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HS-integral and Eisenstein integral mixed circulant graphs

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Abstract

A mixed graph is called *second kind hermitian integral* (*HS-integral*) if the eigenvalues of its Hermitian-adjacency matrix of the second kind are integers. A mixed graph is called *Eisenstein integral* if the eigenvalues of its (0, 1)-adjacency matrix are Eisenstein integers. We characterize the set S for which a mixed circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral. We also show that a mixed circulant graph is Eisenstein integral if and only if it is HS-integral. Further, we express the eigenvalues and the HS-eigenvalues of unitary oriented circulant graphs in terms of generalized Möbius function.

Keywords. integral graphs; HS-integral mixed graph; Eisenstein integer; mixed circulant graph.
Mathematics Subject Classifications: 05C50, 05C25

1 Introduction

A *mixed graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set and $E(G)$ is a subset of $(V(G) \times V(G)) \setminus \{(u,u) \mid u \in V(G)\}$. The sets $V(G)$ and $E(G)$ are called the vertex set and the edge set of G, respectively. If $(v,u) \in E(G)$ if and only if $(u,v) \in E(G)$, then G is called a *simple graph*. If $(v,u) \notin E(G)$ whenever $(u,v) \in E(G)$, then G is called an *oriented graph*.

Let G be a mixed graph on n vertices. The $(0,1)$-*adjacency matrix* and the *Hermitian-adjacency matrix of the second kind* of G are denoted by $\mathcal{A}(G) := (a_{uv})_{n\times n}$ and $\mathcal{H}(G) := (h_{uv})_{n\times n}$, respectively, where

$$a_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad h_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E \text{ and } (v,u) \in E \\ \frac{1+i\sqrt{3}}{2} & \text{if } (u,v) \in E \text{ and } (v,u) \notin E \\ \frac{1-i\sqrt{3}}{2} & \text{if } (u,v) \notin E \text{ and } (v,u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The Hermitian-adjacency matrix of the second kind was introduced by Bojan Mohar [10]. The matrix obtained by replacing the numbers $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$ of $\mathcal{H}(G)$ by the numbers i and $-i$, respectively, is called the *Hermitian-adjacency* matrix (of the first kind) of G. Indeed, these Hermitian-adjacency matrices of mixed graphs are special cases of the adjacency matrix of weighted directed graphs discussed in [3].

By an *eigenvalue* (resp. *HS-eigenvalue*) of G, we mean an eigenvalue of $\mathcal{A}(G)$ (resp. $\mathcal{H}(G)$). Similarly, the spectrum (resp. *HS-spectrum*) of G is the multi-set of the eigenvalues (resp. HS-eigenvalues) of G.

A simple graph is said to be *integral* if all of its eigenvalues are integers. A mixed graph G is said to be *HS-integral* if all of its HS-eigenvalues are integers. A mixed graph G is said to be *Eisenstein integral* if all of its eigenvalues are Eisenstein integers. Note that complex numbers of the form $a + b\omega_3$, where $a, b \in \mathbb{Z}, \omega_3 = \frac{-1+i\sqrt{3}}{2}$, are known as *Eisenstein integers*. If G is a simple graph, then $\mathcal{A}(G) = \mathcal{H}(G)$. As a result, the terms HS-eigenvalue, HS-spectrum, and HS-integrality of G have the same meaning with the terms eigenvalue, spectrum, and integrality of G in the case of a simple graph G.

In 1974, Harary and Schwenk [6] proposed a characterization of integral graphs. This problem has inspired a lot of interest over the last decades. For more results on integral graphs, we refer the reader to [1, 2, 5, 14, 15].

We consider Γ to be a finite group throughout the paper. Let S be a subset of Γ that does not contain the identity element. If S is closed under inverse (resp. $a^{-1} \not\in S$ for all $a \in S$), it is said to be *symmetric* (resp. *skew-symmetric*). Define $\overline{S} = \{u \in S : u^{-1} \not\in S\}$. Then $S \setminus \overline{S}$ is symmetric, while $\overline{S}$ is skew-symmetric. The *mixed Cayley graph* $\text{Cay}(\Gamma, S)$ is a mixed graph with $V(\text{Cay}(\Gamma, S)) = \Gamma$ and $E(\text{Cay}(\Gamma, S)) = \{(a, b) : a, b \in \Gamma, ba^{-1} \in S\}$. If S is symmetric (resp. skew-symmetric), then $\text{Cay}(\Gamma, S)$ is a *simple Cayley graph* (resp. *oriented Cayley graph*). If $G = \mathbb{Z}_n$, then the graph $\text{Cay}(\Gamma, S)$ is called a *circulant* graph, and it is denoted by $\text{Circ}(\mathbb{Z}_n, S)$.

