((t(u,v))^{1/\theta_1}u_{m_j}, (t(u,v))^{1/\theta_2}v_{m_j}),
\end{aligned}$$

and hence letting $j \to \infty$, we obtain

$$\widehat{J}(t_*^{1/\theta_1}u, t_*^{1/\theta_2}v) \geq \widehat{J}((t(u,v))^{1/\theta_1}u, (t(u,v))^{1/\theta_2}v).$$

From (1), we have $t_* = t(u,v)$. Thus $t(u,v)$ is continuous. □

Proof of Theorem 3.2. Let $\{(u_n, v_n)\} \subset \mathcal{N}$ be a minimizing sequences of $\widehat{J}$, that is

$$\lim_{n\to+\infty} \widehat{J}(u_n, v_n) = J^\infty > 0.$$

Similar to the proof of Lemma 3.5, we can see $\{(u_n, v_n)\}$ is bounded in X. Thus there exists a positive constant $\kappa > 1$ such that

$$\int_{\mathbb{R}^N} (|u_n|^{p(x)} + |v_n|^{q(x)})\,dx \leq \kappa, n = 1, 2, \dots. \tag{3.35}$$

We claim that for any fixed $\delta > 0$ and$p(\cdot) << \alpha(\cdot) << p^*(\cdot)$, $q(\cdot) << \beta(\cdot) << q^*(\cdot)$, there exist a $\varepsilon_o > 0$ such that

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,\delta)} |u_n|^{\alpha(x)}\,dx + \sup_{y\in\mathbb{R}^N} \int_{B(y,\delta)} |v_n|^{\beta(x)}\,dx \geq 2\varepsilon_o, \quad n = 1, 2, \dots. \tag{3.36}$$

Indeed, suppose otherwise. Then it follows from Proposition 2.7 that

$$u_n \to 0 \quad \text{in } L^{\alpha(\cdot)}(\mathbb{R}^N),\ \forall p(\cdot) << \alpha(\cdot) << p^*(\cdot), \tag{3.37}$$

$$v_n \to 0 \quad \text{in } L^{\beta(\cdot)}(\mathbb{R}^N),\ \forall q(\cdot) << \beta(\cdot) << q^*(\cdot). \tag{3.38}$$

We claim that

$$\int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n)u_n\,dx \to 0, \quad n \to +\infty. \tag{3.39}$$

For any $\varepsilon > 0$, (A0)–(A1) imply

$$|\widetilde{F}_u(u_n, v_n)u_n| \le \frac{\varepsilon}{2\kappa}(|u_n|^{p(x)} + |v_n|^{q(x)}) + C(\varepsilon)(|u_n|^{\alpha(x)} + |v_n|^{\beta(x)}),$$

and

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n)u_n\,dx\Big| \\ &\le \frac{\varepsilon}{2\kappa}\int_{\mathbb{R}^N}(|u_n|^{p(x)} + |v_n|^{q(x)})\,dx + C(\varepsilon)\int_{\mathbb{R}^N}(|u_n|^{\alpha(x)} + |v_n|^{\beta(x)})\,dx. \end{aligned} \tag{3.40}$$

Combining (3.37) and (3.38), there exist $N_0 > 0$ such that

$$C(\varepsilon)\int_{\mathbb{R}^N}(|u_n|^{\alpha(x)} + |v_n|^{\beta(x)})\,dx \le \frac{\varepsilon}{2}, \quad n \ge N_0. \tag{3.41}$$

From (3.35), (3.40) and (3.41), we have

$$\Big|\int_{\mathbb{R}^N} \widetilde{F}_u(u_n, v_n)u_n\,dx\Big| \le \varepsilon, \forall n \ge N_0.$$

Thus (3.39) is valid. Similarly, we can get

$$\int_{\mathbb{R}^N} \widetilde{F}_v(u_n, v_n)v_n\,dx \to 0, \quad n \to +\infty. \tag{3.42}$$

Note that $(u_n, v_n) \in \mathcal{N}$. It follows from (3.39), (3.42) and $\|(u_n, v_n)\| \to 0$ that

$$\widehat{J}(u_n, v_n) \to 0.$$

This is a contradiction to $\lim_{n\to+\infty} \widehat{J}(u_n, v_n) = J^\infty > 0$. Thus (3.36) is valid.

