5-2015

Families of Weighted Sum Formulas for Multiple Zeta Values

Li Guo
Rutgers University

Peng Lei
Lanzhou University

Jianqiang Zhao
Georgia Southern University, zhaoj8888@gmail.com

Recommended Citation

Guo, Li, Peng Lei, Jianqiang Zhao. 2015. "Families of Weighted Sum Formulas for Multiple Zeta Values." *International Journal of Number Theory*, 11 (3): 997. doi: 10.1142/S1793042115500530 source: http://arxiv.org/abs/1401.6461
https://digitalcommons.georgiasouthern.edu/math-sci-facpubs/415

FAMILIES OF WEIGHTED SUM FORMULAS FOR MULTIPLE ZETA VALUES

LI GUO, PENG LEI, AND JIANQIANG ZHAO

Abstract. Euler's sum formula and its multi-variable and weighted generalizations form a large class of the identities of multiple zeta values. In this paper we prove a family of identities involving Bernoulli numbers and apply them to obtain infinitely many weighted sum formulas for double zeta values and triple zeta values where the weight coefficients are given by symmetric polynomials.

Contents

1. Introduction

Multiple zeta functions are multiple variable generalizations of the Riemann zeta function. For fixed positive integer d and d-tuple of complex numbers $\mathbf{s} = (s_1, \dots, s_d)$, the multiple zeta function is defined by

$$\zeta(\mathbf{s}) = \sum_{k_1 > \cdots > k_d > 0} k_1^{-s_1} \cdots k_d^{-s_d} \tag{1}$$

where $\mathbf{s}$ satisfies $\mathrm{Re}(s_1 + \cdots + s_j) > j$ for all $j = 1, \dots, d$. The number d is called the *depth* (or *length*) and $s_1 + \cdots + s_d$ the *weight*, denoted by $|\mathbf{s}|$. Their convergent special values at positive integers are called multiple zeta values. These values can be traced back to a series of correspondences between Leonhard Euler and Christian Goldbach [6]. On the Christmas Eve of 1742, with different notation Goldbach wrote down some special cases of the following infinite sum on a letter to Euler:

$$\sum_{a \geq b \geq 1} \frac{1}{a^m b^n},$$

2010 *Mathematics Subject Classification.* 11M32, and 11B68.

where m and n are positive integer. Using our notation this is $\zeta(m,n)+\zeta(m+n)$ where $\zeta(m,n)$ is a double zeta value (DZV for short). Later Euler discovered the following sum formula

$$\sum_{k=2}^{n-1}\zeta(k,n-k)=\zeta(n),\quad n\geq 3, \tag{2}$$

and decomposition formula

$$2\zeta(n,1)=n\zeta(n+1)-\sum_{i=1}^{n-2}\zeta(n-i)\zeta(i+1),\quad n\geq 2. \tag{3}$$

There are many generalizations and variations of the sum formula in the literature. Ohno and Zudilin [8] proved a weighted form of Euler's sum formula

$$\sum_{k=2}^{w-1}2^k\zeta(k,w-k)=(w+1)\zeta(w),\quad w\geq 3. \tag{4}$$

Later this was generalized by Guo and Xie [4] to arbitrary depths. During their study of DZVs and modular forms Gangl et al. [3] made the following discovery: For all $n\geq 2$ we have

$$\sum_{k=1}^{n-1}\zeta(2k,2n-2k)=\frac{3}{4}\zeta(2n), \tag{5}$$

$$\sum_{k=1}^{n-1}\zeta(2k+1,2n-2k-1)=\frac{1}{4}\zeta(2n). \tag{6}$$

Recently, Hoffman [5] extended Eq. (5) to arbitrary depths. Of course these can also be regarded as weighted sum formulas. Some more complicated identities in depth two can be found in [7]

$$\sum_{k=1}^{n-1}(4^k+4^{n-k})\zeta(2k,2n-2k)=\left(n+\frac{4}{3}+\frac{4^n}{6}\right)\zeta(2n), \tag{7}$$

$$\sum_{k=2}^{n-2}(2k-1)(2n-2k-1)\zeta(2k,2n-2k)=\frac{3}{4}(n-3)\zeta(2n). \tag{8}$$

Nakamura's idea to prove Eq. (8) is to show the following identity of Bernoulli numbers:

$$6\sum_{i,j\geq 4,i+j=k}(i-1)(j-1)B_iB_j\binom{k}{i}=-(k-1)(k^2-5k-6)B_k, \tag{9}$$

where B_j is a Bernoulli number with generating function

$$\frac{t}{e^t-1}=\sum_{j=0}^{\infty}B_j\frac{t^j}{j!}.$$

Eq. (9) quickly leads to

$$6\sum_{k=2}^{n-2}(2k-1)(2n-2k-1)\zeta(2k)\zeta(2n-2k)=(n-3)(4n^2-1)\zeta(2n) \tag{10}$$

by Euler's famous evaluation

$$\zeta(2n)=-\frac{B_{2n}}{2(2n)!}(2\pi i)^{2n},\quad \text{and}\quad \zeta(1-2n)=-\frac{B_{2n}}{2n}. \tag{11}$$

The identity in Eq. (9) relating Bernoulli numbers have been obtained by a few diverse methods. Rademacher [9, p. 121] derived it as a consequence of an identity among Eisenstein series of different weights. Shimura essential did the same in his book [11, (11.10)]. Eie [1] proved it using the zeta function associated with some polynomials. We use his method in this paper to obtain infinitely many families of Bernoulli number identities similar to Eq. (9) in Section 2. These identities lead to infinitely many different weighted sum formulas for double and triple zeta values with symmetric polynomial coefficients. We will consider the case of double zeta values in Section 3 and the case of triple zeta values in Section 4. We end the paper with a conjecture for the general case.

2. Weighted sum of products of Bernoulli numbers

In this section we shall prove a sum formula for products $B_{2j}B_{2k}$ for fixed $j+k$ with some weight coefficients. This in turn will lead to a sum formula for products $\zeta(2j)\zeta(2k)$ with weight coefficients given by arbitrary polynomials in j and k. First we define a zeta function that will be useful in both depth two and depth three cases:

$$Z_2(s;r_1,r_2)=\sum_{m_1,m_2=1}^{\infty} m_1^{r_1}m_2^{r_2}(m_1+m_2)^{-s}.$$

The basic idea of Eie in [1, 2] is to compute the special value of this function when s is some appropriate negative integer using two different methods. By comparing the two expressions one can derive an identity of Bernoulli numbers which yields the desired identity of multiple zeta values.

First we want to find a useful expression of $Z_2(s;r_1,r_2)$. We have

$$\begin{aligned}
Z_2(s;r_1,r_2)&=\sum_{m=1}^{\infty}\sum_{m_1=1}^{m-1} m_1^{r_1}(m-m_1)^{r_2}m^{-s}\\
&=\sum_{m=1}^{\infty}\sum_{m_1=1}^{m-1} m_1^{r_1}\sum_{i=0}^{r_2}\binom{r_2}{i}(-1)^i m^{r_2-i}m_1^i m^{-s}\\
&=\sum_{m=1}^{\infty}\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\left(\sum_{k=0}^{r_1+i}\binom{r_1+i+1}{k}\frac{(-1)^kB_k}{r_1+i+1}m^{r_1+i+1-k}-m^{r_1+i}\right)\cdot m^{r_2-i-s}
\end{aligned}$$

$$=\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\sum_{k=0}^{r_1+i}\binom{r_1+i+1}{k}\frac{(-1)^k B_k}{r_1+i+1}f_k(s;r_1,r_2)-\delta_{r_2,0}\zeta(s-r_1-r_2), \tag{12}$$

where $f_k(s;r_1,r_2)=\zeta(s+k-r_1-r_2-1)$. To simplify this further we need the following combinatorial lemmas.

Lemma 2.1. *Let k, r_1, and r_2 be nonnegative integers. Then*

$$\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\binom{r_1+i}{k-1}=(-1)^{r_2}\binom{r_1}{k-r_2-1}.$$

Proof. We have

$$\begin{aligned}
&\sum_{k=1}^{r_1+r_2+1}\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\binom{r_1+i}{k-1}x^k\\
=&\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\sum_{k=0}^{r_1+r_2}\binom{r_1+i}{k}x^{k+1}=\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}x(1+x)^{r_1+i}\\
=&x(1+x)^{r_1}(1-(1+x))^{r_2}=\sum_{l=0}^{r_1}\binom{r_1}{l}(-1)^{r_2}x^{r_2+l+1}=\sum_{k=0}^{r_1+r_2+1}(-1)^{r_2}\binom{r_1}{k-r_2-1}x^k.
\end{aligned}$$

Comparing the coefficients we obtain the lemma immediately. □

Lemma 2.2. *Let r_1 and r_2 be two nonnegative integers. Then*

$$\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\frac{1}{r_1+i+1}=\frac{r_1!r_2!}{(r_1+r_2+1)!}.$$

Proof. Define

$$F(x)=\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\frac{x^{r_1+i+1}}{r_1+i+1}.$$

Then

$$F'(x)=\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}x^{r_1+i}=x^{r_1}(1-x)^{r_2}.$$

Thus using the beta function $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ we have

$$\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\frac{1}{r_1+i+1}=\int_0^1 x^{r_1}(1-x)^{r_2}\,dx=B(r_1+1,r_2+1)=\frac{r_1!r_2!}{(r_1+r_2+1)!}$$

as desired. □

Given any function $f(x,y)$ we define

$$\operatorname*{Cyc}_{x,y} f(x,y)=f(x,y)+f(y,x).$$

Proposition 2.3. *For all nonnegative integers r_1 and r_2 we have*

$$Z_2(s;r_1,r_2)=\frac{r_1!r_2!}{(r_1+r_2+1)!}f_0(s;r_1,r_2)+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1}\sum_{k=r_1+1}^{r_1+r_2+1}\binom{r_2}{k-r_1-1}\beta_k f_k(s;r_1,r_2),$$

where $\beta_k=B_k/k$.

