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New Families of Weighted Sum Formulas for Multiple Zeta Values

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Abstract. In this paper we use the generating functions and the double shuffle relations satisfied by the multiple zeta values to derive some new families of identities.

1 Introduction

In recent years there is a flux of research on the multiple zeta functions and their special values due to their deep connections with many branches of mathematics and physics. For any positive integer d (called the *depth*) and $s_1, \dots, s_d$ with $s_1 > 1$ the multiple zeta values (MZVs) are defined by

$$\zeta(s_1, \dots, s_d) = \sum_{k_1 > \dots > k_d > 0} \frac{1}{k_1^{s_1} \cdots k_d^{s_d}}.$$

These values are easily seen to satisfy the so called stuffle relation. For example,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2). \tag{1}$$

Euler [2] first studied the depth two case and obtained the following decomposition formula by using partial fraction techniques.

$$\zeta(s_1)\zeta(s_2) = \sum_{\substack{t_1 \geq 2, t_2 \geq 1 \\ t_1 + t_2 = s_1 + s_2}} \left[\binom{t_1 - 1}{s_1 - 1} + \binom{t_2 - 1}{s_2 - 1} \right] \zeta(t_1, t_2), \quad s_1, s_2 \geq 2. \tag{2}$$

Although he did not consider divergence problem his approach has been made rigorous using modern techniques of regularization. Similarly to (2) one can show that for all $s_1, s_3 \geq 2$ and $s_2 \geq 1$,

$$\begin{aligned} \zeta(s_1, s_2)\, \zeta(s_3) = & \sum_{\substack{t_1 \geq 2, t_2 \geq 1 \\ t_1 + t_2 = s_1 + s_3}} \binom{t_1 - 1}{s_3 - 1} \zeta(t_1, t_2, s_2) \\ & + \sum_{\substack{t_1 \geq 2, t_2, t_3 \geq 1 \\ t_1 + t_2 + t_3 \\ = s_1 + s_2 + s_3}} \binom{t_1 - 1}{s_1 - 1} \left[\binom{t_2 - 1}{s_2 - t_3} + \binom{t_2 - 1}{s_2 - 1} \right] \zeta(t_1, t_2, t_3). \end{aligned} \tag{3}$$

In fact, nowadays this can be derived easily by the shuffle relations satisfied by the iterated integral expression of MZVs (see [13, p. 510]). By combining the stuffle and the shuffle relations one can obtain the so called double shuffle relations (see [9] for details).

Let d be any positive integer and define the generating function

$$G_d(x_1,\dots,x_d)=\sum_{s_1,\dots,s_d\in\mathbb{N},s_1>1} x_1^{s_1-1}\cdots x_d^{s_d-1}\zeta(s_1,\dots,s_d). \tag{4}$$

It is well-known that

$$G_1(x)=-\gamma-\psi(1-x)$$

where γ is Euler's constant and $\psi(x)$ is the digamma function, i.e., the logarithmic derivative of the gamma function. In [3], Gangl, Kaneko and Zagier used the double shuffle relations of (1) and (2) to derive the following equation:

$$G_2(x+y,x)+G_2(x+y,y)-G_2(x,y)-G_2(y,x)=\frac{G_1(x)-G_1(y)}{x-y}, \tag{5}$$

and proved some families of MZV identities. Machide [11] generalized this to depth three case using the extended (also called regularized) double shuffle relations.

It is well-known that in order to get complete linear relations between MZVs one should consider regularized double shuffle relations. For example, the weighted sum formula of Ohno and Zudilin [12] states that

$$\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n}} 2^j\zeta(j,k)=(n+1)\zeta(n). \tag{6}$$

Later, Guo and Xie [6] generalized (6) to arbitrary depths using regularized double shuffle relations (they in fact also used the sum formula which is another consequence of the regularized double shuffle relations [9]).

In this paper we shall use the generating functions of MZVs (4) to reformulate double shuffle relations and derive some new identities of MZVs. Notice that we do not use the extended double shuffle relations, which makes the computation a little easier. All the identities obtained this way are therefore finite extended double shuffle relations in the sense of [9]. For example, we get the following interesting result as a corollary (see Corollary 3.3) in depth three:

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} 2^{j-1}\zeta(j,k,l)+\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} 2^j\zeta(j,k,1)$$
$$=n\zeta(n-1,1)+3\zeta(n-2,2)+\zeta(2,n-2)+2\zeta(n).$$

for every positive integer$n\ge 2$. For a new result in depth four please see Corollary 5.2.

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2 Depth 2: some new identities

In this section we will derive some new identities of double zeta values using the generating function G_2. To begin with, we recall the famous sum formula essentially known to Euler [2]:

$$\sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}} \zeta(j,k)=\zeta(n). \tag{7}$$

Using generating functions A. Granville [4] and D. Zagier proved the following generalization to arbitrary depth first conjectured by Moen (see [7]):

$$\sum_{k_1\geq 2,k_2,\dots,k_d\geq 1,k_1+\cdots+k_d=w} \zeta(k_1,k_2,\dots,k_d)=\zeta(w). \tag{8}$$

Our first result provides a weighted sum formula similar to but different from (6).

Theorem 2.1. *Let n be a positive integer. Then*

$$\sum_{k=2}^{n-1} k\zeta(k,n-k) =\zeta(2,n-2)+2\zeta(n)-(n-2)\zeta(n-1,1), \quad \forall n\geq 3, \tag{9}$$

$$\begin{aligned}\sum_{k=2}^{n-1} k^2\zeta(k,n-k) =&3\zeta(2,n-2)+2\zeta(3,n-3)+6\zeta(n)\\ &-(2n-6)\zeta(n-2,2)-n(n-2)\zeta(n-1,1), \quad \forall n\geq 4.\end{aligned} \tag{10}$$

Proof. Making the substitutions $x\to xt$ and $y\to yt$ in (5) and comparing the coefficients of t^{n-2} we get

$$\sum_{k=2}^{n-1}\Big[(x+y)^{k-1}x^{j-1}+(x+y)^{k-1}y^{j-1}-x^{k-1}y^{j-1}-y^{k-1}x^{j-1}\Big]\zeta(k,j)=\left(\frac{x^{n-1}-y^{n-1}}{x-y}\right)\zeta(n),$$

where $j=n-k$. Differentiating this equation with respect to x we have

$$\begin{aligned}\sum_{k=2}^{n-1}\Big[&(k-1)(x+y)^{k-2}x^{j-1}+(j-1)(x+y)^{k-1}x^{j-2}+(k-1)(x+y)^{k-2}y^{j-1}\\ &-(k-1)x^{k-2}y^{j-1}-(j-1)y^{k-1}x^{j-2}\Big]\zeta(k,j)=\left(\frac{(n-1)x^{n-2}}{x-y}-\frac{x^{n-1}-y^{n-1}}{(x-y)^2}\right)\zeta(n).\end{aligned} \tag{11}$$

Specializing to $(x,y)=(0,1)$ we find easily that

$$(n-2)\zeta(n-1,1)+\sum_{k=2}^{n-1}(k-1)\zeta(k,n-k)-\zeta(2,n-2)=\zeta(n).$$

So (9) follows from the sum formula (7).

Now multiplying (11) by $x+y$, differentiating with respect to x, and then specializing to $(x,y)=(0,1)$ we get

$$(n-2)^2\zeta(n-1,1)+(2n-6)\zeta(n-2,2)+\sum_{k=2}^{n-1}(k-1)^2\zeta(k,n-k)$$
$$-\zeta(2,n-2)-2\zeta(3,n-3)=3\zeta(n).$$

Hence (10) quickly follows from (9) and the sum formula (7). This completes the proof of the theorem. □

Remark 2.2. It is conceivable that for every fixed positive integer d a compact formula of $\sum_{k=2}^{n-1}k^d\zeta(k,n-k)$ can be obtained by differentiating (11) repeatedly, similar to what we have done in Theorem 2.1. However, it seems to be a difficult problem to find a general formula for all d. Guo, Lei and the second author recently have made some progress along this direction, see [5].