In this paper, we characterize the set S for which a mixed circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral. We also show that a mixed circulant graph is Eisenstein integral if and only if it is HS-integral. Indeed, the characterizations of this paper are special cases of that in [8], in which we considered the group to be abelian. However, due to the speciality of the group $\mathbb{Z}_n$, intermediary results and their proof techniques in this paper are different than that in [8]. In this paper, we also express the eigenvalues and the HS-eigenvalues of unitary oriented circulant graphs in terms of generalized Möbius function, which was not considered in [8]. We discussed integrality of the eigenvalues of the Hermitian-adjacency matrix (of the first kind) of mixed circulant graphs in [9]. The results in this paper are influenced from that in [9]. As a result, proof techniques and flow of results in this paper have some similarities with that in [9].

This paper is organized as follows. In Section 2, some preliminary concepts and results are discussed. In particular, we express the HS-eigenvalues of a mixed circulant graph as a sum of HS-eigenvalues of a simple circulant graph and an oriented circulant graph. In Section 3, we obtain a sufficient condition on the connection set for the HS-integrality of an oriented circulant graph. In Section 4, we first characterize HS-integrality of oriented circulant graphs by proving the necessity of the condition obtained in Section 3. After that, we extend this characterization to mixed circulant graphs. In Section 5, we prove that a mixed circulant graph is Eisenstein integral if and only if it is HS-integral. In the last section, we express the eigenvalues and the HS-eigenvalues of unitary oriented circulant graphs in terms of generalized Möbius function.

2 Preliminaries

An $n \times n$ *circulant matrix* C have the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix},$$

where each row is a cyclic shift of the row above it. The circulant matrix C is also denoted by $\text{Circ}(c_0, c_1, \ldots, c_{n-1})$. Note that the (j, k)-th entry of C is $c_{k-j \pmod{n}}$. A circulant matrix

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is diagonalizable by the matrix F whose j-th column is given by

$$F_j = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & \omega_n^j & \dots & \omega_n^{(n-1)j} \end{bmatrix}^T,$$

where $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$ and $0 \leq j \leq n-1$. The eigenvalues of C are given by

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega_n^{jk} \text{ for } j \in \{0, 1, ..., n-1\}. \tag{1}$$

Lemma 2.1. *Let S be a subset of $\mathbb{Z}_n$ such that $0 \notin S$. Then the HS-spectrum of the mixed circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is $\{\gamma_0, \gamma_1, ..., \gamma_{n-1}\}$, where $\gamma_j = \lambda_j + \mu_j$,*

$$\lambda_j = \sum_{k \in S \setminus \overline{S}} \omega_n^{jk} \quad \text{and} \quad \mu_j = \sum_{k \in \overline{S}} (\omega_6 \omega_n^{jk} + \omega_6^5 \omega_n^{-jk}) \text{ for } j \in \{0, 1, ..., n-1\}.$$

Proof. Let

$$c_s = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S} \\ \omega_6 & \text{if } s \in \overline{S} \\ \omega_6^5 & \text{if } s \in \overline{S}^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the HS-spectrum of $\text{Circ}(\mathbb{Z}_n, S)$ is same as the spectrum of $\text{Circ}(c_0, ..., c_{n-1})$. Now the proof follows from Equation (1). □

From Lemma 2.1, we see that the HS-eigenvalues of a mixed circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ are the sum of the HS-eigenvalues of the mixed graphs $\text{Circ}(\mathbb{Z}_n, S \setminus \overline{S})$ and $\text{Circ}(\mathbb{Z}_n, \overline{S})$. Note that $\omega_6 = -\omega_3^2$ and $\omega_6^5 = -\omega_3$. Therefore the eigenvalue μ_j can also be written as $\mu_j = -\sum_{k \in \overline{S}} (\omega_3^2 \omega_n^{jk} + \omega_3 \omega_n^{-jk})$. Further, one can see that the eigenvalues of $\text{Circ}(\mathbb{Z}_n, S)$ are given by $\sum_{k \in S} \omega_n^{jk}$ for each $j \in \{0, 1, ..., n-1\}$.