Proof. By Eq. (12) we get

$$\begin{aligned}Z_2(s;r_1,r_2)=&\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\sum_{k=0}^{r_1+r_2+1}\binom{r_1+i+1}{k}\frac{(-1)^kB_k}{r_1+i+1}f_k(s;r_1,r_2)\\&+\sum_{i=0}^{r_2}(-1)^{r_1}\binom{r_2}{i}\frac{B_{r_1+i+1}}{r_1+i+1}f_{r_1+i+1}(s;r_1,r_2)-\delta_{r_2,0}\zeta(s-r_1-r_2)\\=&\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\frac{1}{r_1+i+1}f_0(s;r_1,r_2)+\sum_{i=0}^{r_2}(-1)^{r_1}\binom{r_2}{i}\beta_{r_1+i+1}f_{r_1+i+1}(s;r_1,r_2)\\&+\sum_{i=0}^{r_2}(-1)^i\binom{r_2}{i}\sum_{k=1}^{r_1+r_2+1}\binom{r_1+i}{k-1}(-1)^k\beta_kf_k(s;r_1,r_2)-\delta_{r_2,0}\zeta(s-r_1-r_2).\end{aligned}$$

By Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}Z_2(s;r_1,r_2)=&\frac{r_1!r_2!}{(r_1+r_2+1)!}f_0(s;r_1,r_2)+(-1)^{r_1}\sum_{k=r_1+1}^{r_1+r_2+1}\binom{r_2}{k-r_1-1}\beta_kf_k(s;r_1,r_2)\\&+(-1)^{r_2}\sum_{k=r_2+1}^{r_1+r_2+1}\binom{r_1}{k-r_2-1}(-1)^k\beta_kf_k(s;r_1,r_2)-\delta_{r_2,0}\zeta(s-r_1-r_2).\end{aligned}$$

Observe that $(-1)^k\beta_k=\beta_k$ unless $k=1$, and this term occurs if and only if $r_2=0$. Further $\beta_1=-\beta_1-1$. So by combining the last two parts of the above equation, we can get the result of the proposition. $\square$

In order to find the weighted sum formulas for DZVs we first consider the corresponding result for Bernoulli numbers.

Theorem 2.4. *For all nonnegative integers r_1, r_2 and $n\ge r_1+r_2+2$ we have*

$$\begin{aligned}&\sum_{k=1}^{2n-1}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}\prod_{a=1}^{r_1}(k-a)\prod_{b=1}^{r_2}(2n-k-b)\\=&\sum_{k=r_1+1}^{2n-r_2-1}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}\prod_{a=1}^{r_1}(k-a)\prod_{b=1}^{r_2}(2n-k-b)\\=&-r_1!r_2!\left(\binom{2n-1}{r_1+r_2+1}+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\binom{2n-1}{r_1}\right)\frac{B_{2n}}{(2n)!}\\&-r_1!r_2!\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1}\sum_{k=r_1+1}^{r_1+r_2+1}\binom{2n-1-k}{r_1+r_2+1-k}\binom{k-1}{r_1}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}.\end{aligned}$$

Proof. Let $s = r_1 + r_2 + 2 - 2n$. By Proposition 2.3 we have

$$Z_2(s; r_1, r_2) = \frac{-r_1! r_2! \beta_{2n}}{(r_1 + r_2 + 1)!} - \operatorname*{Cyc}_{r_1,r_2} (-1)^{r_1} \sum_{k=r_1+1}^{r_1+r_2+1} \binom{r_2}{k - r_1 - 1} \beta_k \beta_{2n-k}. \tag{13}$$

On the other hand Eie [1, 2] showed that functions like $Z_r(s)$ have analytic continuations over the whole complex plane and further they are defined at negative integers. In fact our expressions in (13) shows clearly that this can be done using the Riemann zeta function. More importantly, Eie showed that these special values at negative integers can also be computed using some integrals over clearly specified simplices. In our situation Eie's theory implies that

$$Z_2(s; r_1, r_2) = J^2\Big(x^{r_1} y^{r_2} (x + y)^{-s}\Big) + \operatorname*{Cyc}_{r_1,r_2} J^1\Big(\int_0^{-x} x^{r_1} y^{r_2} (x + y)^{-s}\, dy\Big), \tag{14}$$

where for positive integers a_j $(j = 1, \ldots, m)$

$$J^m(x_1^{a_1} \ldots x_m^{a_m}) = \prod_{j=1}^{m} (-1)^{a_j} \beta_{a_j+1}. \tag{15}$$

For the integral we may use substitution $y = -xt$ to get

$$\int_0^{-x} x^{r_1} y^{r_2} (x + y)^{2n-r_1-r_2-2}\, dy = (-1)^{r_2+1} x^{2n-1} \int_0^1 t^{r_2} (1 - t)^{2n-r_1-r_2-2}\, dt \tag{16}$$

$$= (-1)^{r_2+1} x^{2n-1} \frac{r_2!(2n - r_1 - r_2 - 2)!}{(2n - r_1 - 1)!}. \tag{17}$$

Applying the operator J we get

$$Z_2(s; r_1, r_2) = \sum_{a+b=-s} \frac{(-s)!}{a!b!} \beta_{a+r_1+1} \beta_{b+r_2+1} + \operatorname*{Cyc}_{r_1,r_2} \frac{(-1)^{r_2} r_2!(-s)!}{(2n - r_1 - 1)!} \beta_{2n}.$$

By comparing this with Eq. (14) we have

$$\begin{aligned}
&\sum_{k=r_1+1}^{2n-r_2-1} \frac{(2n - r_1 - r_2 - 2)!}{(k - r_1 - 1)!(2n - k - r_2 - 1)!} \frac{B_k}{k} \frac{B_{2n-k}}{2n - k} \\
=& \left(\frac{-r_1! r_2!}{(r_1 + r_2 + 1)!} - \operatorname*{Cyc}_{r_1,r_2} \frac{(-1)^{r_2} r_2!(2n - r_1 - r_2 - 2)!}{(2n - r_1 - 1)!} \right) \frac{B_{2n}}{2n} \\
&- \operatorname*{Cyc}_{r_1,r_2} (-1)^{r_1} \sum_{k=r_1+1}^{r_1+r_2+1} \binom{r_2}{k - r_1 - 1} \frac{B_k}{k} \frac{B_{2n-k}}{2n - k}.
\end{aligned}$$

The theorem follows immediately. □

Corollary 2.5. *For all nonnegative integers r_1, r_2 and $n \geq r_1 + r_2 + 2$ we have*

$$\sum_{k=\lceil (r_1+r_2)/2 \rceil+1}^{n-\lceil (r_1+r_2)/2 \rceil-1} \frac{B_{2k}}{(2k)!} \frac{B_{2j}}{(2j)!} \binom{2k-1}{r_1} \binom{2j-1}{r_2} \qquad \textit{(here } j = n - k\textit{)}$$

$$
\begin{aligned}
&= -\left(\binom{2n-1}{r_1+r_2+1} + \operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\binom{2n-1}{r_1}\right)\frac{B_{2n}}{(2n)!}\\
&\quad -\operatorname*{Cyc}_{r_1,r_2}\sum_{k=\lceil r_1/2\rceil+1}^{\lfloor (r_1+r_2+1)/2\rfloor}\binom{2k-1}{r_1}\left((-1)^{r_1}\binom{2j-1}{r_1+r_2+1-2k}+\binom{2j-1}{r_2}\right)\frac{B_{2k}}{(2k)!}\frac{B_{2j}}{(2j)!}.
\end{aligned}
$$

Proof. By breaking the left sum in Theorem 2.4 as

$$
\sum_{k=r_1+1}^{2n-r_2-1} = \sum_{k=r_1+1}^{r_1+r_2+1} + \sum_{k=r_1+r_2+2}^{2n-r_1-r_2-2} + \sum_{k=2n-r_1-r_2-1}^{2n-r_2-1}
$$

we see easily that

$$
\begin{aligned}
&\sum_{k=\lceil (r_1+r_2)/2\rceil+1}^{n-\lceil (r_1+r_2)/2\rceil-1}\frac{B_{2k}}{(2k)!}\frac{B_{2n-2k}}{(2n-2k)!}\prod_{a=1}^{r_1}(2k-a)\prod_{b=1}^{r_2}(2n-2k-b)\\
&= -r_1!r_2!\left(\binom{2n-1}{r_1+r_2+1} + \operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\binom{2n-1}{r_1}\right)\frac{B_{2n}}{(2n)!}\\
&\quad -r_1!r_2!\operatorname*{Cyc}_{r_1,r_2}\sum_{k=\lceil (r_1+1)/2\rceil}^{\lfloor (r_1+r_2+1)/2\rfloor}\binom{2k-1}{r_1}\left((-1)^{r_1}\binom{2j-1}{r_1+r_2+1-2k}+\binom{2j-1}{r_2}\right)\frac{B_{2k}}{(2k)!}\frac{B_{2j}}{(2j)!}.
\end{aligned}
$$

Notice that in the last summation if r_1 is odd and $k=(r_1+1)/2$ then the corresponding term happens to be zero. Thus we can improve the lower limit of k from $\lceil (r_1+1)/2\rceil$ to $\lceil r_1/2\rceil+1$. This finishes the proof of the corollary. ∎

By Eq. (11) and Corollary 2.5, we can get the following Corollary

Corollary 2.6. *For all nonnegative integers r_1, r_2 and $n\ge r_1+r_2+2$ we have*

$$
\begin{aligned}
&\sum_{k=\lceil (r_1+r_2)/2\rceil+1}^{n-\lceil (r_1+r_2)/2\rceil-1}\binom{2k-1}{r_1}\binom{2(n-k)-1}{r_2}\zeta(2(n-k))\zeta(2k)\\
&=\frac{1}{2}\left(\binom{2n-1}{r_1+r_2+1} + \operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\binom{2n-1}{r_1}\right)\zeta(2n)\\
&\quad -\operatorname*{Cyc}_{r_1,r_2}\sum_{k=\lceil r_1/2\rceil+1}^{\lfloor (r_1+r_2+1)/2\rfloor}\binom{2k-1}{r_1}\left((-1)^{r_1}\binom{2(n-k)-1}{r_1+r_2+1-2k}+\binom{2(n-k)-1}{r_2}\right)\zeta(2(n-k))\zeta(2k).
\end{aligned}
$$

In the literature there are many different types of sum formulas for DZV. The following statement is a kind of weighted sum formula for a product of two Riemann zeta values with the weight coefficients given by arbitrary polynomials. To derive sum formulas for DZV one has to symmetrize the coefficient which will be done in the next section.

Theorem 2.7. *Let $F(x,y)\in\mathbb{Q}[x,y]$ be a polynomial of degree d. Then for every positive integer $n\ge d+2$ we have*

$$
\sum_{j+k=n} F(j,k)\zeta(2j)\zeta(2k) = \sum_{k=0}^{\lfloor (d+1)/2\rfloor} K_{F,k}(n)\zeta(2k)\zeta(2n-2k),
$$

where $K_{f,k}(x)$ is a polynomial in x depending only on F and k whose degree is at most $d+1$.