3 Depth 3: product of three Riemann zeta values

In the following we use three different methods to compute the generating function of the product of three Riemann zeta values:

$$\sum_{s_1,s_2,s_3\ge 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1)\zeta(s_2)\zeta(s_3).$$

3.1 First method.

Combining (2) and (3) we have

$$\begin{aligned}\zeta(s_1)\zeta(s_2)\zeta(s_3)=&\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_2-1}{s_2-1}\right]\cdot\Bigg\{\sum_{\substack{r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\binom{r_1-1}{s_3-1}\zeta(r_1,r_2,t_2)\\ &+\sum_{\substack{r_1\ge 2,r_2,r_3\ge 1\\ r_1+r_2+r_3\\ =t_1+t_2+s_3}}\binom{r_1-1}{t_1-1}\left[\binom{r_2-1}{t_2-r_3}+\binom{r_2-1}{t_2-1}\right]\zeta(r_1,r_2,r_3)\Bigg\}\\ =&\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\binom{t_1-1}{s_1-1}\binom{r_1-1}{s_3-1}\zeta(r_1,r_2,t_2)\end{aligned}\tag{12}$$

$$+\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\binom{t_2-1}{s_2-1}\binom{r_1-1}{s_3-1}\zeta(r_1,r_2,t_2) \tag{13}$$

$$+\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2,r_3\ge 1\\ r_1+r_2+r_3\\ =t_1+t_2+s_3}}\binom{t_1-1}{s_1-1}\binom{r_1-1}{t_1-1}\binom{r_2-1}{t_2-r_3}\zeta(r_1,r_2,r_3) \tag{14}$$

$$+\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2,r_3\ge 1\\ r_1+r_2+r_3\\ =t_1+t_2+s_3}}\binom{t_2-1}{s_2-1}\binom{r_1-1}{t_1-1}\binom{r_2-1}{t_2-r_3}\zeta(r_1,r_2,r_3) \tag{15}$$

$$+\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2,r_3\ge 1\\ r_1+r_2+r_3\\ =t_1+t_2+s_3}}\binom{t_1-1}{s_1-1}\binom{r_1-1}{t_1-1}\binom{r_2-1}{t_2-1}\zeta(r_1,r_2,r_3) \tag{16}$$

$$+\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2,r_3\ge 1\\ r_1+r_2+r_3\\ =t_1+t_2+s_3}}\binom{t_2-1}{s_2-1}\binom{r_1-1}{t_1-1}\binom{r_2-1}{t_2-1}\zeta(r_1,r_2,r_3) \tag{17}$$

We first treat (12) to (17) using the binomial identities repeatedly to derive formulas involving the generating functions G_3. To save space we only compute (12) in details and leave the others to the interested reader. Also we will use the shorthand $x_{ij}=x_j+x_j$ and $x_{ijk}=x_j+x_j+x_k$ in what follows. We also use the shorthand $\sum(12)$ to stand for $\sum_{s_1,s_2,s_3\ge 1}x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}$ (12). Now

$$\begin{aligned}
\sum(12) &= \sum_{s_1,s_2,s_3\ge 2}\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\binom{t_1-1}{s_1-1}\binom{r_1-1}{s_3-1}x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(r_1,r_2,t_2)\\
&= \sum_{s_1,s_2,s_3\ge 2}\sum_{\substack{t_1\ge 2,t_2\ge 1\\ t_1+t_2=s_1+s_2}}\sum_{\substack{r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\binom{t_1-1}{s_1-1}\binom{r_1-1}{s_3-1}x_1^{s_1-1}x_2^{t_1-s_1}x_2^{t_2-1}x_3^{s_3-1}\zeta(r_1,r_2,t_2)\\
&= \sum_{\substack{t_1,s_3,r_1\ge 2,t_2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\Big[(x_{12})^{t_1-1}-x_2^{t_1-1}\Big]x_3^{s_3-1}\binom{r_1-1}{s_3-1}x_2^{t_2-1}\zeta(r_1,r_2,t_2)\\
&\qquad -\sum_{\substack{t_1,s_3,r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}x_1^{t_1-1}x_3^{s_3-1}\binom{r_1-1}{s_3-1}\zeta(r_1,r_2,1)\\
&= \sum_{\substack{t_1,s_3,r_1\ge 2,t_2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}\Big[(x_{12})^{r_2-1}(x_{12})^{r_1-s_3}-x_2^{r_2-1}x_2^{r_1-s_3}\Big]x_3^{s_3-1}\binom{r_1-1}{s_3-1}x_2^{t_2-1}\zeta(r_1,r_2,t_2)\\
&\qquad -\sum_{\substack{t_1,s_3,r_1\ge 2,r_2\ge 1\\ r_1+r_2=t_1+s_3}}x_1^{r_2-1}x_1^{r_1-s_3}x_3^{s_3-1}\binom{r_1-1}{s_3-1}\zeta(r_1,r_2,1)\\
&= \sum_{r_1\ge 1,t_2,r_2\ge 1}\Big\{(x_{12})^{r_2-1}\big[(x_{123})^{r_1}-(x_{12})^{r_1}\big]-x_2^{r_2-1}\big[(x_{23})^{r_1}-x_2^{r_1}\big]\Big\}x_2^{t_2-1}\zeta(r_1+1,r_2,t_2)
\end{aligned}$$

$$- \sum_{\substack{t_1,s_3,r_1\geq 2,r_2\geq 1\\ r_1+r_2=t_1+s_3}} x_1^{r_2-1}\left[(x_{13})^{r_1-1}-x_1^{r_1-1}\right]\zeta(r_1,r_2,1)+\sum_{r_1\geq 2} x_3^{r_1-1}\zeta(r_1,1,1)$$

$$\begin{aligned}=&G_3(x_{123},x_{12},x_2)-G_3(x_{12},x_{12},x_2)-G_3(x_{23},x_2,x_2)+G_3(x_2,x_2,x_2)\\ &\quad - \sum_{\substack{t_1,s_3,r_1\geq 2,r_2\geq 1\\ r_1+r_2=t_1+s_3}} x_1^{r_2-1}\left[(x_{13})^{r_1-1}-x_1^{r_1-1}\right]\zeta(r_1,r_2,1)+\sum_{r_1\geq 2} x_3^{r_1-1}\zeta(r_1,1,1)\end{aligned}$$

Similarly we find

$$\begin{aligned}\sum(14)=&G_3(x_{123},x_{23},x_2)-G_3(x_{23},x_{23},x_2)-G_3(x_{12},x_2,x_2)+G_3(x_2,x_2,x_2)\\ &\quad - \sum_{r_1\geq 2,r_2\geq 1}\left[(x_{13})^{r_1-1}-x_3^{r_1-1}\right]x_3^{r_2-1}\zeta(r_1,r_2,1)+\sum_{r_1\geq 2} x_1^{r_1-1}\zeta(r_1,1,1)\end{aligned}$$

which is just $\sum(12)$ under the operation $x_1 \leftrightarrow x_3$. Further,

$$\begin{aligned}\sum(16)=&G_3(x_{123},x_{23},x_3)-G_3(x_{23},x_{23},x_2)-G_3(x_{13},x_3,x_3)+G_3(x_3,x_3,x_3)\\ &\quad - \sum_{r_1\geq 2,r_2\geq 1}\left[(x_{12})^{r_1-1}-x_2^{r_1-1}\right]x_2^{r_2-1}\zeta(r_1,r_2,1)+\sum_{r_1\geq 2} x_1^{r_1-1}\zeta(r_1,1,1)\end{aligned}$$

which is $\sum(12)$ under the operation $\mathrm{Cyc}(x_1 \to x_2 \to x_3 \to x_1)$. By the same argument we easily find that $\sum(13)$, $\sum(15)$ and $\sum(17)$ can all be obtained from $\sum(12)$ under different permutations of x_1, x_2 and x_3. Therefore

$$\begin{aligned}&\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1)\zeta(s_2)\zeta(s_3)\\ &=\bigoplus_{\mathcal{S}(x_1,x_2,x_3)}\Big\{G_3(x_{123},x_{12},x_2)-G_3(x_{12},x_{12},x_2)-G_3(x_{23},x_2,x_2)+G_3(x_2,x_2,x_2)\\ &\qquad - \sum_{\substack{t_1,s_3,r_1\geq 2,r_2\geq 1\\ r_1+r_2=t_1+s_3}} x_1^{r_2-1}\left[(x_{13})^{r_1-1}-x_1^{r_1-1}\right]\zeta(r_1,r_2,1)+\sum_{r_1\geq 2} x_3^{r_1-1}\zeta(r_1,1,1)\Big\}\end{aligned}$$

where, for a function $f(x_1,\ldots,x_k)$ we define

$$\bigoplus_{\mathcal{S}(x_1,\ldots,x_k)} f(x_1,\ldots,x_k)=\sum_{\sigma\text{: permutations of }1,\ldots,k} f(x_{\sigma(1)},\ldots,x_{\sigma(k)}).$$

3.2 Second method.

Multiplying (2) by $\zeta(s_3)$ and using the stuffle relations we get

$$\begin{aligned}&\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1)\zeta(s_2)\zeta(s_3)\\ &=\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_1-1}{s_2-1}\right]\zeta(t_1,t_2,s_3) \qquad (18)\end{aligned}$$

$$+\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_1-1}{s_2-1}\right]\zeta(t_1,s_3,t_2) \tag{19}$$

$$+\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_1-1}{s_2-1}\right]\zeta(s_3,t_1,t_2) \tag{20}$$

$$+\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_1-1}{s_2-1}\right]\zeta(t_1,t_2+s_3) \tag{21}$$

$$+\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_2}}\left[\binom{t_1-1}{s_1-1}+\binom{t_1-1}{s_2-1}\right]\zeta(t_1+s_3,t_2). \tag{22}$$