Let $n \geq 2$ be a fixed positive integer. We review some basic definitions and notations from [13]. For a divisor d of n, define

$$M_n(d) = \{dk : 1 \leq dk \leq n-1\} \text{ and}$$
$$G_n(d) = \{dk : 1 \leq dk \leq n-1, \gcd(dk, n) = d\}.$$

It is clear that $M_n(n) = G_n(n) = \emptyset$, $M_n(d) = dM_{\frac{n}{d}}(1)$ and $G_n(d) = dG_{\frac{n}{d}}(1)$.

Lemma 2.2. *[13] If $n = dg$ for some $d, g \in \mathbb{Z}$ then $M_n(d) = \bigcup_{h \mid g} G_n(hd)$.*

Wasin So [13] characterized integral circulant graphs in the following theorem.

Theorem 2.3. *[13] The simple circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is integral if and only if $S = \bigcup_{d \in \mathscr{D}} G_n(d)$, where $\mathscr{D} \subseteq \{d : d \mid n\}$.*

Let $n \equiv 0 \pmod 3$. For a divisor d of $\frac{n}{3}$, $r \in \{0,1,2\}$ and $g \in \mathbb{Z}$, define the following sets:

$$M^r_{n,3}(d) = \{dk : 0 \le dk \le n-1, k \equiv r \pmod 3\},$$
$$G^r_{n,3}(d) = \{dk : 1 \le dk \le n-1, \gcd(dk,n) = d, k \equiv r \pmod 3\},$$
$$D_{g,3} = \{k : k \text{ divides } g, k \not\equiv 0 \pmod 3\} \text{ and}$$
$$D^r_{g,3} = \{k : k \text{ divides } g, k \equiv r \pmod 3\}.$$

It is clear $D_{g,3} = D^1_{g,3} \bigcup D^2_{g,3}$.

Lemma 2.4. *Let $n \equiv 0 \pmod 3$, d divide $\frac{n}{3}$ and $g = \frac{n}{3d}$. Then the following hold:*

(i) $M^1_{n,3}(d) \cup M^2_{n,3}(d) = \bigcup_{h \in D_{g,3}} G_n(hd)$;

(ii) $M^0_{n,3}(d) = M_n(3d) \cup \{0\}$.

Proof. (i) Let $dk \in M^1_{n,3}(d) \cup M^2_{n,3}(d)$. Lemma 2.2 gives $M^1_{n,3}(d) \cup M^2_{n,3}(d) \subseteq M_n(d) = \bigcup_{h|3g} G_n(hd)$. So there exists a divisor h of $3g$ such that $dk = \alpha hd$, for some $\alpha \in \mathbb{Z}$ with $\gcd(\alpha, \frac{3g}{h}) = 1$. Now we have $h = \frac{k}{\alpha}$ and k is not a multiple of 3, which imply that $h|g$ and h is not a multiple of 3. Thus $h \in D_{g,3}$, and so $dk \in \bigcup_{h \in D_{g,3}} G_n(hd)$. Conversely, let $x \in \bigcup_{h \in D_{g,3}} G_n(hd)$. Then there exists $h \in D_{g,3}$ such that $x = \alpha hd$, where $\alpha \in \mathbb{Z}$ and $\gcd(\alpha, \frac{3g}{h}) = 1$. Note that α and h are not multiples of 3. Thus $\alpha h \equiv 1$ or $2 \pmod 3$, and so $x \in M^1_{n,3}(d) \cup M^2_{n,3}(d)$.