Proof. It is well-known that for any nonnegative integers $m \geq r$ the Stirling numbers of the second kind $S(m,r)$ are all rational numbers which can be defined by

$$x^m = \sum_{r=0}^{m} r! S(m,r) \binom{x}{r}. \tag{18}$$

Thus for any nonnegative integers r_1 and r_2 we have

$$\begin{aligned} j^{m_1} k^{m_2} \zeta(2j)\zeta(2k) = \frac{1}{2^{m_1+m_2}} \sum_{r_1=0}^{m_1} \sum_{r_2=0}^{m_2} r_1! r_2! S(m_1, r_1) S(m_2, r_2) \\ \left(\binom{2j-1}{r_1} + \binom{2j-1}{r_1-1} \right) \left(\binom{2k-1}{r_2} + \binom{2k-1}{r_2-1} \right) \zeta(2j)\zeta(2k). \end{aligned}$$

The theorem now follows from Corollary 2.6. □

Example 2.8. We can obtain the following weighted sum formulas applying Corollary 2.6:

$$\sum_{\substack{j+k=n \\ j,k\geq 1}} j\zeta(2l)\zeta(2k) = \frac{n(2n+1)}{4}\zeta(2n), \tag{19}$$

$$\sum_{\substack{j+k=n \\ j,k\geq 1}} j^2\zeta(2l)\zeta(2k) = \frac{n(2n+1)(4n+1)}{24}\zeta(2n) - \frac{2n-3}{2}\zeta(2)\zeta(2n-2),$$

$$\sum_{\substack{j+k=n \\ j,k\geq 1}} j^3\zeta(2l)\zeta(2k) = \frac{n^2(2n+1)^2}{16}\zeta(2n) - \frac{3n(2n-3)}{4}\zeta(2)\zeta(2n-2), \tag{20}$$

$$\begin{aligned} \sum_{\substack{j+k=n \\ j,k\geq 1}} j^4\zeta(2l)\zeta(2k) = & \frac{n(2n+1)(4n+1)(12n^2+6n-1)\zeta(2n)}{480} \\ & - \frac{(2n-3)(8n^2-6n+5)}{8}\zeta(2)\zeta(2n-2) - \frac{3(2n-5)}{2}\zeta(4)\zeta(2n-4). \end{aligned}$$

3. Weighted sum formulas for double zeta values

In this section we apply results from the last section to give weighted sum formula for DZVs $\zeta(2j, 2k)$ for fixed $j+k$ when the weight factors are arbitrary symmetric polynomials in j and k. First, by setting $r_1 = r_2$ in Corollary 2.5 we obtain immediately

Proposition 3.1. *For all nonnegative integers r and $n \geq 2r+2$ we have*

$$\sum_{k=r+1}^{n-r-1} \frac{(2n-2r-2)!}{(2k-r-1)!(2n-2k-r-1)!} \frac{B_{2k}}{2k} \frac{B_{2n-2k}}{2n-2k}$$

$$= -\left(\frac{2(-1)^r r!(2n-2r-2)!}{(2n-r-1)!} + \frac{(r!)^2}{(2r+1)!}\right)\frac{B_{2n}}{2n}$$
$$- \sum_{k=\lceil r/2\rceil+1}^{r} \frac{2}{(2k-r-1)!}\left\{\frac{(-1)^r r!}{(2r-2k+1)!} + \frac{(2n-2r-2)!}{(2n-2k-r-1)!}\right\}\frac{B_{2k}}{2k}\frac{B_{2n-2k}}{2n-2k}.$$

Example 3.2. Taking $r=1$ we recover [1, Proposition 1] (notice it has a typo: $(2n-2)!$ on the left numerator should be $(2n-4)!$). Taking $r=2$ we recover [2, Proposition 4.2.2]. Taking $r=3$ in Proposition 3.1 we get

$$\sum_{k=4}^{n-4} \frac{(2n-8)!}{(2k-4)!(2n-2k-4)!}\frac{B_{2k}}{2k}\frac{B_{2n-2k}}{2n-2k}$$
$$= -\frac{(n-6)(2n+1)(2n^2-11n+35)}{140(n-2)(n-3)(2n-5)(2n-7)}\frac{B_{2n}}{2n} - \frac{2n-11}{6}B_6B_{2n-6}, \quad n \geq 8.$$

By Eq. (11) and Proposition 3.1, we obtain

Corollary 3.3. *For all nonnegative integers r and $n \geq 2r+2$ we have*

$$\sum_{k=r+1}^{n-r-1} \zeta(2k)\zeta(2n-2k)\prod_{\alpha=1}^{r}\Big\{(2k-\alpha)(2n-2k-\alpha)\Big\}$$
$$= \left((-1)^r r! + \frac{(r!)^2}{2(2r+1)!}\prod_{\beta=r+1}^{2r+1}(2n-\beta)\right)\zeta(2n)\prod_{\alpha=1}^{r}(2n-\alpha) \tag{21}$$
$$- \sum_{k=\lceil r/2\rceil+1}^{r}\prod_{\alpha=1}^{r}(2k-\alpha)\prod_{\beta=2k+1}^{2r+1}(2n-\beta)\left\{\frac{2(-1)^r r!}{(2r-2k+1)!} + \frac{2(2n-2r-2)!}{(2n-2k-r-1)!}\right\}\zeta(2k)\zeta(2n-2k).$$

Example 3.4. When $r=0$ we get [7, (2.4)] . When $r=1$ we recover the formula in Eq. (10). When $r=2$ we find

$$\sum_{j,k\geq 3, j+k=n} (2j-1)(2j-2)(2k-1)(2k-2)\zeta(2j)\zeta(2k)$$
$$= \frac{1}{15}(n-1)(4n^2-1)(2n^2-13n+30)\zeta(2n) - 24(n-2)(2n-5)\zeta(4)\zeta(2n-4), \quad n \geq 4.$$

When $r=3$ we get

$$\sum_{j,k\geq 4, j+k=n} (2j-1)(2j-2)(2j-3)(2k-1)(2k-2)(2k-3)\zeta(2j)\zeta(2k)$$
$$= \frac{1}{35}(n-6)(2n-3)(n-1)(4n^2-1)(2n^2-11n+35)\zeta(2n)$$
$$- 240(2n-11)(n-3)(2n-7)\zeta(6)\zeta(2n-6), \quad n \geq 6.$$

When $r=4$ we have

$$\sum_{j,k\geq 5, j+k=n} \zeta(2j)\zeta(2k)\prod_{\alpha=1}^{4}\Big\{(2j-\alpha)(2k-\alpha)\Big\}$$

$$=\frac{4}{315}(n-2)(2n-3)(n-1)(4n^2-1)(4n^4-72n^3+521n^2-1923n+3780)\zeta(2n)$$
$$-960(n-3)(n-4)(2n-7)(2n-9)\zeta(6)\zeta(2n-6)$$
$$-6720(n-4)(2n-9)(2n^2-25n+81)\zeta(8)\zeta(2n-8),\quad n\geq 8.$$

To prepare for the next theorem concerning DZVs we need the following combinatorial statement.

Lemma 3.5. *Let r be a nonnegative integer and let n be an integer variable such that $n \geq 2r+1$. Then as a polynomial in n*

$$\varphi_r(n)=\binom{2n-1}{2r+1}-2\sum_{k=r+1}^{n-r-1}\binom{2k-1}{r}\binom{2n-2k-1}{r}$$
$$=(-1)^r\binom{n-1}{r}+4\sum_{k=1}^{r}\binom{2k-1}{r}\binom{2n-2k-1}{r} \tag{22}$$

has degree less than or equal to r.

Proof. First we prove that for all integer $m > 2r$

$$\binom{m+1}{2r+1}=\sum_{k=0}^{m}\binom{k}{r}\binom{m-k}{r}. \tag{23}$$

Let S be a set of $m+1$ distinct points on a horizonal line. It is not hard to see there is a one to one correspondence between the following two operations: (i) choose $2r+1$ points from S whose middle point is denoted by P; (ii) choose a point $P \in S$ and then choose r points from the k points to the left of P and r points to the right of P. The two sides of (23) clearly give the number of choices in (i) and (ii), respectively.

Second, we see that for all $m > 2r$, under the substitution $k \to m-k$, we have

$$f(m,r):=\sum_{k=0}^{m}(-1)^k\binom{k}{r}\binom{m-k}{r}=(-1)^m f(m,r).$$

Therefore $f(m,r)=0$ for m odd. We now prove that if $m=2n$ is even then

$$f(2n,r):=\sum_{k=0}^{2n}(-1)^k\binom{k}{r}\binom{2n-k}{r}=(-1)^r\binom{n}{r} \tag{24}$$

by induction on $m+r$. Clearly $f(m,0)=1$. Now for all $r>0$ we have

$$\binom{k}{r}=\binom{k-1}{r}+\binom{k-1}{r-1},\quad \binom{m-k}{r}=\binom{m-k-1}{r}+\binom{m-k-1}{r-1}.$$

Therefore by definition

$$f(m,r)=\sum_{k=0}^{m}(-1)^k\binom{k-1}{r}\binom{m-k-1}{r}+\sum_{k=0}^{m}(-1)^k\binom{k-1}{r-1}\binom{m-k-1}{r}$$

$$
\begin{aligned}
&+\sum_{k=0}^{m}(-1)^k\binom{k-1}{r}\binom{m-k-1}{r-1}+\sum_{k=0}^{m}(-1)^k\binom{k-1}{r-1}\binom{m-k-1}{r-1}\\
=&-f(m-2,r)-f(m-2,r-1)+2\sum_{k=0}^{m}(-1)^k\binom{k-1}{r}\binom{m-k-1}{r-1},
\end{aligned}
$$

since the middle two sums are the same by the substitution $k\to m-k$. It is easy to see that

$$
\binom{k-1}{r}=\sum_{j=1}^{k-r}\binom{k-j-1}{r-1}.
$$

Therefore by induction we get

$$
\begin{aligned}
f(2n,r)=&-f(2n-2,r)-f(2n-2,r-1)+2\sum_{k=0}^{2n}\sum_{j=1}^{k-r}(-1)^k\binom{k-j-1}{r-1}\binom{2n-k-1}{r-1}\\
=&(-1)^r\binom{n-1}{r-1}-(-1)^r\binom{n-1}{r}+2\sum_{j=1}^{2n}\sum_{k=j+r}^{2n}(-1)^k\binom{k-j-1}{r-1}\binom{2n-k-1}{r-1}\\
=&(-1)^r\binom{n-1}{r-1}-(-1)^r\binom{n-1}{r}-2\sum_{j=1}^{2n}(-1)^j f(2n-2-j,r-1).
\end{aligned}
$$

