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New Diagonal Graph Ramsey Numbers of Unicyclic Graphs

Richard M. Low
San Jose State University, richard.low@sjsu.edu

Ardak Kapbasov
San Jose State University, ardak.kapbasov@yahoo.com

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Cover Page Footnote

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Abstract

Grossman conjectured that $R(G,G) = 2 \cdot |V(G)| - 1$, for all simple connected unicyclic graphs G of odd girth and $|V(G)| \geq 4$. In this note, we prove his conjecture for various classes of G containing a triangle. In addition, new diagonal graph Ramsey numbers are calculated for some classes of simple connected unicyclic graphs of even girth.

1 Introduction

In 1929, Frank Ramsey [18] established an innocuous-looking result in his groundbreaking paper on formal logic. Although it was not apparent at the time, his theorem would eventually form the cornerstone of Ramsey theory, a vibrant and rich area of extremal combinatorics.

The following general question [10] is investigated in Ramsey theory.

- If a particular mathematical structure (e.g., algebraic, combinatorial, or geometric) is arbitrarily partitioned into finite many classes, what kinds of substructures must always remain intact in at least one of the classes?

Over many decades, Ramsey-type questions in mathematical structures such as the integers, graphs, and Euclidean space have been investigated. As of this writing, a keyword search for "Ramsey" yields 2992 entries in the MathSciNet database. The interested reader is directed to [10, 11] for a comprehensive overview of Ramsey theory. For a gentle introduction to Ramsey theory, [20] is recommended.

The reader should note that the seeds of Ramsey theory were planted even before Ramsey introduced his theorem. Soifer's [23] beautifully written book is filled with deep mathematics and also provides a rich historical context of Ramsey theory.

Interesting applications of Ramsey theory can be found in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and computer science. The reader is directed to Rosta's [21] survey for a detailed exposition of some of these applications.

2 Preliminaries

The focus of this paper is on calculating new diagonal Ramsey numbers in graph Ramsey theory.

First, we recall some standard definitions and notation from graph theory. All graphs are finite, simple and connected, unless otherwise specified. For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *order* and *size* of G are defined to be $|V(G)|$ and $|E(G)|$, respectively. The *cycle* and *path* on n vertices are denoted by C_n and P_n, respectively. A graph G is *unicyclic* if it contains exactly one cycle. The *girth* of a graph G containing cycles is the length of a shortest cycle in G. The *complete graph* K_n is the graph on n (≥ 2) vertices, where every pair of vertices are adjacent. Any notation and terminology which are not explicitly defined in this paper can be found in [5, 10].

In graph Ramsey theory, the following definitions and notation are used.

Definition. Let $k \geq 2$. A *k-coloring* of G is a coloring of $E(G)$, using a maximum of k colors.

Let G and H be simple connected graphs. If every 2-coloring of the edges of K_n yields a monochromatic subgraph G or a monochromatic subgraph H in K_n, then this is denoted by $K_n \to (G, H)$. If that is not the case, then the notation $K_n \nrightarrow (G, H)$ is used.

Definition. The *Ramsey number* $R(G, H)$ is defined to be the minimum n, where $K_n \to (G, H)$.

Using graph-theoretic language, the finite version of Ramsey's theorem can be stated in the following way.

Theorem A. (*Ramsey* [18]). *Let $s, t \geq 2$. Then, there exists a minimal positive integer n such that every edge coloring of K_n (using two colors) contains a monochromatic K_s or a monochromatic K_t.*

Considerable work has been done in graph Ramsey theory. In addition to the calculation of Ramsey numbers in the classical theory, many different concepts have been introduced over time. They include Ramsey functions on graphs, many kinds of mixed Ramsey numbers, size Ramsey numbers, connected Ramsey numbers, anti-Ramsey numbers and Gallai-Ramsey numbers. These topics (and many others) can be found within the extensive mathematical literature. For an overview of classical graph Ramsey theory, the general surveys of Burr [1, 2], Radziszowski [17], Read and Wilson [19], and Sudakov [24] are invaluable. New directions and additional open questions in graph Ramsey theory are addressed in [6, 26, 27].

3 Some Known Results

Several classical lower bounds for $R(G, G)$ can be found in [3, 7, 8]. In particular, the following result of Burr and Erdős will help us in our calculations.