Using the techniques similar to the one used in the proceeding subsection we get

$$\begin{aligned}
(18) = \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ & G_3(x_{12},x_2,x_3) - \sum_{t_1\geq 2,s_3\geq 1} x_1^{t_1-1}x_3^{s_3-1}\zeta(t_1,1,s_3) + \sum_{t_1\geq 2} x_1^{t_1-1}\zeta(t_1,1,1) \\
& - \sum_{t_1\geq 2,t_2\geq 1} (x_{12})^{t_1-1}x_2^{t_2-1}\zeta(t_1,t_2,1) - G_3(x_2,x_2,x_3) + \sum_{t_1\geq 2,t_2\geq 1} x_2^{t_1+t_2-2}\zeta(t_1,t_2,1)\Big\} \\
(19) = \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ & G_3(x_{12},x_3,x_2) - \sum_{t_1\geq 2,s_3\geq 1} x_1^{t_1-1}x_3^{s_3-1}\zeta(t_1,s_3,1) + \sum_{t_1\geq 2} x_1^{t_1-1}\zeta(t_1,1,1) \\
& - \sum_{t_1\geq 2,t_2\geq 1} (x_{12})^{t_1-1}x_2^{t_2-1}\zeta(t_1,1,t_2) - G_3(x_2,x_3,x_2) + \sum_{t_1\geq 2,t_2\geq 1} x_2^{t_1+t_2-2}\zeta(t_1,1,t_2)\Big\} \\
(20) = \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ & G_3(x_3,x_{12},x_2) - G_3(x_3,x_2,x_2) \\
& - \sum_{s_3\geq 2,t_1\geq 1} x_1^{t_1-1}x_3^{s_3-1}\zeta(s_3,t_1,1) + \sum_{s_3\geq 2} x_3^{s_3-1}\zeta(s_3,1,1)\Big\} \\
(21) = \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ & \frac{G_2(x_{12},x_3)-G_2(x_{12},x_2)}{x_3-x_2} - \frac{1}{x_2}G_2(x_{12},x_2) \\
& - \frac{G_2(x_2,x_3)-G_2(x_2,x_2)}{x_3-x_2} + \frac{1}{x_2}G_2(x_2,x_2) - \frac{1}{x_3}G_2(x_1,x_3) \\
& + \frac{1}{x_2}\sum_{t_1\geq 2}\Big((x_{12})^{t_1-1}-x_2^{t_1-1}\Big)\zeta(t_1,1) + \frac{1}{x_3}\sum_{t_1\geq 1} x_1^{t_1-1}\Big(\zeta(t_1,1)+x_3\zeta(t_1,2)\Big)\Big\} \\
(22) = \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ & \frac{G_2(x_3,x_2)-G_2(x_{12},x_2)}{x_3-x_{12}} - \frac{G_2(x_3,x_2)-G_2(x_2,x_2)}{x_3-x_2} + \frac{1}{x_2}G_2(x_2,x_2) \\
& - \frac{1}{x_{12}}G_2(x_{12},x_2) - \sum_{t\geq 2}\left(\frac{x_3^{t-1}-x_1^{t-1}}{x_3-x_1} - x_3^{t-2}-x_1^{t-2}\right)\zeta(t,1) - \zeta(2,1)\Big\}
\end{aligned}$$

Therefore

$$\sum_{s_1,s_2,s_3\geq 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1)\zeta(s_2)\zeta(s_3)$$

$$
\begin{aligned}
&= \bigoplus_{\mathcal{S}(x_1,x_2)} \Big\{ G_3(x_{12},x_2,x_3) - \sum_{t_1\ge 2,t_2\ge 1} x_1^{t_1-1}x_3^{t_2-1}\zeta(t_1,1,t_2) + 2\sum_{t_1\ge 2} x_1^{t_1-1}\zeta(t_1,1,1) \\
&- \sum_{t_1\ge 2,t_2\ge 1} (x_{12})^{t_1-1}x_2^{t_2-1}\zeta(t_1,t_2,1) - \sum_{t_1\ge 2,t_2\ge 1} x_3^{t_1-1}x_1^{t_2-1}\zeta(t_1,t_2,1) + \sum_{t_1\ge 2,t_2\ge 1} x_2^{t_1+t_2-2}\zeta(t_1,t_2,1) \\
&+ \sum_{t_1\ge 2,t_2\ge 1} x_2^{t_1+t_2-2}\zeta(t_1,1,t_2) - \sum_{t_1\ge 2,t_2\ge 1} x_1^{t_1-1}x_3^{t_2-1}\zeta(t_1,t_2,1) - \sum_{t_1\ge 2,t_2\ge 1} (x_{12})^{t_1-1}x_2^{t_2-1}\zeta(t_1,1,t_2) \\
&+ G_3(x_{12},x_3,x_2) - G_3(x_2,x_3,x_2) + G_3(x_3,x_{12},x_2) - G_3(x_3,x_2,x_2) - G_3(x_2,x_2,x_3) \\
&+ \frac{G_2(x_{12},x_3)-G_2(x_{12},x_2)}{x_3-x_2} - \frac{1}{x_2}G_2(x_{12},x_2) - \frac{G_2(x_2,x_3)-G_2(x_2,x_2)}{x_3-x_2} \\
&+ \frac{2}{x_2}G_2(x_2,x_2) - \frac{1}{x_3}G_2(x_1,x_3) + \frac{G_2(x_3,x_2)-G_2(x_{12},x_2)}{x_3-x_2-x_1} - \frac{1}{x_{12}}G_2(x_{12},x_2) \\
&+ \frac{1}{x_2}\sum_{t_1\ge 2}(x_{12})^{t_1-1}\zeta(t_1,1) + \frac{1}{x_3}\sum_{t_1\ge 2} x_1^{t_1-1}\Big(\zeta(t_1,1)+x_3\zeta(t_1,2)\Big) + \sum_{t_1\ge 2} x_3^{t_1-1}\zeta(t_1,1,1) \\
&- \frac{G_2(x_3,x_2)-G_2(x_2,x_2)}{x_3-x_2} - \sum_{t\ge 2}\left(\frac{x_3^{t-1}-x_1^{t-1}}{x_3-x_1} - x_3^{t-2}\right)\zeta(t,1) - \zeta(2,1)\Big\}
\end{aligned}
$$

3.3 Third method.

Repeated use of stuffle relations yields

$$
\begin{aligned}
\zeta(s_1)\zeta(s_2)\zeta(s_3) &= \zeta(s_1,s_2,s_3) + \zeta(s_1,s_3,s_2) + \zeta(s_3,s_1,s_2) + \zeta(s_1,s_2+s_3) \\
&\quad + \zeta(s_1+s_3,s_2) + \zeta(s_2,s_1,s_3) + \zeta(s_2,s_3,s_1) + \zeta(s_3,s_2,s_1) \\
&+ \zeta(s_2,s_1+s_3) + \zeta(s_2+s_3,s_1) + \zeta(s_1+s_2,s_3) + \zeta(s_3,s_1+s_2) + \zeta(s_1+s_2+s_3). \quad (23)
\end{aligned}
$$

On the right hand side of the above there are essentially four types of MZVs. Similar computation as above leads to the following four expressions of their generating functions:

$$
\begin{aligned}
&\sum_{s_1,s_2,s_3\ge 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1,s_2,s_3) = G_3(x_1,x_2,x_3) - \sum_{s_1\ge 2,s_2\ge 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,s_2,1) \\
&- \sum_{s_1\ge 2,s_3\ge 1} x_1^{s_1-1}x_3^{s_3-1}\zeta(s_1,1,s_3) + \sum_{s_1\ge 2} x_1^{s_1-1}\zeta(s_1,1,1) \\
&\sum_{s_1,s_2,s_3\ge 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1,s_2+s_3) = \frac{G_2(x_1,x_3)-G_2(x_1,x_2)}{x_3-x_2} - \frac{G_2(x_1,x_2)}{x_2} \\
&- \frac{G_2(x_1,x_3)}{x_3} + \sum_{s_1\ge 2} x_1^{s_1-1}\zeta(s_1,2) + \left(\frac{1}{x_3}+\frac{1}{x_2}\right)\sum_{s_1\ge 2} x_1^{s_1-1}\zeta(s_1,1) \\
&\sum_{s_1,s_2,s_3\ge 2} x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_2+s_3,s_1) = \frac{G_2(x_3,x_1)-G_2(x_2,x_1)}{x_3-x_2} - \frac{1}{x_2}G_2(x_2,x_1)
\end{aligned}
$$

$$-\frac{1}{x_3}G_2(x_3,x_1)+\sum_{s_1\geq 1}x_1^{s_1-1}\zeta(2,s_1)-\sum_{s\geq 2}\left(\frac{x_3^{s-1}-x_2^{s-1}}{x_3-x_2}-x_3^{s-2}-x_2^{s-2}+\delta_{s,2}\right)\zeta(s,1)$$

$$\sum_{s_1,s_2,s_3\geq 2}x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1+s_2+s_3)=\frac{1}{x_3}\left(\frac{G_1(x_3)}{x_3}+\frac{G_1(x_1)}{x_1}-\frac{G_1(x_3)-G_1(x_1)}{x_3-x_1}\right)$$
$$+\frac{1}{x_3-x_2}\left(\frac{G_1(x_3)-G_1(x_1)}{x_3-x_1}-\frac{G_1(x_3)}{x_3}-\frac{G_1(x_2)-G_1(x_1)}{x_2-x_1}+\frac{G_1(x_2)}{x_2}\right)--\frac{\zeta(2)}{x_3}$$
$$-\frac{1}{x_2}\left(\frac{G_1(x_2)-G_1(x_1)}{x_2-x_1}-\frac{G_1(x_2)}{x_2}-\frac{G_1(x_1)}{x_1}+\zeta(2)\right)+\frac{G_1(x_1)}{x_1^2}-\zeta(3)-\frac{\zeta(2)}{x_1}$$