Noticing $f(2n-2-j,r-1)=0$ for odd j we get by induction

$$
\begin{aligned}
f(2n,r)=&(-1)^r\binom{n-1}{r-1}-(-1)^r\binom{n-1}{r}+2(-1)^r\sum_{j=1}^{n-1}\binom{n-1-j}{r-1}\\
=&(-1)^r\binom{n-1}{r-1}+(-1)^r\binom{n-1}{r}\\
=&(-1)^r\binom{n}{r}.
\end{aligned}
$$

Combining (23) and (24) we see that

$$
\begin{aligned}
&\binom{2n-1}{2r+1}-2\sum_{k=0}^{n}\binom{2k-1}{r}\binom{2n-2k-1}{r}\\
&\qquad=\binom{2n-1}{2r+1}-\sum_{k=0}^{2n-2}\binom{k}{r}\binom{2n-2-k}{r}\Big(1-(-1)^k\Big)=(-1)^r\binom{n-1}{r}.
\end{aligned}
$$

This yields the lemma quickly. □

Theorem 3.6. *For all positive integers r and $n\geq 2r+2$ we have*

$$
\sum_{k=r+1}^{n-r-1}\zeta(2k,2n-2k)\prod_{\alpha=1}^{r}\Big\{(2k-\alpha)(2n-2k-\alpha)\Big\}=\sum_{k=0}^{r}c_{r,k}(n)\zeta(2k)\zeta(2n-2k),
$$

where $\zeta(0) = -1/2$, $c_{r,k}(x) \in \mathbb{Q}[x]$ depend only on r and k and have degrees less than or equal to r. More precisely, $c_{r,k} = 0$ for $1 \le k \le \lceil r/2 \rceil$,

$$c_{r,0}(n) = r!^2 \left(\frac{(-1)^r}{2} \binom{2n-1}{r} + \frac{1}{4} \varphi_r(n) \right),$$

where $\varphi_r(n)$ is defined in Lemma 3.5 and for all $k > \lceil r/2 \rceil$

$$c_{r,k}(n) = -\prod_{\alpha=1}^{r} (2k-\alpha) \prod_{\beta=2k+1}^{2r+1} (2n-\beta) \left\{ \frac{(-1)^r r!}{(2r-2k+1)!} + \frac{(2n-2r-2)!}{(2n-2k-r-1)!} \right\}.$$

Proof. We have the stuffle relation

$$\zeta(2k)\zeta(2n-2k) = \zeta(2k, 2n-2k) + \zeta(2n-2k, 2k) + \zeta(2n).$$

Hence the theorem easily follows from Corollary 3.3 together with Lemma 3.5. $\square$

Example 3.7. For all $n \ge 4$ we have

$$\begin{aligned} &\sum_{k=3}^{n-3} \zeta(2k, 2n-2k) \prod_{\alpha=1}^{2} \Big\{ (2n-2k-\alpha)(2k-\alpha) \Big\} \\ =&\frac{1}{2}(57n^2 - 279n + 366)\zeta(2n) - 12(n-2)(2n-5)\zeta(4)\zeta(2n-4). \end{aligned} \tag{25}$$

For all $n \ge 6$

$$\begin{aligned} &\sum_{k=4}^{n-4} \zeta(2k, 2n-2k) \prod_{\alpha=1}^{3} \Big\{ (2n-2k-\alpha)(2k-\alpha) \Big\} \\ =&\frac{1}{2}(1005n^3 - 12222n^2 + 48243n - 62946)\zeta(2n) \\ &- 120(2n-11)(n-3)(2n-7)\zeta(6)\zeta(2n-6). \end{aligned} \tag{26}$$

For all $n \ge 8$

$$\begin{aligned} &\sum_{k=5}^{n-5} \zeta(2k, 2n-2k) \prod_{\alpha=1}^{4} \Big\{ (2n-2k-\alpha)(2k-\alpha) \Big\} \\ =&\frac{1}{2}(31116n^4 - 631800n^3 + 4846020n^2 - 16543800n + 21168864)\zeta(2n) \\ &- 480(n-3)(n-4)(2n-7)(2n-9)\zeta(6)\zeta(2n-6) \\ &- 3360(n-4)(2n-9)(2n^2 - 25n + 81)\zeta(8)\zeta(2n-8). \end{aligned} \tag{27}$$

Another corollary is a result used by Shen and Cai [10, Lemma 5] which they derived by some complicated method using Bernoulli numbers. We can now prove this result rather quickly.

Corollary 3.8. *We have for all $n \geq 2$*

$$\sum_{k=1}^{n-1} k(n-k)\zeta(2k, 2n-2k) = \frac{n}{16}\zeta(2n) + \frac{2n-3}{4}\zeta(2)\zeta(2n-2). \tag{28}$$

Proof. Let S be the left hand side of Eq. (28). By expanding $(2k-1)(2n-2k-1)$ in Eq. (8) we see that

$$\begin{aligned}&\sum_{k=1}^{n-1}(4k(n-k)-2n+1)\zeta(2k, 2n-2k)\\ =&\frac{3}{4}(n-3)\zeta(2n) + (2n-3)(\zeta(2, 2n-2) + \zeta(2n-2, 2))\\ =&\frac{3-5n}{4}\zeta(2n) + (2n-3)\zeta(2)\zeta(2n-2)\end{aligned}$$

By Eq. (5) we have

$$\begin{aligned}4S =&(2n-1)\sum_{k=1}^{n-1}\zeta(2k, 2n-2k) + \frac{3-5n}{4}\zeta(2n) + (2n-3)\zeta(2)\zeta(2n-2)\\ =&\frac{n}{4}\zeta(2n) + (2n-3)\zeta(2)\zeta(2n-2)\end{aligned}$$

which yields the corollary at once. □

Similarly we can replace the factor $k(n-k)$ in Corollary 3.8 by $k^r(n-k)^r$ for any positive integer r. We can even generalize the factor to an arbitrary symmetric function of two variables evaluated at k and $n-k$.

Lemma 3.9. *Let n and r be two nonnegative integers such that $n \geq r+1$. Then*

$$\sum_{k=1}^{n-1} k^r(n-k)^r\zeta(2k, 2n-2k) = \sum_{k=0}^{r} C_{r,k}(n)\zeta(2k)\zeta(2n-2k), \tag{29}$$

where $C_{r,k}(x) \in \mathbb{Q}[x]$ depend only on r and k and have degrees less or equal to r.

Proof. Let

$$g_r(x, y) = \prod_{\alpha=1}^{r}(x-\alpha)(y-\alpha).$$

Define the following recursive sequence of polynomials:

$$f_1(x) = x - 2n + 1, \quad f_{d+1}(x) = f_1(x)f_d(x-2n+1), \quad \forall d \geq 1.$$

It is not hard to see that

$$f_d(x) = \prod_{j=1}^{d}(x - 2jn + j) = \sum_{j=0}^{d} a_{d,j}(n)x^j, \tag{30}$$

where $a_{d,d}(n) = 1$ and all the coefficients $a_{d,j}(n)$ are polynomials in n with integer coefficients of degree $d - j$. Now we claim that for all $d \geq 1$ and $x + y = 2n$ we have

$$g_d(x, y) = f_d(xy). \tag{31}$$

The case $d = 1$ is obvious. By induction

$$\begin{aligned} g_{r+1}(x, y) =&(x-1)(y-1)g_r(x-1, y-1) \\ =&(xy - 2n + 1)f_r((x-1)(y-1)) \\ =&(xy - 2n + 1)f_r(xy - 2n + 1) \\ =&f_{r+1}(xy). \end{aligned}$$

Hence Eq. (31) is proved. Together with Eq. (30) this implies

$$(JK)^r = g_r(J, K) - \sum_{i=0}^{r-1} a_{r,i}(n)(JK)^i$$

for all even numbers $J = 2j$ and $K = 2k$ with $J + K = 2n$. Therefore

$$\sum_{\substack{J,K \text{ even,} \\ J+K=2n}} (JK)^r \zeta(J)\zeta(K) = \sum_{\substack{J,K \text{ even,} \\ J+K=2n}} g_r(J, K)\zeta(J)\zeta(K) - \sum_{i=0}^{r-1} a_{r,i}(n) \sum_{\substack{J,K \text{ even,} \\ J+K=2n}} (JK)^i \zeta(J)\zeta(K)$$

By an easy induction on r and the fact that $\deg_n a_{r,j}(n) = r - j$ the lemma now follows from Theorem 3.6. Notice in particular that the difference in the summation range does not bring in any polynomial coefficients of degree less than or equal to r. This completes the proof of the lemma. □

Example 3.10. We have for all $n \geq 2$

$$\sum_{k=1}^{n-1} k^2(n-k)^2 \zeta(2k, 2n-2k) = \frac{3}{32}(3n-2)(n-1)\zeta(2n) - \frac{3}{4}\zeta(4)\zeta(2n-4), \tag{32}$$

$$\begin{aligned} &\sum_{k=1}^{n-1} k^3(n-k)^3 \zeta(2k, 2n-2k) \\ =& -\frac{1}{256}n(2n^2-3)\zeta(2n) + \frac{1}{64}(2n-3)(28n^2-48n+21)\zeta(2)\zeta(2n-2) \\ &+ \frac{3}{16}(2n-5)(2n^2-25n+35)\zeta(4)\zeta(2n-4) + \frac{45}{8}(2n-7)\zeta(6)\zeta(2n-6), \end{aligned} \tag{33}$$

$$\begin{aligned} &\sum_{k=1}^{n-1} k^4(n-k)^4 \zeta(2k, 2n-2k) \\ =& -\frac{1}{1024}n(16n^2-17)\zeta(2n) + \frac{1}{256}(2n-3)(6n-5)(20n^2-36n+17)\zeta(2)\zeta(2n-2) \\ &+ \frac{3}{64}(2n-5)(40n^3-420n^2+1050n-777)\zeta(4)\zeta(2n-4) \end{aligned}$$

$$-\frac{15}{16}(2n-7)(8n^2-98n+189)\zeta(6)\zeta(2n-6)-\frac{315}{4}(2n-9))\zeta(8)\zeta(2n-8). \tag{34}$$

To show Eq. (32) we suppose $j+k=n$. It is easy to see that

$$\prod_{\alpha=1}^{2}\Big\{(2j-\alpha)(2k-\alpha)\Big\}=16j^2k^2+4(5-6n)jk+4(2n-1)(n-1).$$

So Eq. (32) follows from Eqs. (5), (25), and (28) immediately. The other formulas can be proved similarly.