Theorem B. (*Burr-Erdős* [3]). *Let $|V(G)| \geq 4$. Then, $R(G, G) \geq \lfloor (4 \cdot |V(G)| - 1)/3 \rfloor$ for any connected G, and $R(G, G) \geq 2 \cdot |V(G)| - 1$ for any connected non-bipartite G.*

In addition to [19], the following summary of known diagonal Ramsey numbers is given on page 62 of [17]:

- $R(G, G)$, for all G without isolates on at most 4 vertices.
- $R(G, G)$, for all G without isolates and with at most 7 edges.
- $R(G, G)$, for all G on 5 vertices and with 7 or 8 edges.

New diagonal graph Ramsey numbers are computed in [16]. For current upper bounds on $R(K_s, K_s)$, the reader is directed to [9, 22].

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Generally speaking, calculating Ramsey numbers is very difficult. In 1979, Grossman made the following remarkable conjecture.

Conjecture 1. (*Grossman* [12]). *Let G be a connected unicyclic graph of odd girth and $|V(G)| \geq 4$. Then, $R(G, G) = 2 \cdot |V(G)| - 1$.*

This conjecture has been shown to be true for various classes of connected unicyclic graphs of odd girth. See [4, 12, 14, 15, 25].

Theorem C. (*Grossman* [12]). *Conjecture 1 holds for the following: (i) a triangle with stars emanating from its three vertices; (ii) a triangle with a star emanating from one vertex and both a star and a path emanating from another.*

Note that a C_3 with a path attached to a single vertex is a special case of Theorem C.

Theorem D. (*Köhler* [14]). *Let G be an odd cycle with $k \geq 1$ pendant edges at a single vertex of G. Then, $R(G, G) = 2 \cdot |V(G)| - 1$.*

Theorem E. (*Krasikov, Roditty* [15]). *Let G be an odd cycle with a pendant edge at two adjacent vertices of G. Then, Conjecture 1 holds.*

Since we wish to calculate new diagonal Ramsey numbers, our attention is focused on simple connected unicyclic graphs G, where $|V(G)| \geq 6$ and $|E(G)| \geq 8$. In doing so, we provide further evidence that Grossman's conjecture is true.

Lemma 1. *Let G be a connected unicyclic graph of odd girth and $|V(G)| \geq 4$. Then, $2 \cdot |V(G)| - 1 \leq R(G, G)$.*

Proof. This follows from Theorem B. □

Notation. Let $C_k *_1 H$ be the graph obtained by identifying a vertex u of C_k with a degree-one vertex of H. If H is disconnected, then a degree-one vertex in each component of H is identified with u.

Theorem 1. *Let $n \geq 1$ and $G = C_3 *_1 K_{1,n}$. Then, $R(G, G) = 2 \cdot |V(G)| - 1 = 2n + 5$.*

Proof. Note that $|V(C_3 *_1 K_{1,n})| = n + 3$. Since G is a connected non-bipartite graph, $R(G, G) \geq 2 \cdot |V(G)| - 1 = 2n + 5$ by Theorem B. We use mathematical induction to prove that $R(G, G) \leq 2 \cdot |V(G)| - 1 = 2n + 5$.

Base case. For $n = 1, 2, 3$ and 4, the claim holds [19].

Inductive step. Let H_p denote the graph $C_3 *_1 K_{1,p+1}$. Assume the claim holds for some fixed $p \geq 3$. Then, $R(H_p, H_p) = 2(p + 1) + 5 = 2p + 7$.

Now consider H_{p+1} with $|V(H_{p+1})| = (p+1)+4$. We want to show that $R(H_{p+1}, H_{p+1}) \leq 2(p+5)-1 = 2p+9$. Assume that $R(H_{p+1}, H_{p+1}) > 2p+9$. Then, there exists a 2-coloring $\mathcal{C}$ of K_{2p+9} with no monochromatic H_{p+1}. By the induction hypothesis, there is a monochromatic (say, red) H_p in $\mathcal{C}$. Let $V(H_p) = \{v_1, v_2, \ldots, v_{p+4}\}$ and $\alpha = \{v_i \in V(H_p) \mid \deg(v_i) = 1\} = \{v_5, v_6, \ldots, v_{p+4}\}$; $|\alpha| = p$. Then in $\mathcal{C}$, the edges $\{v_4v_{p+5}, v_4v_{p+6}, \ldots, v_4v_{2p+9}\}$ are all blue. See Figure 1.