(ii) By definition, we have

$$\begin{aligned} M^0_{n,3}(d) &= \{dk : 0 \le dk \le n-1, k = 3\alpha \text{ for some } \alpha \in \mathbb{Z}\} \\ &= \{3\alpha d : 1 \le 3\alpha d \le n-1 \text{ for some } \alpha \in \mathbb{Z}\} \cup \{0\} \\ &= M_n(3d) \cup \{0\}. \end{aligned}$$

□

We now prove that $G_n(d)$ is a disjoint union of $G^1_{n,3}(d)$ and $G^2_{n,3}(d)$.

Lemma 2.5. *Let $n \equiv 0 \pmod 3$, d divide $\frac{n}{3}$ and $g = \frac{n}{3d}$. Then the following hold:*

(i) $G^1_{n,3}(d) \cap G^2_{n,3}(d) = \emptyset$;

(ii) $G_n(d) = G^1_{n,3}(d) \cup G^2_{n,3}(d)$;

(iii) $M^1_{n,3}(d) = \left(\bigcup_{h \in D^1_{g,3}} G^1_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^2_{n,3}(hd) \right)$;

(iv) $M^2_{n,3}(d) = \left(\bigcup_{h \in D^1_{g,3}} G^2_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^1_{n,3}(hd) \right)$.

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Proof. (i) It is clear from the definitions of $G^1_{n,3}(d)$ and $G^2_{n,3}(d)$ that $G^1_{n,3}(d) \cap G^2_{n,3}(d) = \emptyset$.

(ii) Since $G^1_{n,3}(d) \subseteq G_n(d)$ and $G^2_{n,3}(d) \subseteq G_n(d)$, we have $G^1_{n,3}(d) \cup G^2_{n,3}(d) \subseteq G_n(d)$. Conversely, let $x \in G_n(d)$. Then $x = d\alpha$ for some α satisfying $\gcd(\alpha, 3g) = 1$, and so $\alpha \equiv 1$ or $2 \pmod 3$. Hence $G_n(d) \subseteq G^1_{n,3}(d) \cup G^2_{n,3}(d)$.

(iii) Let $dk \in M^1_{n,3}(d)$ so that $k \equiv 1 \pmod 3$. By Lemma 2.4, there exists $h \in D_{g,3}$ satisfying $dk \in G_n(hd)$. Thus $dk \in G^1_{n,3}(hd)$ or $dk \in G^2_{n,3}(hd)$.
Case 1. Assume that $h \equiv 1 \pmod 3$. Let, if possible, $dk \in G^2_{n,3}(hd)$, that is, $dk = \alpha hd$ for some $\alpha \equiv 2 \pmod 3$ satisfying $\gcd(\alpha, \frac{3g}{h}) = 1$. Then we have $k = \alpha h \equiv 2 \pmod 3$, a contradiction. Hence $dk \in G^1_{n,3}(hd)$.

Case 2. Assume that $h \equiv 2 \pmod 3$. Proceeding as in Case 1, we get $k \equiv 2 \pmod 3$, a contradiction. Hence $dk \in G^2_{n,3}(hd)$. Thus $M^1_{n,3}(d) \subseteq \left(\bigcup_{h \in D^1_{g,3}} G^1_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^2_{n,3}(hd) \right)$.

Conversely, if $\alpha hd \in G^1_{n,3}(hd)$, where $h \in D^1_{g,3}$ and $\alpha \equiv 1 \pmod 3$, we get $\alpha h \equiv 1 \pmod 3$, that is, $\alpha hd \in M^1_{n,3}(d)$. Similarly, $\beta hd \in G^2_{n,3}(hd)$, where $h \in D^2_{g,3}$ and $\beta \equiv 1 \pmod 3$, imply that $\beta hd \in M^1_{n,3}(d)$. Therefore $\left(\bigcup_{h \in D^1_{g,3}} G^1_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^2_{n,3}(hd) \right) \subseteq M^1_{n,3}(d)$.

(iv) The proof of this part is similar to the proof of Part (iii).

□

The *cyclotomic polynomial* $\Phi_n(x)$ is the monic polynomial whose zeros are the primitive n^{th} roots of unity. That is,

$$\Phi_n(x) = \prod_{a \in G_n(1)} (x - \omega_n^a).$$

Clearly, the degree of $\Phi_n(x)$ is $\varphi(n)$, where φ denotes the Euler φ-function. It is well known that the cyclotomic polynomial $\Phi_n(x)$ is monic and irreducible in $\mathbb{Z}[x]$. See [7] for more details on cyclotomic polynomials.

The polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\omega_3)$ if and only if $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \varphi(n)$. Also, $\mathbb{Q}(\omega_n)$ does not contain the number ω_3 if and only if $n \not\equiv 0 \pmod 3$. Thus, if $n \not\equiv 0 \pmod 3$ then $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_n)] = 2 = [\mathbb{Q}(\omega_3), \mathbb{Q}]$, and therefore

$$[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_n)] \times [\mathbb{Q}(\omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = [\mathbb{Q}(\omega_n) : \mathbb{Q}] = \varphi(n).$$

Further, if $n \equiv 0 \pmod 3$ then $\mathbb{Q}(\omega_3, \omega_n) = \mathbb{Q}(\omega_n)$, and so

$$[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = \frac{\varphi(n)}{2}.$$

Note that $\mathbb{Q}(\omega_3) = \mathbb{Q}(\omega_6) = \mathbb{Q}(i\sqrt{3})$. Therefore $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\omega_3), \mathbb{Q}(\omega_6)$ or $\mathbb{Q}(i\sqrt{3})$ if and only if $n \not\equiv 0 \pmod 3$.

Let $n \equiv 0 \pmod 3$. From Lemma 2.5, we know that $G_n(1)$ is a disjoint union of $G^1_{n,3}(1)$ and $G^2_{n,3}(1)$. Define

$$\Phi^1_{n,3}(x) = \prod_{a \in G^1_{n,3}(1)} (x - \omega_n^a) \quad \text{and} \quad \Phi^2_{n,3}(x) = \prod_{a \in G^2_{n,3}(1)} (x - \omega_n^a).$$

It is clear from the definition that $\Phi_n(x) = \Phi^1_{n,3}(x)\Phi^2_{n,3}(x)$.

Lemma 2.6. *If $n \equiv 0 \pmod 3$ then the following hold:*

(i) $x^{\frac{n}{3}} - \omega_3 = \prod_{h \in D^1_{\frac{n}{3},3}} \Phi^1_{\frac{n}{h},3}(x) \prod_{h \in D^2_{\frac{n}{3},3}} \Phi^2_{\frac{n}{h},3}(x)$*; and*

(ii) $x^{\frac{n}{3}} - \omega_3^2 = \prod_{h \in D^1_{\frac{n}{3},3}} \Phi^2_{\frac{n}{h},3}(x) \prod_{h \in D^2_{\frac{n}{3},3}} \Phi^1_{\frac{n}{h},3}(x)$.

Proof. (i) We have $|M^1_{n,3}(1)| = \frac{n}{3}$ and ω_n^a is a root of $x^{\frac{n}{3}} - \omega_3$ for each $a \in M^1_{n,3}(1)$. Therefore

$$\begin{aligned}
x^{\frac{n}{3}} - \omega_3 &= \prod_{a \in M^1_{n,3}(1)} (x - \omega_n^a) \\
&= \prod_{h \in D^1_{\frac{n}{3},3}} \prod_{a \in G^1_{n,3}(h)} (x - \omega_n^a) \prod_{h \in D^2_{\frac{n}{3},3}} \prod_{a \in G^2_{n,3}(h)} (x - \omega_n^a), \qquad \text{using Lemma 2.5} \\
&= \prod_{h \in D^1_{\frac{n}{3},3}} \prod_{a \in hG^1_{\frac{n}{h},3}(1)} (x - \omega_n^a) \prod_{h \in D^2_{\frac{n}{3},3}} \prod_{a \in hG^2_{\frac{n}{h},3}(1)} (x - \omega_n^a) \\
&= \prod_{h \in D^1_{\frac{n}{3},3}} \prod_{a \in G^1_{\frac{n}{h},3}(1)} (x - (\omega_n^h)^a) \prod_{h \in D^2_{\frac{n}{3},3}} \prod_{a \in G^2_{\frac{n}{h},3}(1)} (x - (\omega_n^h)^a) \\
&= \prod_{h \in D^1_{\frac{n}{3},3}} \Phi^1_{\frac{n}{h},3}(x) \prod_{h \in D^2_{\frac{n}{3},3}} \Phi^2_{\frac{n}{h},3}(x).
\end{aligned}$$

In the last equality, we have used the fact that $\omega_n^h = \exp(\frac{2\pi i}{n/h})$ is a primitive $\frac{n}{h}$-th root of unity.