Theorem 3.11. *Let $F(x,y)=F(y,x)\in\mathbb{Q}[x,y]$ be a symmetric polynomial of degree r. Suppose $d=\deg_x F(x,y)$. Then for every positive integer $n\geq 2$ we have*

$$\sum_{k=1}^{n-1}F(k,n-k)\zeta(2k,2n-2k)=\sum_{k=0}^{\lfloor r/2\rfloor}c_{F,k}(n)\zeta(2k)\zeta(2n-2k),$$

where $c_{F,k}(x)\in\mathbb{Q}[x]$ depends only on k and F and has degrees less than or equal to d.

Proof. It is a well-known fact that any symmetric polynomial $F(x,y)\in\mathbb{Q}[x,y]$ of degree r is a linear combination of symmetric binomials $x^dy^{r-d}+x^{r-d}y^d$ for $d=\lceil r/2\rceil,\dots,r$. Let $\sigma_1=x+y$, $\sigma_2=xy$ and $a=r-d$. By induction on the difference $d-a$ it is easy to show that

$$x^dy^a+x^ay^d=\sigma_1^{d-a}\sigma_2^a+\sum_{j=a+1}^{\lfloor r/2\rfloor}c_j\sigma_1^{r-2j}\sigma_2^j$$

for some $c_j\in\mathbb{Z}$. Hence the theorem follows from Lemma 3.9 immediately. □

4. Weighted sum formulas for triple zeta values

In this section we will derive weighted sum formula for triple zeta values $\zeta(2i,2j,2k)$ for fixed $i+j+k$ when the weight factors are arbitrary symmetric polynomials in i, j and k. Similar to the double zeta case we will first consider weighted sum formula for triple products of Bernoulli numbers which leads to weighted sum formula for triple products of Riemann zeta values. Then we may apply the stuffle relations to derive the desired weighted sum formula for triple zeta values.

We begin with the following triple sum:

$$Z_3(s;r_1,r_2,0)=\sum_{m_1,m_2,m_3=1}^{\infty}m_1^{r_1}m_2^{r_2}(m_1+m_2+m_3)^{-s}.$$

By the substitution $m\to m-m_1-m_2$ we see easily that

$$\begin{aligned}Z_3(s;r_1,r_2,0)&=\sum_{m=1}^{\infty}\sum_{m_1=1}^{m-1}\sum_{m_2=1}^{m-m_1-1}m_1^{r_1}m_2^{r_2}m^{-s}\\&=\sum_{m=1}^{\infty}m^{-s}\sum_{m_1=1}^{m-1}m_1^{r_1}\sum_{m_2=1}^{m-m_1-1}m_2^{r_2}\end{aligned}$$

$$=\sum_{m=1}^{\infty} m^{-s}\sum_{m_1=1}^{m-1} m_1^{r_1}\left(\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^k B_k}{r_2+1}(m-m_1)^{r_2+1-k}-(m-m_1)^{r_2}\right)$$
$$=\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^k B_k}{r_2+1}\sum_{m=1}^{\infty} m^{-s}\sum_{m_1=1}^{m-1} m_1^{r_1}(m-m_1)^{r_2+1-k}-Z_2(s;r_1,r_2)$$
$$=\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^k B_k}{r_2+1}Z_2(s;r_1,r_2+1-k)-Z_2(s;r_1,r_2). \tag{35}$$

By Proposition 2.3, the first sigma sum of the above equation equals

$$\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^k B_k}{r_2+1}\tau(r_1,r_2+1-k)\zeta(s+k-r_1-r_2-2) \tag{36}$$
$$+\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^{r_2+1} B_k}{r_2+1}\sum_{l=r_2+2-k}^{r_1+r_2+2-k}\binom{r_1}{l+k-r_2-2}\beta_l\zeta(s+k+l-r_1-r_2-2) \tag{37}$$
$$+\sum_{k=0}^{r_2}\binom{r_2+1}{k}\frac{(-1)^{r_1+k} B_k}{r_2+1}\sum_{l=r_1+1}^{r_1+r_2+2-k}\binom{r_2+1-k}{l-r_1-1}\beta_l\zeta(s+k+l-r_1-r_2-2). \tag{38}$$

For the first part of the above equation, breaking away the term for $k=0$ we get

$$(36)=\frac{r_1!r_2!}{(r_1+r_2+2)!}\zeta(s-r_1-r_2-2)+\sum_{k=1}^{r_2}\frac{(-1)^k r_1!r_2!}{(r_1+r_2-k+2)!}\frac{B_k}{k!}\zeta(s-r_1-r_2+k-2).$$

For the second part of the equation,

$$(37)=-\frac{(-1)^{r_2}}{r_2+1}\sum_{j=r_2+2}^{r_1+r_2+2}\binom{r_1}{j-r_2-2}\frac{B_j}{j}\zeta(s+j-r_1-r_2-2)$$
$$-\sum_{j=r_2+2}^{r_1+r_2+2}(-1)^{r_2}\sum_{k=1}^{r_2}\binom{r_2}{k-1}\binom{r_1}{j-r_2-2}\frac{B_kB_{j-k}}{k(j-k)}\zeta(s+j-r_1-r_2-2).$$

For the third part of the equation,

$$(38)=\sum_{j=r_1+1}^{r_1+r_2+2}\frac{(-1)^{r_1}}{r_2+1}\binom{r_2+1}{j-r_1-1}\frac{B_j}{j}\zeta(s+j-r_1-r_2-2)$$
$$+\sum_{k=1}^{r_2}\sum_{j=r_1+1+k}^{r_1+r_2+2}(-1)^{r_1+k}\binom{r_2}{k-1}\binom{r_2+1-k}{j-k-r_1-1}\frac{B_kB_{j-k}}{k(j-k)}\zeta(s+j-r_1-r_2-2).$$

Setting $s=r_1+r_2+3-2n$ in the above equations and using Proposition 2.3 and Eq. (11), from Eq. (35) we can get

$$Z_3(r_1+r_2+3-2n;r_1,r_2,0)+Z_2(r_1+r_2+3-2n;r_1,r_2)$$
$$=-\frac{r_1!r_2!}{(r_1+r_2+2)!}\frac{B_{2n}}{2n}-\sum_{k=1}^{r_2}\frac{(-1)^k r_1!r_2!}{(r_1+r_2-k+2)!}\frac{B_k}{k!}\frac{B_{2n-k}}{2n-k}$$

$$+\sum_{k=0}^{r_2}\sum_{j=r_2+2}^{r_1+r_2+2}\frac{(-1)^{r_2}}{r_2+1}\binom{r_2+1}{k}\binom{r_1}{j-r_2-2}\frac{B_kB_{j-k}}{j-k}\frac{B_{2n-j}}{2n-j}$$
$$-\sum_{k=0}^{r_2}\sum_{j=r_1+1+k}^{r_1+r_2+2}\frac{(-1)^{r_1+k}}{r_2+1}\binom{r_2+1}{k}\binom{r_2+1-k}{j-k-r_1-1}\frac{B_kB_{j-k}}{j-k}\frac{B_{2n-j}}{2n-j}. \tag{39}$$

On the other hand, we can compute the value of $Z_3(s;r_1,r_2,r_3)$ using some integrals:

$$\begin{aligned}Z_3(s;r_1,r_2,r_3) =&J^3\Big(x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\Big)\\ &+\sum_{\mathrm{cyc}(r_1,r_2,r_3)}J^2\Big(\int_0^{-x_1-x_2}x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_3\Big)\\ &+\sum_{\mathrm{cyc}(r_1,r_2,r_3)}J^1\Big(\int_0^{-x_1}\int_0^{-x_1-x_2}x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_3dx_2\Big).\end{aligned}$$

For J^3 we use (15) to get

$$\begin{aligned}&J^3\Big(\sum_{i+j+k=-s}\binom{-s}{i,j,k}x_1^{r_1+i}x_2^{r_2+j}x_3^{r_3+k}\Big)\\ =&\sum_{i+j+k=-s}\binom{-s}{i,j,k}(-1)^{r_1+r_2+r_3-s}\beta_{r_1+i+1}\beta_{r_2+j+1}\beta_{r_3+k+1}.\end{aligned}$$

For the integral in J^2 we use the substitution $x_3=-(x_1+x_2)t$ and Eq. (15) to get

$$\begin{aligned}&J^2\Big(\int_0^{-x_1-x_2}x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_3\Big)\\ =&J^2\Big(\int_0^1 x_1^{r_1}x_2^{r_2}(-x_1-x_2)^{r_3}t^{r_3}(x_1+x_2)^{-s}(1-t)^{-s}(-x_1-x_2)\,dt\Big)\\ =&J^2\Big((-1)^{r_3+1}x_1^{r_1}x_2^{r_2}(x_1+x_2)^{r_3+1-s}\int_0^1 t^{r_3}(1-t)^{-s}\,dt\Big)\\ =&J^2\Big(x_1^{r_1}x_2^{r_2}(x_1+x_2)^{r_3+1-s}\Big)(-1)^{r_3+1}\tau(r_3,-s)\\ =&\sum_{i+j=r_3+1-s}\binom{r_3+1-s}{i,j}(-1)^{r_1+r_2-s}\beta_{r_1+i+1}\beta_{r_2+j+1}\tau(r_3,-s).\end{aligned}$$

So

$$\begin{aligned}&J^2\Big(\int_0^{-x_1-x_3}x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_2\Big)\\ =&\sum_{i+j=r_2+1-s}\binom{r_2+1-s}{i,j}(-1)^{r_1+r_3-s}\beta_{r_1+i+1}\beta_{r_3+j+1}\tau(r_2,-s),\end{aligned}$$

and

$$J^2\Big(\int_0^{-x_2-x_3}x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_2\Big)$$

$$= \sum_{i+j=r_1+1-s} \binom{r_1+1-s}{i,j} (-1)^{r_2+r_3-s} \beta_{r_2+i+1}\beta_{r_3+j+1}\tau(r_1,-s).$$

For the integral in J^1 we use substitution $x_3 = -(x_1+x_2)t_1$, and (16) to get

$$\begin{aligned}
&J^1\Big(\int_0^{-x_1}\int_0^{-x_1-x_2} x_1^{r_1}x_2^{r_2}x_3^{r_3}(x_1+x_2+x_3)^{-s}\,dx_3dx_2\Big)\\
=&J^1\Big(\int_0^{-x_1}(-1)^{r_3+1}x_1^{r_1}x_2^{r_2}(x_1+x_2)^{r_3+1-s}dx_2\int_0^1 t^{r_3}(1-t)^{-s}\,dt_1\Big)\\
=&(-1)^{r_1-s+r_3+1}\tau(r_2,-s+r_3+1)\beta_{r_1+r_2-s+r_3+1+2}(-1)^{r_3+1}\tau(r_3,-s)\\
=&(-1)^{r_1-s}\tau(r_2,-s+r_3+1)\tau(r_3,-s)\beta_{r_1+r_2+r_3-s+3}\\
=&(-1)^{r_1-s}\frac{r_2!r_3!(-s)!}{(r_2+r_3-s+2)!}\beta_{r_1+r_2+r_3-s+3}.
\end{aligned}$$