Now, let $\beta = \{v_{p+5}, v_{p+6}, \ldots, v_{2p+9}\}$; $|\beta| = p + 5$, and let G15 (as found in [19]) be the 3-cycle with a pendant edge. Since $|\beta| \geq 3 + 5$, let

$$E(\text{G15}) = \{v_{2p+9}v_{2p+8}, v_{2p+9}v_{2p+7}, v_{2p+8}v_{2p+7}, v_{2p+7}v_{2p+6}\},$$

without loss of generality. Note that $R(\text{G15}, \text{G15}) = 7$. So, G15 is monochromatic in $\mathcal{C}$. If G15 is blue, then a blue H_{p+1} exists in $\mathcal{C}$, giving us a desired contradiction. Thus, G15 is red.

Let $\gamma = V(K_{2p+9}) - \{v_4, v_{2p+6}, v_{2p+7}, v_{2p+8}, v_{2p+9}\}$; $|\gamma| = 2p+4$. Then at least $p+4$ edges from v_{2p+6} to vertices in γ must be blue (otherwise, a red H_{p+1} exists). Let the vertex set (not including v_{2p+6}) of these particular blue edges be denoted by $N(v_{2p+6})_B$. If any pair of vertices in $N(v_{2p+6})_B$ are connected by a blue edge, then a blue H_{p+1} exists. Thus, a red K_{p+4} exists in $\mathcal{C}$, where $V(K_{p+4}) \subseteq N(v_{2p+6})_B$. In $\mathcal{C}$, if any red edges are adjacent to the red K_{p+4}, then a red H_{p+1} exists. We see that this is unavoidable, as $|\beta - V(\text{G15})| = p + 1$, which implies that at least three vertices in K_{p+4} belong to $\alpha \cup \{v_1, v_2, v_3\}$. Therefore, $R(H_{p+1}, H_{p+1}) \leq 2 \cdot |V(H_{p+1})| - 1$. $\square$

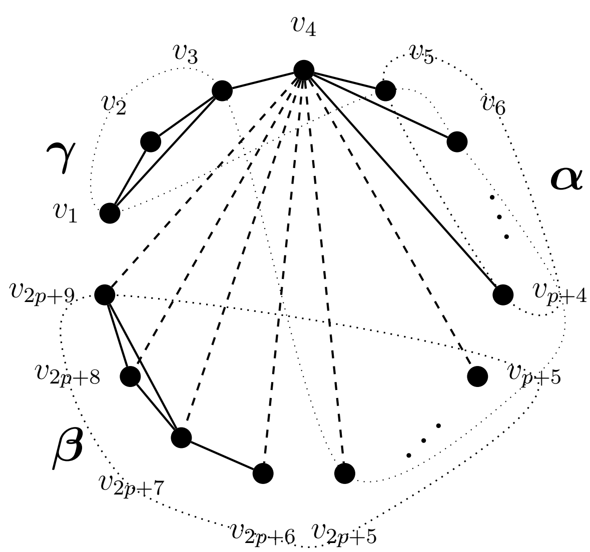

Figure 1: Partial 2-coloring of K_{2p+9} in the proof of Theorem 1. The solid edges are red and the dashed edges are blue.

For $n \geq 0$, let nP_2 denote the graph consisting of n copies of P_2. Generalizing Theorem 1, we obtain

Theorem 2. *Let $n \geq 0$, $m \geq 2$, and $G = C_3 *_1 (nP_2 \cup K_{1,m})$. Then,* $R(G, G) = 2 \cdot |V(G)| - 1 = 2n + 2m + 5$.

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Proof. Since G is a connected non-bipartite graph, $R(G,G) \geq 2 \cdot |V(G)| - 1 = 2n+2m+5$ by Theorem B. We use mathematical induction on n to prove that $R(G,G) \leq 2 \cdot |V(G)| - 1 = 2n + 2m + 5$.

<u>Base case</u>. For $n = 0$ and all $m \geq 2$, the claim holds by Theorem 1.

<u>Inductive step</u>. For some fixed $p \geq 1$ and all $q \geq 2$, assume the claim holds for $G_{p-1} = C_3 *_1 ((p-1)P_2 \cup K_{1,q})$. We want to show that $R(G_p, G_p) = 2p + 2q + 5$.

Let $\mathcal{C}$ be a 2-coloring of $K_{2p+2q+5}$. In $\mathcal{C}$, let w be a vertex with a maximum number of blue edges adjacent to it (say, $\deg_B(w) = \Delta_B(K_{2p+2q+5})$), and y be a vertex with a maximum number of red edges to it (say, $\deg_R(y) = \Delta_R(K_{2p+2q+5})$). By the induction hypothesis, there is a monochromatic (say, red, without loss of generality) in $K_{2p+2q+5} - \{w, y\}$. Let x be the vertex of this red G_{p-1} that connects C_3 to $((p-1)P_2 \cup K_{1,q})$ and $\alpha = \{v_1, v_2, x, v_4, \ldots, v_{p+q+2}\}$ be the set of vertices of this G_{p-1}. Let β be the set of the remaining $p+q+3$ vertices (including w and y) in $K_{2p+2q+5}$. See Figure 2.