(ii) We have $|M^2_{n,3}(1)| = \frac{n}{3}$ and ω_n^a is a root of $x^{\frac{n}{3}} - \omega_3^2$ for each $a \in M^2_{n,3}(1)$. Now the proof is similar to the proof of Part (i).

□

Corollary 2.7. *Let $n \equiv 0 \pmod 3$. Then $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are monic polynomials in $\mathbb{Z}(\omega_3)[x]$ of degree $\varphi(n)/2$.*

Proof. By definition, $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are monic polynomials. Also, $G_n(1) = G^1_{n,3}(1) \cup G^2_{n,3}(1)$ is a disjoint union and that $|G^1_{n,3}(1)| = |G^2_{n,3}(1)|$. Therefore the polynomials $\Phi^1_{n,3}(x)$

and $\Phi^2_{n,3}(x)$ are of degree $\frac{\varphi(n)}{2}$. Now apply induction on n to show that $\Phi^1_{n,3}(x), \Phi^2_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$. For $n = 3$, the polynomials $\Phi^1_{3,3}(x) = x - \omega_3$ and $\Phi^2_{3,3}(x) = x - \omega_3^2$ are clearly in $\mathbb{Z}(\omega_3)[x]$. Assume that $\Phi^1_{k,3}(x)$ and $\Phi^2_{k,3}(x)$ are in $\mathbb{Z}(\omega_3)[x]$ for each $k < n$ and $k \equiv 0 \pmod 3$. By Lemma 2.6, $\Phi^1_{n,3}(x) = \frac{x^{\frac{n}{3}} - \omega_3}{f(x)}$ and $\Phi^2_{n,3}(x) = \frac{x^{\frac{n}{3}} - \omega_3^2}{g(x)}$. By induction hypothesis, $f(x)$ and $g(x)$ are monic polynomials in $\mathbb{Z}(\omega_3)[x]$. It follows by "long division" that $\Phi^1_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$ and $\Phi^2_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$. □

Theorem 2.8. *Let $n \equiv 0 \pmod 3$. Then $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are irreducible monic polynomials in $\mathbb{Q}(\omega_3)[x]$ of degree $\frac{\varphi(n)}{2}$.*

Proof. In view of Corollary 2.7, we only need to show the irreducibility of $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ in $\mathbb{Q}(\omega_3)$. For $n \equiv 0 \pmod 3$, we have $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{\varphi(n)}{2}$. So there is a unique irreducible monic polynomial $p(x) \in \mathbb{Q}(\omega_3)[x]$ of degree $\frac{\varphi(n)}{2}$ having ω_n as a root. Since ω_n is also a root of $\Phi^1_{n,3}(x)$, we have that $\Phi^1_{n,3}(x) = p(x)f(x)$ for some $f(x) \in \mathbb{Q}(\omega_3)[x]$. Also $\Phi^1_{n,3}(x)$ is a monic polynomial of degree $\varphi(n)/2$, and so $f(x) = 1$. Hence $\Phi^1_{n,3}(x) = p(x)$ is irreducible. Similarly, $[\mathbb{Q}(\omega_3, \omega_n^a) : \mathbb{Q}(\omega_3)] = \frac{\varphi(n)}{2}$ for $a \in G^2_{n,3}(1)$. This, along with Corollary 2.7, give that $\Phi^2_{n,3}(x)$ is irreducible. □

3 A sufficient condition for the HS-integrality of oriented circulant graphs

In this section, we obtain a sufficient condition for the HS-integrality of oriented circulant graphs on n vertices. Throughout this section, we consider S to be a skew-symmetric subset of $\mathbb{Z}_n$.