Combining the above equations by setting $r_3=0, s=r_1+r_2+3-2n$ we find

$$\begin{aligned}
&Z_3(r_1+r_2+3-2n;r_1,r_2,0)\\
=&-\sum_{i+j+k=2n-3-r_1-r_2}\binom{2n-3-r_1-r_2}{i,j,k}\beta_{r_1+i+1}\beta_{r_2+j+1}\beta_{k+1}\\
&-\sum_{i+j=2n-2-r_1-r_2}\binom{2n-2-r_1-r_2}{i,j}\beta_{r_1+i+1}\beta_{r_2+j+1}\tau(0,2n-3-r_1-r_2)\\
&-\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\sum_{i+j=2n-2-r_1}\binom{2n-2-r_1}{i,j}\beta_{r_1+i+1}\beta_{j+1}\tau(r_2,2n-3-r_1-r_2)\\
&+(-1)^{-s}\frac{r_1!r_2!(-s)!}{(r_1+r_2-s+2)!}\beta_{2n}+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1-s}\frac{r_2!r_3!(-s)!}{(r_2+r_3-s+2)!}\beta_{2n}\\
=&-\sum_{i+j+k=2n-3-r_1-r_2}\binom{2n-3-r_1-r_2}{i,j,k}\frac{B_{r_1+i+1}B_{r_2+j+1}B_{k+1}}{(r_1+i+1)(r_2+j+1)(k+1)}\\
&-\frac{1}{2n-2-r_1-r_2}\sum_{i+j=2n-2-r_1-r_2}\binom{2n-2-r_1-r_2}{i,j}\frac{B_{r_1+i+1}}{r_1+i+1}\frac{B_{r_2+j+1}}{r_2+j+1}\\
&-\operatorname*{Cyc}_{r_1,r_2}\frac{(-1)^{r_2}r_2!(2n-3-r_1-r_2)!}{(2n-2-r_1)!}\sum_{i+j=2n-2-r_1}\binom{2n-2-r_1}{i,j}\frac{B_{r_1+i+1}}{r_1+i+1}\frac{B_{j+1}}{j+1}\\
&-\left((-1)^{r_1+r_2}\frac{r_1!r_2!(2n-3-r_1-r_2)!}{(2n-1)!}+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\frac{r_2!(2n-3-r_1-r_2)!}{(2n-1-r_1)!}\right)\frac{B_{2n}}{2n}.
\end{aligned}$$

By comparing this with Eq. (39) we obtain

$$(2n-3-r_1-r_2)!\sum_{\substack{i+j+k=2n\\ i,j,k\geq 1}}\frac{B_iB_jB_k}{i!j!k!}\left\{\prod_{a=1}^{r_1}(i-a)\prod_{b=1}^{r_2}(j-b)\right\}$$

$$
\begin{aligned}
&+(2n-3-r_1-r_2)!\sum_{\substack{i+j=2n\\ i,j\geq 1}}\frac{B_i}{i!}\frac{B_j}{j!}\left\{\prod_{a=1}^{r_1}(i-a)\prod_{b=1}^{r_2}(j-b)\right\}\\
&+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}r_2!(2n-3-r_1-r_2)!\sum_{\substack{i+j=2n\\ i,j\geq 1}}\frac{B_i}{i!}\frac{B_j}{j!}\left\{\prod_{a=1}^{r_1}(i-a)\right\}\\
&+\left((-1)^{r_1+r_2}\frac{r_1!r_2!(2n-3-r_1-r_2)!}{(2n-1)!}+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1}\frac{r_1!(2n-3-r_1-r_2)!}{(2n-1-r_2)!}\right)\frac{B_{2n}}{2n}\\
=&\frac{r_1!r_2!}{(r_1+r_2+2)!}\frac{B_{2n}}{2n}+\sum_{k=1}^{r_2}\frac{(-1)^k r_1!r_2!}{(r_1+r_2-k+2)!}\frac{B_k}{k!}\frac{B_{2n-k}}{2n-k}\\
&-\sum_{k=0}^{r_2}\sum_{j=r_2+2}^{r_1+r_2+2}\frac{(-1)^{r_2}}{r_2+1}\binom{r_2+1}{k}\binom{r_1}{j-r_2-2}\frac{B_kB_{j-k}}{j-k}\frac{B_{2n-j}}{2n-j}\\
&+\sum_{k=0}^{r_2}\sum_{j=r_1+1+k}^{r_1+r_2+2}\frac{(-1)^{r_1+k}}{r_2+1}\binom{r_2+1}{k}\binom{r_2+1-k}{j-k-r_1-1}\frac{B_kB_{j-k}}{j-k}\frac{B_{2n-j}}{2n-j}+Z_2(r_1+r_2+3-2n;r_1,r_2).
\end{aligned}
$$

By Proposition 2.3 we have

$$
\begin{aligned}
Z_2(r_1+r_2+3-2n;r_1,r_2)&=-\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1}\sum_{k=r_1+1}^{r_1+r_2+1}\binom{r_2}{k-r_1-1}\beta_k\beta_{2n-k-1}\\
&=(\delta_{r_1,0}+\delta_{r_2,0})\frac{B_{2n-2}}{4n-4}. \qquad (40)
\end{aligned}
$$

Using Theorem 2.4 and (40) we can obtain the next result.

Theorem 4.1. *For any nonnegative integers r_1 and r_2 and positive integer $n \geq r_1+r_2=2$ we have*

$$
\begin{aligned}
&\frac{1}{r_1!r_2!}\sum_{\substack{i+j+k=2n\\ i,j,k\geq 1}}\frac{B_iB_jB_k}{i!j!k!}\left\{\prod_{a=1}^{r_1}(i-a)\prod_{b=1}^{r_2}(j-b)\right\}\\
=&\operatorname*{Cyc}_{r_1,r_2}\left(\frac{(-1)^{r_1+r_2}B_{r_1+1}B_{2n-r_1-1}}{(r_1+1)!(2n-r_1-1)!}+\frac{\delta_{r_1,0}}{2}\binom{2n-3}{r_2}\frac{B_{2n-2}}{(2n-2)!}\right)\\
&+\left(\binom{2n}{r_1+r_2+2}+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\binom{2n}{r_1+1}+(-1)^{r_1+r_2}\right)\frac{B_{2n}}{(2n)!}\\
&+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_1}\sum_{k=r_1+1}^{r_1+r_2+1}\binom{k-1}{r_1}\binom{2n-k-1}{r_2+r_1+1-k}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}\\
&+\operatorname*{Cyc}_{r_1,r_2}(-1)^{r_2}\sum_{k=1}^{r_1+1}\binom{2n-1-k}{r_1+1-k}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}
\end{aligned}
$$

$$+\sum_{k=1}^{r_2}(-1)^k\binom{2n-1-k}{r_1+r_2+2-k}\frac{B_k}{k!}\frac{B_{2n-k}}{(2n-k)!}$$
$$-\sum_{k=0}^{r_2}\sum_{j=r_2+2}^{r_1+r_2+1}(-1)^{r_2}\binom{j-1-k}{r_2+1-k}\binom{2n-1-j}{r_1+r_2+2-j}\frac{B_kB_{j-k}B_{2n-j}}{k!(j-k)!(2n-j)!}$$
$$+\sum_{k=0}^{r_2}\sum_{j=r_1+1+k}^{r_1+r_2+1}(-1)^{r_1+k}\binom{j-1-k}{r_1}\binom{2n-1-j}{r_1+r_2+2-j}\frac{B_kB_{j-k}B_{2n-j}}{k!(j-k)!(2n-j)!}.$$

Proof. We only need to show that all the terms with $j=r_1+r_2+2$ cancel out. Indeed, for such a term to be nonzero both k and r_1+r_2 must be even. Hence the the sign $(-1)^{r_2}$ in the penultimate sum and the sign $(-1)^{r_1+k}$ in the last sum are the same. Moreover

$$\binom{j-1-k}{r_2+1-k}=\binom{j-1-k}{r_1}$$

when $j=r_1+r_2+2$. Therefore all these terms cancel out. □

Example 4.2. Set $\sum=\sum_{\substack{i+j+k=2n\\ i,j,k\geq 1}}$. For small r_1 and r_2 we may use Theorem 4.1 for $n\geq r_1+r_2+2$ and direct computation for small n to verify the following identities which are valid for all $n\geq 2$:

$$\begin{aligned}
\sum\frac{B_iB_jB_k}{i!j!k!}&=\binom{2n+2}{2}\frac{B_{2n}}{(2n)!}+\frac{B_{2n-2}}{(2n-2)!},\\
\sum i\frac{B_iB_jB_k}{i!j!k!}&=\binom{2n+2}{3}\frac{B_{2n}}{(2n)!}+\frac{2n}{3}\frac{B_{2n-2}}{(2n-2)!},\\
\sum ij\frac{B_iB_jB_k}{i!j!k!}&=\binom{2n+2}{4}\frac{B_{2n}}{(2n)!}-\frac{2n^2-19n+12}{12}\frac{B_{2n-2}}{(2n-2)!},\\
\sum i^2j\frac{B_iB_jB_k}{i!j!k!}&=\frac{4n+1}{5}\binom{2n+2}{4}\frac{B_{2n}}{(2n)!}+\frac{10n^2-11n+6}{12}\frac{B_{2n-2}}{(2n-2)!}-\frac{2n-5}{60}\frac{B_{2n-4}}{(2n-4)!},\\
\sum ijk\frac{B_iB_jB_k}{i!j!k!}&=\binom{2n+2}{5}\frac{B_{2n}}{(2n)!}-\frac{2n^3-9n^2+n+6}{6}\frac{B_{2n-2}}{(2n-2)!}+\frac{2n-5}{30}\frac{B_{2n-4}}{(2n-4)!},\\
\sum i^3\frac{B_iB_jB_k}{i!j!k!}&=\frac{12n^2+12n+1}{10}\binom{2n+2}{3}\frac{B_{2n}}{(2n)!}\\
&\quad+\frac{20n^3-48n^2+35n-6}{6}\frac{B_{2n-2}}{(2n-2)!}+\frac{2n-5}{30}\frac{B_{2n-4}}{(2n-4)!},\\
\sum i^2j^2\frac{B_iB_jB_k}{i!j!k!}&=\frac{8n^2+4n+3}{15}\binom{2n+2}{4}\frac{B_{2n}}{(2n)!}\\
&\quad+\frac{8n^4-72n^3+232n^2-261n+108}{36}\frac{B_{2n-2}}{(2n-2)!}-\frac{(7n+3)(2n-5)}{180}\frac{B_{2n-4}}{(2n-4)!},\\
\sum i^3j\frac{B_iB_jB_k}{i!j!k!}&=\frac{4n^2+2n-1}{5}\binom{2n+2}{4}\frac{B_{2n}}{(2n)!}
\end{aligned}$$

$$+\frac{32n^3-98n^2+107n-36}{12}\frac{B_{2n-2}}{(2n-2)!}-\frac{(n-1)(2n-5)}{60}\frac{B_{2n-4}}{(2n-4)!},$$

$$\sum i^4\frac{B_iB_jB_k}{i!j!k!}=\frac{(8n^2+8n-1)(2n+1)}{10}\binom{2n+2}{3}\frac{B_{2n}}{(2n)!}$$
$$+\frac{40n^4-128n^3+168n^2-119n+36}{6}\frac{B_{2n-2}}{(2n-2)!}+\frac{(3n-1)(2n-5)}{60}\frac{B_{2n-4}}{(2n-4)!}.$$