From (3.36), without loss of generality, we assume that

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,\delta)} |u_n|^{\alpha(x)}\,dx \ge \varepsilon_o, \quad n = 1, 2, \dots$$

We may assume that

$$\begin{gathered} \int_{B(y_n,\delta)} |u_n|^{\alpha(x)}\,dx \ge \frac{1}{2}\sup_{y\in\mathbb{R}^N}\int_{B(y,\delta)} |u_n|^{\alpha(x)}\,dx, \\ \int_{B(\eta_n,\delta)} |v_n|^{\beta(x)}\,dx \ge \frac{1}{2}\sup_{y\in\mathbb{R}^N}\int_{B(y,\delta)} |v_n|^{\beta(x)}\,dx. \end{gathered}$$

From $p(\cdot)$ begin periodic, for any y_n,η_n, there exist $x_n, \xi_n \in Q(x_o, A)$ such that

$$\begin{gathered} p(x) = p(y_n - x_n + x), \forall x \in \mathbb{R}^N, \\ q(x) = q(\eta_n - \xi_n + x), \forall x \in \mathbb{R}^N. \end{gathered}$$

Now, we consider $J(t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))$. Denote $J_1(u,v)=\Phi(u,v)-\int_{\mathbb{R}^N}F(x,u,v)\,dx$. It follows from (A2) that $F(x,t^{\frac{1}{\theta_1}}u,t^{1/\theta_2}v)/t$ is increasing with respect to t. Suppose $t\in(0,1)$, it follows from (3.2) that

$$\begin{aligned} &J_1(t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))\\ &=\Phi(t^{1/\theta_1}u_n, t^{1/\theta_2}v_n)-\int_{\mathbb{R}^N}F(x,t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))\,dx\\ &\geq t^{\max\{\frac{p^+}{\theta_1},\frac{q^+}{\theta_2}\}}\Phi(u_n,v_n)-t\int_{\mathbb{R}^N}F(x,u_n(y_n-x_n+x),v_n(\eta_n-\xi_n+x))\,dx\\ &=t\{t^{\max\{\frac{p^+}{\theta_1},\frac{q^+}{\theta_2}\}-1}\Phi(u_n,v_n)-\int_{\mathbb{R}^N}F(x,u_n(y_n-x_n+x),v_n(\eta_n-\xi_n+x))\,dx\}. \end{aligned} \tag{3.43}$$

From (3.36) and the boundedness of $\{(u_n,v_n)\}$, we can see that there exists positive constants C_1,C_2 such that

$$C_1\leq\|(u_n,v_n)\|\leq C_2. \tag{3.44}$$

Since $\theta_1>p^+$ and $\theta_2>q^+$, there exists a fixed $t_*\in(0,1)$ such that, for any $n=1,2,\dots$, we have

$$\begin{aligned} &t_*^{\max\{\frac{p^+}{\theta_1},\frac{q^+}{\theta_2}\}-1}\Phi(u_n,v_n)-\int_{\mathbb{R}^N}F(x,u_n(y_n-x_n+x),v_n(\eta_n-\xi_n+x))\,dx\\ &\geq\frac{1}{2}\Phi(u_n,v_n)\geq C_3>0. \end{aligned} \tag{3.45}$$

From (3.43) and (3.45), we obtain

$$J_1(t_*^{1/\theta_1}u_n(y_n-x_n+x), t_*^{1/\theta_2}v_n(\eta_n-\xi_n+x))\geq t_*C_3>0,\quad n=1,2,\dots.$$

Suppose λ is small enough. From the above inequality, we have

$$J(t_*^{1/\theta_1}u_n(y_n-x_n+x), t_*^{\frac{1}{\theta_2}}v_n(\eta_n-\xi_n+x))\geq\frac{1}{2}t_*C_3>0,\quad n=1,2,\dots. \tag{3.46}$$

Obviously, $J(0,0)=0$ and $J(t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))\to-\infty$ as $t\to+\infty$. Thus there exist $t_n\in(0,+\infty)$ such that

$$\begin{aligned} &J(t_n^{1/\theta_1}u_n(y_n-x_n+x), t_n^{1/\theta_2}v_n(\eta_n-\xi_n+x))\\ &=\max_{t\geq0}J(t^{\frac{1}{\theta_1}}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))>0. \end{aligned} \tag{3.47}$$