In $\mathcal{C}$, all edges from x to the vertices in β are blue. Otherwise, a red G_p exists. Since the number of blue edges containing x is at least $p + q + 3$ and $\deg_B(w) = \Delta_B(K_{2p+2q+5})$, this implies that $\deg_B(w) \geq p+q+3$. Thus, there exists $z \in \beta$ such that edge wz is blue (z could be y). The edges between all vertex pairs in $\beta - \{w\}$ are red. Otherwise, a blue G_p exists in $\mathcal{C}$. So, a red K_{p+q+2} exists with vertex set in β. Let $v \notin \{z, y\}$ be a vertex of this red K_{p+q+2}. Then in $\mathcal{C}$, all edges from v to the vertices in α are blue. Otherwise, a red G_p exists and we are done. Consequently, a blue G_p exists in $\mathcal{C}$. Therefore, $R(G_p, G_p) \leq 2p + 2q + 5$. $\square$

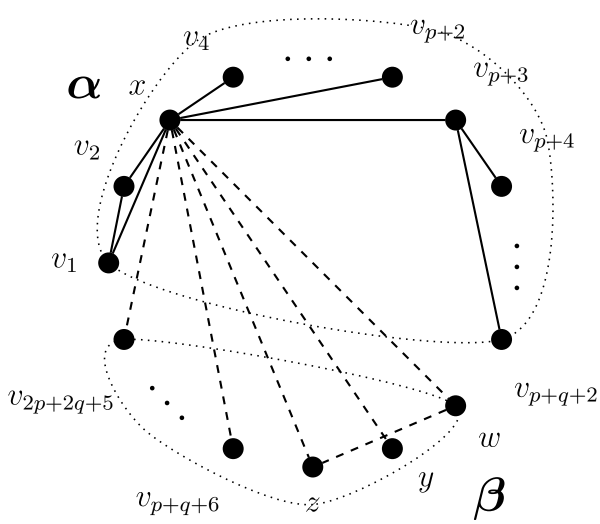

Figure 2: Partial 2-coloring of $K_{2p+2q+5}$ in the proof of Theorem 2. The solid edges are red and the dashed edges are blue.

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Theorem 3. *Let $n \geq 3$ and $G = C_n *_1 P_2$. Then,*

$$R(G, G) = \begin{cases} 2n + 1 & \text{if } n \text{ is odd,} \\ \frac{3n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $n \geq 3$ and $G = C_n *_1 P_2$. In [19], we see that $R(G, G) = 7$ and 6, for $n = 3$ and 4, respectively.

Case 1. $n \geq 6$ and even.

In [19], we have that $R(C_n, C_n) = n - 1 + \frac{n}{2} = \frac{3n}{2} - 1$, for $n \geq 6$ and even. Since C_n is a subgraph of G, $\frac{3n}{2} - 1 \leq R(G, G)$.

First, we show that $R(G, G) \leq \frac{3n}{2}$. Let $j = \frac{3n}{2}$. Let $\mathcal{C}$ be a 2-coloring of K_j. Since $R(C_n, C_n) = j - 1$, there is a subcycle $C_n = (v_1 v_2 \cdots v_n)$ which is red, say. Let $X = \{v_1, \ldots, v_n\}$ and $Y = V(K_j) - X$. Let $B = K_{n,n/2}$ be the complete bipartite graph with partite sets X and Y. If there is a red edge in B, then there is a red G in K_j, as desired.

Thus, all of the edges in B are blue. If we choose $\frac{n}{2}$ vertices from X and all the vertices from Y, then these vertices induce a blue $K_{n/2,n/2}$. Hence, we have a blue n-cycle ${}_bC_n$ containing these n vertices. There is a vertex $v \in X - V({}_bC_n)$ and a blue edge uv in B, where $u \in V({}_bC_n)$. Hence, we have a blue G in K_j. Therefore, $K_j \to (G, G)$ and $R(G, G) \leq \frac{3n}{2}$.