Therefore (23) becomes

$$\sum_{s_1,s_2,s_3\geq 2}x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}\zeta(s_1)\zeta(s_2)\zeta(s_3)=\bigoplus_{\mathcal{S}(x_1,x_2,x_3)}\Big\{G_3(x_1,x_2,x_3)-\frac{G_2(x_1,x_2)}{x_2}$$
$$-\frac{G_2(x_2,x_1)}{x_2}+\frac{G_2(x_1,x_3)+G_2(x_3,x_1)}{x_3-x_2}-\sum_{s_1\geq 2,s_2\geq 1}x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,s_2,1)+\sum_{s\geq 2}x_1^{s-2}\zeta(s,1)$$
$$+\sum_{s\geq 2}x_1^{s-1}\zeta(s,1,1)-\sum_{s_1\geq 2,s_3\geq 1}x_1^{s_1-1}x_3^{s_3-1}\zeta(s_1,1,s_3)+\frac{1}{x_2}\sum_{s\geq 2}x_1^{s-1}\zeta(s,1)\Big\}-\zeta(3)$$
$$+\bigoplus_{\mathcal{C}(x_1,x_2,x_3)}\Big\{\sum_{s\geq 2}x_1^{s-1}\zeta(s,2)+\sum_{s\geq 2}x_1^{s-1}\zeta(2,s)-\sum_{s\geq 2}\left(\frac{x_3^{s-1}-x_2^{s-1}}{x_3-x_2}\right)\zeta(s,1)+\frac{G_1(x_1)}{x_1^2}-\frac{\zeta(2)}{x_1}\Big\}$$
$$+\bigoplus_{\mathcal{C}(x_2,x_3)}\Big\{\frac{1}{x_3-x_2}\left(\frac{G_1(x_3)-G_1(x_1)}{x_3-x_1}+\frac{G_1(x_2)}{x_2}\right)-\frac{1}{x_2}\left(\frac{G_1(x_2)-G_1(x_1)}{x_2-x_1}-\frac{G_1(x_1)}{x_1}\right)\Big\}$$

Here, for a function $f(x_1,\dots,x_k)$ we define

$$\bigoplus_{\mathcal{C}(x_1,\dots,x_k)}f(x_1,\dots,x_k)=\sum_{i=1}^{k}f(x_i,x_{i+1},\dots,x_{i+k-1})$$

where the subscript is taken modulo k.

By comparing the first and the third method we get

Theorem 3.1. *We have*

$$\bigoplus_{\mathcal{S}(x_1,x_2,x_3)}\Big\{G_3(x_{123},x_{12},x_2)-G_3(x_{12},x_{12},x_2)-G_3(x_{23},x_2,x_2)$$
$$+G_3(x_2,x_2,x_2)-G_3(x_1,x_2,x_3)-\frac{G_2(x_1,x_3)+G_2(x_3,x_1)}{x_3-x_2}+\frac{G_2(x_1,x_2)}{x_2}+\frac{G_2(x_2,x_1)}{x_2}\Big\}$$
$$-\bigoplus_{\mathcal{C}(x_2,x_3)}\Big\{\frac{1}{x_3-x_2}\left(\frac{G_1(x_3)-G_1(x_1)}{x_3-x_1}+\frac{G_1(x_2)}{x_2}\right)-\frac{G_1(x_2)-G_1(x_1)}{x_2(x_2-x_1)}+\frac{G_1(x_1)}{x_1x_2}\Big\}$$
$$=\bigoplus_{\mathcal{S}(x_1,x_2,x_3)}\Big\{\sum_{r_1\geq 2,r_2\geq 1}x_1^{r_2-1}\big[(x_{13})^{r_1-1}-x_1^{r_1-1}-x_3^{r_1-1}\big]\zeta(r_1,r_2,1)+\sum_{s\geq 2}x_1^{s-2}\zeta(s,1)$$

$$- \sum_{s_1\geq 2, s_3\geq 1} x_1^{s_1-1}x_3^{s_3-1}\zeta(s_1,1,s_3) + \frac{1}{x_2}\sum_{s\geq 2} x_1^{s-1}\zeta(s,1) - \sum_{s\geq 2}\left(\frac{x_3^{s-1}}{x_3-x_2}\right)\zeta(s,1)\Big\}$$

$$+ \bigoplus_{\mathcal{C}(x_1,x_2,x_3)} \Big\{ \sum_{s\geq 2} x_1^{s-1}\zeta(s,2) + \sum_{s\geq 2} x_1^{s-1}\zeta(2,s) + \frac{G_1(x_1)}{x_1^2} - \frac{\zeta(2)}{x_1}\Big\} - \zeta(3)$$

Proof. Clear. $\square$

This theorem is equivalent to the following result which can be regarded as a parametric family of weighted sum formulas.

Theorem 3.2. *Let a, b and c be any real numbers and let $\sigma = a+b$. Then for any positive integer $n \geq 4$ we have*

$$\bigoplus_{\mathcal{S}(a,b,c)} \Big\{ \sum_{\substack{j\geq 2, k,l\geq 1\\ j+k+l=n}} \Big(ac(a+b+c)^{j-1}(a+b)^{k-1}b^l - ac(a+b)^{j+k-2}b^l - ac(b+c)^{j-1}b^{k+l-1}$$

$$+ acb^{j+k+l-2} - a^jb^kc^l\Big)\zeta(j,k,l) + \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}}\Big(ca^jb^{k-1} + ca^kb^{j-1} - \frac{b(a^jc^k+a^kc^j)}{c-b}\Big)\zeta(j,k)\Big\}$$

$$- \bigoplus_{\mathcal{C}(b,c)} \Big\{\frac{1}{c-b}\left(\frac{abc^n - bca^n}{c-a} + acb^{n-1}\right) - \frac{ac(b^{n-1}-a^{n-1})}{b-a} + ca^{n-1}\Big\}\zeta(n)$$

$$= \bigoplus_{\mathcal{S}(a,b,c)} \Big\{ \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n-1}} \Big(a^kbc\big[(a+c)^{j-1} - a^{j-1} - c^{j-1}\big]\zeta(j,k,1) - a^jbc^k\zeta(j,1,k)\Big)$$

$$+\Big(a^{n-1}c + a^{n-2}bc - \frac{abc^{n-1}}{c-b}\Big)\zeta(n-1,1) + \frac{1}{2}a^{n-2}bc\big(\zeta(n-2,2)+\zeta(2,n-2)\big) + \frac{1}{2}a^{n-2}bc\zeta(n)\Big\}.$$

Proof. In Theorem 3.1 we first set $x_1 = at$, $x_2 = bt$ and $x_3 = ct$. Then by comparing the coefficient of t^{n-3} we get Theorem 3.2. $\square$

The following weighted sum formula seems to be new.