It follows from (3.46) and the boundedness of $\{(u_n,v_n)\}$ that there exist a positive constant ϵ such that

$$t_n\geq\epsilon,\quad n=1,2,\dots. \tag{3.48}$$

Denote

$$g(t)=\widehat{J}(t^{1/\theta_1}u_n(y_n-x_n+x), t^{\frac{1}{\theta_2}}v_n(\eta_n-\xi_n+x)).$$

Since $\{(u_n,v_n)\}\subset\mathcal{N}$, Lemma 3.7 implies

$$\begin{aligned} &\widehat{J}(u_n(y_n-x_n+x),v_n(\eta_n-\xi_n+x))\\ &=\max_{t\geq0}\widehat{J}(t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x)). \end{aligned} \tag{3.49}$$

Suppose λ is small enough. From (3.3), (3.47), (3.48) and (3.49), we have

$$\max_{t\geq0}J(t^{1/\theta_1}u_n(y_n-x_n+x), t^{1/\theta_2}v_n(\eta_n-\xi_n+x))$$

$$
\begin{aligned}
&= J(t_n^{1/\theta_1} u_n(y_n - x_n + x), t_n^{1/\theta_2} v_n(\eta_n - \xi_n + x)) \\
&\le \widehat{J}(t_n^{1/\theta_1} u_n(y_n - x_n + x), t_n^{1/\theta_2} v_n(\eta_n - \xi_n + x)) \\
&\quad + \int_{\mathbb{R}^N} \lambda[\frac{|a(x)|}{\gamma(x)} |t_n^{1/\theta_1} u_n(y_n - x_n + x)|^{\gamma(x)} + \frac{|b(x)|}{\delta(x)} |t_n^{1/\theta_2} v_n(\eta_n - \xi_n + x)|^{\delta(x)}]\,dx \\
&\quad - \frac{\tau}{2\alpha_o^+} \int_{B(x_n,\delta)} |t_n^{\frac{1}{\theta_1}} u_n(y_n - x_n + x)|^{\alpha_o(x)}\,dx \\
&\quad - \frac{\tau}{2\beta_o^+} \int_{B(\xi_n,\delta)} |t_n^{\frac{1}{\theta_2}} v_n(\eta_n - \xi_n + x)|^{\beta_o(x)}\,dx \\
&\le \widehat{J}(t_n^{1/\theta_1} u_n, t_n^{\frac{1}{\theta_2}} v_n) - \frac{\tau\varepsilon_0 \epsilon^{\max\{\alpha_o^+, \beta_o^+\}}}{2\max\{\alpha_o^+, \beta_o^+\}} \\
&\quad + \int_{\mathbb{R}^N} \lambda[\frac{|a(x)|}{\gamma(x)} |t_n^{1/\theta_1} u_n(y_n - x_n + x)|^{\gamma(x)} + \frac{|b(x)|}{\delta(x)} |t_n^{1/\theta_2} v_n(\eta_n - \xi_n + x)|^{\delta(x)}]\,dx \\
&\le \widehat{J}(u_n, v_n) - \frac{\tau\varepsilon_0 \epsilon^{\max\{\alpha_o^+, \beta_o^+\}}}{4\max\{\alpha_o^+, \beta_o^+\}} < J^\infty \quad \text{where } \zeta \in Q(x_o, A)\,.
\end{aligned}
$$

This completes the proof. $\square$

3.3. **Proof of Theorem 3.3.** According to the [17, Theorems 2.2 and 3.2], u and v are locally bounded. From [10, Theorem 1.2], u and v are locally $C^{1,\alpha}$ continuous. Similar to the proof of [13, Proposition 2.5], we obtain that $u, v \in C^{1,\alpha}(\mathbb{R}^N)$, $u(x) \to 0$, $|\nabla u(x)| \to 0$, $v(x) \to 0$ and $|\nabla v(x)| \to 0$ as $|x| \to \infty$.