Now, we show that $R(G, G) > \frac{3n}{2} - 1$. In particular, we construct a 2-coloring of K_l where $l = \frac{3n}{2} - 1$, which does not contain a monochromatic G. Let $\mathcal{Q}_1 = \{w_1, w_2, \ldots, w_n\}$ be the vertices of a red C_n in K_l and $\mathcal{Q}_2 = V(K_l) - \mathcal{Q}_1$. Let $B' = K_{n,n/2-1}$ be the complete bipartite graph with partite sets $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Color the edges of B' with blue. Since $|\mathcal{Q}_2| = \frac{n}{2} - 1$, the largest blue cycle in B' is a C_{n-2}. In particular, there is no blue G in B'. Finally, color all of the edges in the complete subgraphs $K_{|\mathcal{Q}_1|}$ and $K_{|\mathcal{Q}_2|}$, induced by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively, red. Since $K_{|\mathcal{Q}_1|}$ and $K_{|\mathcal{Q}_2|}$ each contain less vertices than G, those complete subgraphs do not contain a red G. Thus, $R(G, G) > \frac{3n}{2} - 1$.

Hence, $\frac{3n}{2} - 1 < R(G, G) \leq \frac{3n}{2}$ and we conclude that $R(G, G) = \frac{3n}{2}$.

Case 2. $n \geq 5$ and odd.

This case follows from Theorem D. □

Lemma 2. *Let $k \geq 1$ and $C_4^k = C_4 *_1 kP_2$. Then,*

$$R(C_4^k, C_4^k) > \begin{cases} 2(2 + k) - 1 & \text{if } k \text{ is odd,} \\ 2(2 + k) - 2 & \text{if } k \text{ is even.} \end{cases}$$

Proof. The maximum degree of C_4^k is $\Delta(C_4^k) = k + 2$.

Case 1. $k \geq 1$ and odd.

We want to show that $R(C_4^k, C_4^k) > 2k+3$. In particular, we construct a 2-coloring $\mathcal{C}$ of K_{2k+3}, which does not contain a monochromatic C_4^k. It is known [5] that for every $s \geq 1$, K_{2s+1} can be factored into s Hamiltonian cycles. Superimposing $\frac{k+1}{2}$ Hamiltonian cycles yield a $(k+1)$-regular subgraph G on the $2k+3$ vertices of K_{2k+3}. Let $E(G)$ be the red edges of $\mathcal{C}$. Note that the complement of G, namely $\overline{G}$, is also a $(k+1)$-regular subgraph on the $2k+3$ vertices of K_{2k+3}. Let $E(\overline{G})$ be the blue edges of $\mathcal{C}$. Since $\Delta(G) = \Delta(\overline{G}) = k+1$, no vertex of $\mathcal{C}$ has degree $k+2$ with respect to red or blue edges. Hence, $\mathcal{C}$ does not contain a monochromatic C_4^k.

Case 2. $k \geq 2$ and even.

We want to show that $R(C_4^k, C_4^k) > 2k+2$. In particular, we construct a 2-coloring $\mathcal{C}$ of K_{2k+2}, which does not contain a monochromatic C_4^k. It is known [5] that for every $s \geq 1$, K_{2s} is the edge sum of $s-1$ Hamiltonian cycles and a 1-factor. Superimposing $\frac{k}{2}$ Hamiltonian cycles and a 1-factor yield a $(k+1)$-regular subgraph G on the $2k+2$ vertices of K_{2k+2}. Let $E(G)$ be the red edges of $\mathcal{C}$. Note that the complement of G, namely $\overline{G}$, is a k-regular subgraph on the $2k+2$ vertices of K_{2k+2}. Let $E(\overline{G})$ be the blue edges of $\mathcal{C}$. Since $\Delta(G)$ and $\Delta(\overline{G})$ are less than $k+2$, no vertex of $\mathcal{C}$ has degree $k+2$ with respect to red or blue edges. Hence, $\mathcal{C}$ does not contain a monochromatic C_4^k. □

Notation. We now introduce some notation which will be used in the proofs of Lemmas 3 and 4. Suppose that we have a k-coloring of a simple graph G. Let vertex $v \in V(G)$, $\mathcal{S} \subseteq V(G)$, and $\mathfrak{c}$ be some fixed color. Then, $\deg_{[\mathcal{S},\mathfrak{c}]}(v)$ denotes the number of vertices in $\mathcal{S}$ that are adjacent to v by a $\mathfrak{c}$-colored edge (i.e. it is the $\mathfrak{c}$-degree of v with respect to the set $\mathcal{S}$).