Corollary 3.3. *For any positive integer $n \geq 2$ we have*

$$\sum_{\substack{j\geq 2,k,l\geq 1\\ j+k+l=n}} 2^{j-1}\zeta(j,k,l) + \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n-1}} 2^j\zeta(j,k,1)$$

$$= n\zeta(n-1,1) + 3\zeta(n-2,2) + \zeta(2,n-2) + 2\zeta(n).$$

Proof. Setting $a=1$ and letting $b\to 1$ and then $c\to 1$ in Theorem 3.2 we get

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} 6\Big(3^{j-1}2^{k-1}-2^{j+k-2}-2^{j-1}\Big)\zeta(j,k,l)+\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n}} 3(6-n)\zeta(j,k)-\frac{n^2-9n+20}{2}\zeta(n)$$
$$=\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} 6\Big((2^{j-1}-2)\zeta(j,k,1)-\zeta(j,1,k)\Big)+3(6-n)\zeta(n-1,1)+3\zeta(n-2,2)+3\zeta(2,n-2).$$

The following two special type sum formulas are special cases of [8, Thm. 2.3] and [7, Thm. 5.1], respectively:

$$\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} \zeta(j,1,k) =\zeta(n-1,1)+\zeta(2,n-2) \tag{24}$$
$$\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} \zeta(j,k,1) =\zeta(n-1,1)+\zeta(n-2,2). \tag{25}$$

Notice that (25) is a also special case of Eie's generalized sum formula in [1]. Combining with sum formula (7) we get

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} 6\Big(3^{j-1}2^{k-1}-2^{j+k-2}-2^{j-1}\Big)\zeta(j,k,l)-\frac{n^2-3n-16}{2}\zeta(n)$$
$$=3\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} 2^j\zeta(j,k,1)-3n\zeta(n-1,1)-9\zeta(n-2,2)-3\zeta(2,n-2).$$

Dividing by 3 throughout we get

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} \Big(3^{j-1}2^{k}-2^{j+k-1}-2^{j}\Big)\zeta(j,k,l)-\frac{n^2-3n-16}{6}\zeta(n)$$
$$=\sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} 2^j\zeta(j,k,1)-n\zeta(n-1,1)-3\zeta(n-2,2)-\zeta(2,n-2).$$

Hence the corollary follows from [11, Cor. 4.1]. ☐

By comparing the second and the third method we can get another identity involving the generating function G_3. However, it is quite long so here we just write down the following version concerning a parametric family of weighted sum formulas.

Theorem 3.4. *Let a, b and c be any real numbers and let $\sigma=a+b$. Then for any positive integer $n\ge 2$ we have*

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} \bigoplus_{\mathcal{C}(a,b)} \Big(a\sigma^{j-1}(b^kc^l+b^lc^k)+a\sigma^{k-1}b^lc^j \; - \bigoplus_{\mathcal{C}(j,k,l)} \big(a^{j+k-1}bc^l+a^jb^kc^l\big)\Big)\zeta(j,k,l)$$

$$- \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} \bigoplus_{\mathcal{C}(a,b)} \big(a\sigma^{j-1}b^kc-a^{j+k-1}bc-a^jb^kc\big)\zeta(j,k,1)$$

$$- \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} \bigoplus_{\mathcal{C}(a,b)} \big(a\sigma^{j-1}b^kc-a^{j+k-1}bc-a^kbc^j-a^kcb^j\big)\zeta(j,1,k)$$

$$= \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n}} \Bigg(\bigoplus_{\mathcal{C}(a,b,c)} \bigoplus_{\mathcal{C}(j,k)} \Big(\frac{a^j(bc^k-cb^k)}{c-b} - a^jb^{k-1}c - a^jbc^{k-1}\Big) - \bigoplus_{\mathcal{C}(a,b)} \Big(\frac{\sigma^{j-1}a(bc^k-cb^k)}{c-b} - a^jbc^{k-1}$$

$$-\sigma^{j-1}b^{k-1}ac + \frac{ab^kc^j-\sigma^{j-1}acb^k}{c-b-a} - \sigma^{j-2}b^kac + \bigoplus_{\mathcal{C}(j,k)} \Big(acb^{j+k-2} - \frac{b^{k-1}a(bc^j-cb^j)}{c-b}\Big)\Big)\Bigg)\zeta(j,k)$$

$$- \bigoplus_{\mathcal{C}(a,b)} \Big(ac\sigma^{n-2}-a^{n-1}c-c^{n-1}a-2ab^{n-2}c+\frac{acb^{n-1}-bca^{n-1}}{2(b-a)}\Big)\zeta(n-1,1)$$

$$+ \bigoplus_{\mathcal{C}(b,c)} \Big(\frac{b^2(a^2c^{n-1}-a^nc)}{c(c-b)(c-a)}+a^{n-1}b\Big)\zeta(n)+bca^{n-2}\zeta(n)+abc^{n-2}\zeta(n-2,2)$$

$$+ \bigoplus_{\mathcal{C}(a,b,c)} a^{n-2}bc\zeta(2,n-2) - \delta_{n,2}(ab+ac+bc)\zeta(2) - \delta_{n,3}abc(\zeta(2,1)+\zeta(3))$$

Proof. Similar to that of Theorem 3.2. We leave the details to the interested reader. □

Corollary 3.5. *Let n be any positive integer. Then*

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} \big(2^{j+1}+2^k\big)\zeta(j,k,l) - \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} 2^j\big(\zeta(j,1,k)+\zeta(j,k,1)\big) + \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n}} 2^j\cdot k\cdot\zeta(j,k)$$

$$= \frac{(n+3)(n+1)}{2}\zeta(n) - 3\zeta(n-2,2) - (2^{n-1}+n)\zeta(n-1,1) - 3\zeta(2,n-2).$$

Proof. Setting $a=1$ and letting $b,c\to 1$ in Theorem 3.4 we get

$$\sum_{\substack{j\ge 2,k,l\ge 1\\ j+k+l=n}} \big(2^{j+1}+2^k-6\big)\zeta(j,k,l) - \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n-1}} \big[(2^j-6)\zeta(j,1,k)+(2^j-4)\zeta(j,k,1)\big]$$

$$+ \sum_{\substack{j\ge 2,k\ge 1\\ j+k=n}} \big(2^j(k-1)-2^{j-1}-5n+22\big)\zeta(j,k)$$

$$= \frac{n^2-9n+32}{2}\zeta(n) + \zeta(n-2,2) - (2^{n-1}+n-10)\zeta(n-1,1) + 3\zeta(2,n-2).$$

Using the weighted sum formula (6), the sum formulas (8), (24), and (25) we can derive our corollary quickly. □

4 Depth 3: product of zeta and double zeta

Set $\mathbf{x}^{\mathbf{s-1}} = x_1^{s_1-1}x_2^{s_2-1}x_3^{s_3-1}$ throughout this section. By (3) for all $s_1, s_3 \geq 2$ and $s_2 \geq 1$,

$$\begin{aligned}
&\sum_{s_1,s_3\geq 2,s_2\geq 1} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_2)\,\zeta(s_3) \\
=& \sum_{s_1,s_3\geq 2,s_2\geq 1} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\geq 2,t_2\geq 1\\ t_1+t_2=s_1+s_3}} \binom{t_1-1}{s_3-1}\zeta(t_1,t_2,s_2) && (26)\\
&+ \sum_{s_1,s_3\geq 2,s_2\geq 1} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\geq 2,t_2,t_3\geq 1\\ t_1+t_2+t_3\\ =s_1+s_2+s_3}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-t_3}\zeta(t_1,t_2,t_3) && (27)\\
&+ \sum_{s_1,s_3\geq 2,s_2\geq 1} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\geq 2,t_2,t_3\geq 1\\ t_1+t_2+t_3\\ =s_1+s_2+s_3}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\zeta(t_1,t_2,t_3). && (28)
\end{aligned}$$

Similar to the last section we can get

$$\begin{aligned}
(26) =& G_3(x_{13},x_1,x_2) - G_3(x_1,x_1,x_2) - \sum_{s_1\geq 2,s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,1,s_2),\\
(27) =& G_3(x_{13},x_{23},x_2) - G_3(x_3,x_{23},x_2) - G_3(x_1,x_2,x_2),\\
(28) =& G_3(x_{13},x_{23},x_3) - G_3(x_3,x_{23},x_3) - \sum_{s_1\geq 2,s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,s_2,1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\sum_{s_1\geq 2,s_2\geq 1,s_3\geq 2} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_3,s_2) =& G_3(x_1,x_3,x_2) - \sum_{s_1\geq 2,s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,1,s_2) && (29)\\
\sum_{s_1\geq 2,s_2\geq 1,s_3\geq 2} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_1,s_2) =& G_3(x_3,x_1,x_2) - \sum_{s_1\geq 2,s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,1,s_2) && (30)\\
\sum_{s_1\geq 2,s_2\geq 1,s_3\geq 2} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_2,s_3) =& G_3(x_1,x_2,x_3) - \sum_{s_1\geq 2,s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,s_2,1). && (31)
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{s_1\geq 2,s_2\geq 1,s_3\geq 2} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1+s_3,s_2)\\
=& \sum_{s_2\geq 1} x_2^{s_2-1} \sum_{s\geq 2}\left(\sum_{s_1\geq 2,s_3\geq 2,s_1+s_3=s} x_1^{s_1-1}x_3^{s_3-1}\right)\zeta(s,s_2)\\
=& \sum_{s_2\geq 1} x_2^{s_2-1} \sum_{s\geq 2}\left(\frac{x_3^{s-1}-x_1^{s-1}}{x_3-x_1} - x_3^{s-2} - x_1^{s-2} + \delta_{s,2}\right)\zeta(s,s_2)\\
=& \frac{1}{x_3-x_1}\Big(G_2(x_3,x_2)-G_2(x_1,x_2)\Big) - \frac{1}{x_1}G_2(x_1,x_2) - \frac{1}{x_3}G_2(x_3,x_2) + \sum_{s\geq 1} x_2^{s-1}\zeta(2,s)
\end{aligned} \tag{32}$$