Note 1. Let us consider the existence of solutions for the system

$$
\begin{gathered}
-\operatorname{div} |\nabla u_i|^{p_i(x)-2} \nabla u_i) + |u_i|^{p_i(x)-2} u_i \\
= \lambda a_i(x) |u_i|^{\gamma_i(x)-2} u_i + F_{u_i}(x, u_1, \dots, u_n) \quad \text{in } \mathbb{R}^N, \\
u_i \in W^{1,p_i(\cdot)}(\mathbb{R}^N),
\end{gathered}
$$

$i = 1, \dots, n$, where $u = (u_1, \dots, u_n)$, suppose λ is small enough, then the system has a nontrivial solution if it satisfies the following assumptions:

(H0) $p_i(\cdot)$ are Lipschitz continuous, $1 << p_i(\cdot) << N$, $1 << \gamma_i(\cdot) << p_i(\cdot)$, $a_i(\cdot) \in L^{\frac{p_i(\cdot)}{p_i(\cdot)-\gamma_i(\cdot)}}(\mathbb{R}^N)$, $F \in C^1(\mathbb{R}^N \times \mathbb{R}^n, \mathbb{R})$ and satisfies

$$
\begin{aligned}
&|F_{u_i}(x, u_1, \dots, u_n)| \\
&\le C(|u_i|^{p_i(x)-1} + |u_i|^{\alpha_i(x)-1} + \sum_{1 \le j \le n,\, j \ne i} [|u_j|^{p_j(x)/\alpha_i^0(x)} + |u_j|^{\alpha_j(x)/\alpha_i^0(x)}]),
\end{aligned}
$$

where $F_{u_i} = \frac{\partial}{\partial u_i} F$, $\alpha_i \in C(\mathbb{R}^N)$, and $p_i(\cdot) \le \alpha_i(\cdot) << p_i^*(\cdot)$, where

$$
p_i^*(x) = \begin{cases} N p_i(x)/(N - p_i(x)), & p_i(x) < N, \\ \infty, & p(x) \ge N, \end{cases}
$$

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^n)$ and satisfies the following conditions

$$
0 \le s_i F_{s_i}(x, s_1, \dots, s_n), \quad \forall (x, s_1, \dots, s_n) \in \mathbb{R}^N \times \mathbb{R}^n,\ i = 1, \dots, n,
$$

$$
0 < F(x, s_1, \dots, s_n) \le \sum_{1 \le i \le n} \frac{1}{\theta_i} s_i F_{s_i}(x, s_1, \dots, s_n), \quad \forall (x, s_1, \dots, s_n) \in \mathbb{R}^N \times \mathbb{R}^n;
$$

(H2) For any $(s,t)\in(\mathbb{R}\times\mathbb{R})$, $F_{s_i}(x,\tau^{1/\theta_1}s_1,\dots,\tau^{1/\theta_n}s_n)/\tau^{\frac{\theta_i-1}{\theta_i}}$ $(i=1,\dots,n)$ are increasing respect to $\tau>0$;

(H3) There is a measurable function $\widetilde{F}(s_1,\dots,s_n)$ such that
$$\lim_{|x|\to+\infty}F(x,s_1,\dots,s_n)=\widetilde{F}(s_1,\dots,s_n)$$
for bounded $\sum_{1\le i\le n}|s_i|$ uniformly,
$$|\widetilde{F}(s_1,\dots,s_n)|+|\sum_{1\le i\le n}s_i\widetilde{F}_{s_i}(s_1,\dots,s_n)|\le C\sum_{1\le i\le n}(|s_i|^{p_i^+}+|s_i|^{\alpha_i^-}),$$
for all $(s_1,\dots,s_n)\in\mathbb{R}^n$, and
$$|F(x,s_1,\dots,s_n)-\widetilde{F}(s_1,\dots,s_n)|\le\varepsilon(R)\sum_{1\le i\le n}(|s_i|^{p_i(x)}+|s_i|^{p_i^*(x)})\quad\text{when } |x|\ge R,$$
$$\begin{aligned}&|F_{s_i}(x,s_1,\dots,s_n)-\widetilde{F}_{s_i}(s_1,\dots,s_n)|\\&\le\varepsilon(R)\{|s_i|^{p_i(x)-1}+|s_i|^{p_i^*(x)-1}\\&\quad+\sum_{1\le j\le n,j\ne i}[|s_j|^{p_j(x)(p_i^*(x)-1)/p_i^*(x)}+|s_j|^{p_j^*(x)(p_i^*(x)-1)/p_i^*(x)}]\}\quad\text{when } |x|\ge R,\end{aligned}$$
where $\varepsilon(R)$ satisfies $\lim_{R\to+\infty}\varepsilon(R)=0$.