Lemma 3. *Let $k \geq 1$ be odd and $C_4^k = C_4 *_1 kP_2$. Then,*

$$R(C_4^k, C_4^k) \leq 2(2+k).$$

Proof. We induct on k to establish the claim.

Base case. In [19], we see that $R(C_4^k, C_4^k) = 2(2+k)$, for $k = 1$ and 3.

Inductive step. For some fixed odd $p \geq 3$, assume the claim holds and $R(C_4^p, C_4^p) \leq 2(2+p)$. We want to show that $R(C_4^{p+2}, C_4^{p+2}) \leq 2(2+(p+2)) = 2p+8$.

Let $\mathcal{C}$ be a 2-coloring of K_{2p+8}. By the induction hypothesis, there is a monochromatic (say, red) C_4^p in K_{2p+8}. Let $\{v_1, v_2, \ldots, v_{p+4}\}$ be the vertex set of this red C_4^p, where $C_4 = v_1v_2, v_2v_3, v_3v_4, v_4v_1$, $\deg(v_1) = \Delta(C_4^p)$, and $\alpha = \{v_5, v_6, \ldots, v_{p+4}\}$. Lastly, let $\beta = V(K_{2p+8}) - \{v_1, v_2, v_3, v_4\} - \alpha$. Note that $|\beta| = p+4$.

Now, consider the edges from v_1 to β, and from v_4 to $\alpha \cup \beta$. If v_1 has two additional incident red edges to β, then a red C_4^{p+2} exists. Thus, $\deg_{[\beta,R]}(v_1) \leq 1$ and $\deg_{[\beta,B]}(v_1) \geq p+3$. If v_4 has $p+2$ (or more) additional incident red edges to $\alpha \cup \beta$, then a red C_4^{p+2} exists. Thus, $\deg_{[\alpha\cup\beta,R]}(v_4) \leq p+1$ and $\deg_{[\alpha\cup\beta,B]}(v_4) \geq p+3$. By the Pigeonhole principle, at least two vertices $v_x, v_y \in \beta$ will be blue adjacent to both v_1 and v_4, which forms a blue C_4 with vertex set $\{v_1, v_4, v_x, v_y\}$. If $\deg_{[\beta,B]}(v_1) > p+3$, then a blue C_4^{p+2} exists. Thus, $\deg_{[\beta,B]}(v_1) \leq p+3$.

This implies that $\deg_{[\beta,B]}(v_1) = p+3$ and $\deg_{[\beta,R]}(v_1) = 1$. Moreover, the edge v_1v_3 is red (otherwise, a blue C_4^{p+2} exists). See Figure 3.

Let $\gamma = V(K_{2p+8}) - \{v_1, v_4, v_x, v_y\}$, and consider the edges from v_y to γ in $\mathcal{C}$. If v_y has more than $p+1$ incident blue edges, then a blue C_4^{p+2} exists and we are done. Thus, $\deg_{[\gamma,B]}(v_y) \leq p+1$ and $\deg_{[\gamma,R]}(v_y) \geq p+3$. Since there are only $p+2$ vertices in $\gamma \cap \beta$, there exists at least one vertex in $\eta = \alpha \cup \{v_2, v_3\}$ that is red adjacent to v_y.

- If $\deg_{[\eta,R]}(v_y) \geq 2$, let $v_w, v_z \in \eta$ be two vertices that are red adjacent to v_y. Then, $v_wv_y, v_yv_z, v_zv_1, v_1v_w$ is a red C_4 and v_1 has $p+3+1-2$ additional adjacent red edges to vertices ($\neq v_w, v_y, v_z$). Thus, a red C_4^{p+2} exists.
- If $\deg_{[\eta,R]}(v_y) = 1$, then $\deg_{[\gamma\cap\beta,R]}(v_y) = p+2 = |\gamma \cap \beta|$. Let $v_w \in \eta$ be red adjacent to v_y. Furthermore, there exists a vertex $v_z \in \gamma \cap \beta$ that is red adjacent to both v_y and v_1. Then, $\{v_1, v_z, v_y, v_w\}$ is a vertex set of a red C_4 and v_1 has $p+3+1-2$ additional adjacent red edges to vertices ($\neq v_w, v_y, v_z$). Thus, a red C_4^{p+2} exists.