where $\delta_{s,2}=1$ is $s=2$ and $\delta_{s,2}=0$ otherwise. Similarly

$$\begin{aligned}
&\sum_{s_1\geq 2, s_2\geq 1, s_3\geq 2} \mathbf{x^{s-1}}\zeta(s_1, s_2+s_3)\\
&=\sum_{s_1\geq 2} x_1^{s_1-1}\sum_{s\geq 1}\left(\sum_{s_2\geq 1, s_3\geq 2, s_2+s_3=s} x_2^{s_2-1}x_3^{s_3-1}\right)\zeta(s_1,s)\\
&=\sum_{s_1\geq 2} x_1^{s_1-1}\sum_{s\geq 1}\left(\frac{x_3^{s-1}-x_2^{s-1}}{x_3-x_1} - x_2^{s-2}+\delta_{s,1}x_2^{-1}\right)\zeta(s_1,s)\\
&=\frac{1}{x_3-x_2}\Big(G_2(x_1,x_3)-G_2(x_1,x_2)\Big)-\frac{1}{x_2}G_2(x_1,x_2)+\frac{1}{x_2}\sum_{s\geq 2}x_1^{s-1}\zeta(s,1) \qquad (33)
\end{aligned}$$

Notice we have the stuffle relation

$$\zeta(s_1,s_2)\,\zeta(s_3)=\zeta(s_1,s_2,s_3)+\zeta(s_1,s_3,s_2)+\zeta(s_3,s_1,s_2)+\zeta(s_1+s_3,s_2)+\zeta(s_1,s_2+s_3).$$

Hence the sum of (29) through (33) equals the sum of (26) throught (28), giving the following result.

Theorem 4.1. *We have*

$$\begin{aligned}
&G_3(x_{13},x_1,x_2)-G_3(x_1,x_1,x_2)+G_3(x_{13},x_{23},x_2)-G_3(x_3,x_{23},x_2)-G_3(x_1,x_2,x_2)\\
&+G_3(x_{13},x_{23},x_3)-G_3(x_3,x_{23},x_3)-G_3(x_1,x_2,x_3)-G_3(x_1,x_3,x_2)-G_3(x_3,x_1,x_2)\\
&=\frac{1}{x_3-x_1}\Big(G_2(x_3,x_2)-G_2(x_1,x_2)\Big)+\frac{1}{x_3-x_2}\Big(G_2(x_1,x_3)-G_2(x_1,x_2)\Big)\\
&\qquad -\frac{1}{x_1}G_2(x_1,x_2)-\frac{1}{x_3}G_2(x_3,x_2)-\frac{1}{x_2}G_2(x_1,x_2)\\
&\qquad -\sum_{s_1\geq 2, s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,1,s_2)+\sum_{s\geq 1}x_2^{s-1}\zeta(2,s)+\frac{1}{x_2}\sum_{s\geq 2}x_1^{s-1}\zeta(s,1). \qquad (34)
\end{aligned}$$

Proof. Clear. ☐

Theorem 4.2. *Let a,b and c be three real numbers. Then we have*

$$\begin{aligned}
&\sum_{\substack{j\geq 2,k,l\geq 1\\ j+k+l=n}}\Big[(a+c)^{j-1}a^kb^lc+a(a+c)^{j-1}(b+c)^{k-1}(cb^l+bc^l)\\
&-a^{j+k-1}b^lc-a^jb^{k+l-1}c-ac^j(b+c)^{k-1}(b^l+bc^{l-1})-a^jb^kc^l-a^jc^kb^l-c^ja^kb^l\Big]\zeta(j,k,l)\\
&=\sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}}\Big[\frac{(ac^j-a^jc)b^k}{c-a}+\frac{a^j(bc^k-b^kc)}{c-b}-a^{j-1}b^kc-ab^kc^{j-1}-a^jb^{k-1}c\Big]\zeta(j,k)\\
&\qquad -\sum_{k=2}^{n-2}a^kb^{n-1-k}c\zeta(k,1,n-1-k)+ab^{n-2}c\zeta(2,n-2)+a^{n-1}c\zeta(n-1,1). \qquad (35)
\end{aligned}$$

Proof. Multiplying $x_1x_2x_3$ on (34), taking $x_1 = at, x_2 = bt, x_3 = ct$ and then comparing the coefficients of t^n we arrive at (35) immediately. $\square$

Notice that Theorem 4.1 is very similar to [11, Thm. 1.1(i)] but not the same. Moreover, by comparing the two results we obtain the following immediately.

Theorem 4.3. *We have*

$$\begin{aligned} &G_3(x_1,x_1,x_2) + G_3(x_1,x_2,x_2) - \frac{1}{x_1}G_2(x_1,x_2) - \frac{1}{x_2}G_2(x_1,x_2) \\ &\quad = \sum_{s_1\geq 2, s_2\geq 1} x_1^{s_1-1}x_2^{s_2-1}\zeta(s_1,1,s_2) - \sum_{s\geq 1} x_2^{s-1}\zeta(2,s) - \frac{1}{x_2}\sum_{s\geq 2} x_1^{s-1}\zeta(s,1). \quad (36) \end{aligned}$$

Proof. The terms appearing in (36) are exactly those that are in Theorem 4.1 but not in [11, Thm. 1.1(i)]. $\square$

Theorem 4.4. *Let a and b be three real numbers. Then we have*

$$\begin{aligned} &\sum_{\substack{j\geq 2, k,l\geq 1\\ j+k+l=n}} \Big[a^{j+k-1}b^l + a^j b^{k+l-1}\Big]\zeta(j,k,l) - \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}} \Big[a^{j-1}b^k + a^j b^{k-1}\Big]\zeta(j,k) \\ &\quad = \sum_{k=2}^{n-2} a^k b^{n-1-k}\zeta(k,1,n-1-k) - ab^{n-2}\zeta(2,n-2) - a^{n-1}\zeta(n-1,1). \quad (37) \end{aligned}$$

Proof. This follows from Theorem 4.3 immediately. $\square$

Corollary 4.5. *Let n be a positive integer such that $n \geq 3$. Then*

$$\sum_{\substack{j\geq 2,k,l\geq 1\\ j+k+l=n}} (2j+k)\zeta(j,k,l) - \sum_{\substack{k\geq 2,l\geq 1\\ k+l=n-1}} k\zeta(k,1,l) = \zeta(2,n-2) + 4\zeta(n) - (3n-5)\zeta(n-1,1). \quad (38)$$

Proof. Differentiating (37) with respect to a and then putting $a = b = 1$ we get

$$\begin{aligned} &\sum_{\substack{j\geq 2,k,l\geq 1\\ j+k+l=n}} (2j+k-1)\zeta(j,k,l) - \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}} (2j-1)\zeta(j,k) \\ &\quad = \sum_{\substack{k\geq 2,l\geq 1\\ k+l=n-1}} k\zeta(k,1,l) - \zeta(2,n-2) - (n-1)\zeta(n-1,1). \end{aligned}$$

The corollary now follows from (9) and the sum formulas (8) for $d = 2, 3$. $\square$

The following corollary provides a sum formula relating some special type triple zeta and double zeta values. It also appeared in [11] as (5.12) which is a special case of [8, Thm. 2.3].

Corollary 4.6. *Let n be a positive integer such that $n \geq 3$. Then*

$$\sum_{k=2}^{n-1} \zeta(k, 1, n-k) = \zeta(2, n-1) + \zeta(n, 1). \tag{39}$$

Proof. Taking $a = b = 1$ and let $c \to 1$ in the Theorem 4.2 we get

$$\sum_{\substack{j\geq 2, k,l\geq 1\\ j+k+l=n}} (2^{j-1} + 2^{j+k-1} + 2^k - 5)\zeta(j,k,l) - (n-5) \sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}} \zeta(j,k)$$
$$= -\sum_{k=2}^{n-2} \zeta(k, 1, n-1-k) + \zeta(2, n-2) + \zeta(n-1, 1).$$

By sum formula (8) we see that

$$\sum_{\substack{j\geq 2,k\geq 1\\ j+k=n}} \zeta(j,k) = \sum_{\substack{j\geq 2,k,l\geq 1\\ j+k+l=n}} \zeta(j,k,l) = \zeta(n).$$

Now (39) quickly follows from the weighted sum formula of Guo and Xie [6, Theorem 1.1] (setting $k = 2$ there). □

5 Depth 4: product of two double zetas

In this last section we turn our attention to depth 4 case and derive some new families of MZV identities using the idea of generating functions developed as above. Throughout this section we set $\mathbf{x}^{\mathbf{s-1}} = x_1^{s_1-1} x_2^{s_2-1} x_3^{s_3-1} x_4^{s_4-1}$. We also use the short hand $t_{ij} = t_i + t_j$, $s_{ij} = s_i + s_j$ and so on.