For all $n\geq 3$ we have

$$\sum i^5\frac{B_iB_jB_k}{i!j!k!}=\frac{(8n^2+8n+1)(4n^2+4n-1)}{14}\binom{2n+2}{3}\frac{B_{2n}}{(2n)!}$$
$$+\frac{80n^5-320n^4+560n^3-520n^2+219n-30}{6}\frac{B_{2n-2}}{(2n-2)!}$$
$$+\frac{40n^4-128n^3+168n^2-119n+36}{6}\frac{B_{2n-4}}{(2n-4)!}$$
$$+\frac{(2n-5)(2n^2-6n+7)}{6}\frac{B_{2n-6}}{(2n-6)!}.$$

Theorem 4.3. *Let $F(x,y,z)\in\mathbb{Q}[x,y,z]$ be a polynomial of degree d. Then for every positive integer $n\geq d+2$ we have*

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}}F(i,j,k)\zeta(2i)\zeta(2j)\zeta(2k)=\sum_{k=0}^{\lfloor (d+1)/2\rfloor}K_{F,k}(n)\zeta(2k)\zeta(2n-2k),\tag{41}$$

where $K_{F,k}(x)$ is a polynomial in x depending only on F and k which can be explicitly given using Theorem 4.1. Moreover, $\deg K_{F,0}(x)=d+2$ and $\deg K_{F,k}(x)\leq d$ for all $k\geq 1$.

Proof. We can apply Euler's identity in Eq. (11) to Theorem 4.1 to derive a weighted sum formula for the triple product of Riemann zeta values. Then we may use Eq. (18) to change the weight factors to any monomial of the form $i^{r_1}j^{r_2}$. Finally we can deduce the theorem by observing that $i^{r_1}j^{r_2}k^{r_3}=i^{r_1}j^{r_2}(n-i-j)^{r_3}$. ☐

Example 4.4. Setting $\sum=\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}}$ we have

$$\sum\zeta(2i)\zeta(2j)\zeta(2k)=\frac{(n+1)(2n+1)}{4}\zeta(2n)-\frac{3}{2}\zeta(2n-2)\zeta(2),\tag{42}$$

$$\sum i\zeta(2i)\zeta(2j)\zeta(2k)=\frac{n(n+1)(2n+1)}{12}\zeta(2n)-\frac{n}{2}\zeta(2n-2)\zeta(2),$$

$$\sum ij\zeta(2i)\zeta(2j)\zeta(2k)=\frac{n(4n^2-1)(n+1)}{96}\zeta(2n)+\frac{(2n-1)(n-3)}{8}\zeta(2n-2)\zeta(2),\tag{43}$$

$$\sum i^2j\zeta(2i)\zeta(2j)\zeta(2k)=\frac{n(4n+1)(4n^2-1)(n+1)}{960}\zeta(2n)$$
$$+\frac{(2n-1)(n-3)}{16}\zeta(2n-2)\zeta(2)+\frac{3(2n-5)}{4}\zeta(2n-4)\zeta(4),$$

$$\begin{aligned}\sum ijk\zeta(2i)\zeta(2j)\zeta(2k) =&\frac{n(n^2-1)(4n^2-1)}{480}\zeta(2n)\\ &+\frac{(n-1)(n-3)(2n-1)}{8}\zeta(2)\zeta(2n-2)+\frac{3}{2}(2n-5)\zeta(4)\zeta(2n-4).\end{aligned}$$

Theorem 4.5. *Let $F(x,y,z)\in\mathbb{Q}[x,y,z]$ be a symmetric polynomial of degree d. Then for every positive integer $n\geq d+2$ we have*

$$\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} F(i,j,k)\zeta(2i,2j,2k)=\sum_{k=0}^{\lfloor (d+1)/2\rfloor} C_{F,k}(n)\zeta(2k)\zeta(2n-2k), \tag{44}$$

where $C_{F,k}(x)$ is a polynomial in x depending only on F and k with $\deg C_{F,k}(x)\leq d+2$ for all $k\geq 0$.

Proof. Let S_3 be the permutation group of $\{1,2,3\}$. For any function $f(x_1,x_2,x_3)$ of three variables we set

$$\operatorname*{Sym}_{x_1,x_2,x_3} f(x_1,x_2,x_3)=\sum_{\sigma\in S_3} f(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}).$$

If $i+j+k=n$ then by the stuffle relation we have

$$2\zeta(2i)\zeta(2j)\zeta(2k)=2\operatorname*{Sym}_{i,j,k}\zeta(2i,2j,2k)+\operatorname*{Sym}_{i,j,k}\zeta(2i+2j)\zeta(2k)-4\zeta(2n).$$

Therefore by setting $\sum=\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}}$ we get

$$\begin{aligned}\sum F(i,j,k)\zeta(2i)\zeta(2j)\zeta(2k)=6&\sum F(i,j,k)\zeta(2i,2j,2k)\\ &+3\sum F(i,j,k)\zeta(2i+2j)\zeta(2k)-2\sum F(i,j,k)\zeta(2n).\end{aligned}$$

An easy computation shows that

$$\begin{aligned}\sum i^a j^b\zeta(2i+2j)\zeta(2k)&=\sum_{l=1}^{n-1}\sum_{i=1}^{l-1} i^a(l-i)^b\zeta(2l)\zeta(2n-2l)=\sum_{l=1}^{n-1} f(l)\zeta(2l)\zeta(2n-2l),\\ \sum i^a k^b\zeta(2i+2j)\zeta(2k)&=\sum_{l=1}^{n-1}\sum_{i=1}^{l-1} i^a(n-l)^b\zeta(2l)\zeta(2n-2l)=\sum_{l=1}^{n-1} g(l,n)\zeta(2l)\zeta(2n-2l),\\ \sum i^a j^b\zeta(2n)&=\sum_{l=1}^{2n-2}\sum_{i=1}^{l-1} i^a(l-i)^b\zeta(2n)=\sum_{l=1}^{n-1} f(l)\zeta(2n)=h(n)\zeta(2n),\end{aligned}$$

where $f(x),g(x,y),h(x)\in\mathbb{Q}[x,y]$ are polynomials of degree at most $a+b+1$, $a+b+1$ and $a+b+2$, respectively. Hence the theorem follows from Theorem 2.7 quickly. □

Example 4.6. Let $e_m(x,y,z)$ be the m-th elementary symmetric polynomial of x,y,z. Setting $\sum=\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}}$ we have

$$\sum\zeta(2i,2j,2k)=\frac{5}{8}\zeta(2n)-\frac{1}{4}\zeta(2)\zeta(2n-2), \tag{45}$$

$$\sum(ij+jk+ki)\,\zeta(2i,2j,2k) = \frac{5n}{64}\zeta(2n)+\frac{4n-9}{16}\zeta(2)\zeta(2n-2), \tag{46}$$

$$\sum(i^2+j^2+k^2)\,\zeta(2i,2j,2k) = \frac{5n(4n-1)}{32}\zeta(2n)-\frac{2n^2+4n-9}{8}\zeta(2)\zeta(2n-2), \tag{47}$$

$$\begin{aligned}\sum\Big(\operatorname*{Sym}_{i,j,k} i^2j\Big)\,\zeta(2i,2j,2k) =& \frac{n(10n-3)}{128}\zeta(2n)+\frac{8n^2-18n+3}{32}\zeta(2)\zeta(2n-2)\\ &-\frac{3(2n-5)}{8}\zeta(4)\zeta(2n-4),\end{aligned} \tag{48}$$

$$\begin{aligned}\sum(i^3+j^3+k^3)\,\zeta(2i,2j,2k) =& \frac{n(80n^2-30n+3)}{128}\zeta(2n)+\frac{3(2n-5)}{8}\zeta(4)\zeta(2n-4)\\ &-\frac{8n^3+24n^2-54n+3}{32}\zeta(2)\zeta(2n-2),\end{aligned} \tag{49}$$

$$\sum ijk\,\zeta(2i,2j,2k) = \frac{n}{128}\zeta(2n)-\frac{1}{32}\zeta(2)\zeta(2n-2)+\frac{2n-5}{8}\zeta(4)\zeta(2n-4). \tag{50}$$

We remark that Eq. (45) is a result of Shen and Cai [10]. We now prove Eq. (46). The others can be proved using the same idea. First we have for all nonnegative integers a and b

$$\begin{aligned}\sum i^aj^b\zeta(2i)\zeta(2j)\zeta(2k) =& \sum\operatorname*{Sym}_{i,j,k} i^aj^b\zeta(2i,2j,2k)+\sum i^aj^b\zeta(2i+2j)\zeta(2k)\\ &+\sum i^ak^b\zeta(2i+2j)\zeta(2k)+\sum i^bk^a\zeta(2i+2j)\zeta(2k)-2\sum i^aj^b\zeta(2n).\end{aligned} \tag{51}$$

For Eq. (46) we note that $\operatorname*{Sym}_{i,j,k} ij = 2e_2(i,j,k)$ so we need the following:

$$\sum ij\,\zeta(2i+2j)\zeta(2k) = \sum_{l=1}^{n-1}\sum_{i=1}^{l-1} i(l-i)\zeta(2l)\zeta(2n-2l) = \frac{1}{6}\sum_{\substack{l+k=n\\ l,k\geq 1}}(l^3-l)\zeta(2l)\zeta(2k),$$

$$\sum ik\,\zeta(2i+2j)\zeta(2k) = \sum_{l=1}^{n-1}\sum_{i=1}^{l-1} i(n-l)\zeta(2l)\zeta(2n-2l) = \frac{1}{2}\sum_{\substack{l+k=n\\ l,k\geq 1}}\big((n+1)l^2-l^3-nl\big)\zeta(2l)\zeta(2k),$$

$$\sum ij\,\zeta(2n) = \sum_{l=1}^{n-1}\sum_{i=1}^{l-1} i(n-l)\zeta(2n) = \binom{n+1}{4}\zeta(2n).$$