Therefore, $R(C_4^{p+2}, C_4^{p+2}) \leq 2(2+(p+2)) = 2p+8$. □

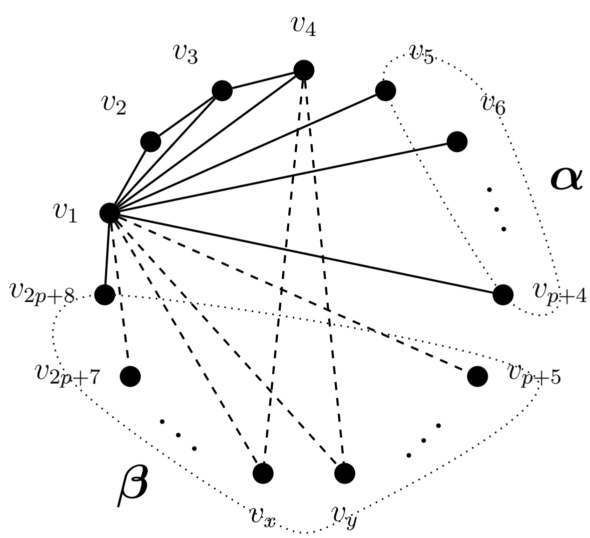

Figure 3: Partial 2-coloring of K_{2p+8} in the proof of Lemma 3. The solid edges are red and the dashed edges are blue.

Lemma 4. *Let $k \geq 2$ be even and $C_4^k = C_4 *_1 kP_2$. Then,*

$$R(C_4^k, C_4^k) \leq 2(2+k) - 1.$$

Proof. In [19], we see that $R(C_4^2, C_4^2) = 7$. So, the lemma holds for $k = 2$. Now, let $k \geq 4$ be even and $\mathcal{C}$ be a 2-coloring of K_{2k+3}, where $V(K_{2k+3}) = \{v_1, v_2, \ldots, v_{2k+3}\}$. Harary [13] proved the following fact:

- $R(K_{1,n}, K_{1,m}) = n + m - \varepsilon$, where $\varepsilon = 1$ for even n and m, and $\varepsilon = 0$ otherwise.

This fact implies that $\mathcal{C}$ contains a monochromatic (say, red) $S = K_{1,k+2}$. Without loss of generality, let $V(S) = \{v_1, v_2, \ldots, v_{k+3}\}$ and $\deg(v_1) = k+2$. Now, let $\alpha = \{v_2, v_3, \ldots, v_{k+3}\}$ and $\beta = \{v_{k+4}, v_{k+5}, \ldots, v_{2k+3}\}$.

If there exists a vertex $v \in \beta$ with $\deg_{[\alpha,R]}(v) \geq 2$, then a red C_4^k exists. If there are two vertices $v, w \in \beta$ with $\deg_{[\alpha,R]}(v) = \deg_{[\alpha,R]}(w) = 0$, then a blue C_4^k exists. If there are two vertices $v, w \in \beta$ with $\deg_{[\alpha,R]}(v) = 1$ and $\deg_{[\alpha,R]}(w) = 0$, then a blue C_4^k exists. Thus for all $v_i \in \beta$, $\deg_{[\alpha,R]}(v_i) = 1$. This implies that for all $v_i \in \beta$, $\deg_{[\alpha,B]}(v_i) = k+1$.

Note that there are many copies of blue C_4^{k-1} with vertex sets in $\gamma = \alpha \cup \beta$. If there exists a vertex $v \in \beta$ where the edge $v_1 v$ is blue, then a blue C_4^k exists. Thus for all $v_i \in \beta$, the edge $v_1 v_i$ is red. Hence, $\mathcal{C}$ contains a red $K_{1,2k+2}$ with central vertex v_1 (when viewed as a star).

If there exists a vertex $v \in \gamma$ with $\deg_{[\gamma-\{v\},R]}(v) \geq 2$, then a red C_4^k exists. Thus for all $v_i \in \gamma$, $\deg_{[\gamma-\{v_i\},R]}(v_i) \leq 1$. This implies that for all $v_i \in \gamma$, $\deg_{[\gamma-\{v_i\},B]}(v_i) \geq 2k$.

Because of the following established assertions:

- For all $v_i \in \beta$, $\deg_{[\alpha,B]}(v_i) = k+1$;
- There are many copies of blue C_4^{k-1} with vertex sets in $\gamma = \alpha \cup \beta$;
- For all $v_i \in \gamma$, $\deg_{[\gamma-\{v_i\},B]}(v_i) \geq 2k$,

$\mathcal{C}$ must contain a blue C_4^k in K_{3+2k}. Therefore, $R(C_4^k, C_4^k) \leq 2(2+k) - 1$. □

Theorem 4. *Let $k \geq 1$ and $C_4^k = C_4 *_1 kP_2$. Then,*

$$R(C_4^k, C_4^k) = \begin{cases} 2k+4 & \text{if } k \text{ is odd,} \\ 2k+3 & \text{if } k \text{ is even.} \end{cases}$$

Proof. This follows immediately from Lemmas 2, 3 and 4. □

We conclude with an open problem. Table 1 is a summary of some known diagonal graph Ramsey numbers in [19], which provides evidence for the truthfulness of Conjecture 2.