First, for integers $s_1, s_3 \geq 2$ and $s_2, s_4 \geq 1$, Guo and Xie proved the following at the end of [6]

$$\begin{aligned}
&\zeta(s_1,s_2)\,\zeta(s_3,s_4)\\
&= \sum_{\substack{t_1\geq 2, t_2,t_3\geq 1\\ t_{123}=s_{123}}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\zeta(t_1,t_2,t_3,s_4)\\
&+ \sum_{\substack{t_1\geq 2, t_2,t_3\geq 1\\ t_{123}=s_{134}}} \binom{t_1-1}{s_3-1}\binom{t_2-1}{s_4-1}\zeta(t_1,t_2,t_3,s_2)\\
&+ \sum_{\substack{t_1\geq 2, t_2,t_3,t_4\geq 1\\ t_{1234}=s_{1234}}} \left[\binom{t_1-1}{s_1-1}\binom{t_2-1}{t_{12}-s_{13}}\left(\binom{t_3-1}{s_4-t_4}+\binom{t_3-1}{s_4-1}\right)\right.\\
&\qquad\left. + \binom{t_1-1}{s_3-1}\binom{t_2-1}{t_{12}-s_{13}}\left(\binom{t_3-1}{s_2-t_4}+\binom{t_3-1}{s_2-1}\right)\right]\zeta(t_1,t_2,t_3,t_4).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_2)\,\zeta(s_3,s_4)\\
&=\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3\ge 1\\ t_{123}=s_{123}}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\zeta(t_1,t_2,t_3,s_4) && (40)\\
&+\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3\ge 1\\ t_{123}=s_{134}}} \binom{t_1-1}{s_3-1}\binom{t_2-1}{s_4-1}\zeta(t_1,t_2,t_3,s_2) && (41)\\
&+\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3,t_4\ge 1\\ t_{1234}=s_{1234}}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_{12}-s_{13}}\binom{t_3-1}{s_4-t_4}\zeta(t_1,t_2,t_3,t_4) && (42)\\
&+\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3,t_4\ge 1\\ t_{1234}=s_{1234}}} \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_{12}-s_{13}}\binom{t_3-1}{s_4-1}\zeta(t_1,t_2,t_3,t_4) && (43)\\
&+\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3,t_4\ge 1\\ t_{1234}=\\ s_{1234}}} \binom{t_1-1}{s_3-1}\binom{t_2-1}{t_{12}-s_{13}}\binom{t_3-1}{s_2-t_4}\zeta(t_1,t_2,t_3,t_4) && (44)\\
&+\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}} \sum_{\substack{t_1\ge 2,t_2,t_3,t_4\ge 1\\ t_{1234}=s_{1234}}} \binom{t_1-1}{s_3-1}\binom{t_2-1}{t_{12}-s_{13}}\binom{t_3-1}{s_2-1}\zeta(t_1,t_2,t_3,t_4). && (45)
\end{aligned}$$

As before we can get

$$(40)=G_4(x_{13},x_{23},x_3,x_4)-G_4(x_3,x_{23},x_3,x_4)-\sum_{\substack{s_1\ge 2\\ s_2,s_4\ge 1}} x_1^{s_1-1}x_2^{s_2-1}x_4^{s_4-1}\zeta(s_1,s_2,1,s_4).$$

$$\begin{aligned}
(42)=&G(x_{13},x_{23},x_{24},x_4)-G(x_3,x_{23},x_{24},x_4)-G(x_1,x_2,x_{24},x_4)\\
(43)=&G(x_{13},x_{23},x_{24},x_2)-G(x_3,x_{23},x_{24},x_2)-G(x_1,x_2,x_{24},x_2)
\end{aligned}$$

$$(41)=\sigma_{x_1,x_3}\sigma_{x_2,x_4}(40),\quad (44)=\sigma_{x_1,x_3}\sigma_{x_2,x_4}(42),\quad (45)=\sigma_{x_1,x_3}\sigma_{x_2,x_4}(43).$$

On the other hand, by the stuffle relation

$$\begin{aligned}
&\zeta(s_1,s_2)\zeta(s_3,s_4)=\zeta(s_3,s_4,s_1,s_2)+\zeta(s_3,s_1,s_4,s_2)+\zeta(s_1,s_3,s_4,s_2)+\zeta(s_1,s_3,s_2,s_4)\\
&+\zeta(s_1,s_2,s_3,s_4)+\zeta(s_3,s_1,s_2,s_4)+\zeta(s_{13},s_2,s_4)+\zeta(s_1,s_{23},s_4)+\zeta(s_{13},s_2,s_4)\\
&+\zeta(s_3,s_{14},s_2)+\zeta(s_1,s_3,s_{24})+\zeta(s_3,s_1,s_{24})+\zeta(s_{13},s_{24}).
\end{aligned}$$

Thus we have

$$\sum_{\substack{s_1,s_3\ge 2\\ s_2,s_4\ge 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_2)\,\zeta(s_3,s_4)$$

$$= \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_4,s_1,s_2) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_1,s_4,s_2) \tag{46}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_3,s_4,s_2) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_3,s_2,s_4) \tag{47}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_2,s_3,s_4) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_1,s_2,s_4) \tag{48}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_{13},s_4,s_2) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_{23},s_4) \tag{49}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_{13},s_2,s_4) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_4+s_1,s_2) \tag{50}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_1,s_3,s_{24}) + \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_3,s_1,s_{24}) \tag{51}$$

$$+ \sum_{\substack{s_1,s_3\geq 2\\ s_2,s_4\geq 1}} \mathbf{x}^{\mathbf{s-1}}\zeta(s_{13},s_{24}). \tag{52}$$

One can verify the following identities as before:

$$(46) = G_4(x_3,x_4,x_1,x_2) + G_4(x_3,x_1,x_4,x_2) - \sum_{\substack{s_3\geq 2\\ s_2,s_4\geq 1}} x_2^{s_2-1}x_3^{s_3-1}x_4^{s_4-1}\zeta(s_3,s_4,1,s_2)$$
$$- \sum_{\substack{s_3\geq 2\\ s_2,s_4\geq 1}} x_2^{s_2-1}x_3^{s_3-1}x_4^{s_4-1}\zeta(s_3,1,s_4,s_2)$$

$$(47) = G_4(x_1,x_3,x_4,x_2) + G_4(x_1,x_3,x_2,x_4) - \sum_{\substack{s_1\geq 2\\ s_2,s_4\geq 1}} x_1^{s_1-1}x_2^{s_2-1}x_4^{s_4-1}\zeta(s_1,1,s_4,s_2)$$
$$- \sum_{\substack{s_1\geq 2\\ s_2,s_4\geq 1}} x_1^{s_1-1}x_2^{s_2-1}x_4^{s_4-1}\zeta(s_1,1,s_2,s_4)$$

$$(48) = G_4(x_1,x_2,x_3,x_4) + G_4(x_3,x_1,x_2,x_4) - \sum_{\substack{s_1\geq 2\\ s_2,s_4\geq 1}} x_1^{s_1-1}x_2^{s_2-1}x_4^{s_4-1}\zeta(s_1,s_2,1,s_4)$$
$$- \sum_{\substack{s_3\geq 2\\ s_2,s_4\geq 1}} x_2^{s_2-1}x_3^{s_3-1}x_4^{s_4-1}\zeta(s_3,1,s_2,s_4)$$

$$(49) = \bigoplus_{\mathcal{C}(x_1,x_3)} \left\{\frac{G_3(x_1,x_4,x_2)}{x_1-x_3} - \frac{G_3(x_1,x_4,x_2)}{x_1}\right\} + \frac{G_3(x_1,x_2,x_4)-G_3(x_1,x_3,x_4)}{x_2-x_3}$$
$$- \frac{G_3(x_1,x_2,x_4)}{x_2} + \sum_{s,t\geq 1} x_2^{s-1}x_4^{t-1}\zeta(2,t,s) + \frac{1}{x_2}\sum_{\substack{s\geq 2\\ t\geq 1}} x_1^{s-1}x_4^{t-1}\zeta(s,1,t)$$

$$(50) = \sigma_{x_1,x_3}\sigma_{x_2,x_4}(49)$$

$$(51)=\bigoplus_{\mathcal{C}(x_2,x_4)}\Big\{\bigoplus_{\mathcal{C}(x_1,x_3)}\Big\{\frac{G_3(x_1,x_3,x_2)}{x_2-x_4}-\frac{1}{x_2-x_4}\sum_{\substack{t\geq 2\\ s\geq 1}}x_1^{t-1}x_2^{s-1}\zeta(t,1,s)\Big\}\Big\}$$

$$(52)=\bigoplus_{\mathcal{C}(x_2,x_4)}\Big\{\bigoplus_{\mathcal{C}(x_1,x_3)}\Big\{\Big(\frac{1}{x_1-x_3}-\frac{1}{x_1}\Big)\frac{G_2(x_1,x_2)}{x_2-x_4}\Big\}+\frac{1}{x_2-x_4}\sum_{t\geq 1}x_2^{t-1}\zeta(2,t)\Big\}.$$

Hence the sum of (40) to (45) is equal to the sum of (46) to (52). This equality establishes an identity involving G_4 and four formal variables $x_1,\dots,x_4$. By taking $x_1=at, x_2=bt, x_3=ct$ and $x_4=dt$ and comparing the coefficient of t^{n-4} we can obtain our last theorem.