Similarly, by using Eq. (51) and the symmetric function

$$\operatorname*{Sym}_{i,j,k} i^2j = e_1(i,j,k)e_2(i,j,k)-3e_3(i,j,k),$$

we see that to show Eqs. (48) and (49) we need the following:

$$\sum_{i=1}^{l-1} i^2(l-i) = \frac{1}{12}(l^4-l^2), \qquad \sum_{i=1}^{l-1} i^3 = \frac{1}{4}(l^4-l^2),$$

$$\sum_{i=1}^{l-1} i^2(n-l) =\frac{1}{6}l(l-1)(2l-1)(n-l) = \frac{1}{6}\Big((2n+3)l^3-(3n+1)l^2+nl-2l^4\Big),$$
$$\sum_{i=1}^{l-1}(n-l)^3 =l^4-(3n+1)l^3+3(n^2+n)l^2-(n^3+3n^2)l+n^3,$$
$$\sum_{i=1}^{l-1} i(n-l)^2 =\frac{1}{2}l(l-1)(n-l)^2 = \frac{1}{2}\Big(l^4-(2n+1)l^3+(n^2+2n)l^2-n^2l\Big),$$
$$\sum i^2 j =\frac{1}{12}\sum_{l=1}^{n-1}(l^4-l^2) = \frac{2n-1}{5}\binom{n+1}{4}, \qquad \sum i^3 = \frac{3(2n-1)}{5}\binom{n+1}{4}.$$

Using Eqs. (19)-(20), (43) and (51) we can get Eqs. (46), (48) and (49).

Now multiplying $(i+j+k)^2=n^2$ on Eq. (45) and comparing with Eq. (46) we can prove Eq. (47) easily. Similarly, by multiplying $(i+j+k)^2=n^3$ on Eq. (45) we can readily deduce Eq. (50) from Eqs. (48) and (49). Exactly the same ideas lead to the following:

$$\sum(i^2jk+j^2ki+k^2ij)\,\zeta(2i,2j,2k) = \frac{n^2}{128}\zeta(2n)-\frac{n}{32}\zeta(2)\zeta(2n-2)+\frac{n(2n-5)}{8}\zeta(4)\zeta(2n-4),$$
$$\sum(i^2j^2+j^2k^2+k^2i^2)\,\zeta(2i,2j,2k) = -\frac{n(2n-5)}{256}\zeta(2n)$$
$$+\frac{3(12n^2-30n+17)}{64}\zeta(2)\zeta(2n-2)-\frac{4n^2+20n-75}{16}\zeta(4)\zeta(2n-4),$$
$$\sum\Big(\operatorname*{Sym}_{i,j,k} i^3j\Big)\,\zeta(2i,2j,2k) = \frac{n(10n^2-3n-5)}{128}\zeta(2n)$$
$$+\frac{8n^3-54n^2+95n-51}{32}\zeta(2)\zeta(2n-2)-\frac{3(2n^2-15n+25)}{8}\zeta(4)\zeta(2n-4),$$
$$\sum(i^4+j^4+k^4)\,\zeta(2i,2j,2k) = \frac{n(80n^3-40n^2+6n+5)}{128}\zeta(2n)$$
$$-\frac{8n^4+32n^3-108n^2+98n-51}{32}\zeta(2)\zeta(2n-2)+\frac{3(4n^2-20n+25)}{8}\zeta(4)\zeta(2n-4).$$

Moreover, one checks easily that a suitable linear combination of the four identities above yields Eq. (45) multiplied by $(i+j+k)^4=n^4$ because

$$(i+j+k)^4 = \operatorname*{Sym}_{i,j,k}\Big(i^4+4i^3j+6i^2j^2+12i^2jk\Big).$$

Finally, when the weight factors have degree five we have

$$\sum\Big(\operatorname*{Sym}_{i,j,k} i^2j^2k\Big)\,\zeta(2i,2j,2k) = \frac{n}{256}\zeta(2n)+\frac{2n^2-6n+3}{64}\zeta(2)\zeta(2n-2)$$
$$+\frac{(5n-9)(2n-5)}{16}\zeta(4)\zeta(2n-4)-\frac{3(2n-7)}{8}\zeta(6)\zeta(2n-6),$$
$$\sum\Big(\operatorname*{Sym}_{i,j,k} i^3jk\Big)\,\zeta(2i,2j,2k) = \frac{n(n-1)(n+1)}{128}\zeta(2n)-\frac{3(n-1)^2}{32}\zeta(2)\zeta(2n-2)$$

$$+\frac{(2n-5)(n^2-5n+9)}{8}\zeta(4)\zeta(2n-4)+\frac{3(2n-7)}{4}\zeta(6)\zeta(2n-6),$$

$$\sum\left(\operatorname*{Sym}_{i,j,k} i^3j^2\right)\zeta(2i,2j,2k)=-\frac{n(2n^2+1-5n)}{256}\zeta(2n)+\frac{3(2n-7)}{8}\zeta(6)\zeta(2n-6)$$

$$+\frac{57n-92n^2+36n^3-3}{64}\zeta(2)\zeta(2n-2)-\frac{(2n-5)(2n^2+20n-9)}{16}\zeta(4)\zeta(2n-4),$$

$$\sum\left(\operatorname*{Sym}_{i,j,k} i^4j\right)\zeta(2i,2j,2k)=\frac{n(20n^3-8n^2-15n+5)}{256}\zeta(2n)$$

$$+\frac{16n^4-144n^3+294n^2-183n+15}{64}\zeta(2)\zeta(2n-2)$$

$$-\frac{(2n-5)(8n^2-70n+45)}{16}\zeta(4)\zeta(2n-4)-\frac{15(2n-7)}{8}\zeta(6)\zeta(2n-6),$$

$$\sum(i^5+j^5+k^5)\,\zeta(2i,2j,2k)=\frac{5n(32n^4-20n^3+4n^2+5n-1)}{256}\zeta(2n)$$

$$-\frac{16n^5+80n^4-360n^3+490n^2-285n+15}{64}\zeta(2)\zeta(2n-2)$$

$$+\frac{5(2n-5)(2n-9)(2n-1)}{16}\zeta(4)\zeta(2n-4)+\frac{15(2n-7)}{8}\zeta(6)\zeta(2n-6).$$

We may check the consistency by using the identity

$$(i+j+k)^5=\operatorname*{Sym}_{i,j,k}\left(i^5+5i^4j+10i^3j^2+20i^3jk+30i^2j^2k\right).$$

We end our paper by the following general conjecture which is supported by the above examples in depth 3, Examples 3.7 and Theorem 3.11 in depth 2.

Conjecture 4.7. *Let $F(x_1,\dots,x_m)\in\mathbb{Q}[x_1,\dots,x_m]$ be a symmetric polynomial of total degree r. Suppose $d=\deg_{x_1}F(x_1,\dots,x_m)$. Then for every positive integer $n\ge m$ we have*

$$\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 1}} F(k_1,\dots,k_m)\zeta(2k_1)\dots\zeta(2k_m)=\sum_{k=0}^{T} e_{F,k}(n)\zeta(2k)\zeta(2n-2k),$$

$$\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 1}} F(k_1,\dots,k_m)\zeta(2k_1,\dots,2k_m)=\sum_{k=0}^{T} c_{F,k}(n)\zeta(2k)\zeta(2n-2k),$$

where $T=\max\{\lfloor(r+m-2)/2\rfloor,\lfloor(m-1)/2\rfloor\}$, $e_{F,k}(x),c_{F,k}(x)\in\mathbb{Q}[x]$ depend only on k and F, $\deg e_{F,k}(x)\le r-1$ and $\deg c_{F,k}(x)\le d$.

Notice that $T=\lfloor(m-1)/2\rfloor$ or $T=\lfloor(r+m-2)/2\rfloor$ depending on whether $r=0$ or $r>0$. When $r=d=0$ the second formula of the conjecture follows from the main result in [5] by Hoffman. When $r=d=1$ then $F(k_1,\dots,k_m)=k_1+,\dots+k_m=n$ so the sum formula reduces to the case $r=d=0$.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (Grant No. 11371178) and the National Science Foundation of US (Grant No. DMS 1001855 and DMS 1162116). Part of the work was done while JZ was visiting the Max-Planck Institute for Mathematics and the Kavli Institute for Theoretical Physics China whose support is gratefully acknowledged.

References

[1] M. Eie, A note on Bernoulli numbers and Shintani generalized Bernoulli polynomials, *Trans. Amer. Math. Soc.* **348** (1996), 1117–1136.

[2] M. Eie, Topics in Number Theory, *Monographs in Number Theory*, vol. **2**, World Scientific, 2009.

[3] H. Gangl, M. Kaneko, D. Zagier, Double zeta values and modular forms, in: Automorphic Forms and Zeta Functions, S. Böcherer et al. eds., World Scientific, Hackensack, NJ (2006), 71–106.

[4] L. Guo and B. Xie, Weighted sum formula for multiple zeta values, *J. Number Theory* **129**(11) (2009), 2747–2765.

[5] M. E. Hoffman, On multiple zeta values of even arguments, arXiv: 1205.7051.

[6] A.P. Juskevic and E. Winter, eds., Leonhard Euler und Christian Goldbach: Briefwechsel 1729–1764, Akademie-Verlag, Berlin, 1965.

[7] T. Nakamura, Restricted and weighted sum formulas for double zeta values of even weight, *Šiauliai Math. Seminar* **4** (2009), 151–155.

[8] Y. Ohno and W. Zudilin, Zeta stars, *Commun. Number Theory Phys.* **2** (2008), 325–347.

[9] H. Rademacher, Topics in Analytic Number Theory (1973), Springer-Verlag.

[10] Z. Shen and T. Cai, Some identities for multiple zeta values, *J. Number Theory* **132** (2012), 314–323.

[11] G. Shimura, Elementary Dirichlet Series and Modular Forms, Springer Monographs in Math., Springer, New York, 2007.

Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA
E-mail address: liguo@rutgers.edu

Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, China
E-mail address: leip@lzu.edu.cn

Department of Mathematics, Eckerd College, St. Petersburg, FL 33711, USA
E-mail address: zhaoj@ekcerd.edu