Conjecture 2. *Let G be a simple connected unicyclic graph of girth 4, where $|V(G)| = m \geq 6$ and $G \neq H = C_4 *_1 (m-4)P_2$. Then,*

$$R(G, G) = \begin{cases} R(H, H) - 1 & \text{if } m \text{ is odd,} \\ R(H, H) + 1 & \text{if } m \text{ is even.} \end{cases}$$

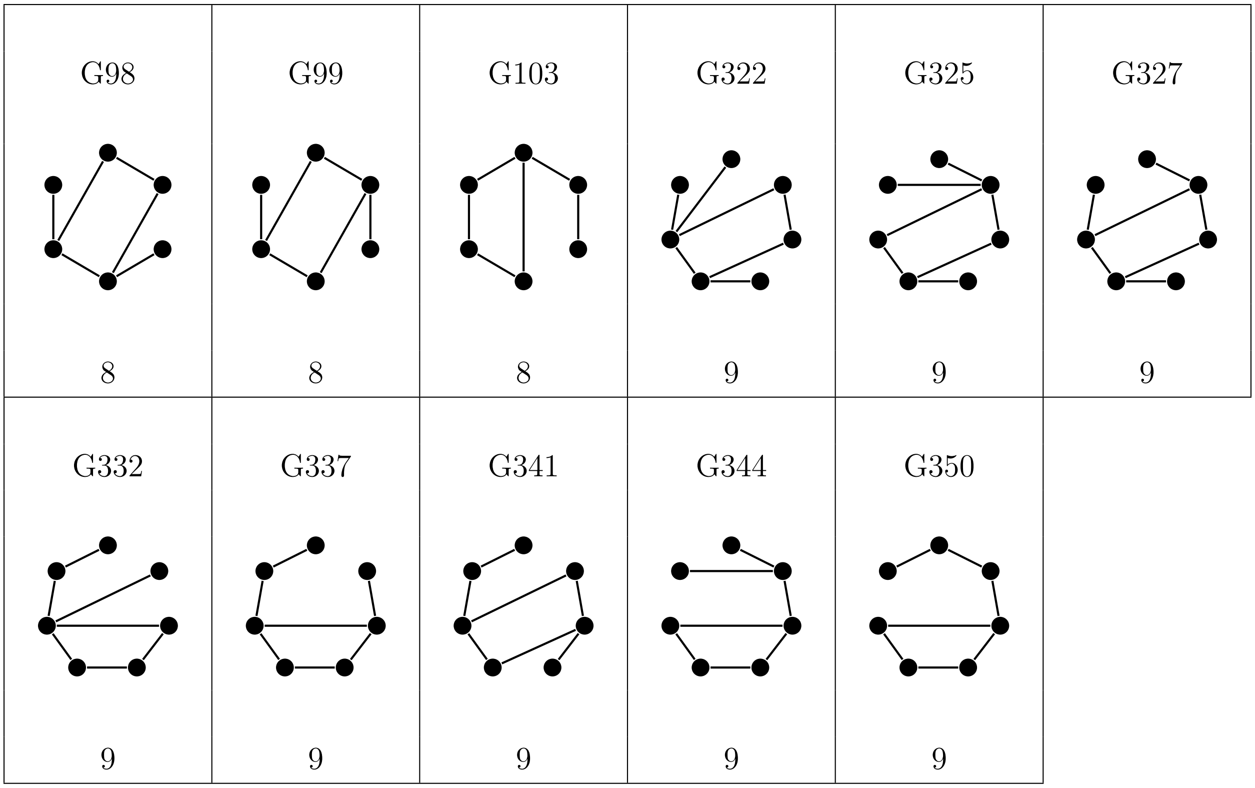

Table 1: $R(C_4^2, C_4^2) = 7$, $R(C_4^3, C_4^3) = 10$. Some known diagonal graph Ramsey numbers in support of Conjecture 2. The "Gxxx" labels in this table correspond to classification numbers used in [19].

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