Let $S_1=\{e,\sigma_{a,c}\sigma_{b,d}\}$, where $\sigma_{a,c}$ (or $\sigma_{b,d}$) denotes the transposition that switches a and c (or b and d) and $S_2=\{e,\sigma_{a,c}\sigma_{b,d},\sigma_{a,c},\sigma_{b,c},\sigma_{b,d},\sigma_{c,d,b,a}\}$. For any subset S of the symmetric group $\mathfrak{S}_4$, let $\bigoplus_S f(a,b,c,d)=\sum_{\sigma\in S}f(\sigma(a),\sigma(b),\sigma(c),\sigma(d))$.

Theorem 5.1. *Let a,b, c and d be any real numbers. Then for any positive integer $n\geq 2$, we have*

$$\begin{aligned}
&\bigoplus_{S_1}\Big\{\sum_{\substack{i\geq 2\\ i+j+k+l=n}}[abc(a+c)^{i-1}(b+c)^{j-1}-abc^i(b+c)^{j-1}-ca^ib^j](b+d)^{k-1}d^l\cdot\zeta(i,j,k,l)\\
&+\sum_{\substack{i\geq 2\\ i+j+k+l=n}}[(a+c)^{i-1}-c^{i-1}]ab(b+c)^{j-1}c^kd^l\zeta(i,j,k,l)+\sum_{\substack{i\geq 2\\ i+j+l=n-1}}(b^jd^l+d^jb^l)ca^i\zeta(i,1,j,l)\\
&+\sum_{\substack{i\geq 2\\ i+j+k+l=n}}[acd(a+c)^{i-1}(b+c)^{j-1}-adc^i(b+c)^{j-1}-cda^ib^{j-1}](b+d)^{k-1}b^l\cdot\zeta(i,j,k,l)\Big\}\\
&=\bigoplus_{S_2}\Big\{\sum_{\substack{i\geq 2\\ i+j+k+l=n}}a^ib^jc^kd^l\zeta(i,j,k,l)\Big\}+\bigoplus_{S_1}\Big\{\sum_{\substack{i\geq 2\\ i+j+k=n}}\Big(\frac{cb^j-bc^j}{b-c}-cb^{j-1}\Big)a^id^k\zeta(i,j,k)\\
&\quad+\sum_{\substack{i\geq 2\\ i+j+k=n}}\Big(\frac{ca^i-ac^i}{a-c}-ca^{i-1}-ac^{i-1}\Big)b^jd^k\zeta(i,j,k)+\sum_{\substack{i\geq 2\\ i+j+k=n}}\frac{db^k-bd^k}{b-d}a^ic^j\zeta(i,j,k)\\
&\quad+\sum_{j+k=n-2}acd^jb^k\zeta(2,j,k)+\sum_{\substack{i\geq 2\\ i+k=n-1}}ca^id^k\zeta(i,1,k)-\sum_{\substack{i\geq 2\\ i+k=n-1}}\frac{db^k-bd^k}{b-d}a^ic\zeta(i,1,k)\Big\}\\
&\quad+\sum_{\substack{i\geq 2,\\ i+j=n}}\Big(\frac{ca^i-ac^i}{a-c}-ca^{i-1}-ac^{i-1}\Big)\frac{db^j-bd^j}{b-d}\zeta(i,j)+\frac{ac(db^{n-2}-bd^{n-2})}{b-d}\zeta(2,n-2).
\end{aligned}$$

Corollary 5.2. *Let $n\geq 5$ be any positive integer. Then*

$$\begin{aligned}
&2\sum_{\substack{i\geq 2\\ i+j+l=n-1}}\zeta(i,1,j,l)-\sum_{\substack{i\geq 2\\ i+j+k+l=n}}(2^{i-1}+2^k)\zeta(i,j,k,l)-\sum_{\substack{k\geq 2\\ k+j=n-1}}k\zeta(k,1,j)\\
&\qquad=2\zeta(2,n-2)+(n-3)\zeta(n-2,2)-(2n-5)\zeta(n-1,1)+\frac{n+5}{2}\zeta(n). \quad (53)
\end{aligned}$$

Proof. Let $a = 1 = b$ and $c \to 1$ and $d \to 1$ in Theorem 5.1. Then we get

$$\begin{aligned}
2 \sum_{\substack{i\geq 2\\ i+j+k+l=n}} (2^{i+j+k-2} - 2^{j+k-1} - 2^k + 2^{i+j-2} - 2^{j-1})\zeta(i,j,k,l) = 6 \sum_{\substack{i\geq 2\\ i+j+k+l=n}} \zeta(i,j,k,l) \\
- 4 \sum_{\substack{i\geq 2\\ i+j+l=n-1}} \zeta(i,1,j,l) + 2 \sum_{\substack{i\geq 2\\ i+j+k=n}} (n-6)\zeta(i,j,k) + 2 \sum_{j+k=n-2} \zeta(2,j,k) \\
- 2 \sum_{\substack{i\geq 2\\ i+k=n-1}} (k-2)\zeta(i,1,k) + \sum_{\substack{i\geq 2\\ i+j=n}} (i-3)(j-1)\zeta(i,j) + (n-3)\zeta(2,n-2).
\end{aligned}$$

By the sum formula and the weighted sum formula of Guo and Xie [6, Theorem 1.1] (setting $k = 4$ there), we get

$$\begin{aligned}
2 \sum_{\substack{i\geq 2\\ i+j+l=n-1}} \zeta(i,1,j,l) - \sum_{\substack{i\geq 2\\ i+j+k+l=n}} (2^{i-1} + 2^k)\zeta(i,j,k,l) = \sum_{j+k=n-2} \zeta(2,j,k) - 3\zeta(n) \\
- \sum_{i=2}^{n-2} (n-i-3)\zeta(i,1,n-i-1) + \frac{1}{2}\sum_{i=2}^{n-1}(i-3)(n-i-1)\zeta(i,n-i) + \frac{n-3}{2}\zeta(2,n-2)
\end{aligned} \tag{54}$$

Taking $l = 2, i_1 = 2, i_2 = n-3$ in [7, Thm. 5.1] we get

$$\sum_{j+k=n-2} \zeta(2,j,k) = \zeta(3,n-3) + \zeta(2,n-2). \tag{55}$$

Combining this with (9), (10), (24) and the sum formula (7) we see easily that (54) can be simplified to (53). This finishes the proof of the corollary. ☐

References

[1] M. Eie, W. Liaw, Y. Ong, A restricted sum formula among multiple zeta values, *J. Number Theory* **129**(4) (2009), pp. 908–921.

[2] L. Euler, Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol.* **20** (1775), pp. 140–186; reprinted in Opera Omnia, Ser. I, vol. **15**, B. Teubner, Berlin, 1927, pp. 217–267.

[3] H. Gangl, M. Kaneko, D. Zagier, Double zeta values and modular forms, in: Automorphic Forms and Zeta Functions, S. Böcherer et al. eds., World Scientific, Hackensack, NJ (2006), pp. 71–106.

[4] A. Granville, A decomposition of Riemanns zeta-function, in: Analytic Number Theory, Y. Motohashi (ed.), London Math. Soc. Lecture Note Ser. **247**, Cambridge, 1997, pp. 95-101.

[5] L. Guo, P. Lei and J. Zhao, New families of weighted sum formulas for multiple zeta values II, preprint.

[6] L. Guo and B. Xie, Weighted sum formula for multiple zeta values, *J. Number Theory* **129**(11) (2009), pp. 2747–2765.

[7] M.E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (1992), pp. 275–290.

[8] M.E. Hoffman and Y. Ohno, Relations of multiple zeta values and their algebraic expression, *J. Algebra* **262** (2003), pp. 332-347.

[9] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, *Compos. Math.* **142** (2006), pp. 307–338.

[10] T. Machide, Weighted sums with two parameters of multiple zeta values and their formulas, *Int. J. Number Theory* **8** (2012), pp. 1903-1921.

[11] T. Machide, Extended double shuffle relations and the generating function of triple zeta values of any fixed weight, arXiv: 1204.4085

[12] Y. Ohno and W. Zudilin, Zeta stars, *Commun. Number Theory Phys.* **2** (2008), pp. 325–347;

[13] D. Zagier, Values of zeta functions and their applications, in: First European Congress of Mathematics (Paris, 1992), vol. **II**, A. Joseph et. al. (eds.), Birkhäuser, Basel, 1994, pp. 497–512.

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