(H4)
$$F(x,s_1,\dots,s_n)\ge\widetilde{F}(s_1,\dots,s_n),\quad\forall(x,s_1,\dots,s_n)\in\mathbb{R}^N\times(\mathbb{R}^+)^n,$$
$$\begin{aligned}&F(x,s_1,\dots,s_n)\\&\ge\widetilde{F}(s_1,\dots,s_n)+\sum_{1\le i\le n}\tau s_i^{\alpha_i^\#(x)-1},\quad\forall(x,s_1,\dots,s_n)\in B(Q(x_o,A),\delta)\times(\mathbb{R}^+)^n,\end{aligned}$$
where $p_i(\cdot)<<\alpha_i^\#(\cdot)<<p_i^*(\cdot)$.

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References

[1] E. Acerbi, G. Mingione; Regularity results for a class of functionals with nonstandard growth, *Arch. Ration. Mech. Anal.,* **156** (2001), 121–140.

[2] C. O. Alves, P. C. Carriä, O. H. Miyagaki; Nonlinear perturbations of a periodic elliptic problem with critical growth, *J. Math. Anal. Appl.* 260 (2001) 133–146.

[3] C. O. Alves, J. Marcos do Ó, O. H. Miyagaki; On perturbations of a class of a periodic m-Laplacian equation with critical growth, *Nonlinear Anal.* 45 (2001) 849–863.

[4] C. O. Alves, J. Marcos do Ó, O. H. Miyagaki; Nontrivial solutions for a class of semilinear biharmonic problems involving critical exponents, *Nonlinear Anal.* 46 (2001) 121–133.

[5] C. O. Alves, J. Marcos do Ó, O. H. Miyagaki; On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth, *Nonlinear Anal.* 56 (2004) 781–791.

[6] K. C. Chang; *Critical point theory and applications*, shanghai Scientific and Technology press, Shanghai, 1986.

[7] Y. Chen, S. Levine, M. Rao; Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66** (2006), No.4,1383–1406.

[8] L. Diening, P. Harjulehto, P. Hästö, M. Růžička; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.

[9] X. L. Fan, D. Zhao; On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.
[10] X. L. Fan; Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form, *J. Diff. Equa.* **235** (2007) 397-417.
[11] X. L. Fan, J. S. Shen, D. Zhao; Sobolev embedding theorems for spaces $W^{k,p(x)}$, *J. Math. and Appl.*, **262** (2001), 749-760.
[12] X. L. Fan, Y. Z. Zhao, D. Zhao; Compact imbedding theorems with symmetry of Strauss-Lions type for the spaces $W^{1,p(x)}$, *J. Math. Anal. Appl.*, **255** (2001), 333-348.
[13] X. L. Fan; $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations, *J. Math. Anal. Appl.*, 341 (2008) 103-119.
[14] X. L. Fan, Q. H. Zhang; Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003), 1843-1852.
[15] X. L. Fan, Q. H. Zhang, D. Zhao; Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005), 306-317.
[16] X. L. Fan; Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.* 312 (2005) 464–477.
[17] X. L. Fan, D. Zhao; The quasi-minimizer of integral functionals with $m(x)$ growth conditions, *Nonlinear Anal.* 39, (2000), 807-816.
[18] A. El Hamidi; Existence results to elliptic systems with nonstandard growth conditions, *J. Math. Anal. Appl.* **300** (2004) 30-42.
[19] M. Růžička; *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Math 1748, Springer-Verlag, Berlin, 2000.
[20] Y. T. Shen, S. S. Yan; *The Variational Methods for Quasilinear Elliptic Equations*, South China Univ. of Technology Press, Guang Zhou, 1995 (in Chinese).
[21] M. Willem; *Minimax Theorems*, Birkhäuser, Boston, 1996.
[22] X. C. Xu, Y. K. An; Existence and multiplicity of solutions for elliptic systems with nonstandard growth condition in $\mathbb{R}^N$, *Nonlinear Analysis*, 68 (2008) 956–968.
[23] Q. H. Zhang; Existence of radial solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **315** (2006), No.2, 506-516.
[24] Q. H. Zhang, Z. M. Qiu, R. Dong; Existence and asymptotic behavior of positive solutions for a variable exponent elliptic system without variational structure, *Nonlinear Analysis*, 72 (2010) 354-363.
[25] Q. H. Zhang; Existence of solutions for $p(x)$-Laplacian equations with singular coefficients in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **348** (2008), No.1, 38-50.
[26] V. V. Zhikov; Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR. Izv.* **29** (1987), 33-36.

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