Lemma 3.1. *Let S be a skew-symmetric subset of $\mathbb{Z}_n$. If $\sum\limits_{k\in S} i\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk}) = 0$ for each $j \in \{0, 1, ..., n-1\}$ then $S = \emptyset$.*

Proof. Let $A_S = (a_{uv})_{n\times n}$ be the matrix whose rows and columns are indexed by elements of $\mathbb{Z}_n$, where

$$a_{uv} = \begin{cases} i\sqrt{3} & \text{if } v-u \in S \\ -i\sqrt{3} & \text{if } v-u \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that A_S is a circulant matrix. Therefore the spectrum of A_S is $\{\alpha_j : j = 0, 1, ..., n-1\}$, where $\alpha_j = \sum\limits_{k\in S} i\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})$. Given that $\alpha_j = 0$ for all $j = 0, 1, ..., n-1$. This implies that all the entries of A_S are zero, and hence $S = \emptyset$. □

Theorem 3.2. *Let $n \not\equiv 0 \pmod 3$. Then the oriented circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is HS-integral if and only if $S = \emptyset$.*

Proof. Let $G = \text{Circ}(\mathbb{Z}_n, S)$ and $Sp_H(G) = \{\mu_0, \mu_1, ..., \mu_{n-1}\}$. Assume that G is HS-integral. By Lemma 2.1,

$$\mu_j = \sum_{k\in S}(\omega_6\omega_n^{jk} + \omega_6^5\omega_n^{-jk}) \in \mathbb{Z} \text{ for } 0 \le j \le n-1.$$

Observe that ω_n is a root of the polynomial $p(x) = \sum_{k\in S}(\omega_6 x^{jk} + \omega_6^5 x^{j(n-k)}) - \mu_j \in \mathbb{Q}(\omega_6)[x]$. Since $n \not\equiv 0 \pmod 3$, the polynomial $\Phi_n(x)$ is irreducible in $\mathbb{Q}(\omega_6)[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi_n(x)$, and so $\omega_n^{-1} = \omega_n^{n-1}$ is also a root of $p(x)$, that is, $\mu_j = \mu_{n-j}$. Thus

$$0 = \mu_j - \mu_{n-j} = \sum_{k\in S}(\omega_6 - \omega_6^5)\omega_n^{kj} + (\omega_6^5 - \omega_6)\omega_n^{-kj} = \sum_{k\in S} i\sqrt{3}(\omega_n^{kj} - \omega_n^{-kj}).$$

Therefore by Lemma 3.1, $S = \emptyset$. Conversely, if $S = \emptyset$ then $\mathrm{Circ}(\mathbb{Z}_n, S)$ has no edges, and hence all its eigenvalues are zero. □

Theorem 3.2 characterizes integral oriented circulant graphs for the case $n \not\equiv 0 \pmod 3$.

Lemma 3.3. *Let S be a skew-symmetric subset of $\mathbb{Z}_n$ and $k \in \mathbb{N}$. Then*

$$\sum_{q\in S}\omega_k\omega_n^{jq} + \sum_{q\in S^{-1}}\omega_k^{k-1}\omega_n^{jq} \in \mathbb{Z} \text{ for each } j \in \{0,1,...,n-1\}$$

if and only if

$$\sum_{q\in S}\omega_k^{k-1}\omega_n^{jq} + \sum_{q\in S^{-1}}\omega_k\omega_n^{jq} \in \mathbb{Z} \text{ for each } j \in \{0,1,...,n-1\}.$$

Proof. Let $H_S = [h_{uv}]_{n\times n}$ be the matrix, whose rows and columns are indexed by elements of $\mathbb{Z}_n$, where

$$h_{uv} = \begin{cases} \omega_k & \text{if } v-u \in S \\ \omega_k^{k-1} & \text{if } v-u \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Since H_S is a circulant matrix, $\sum_{q\in S}\omega_k\omega_n^{jq} + \sum_{q\in S^{-1}}\omega_k^{k-1}\omega_n^{jq}$ is an eigenvalue of H_S for all $j \in \{0,1,...,n-1\}$. Let $\overline{H}_S$ be obtained by taking the complex conjugate of the corresponding entry of H_S. Note that $\overline{H}_S = H_{S^{-1}}$. The result follows from the fact that the eigenvalues of H_S are integers if and only if the eigenvalues of $\overline{H}_S$ are integers. □

Lemma 3.4. *Let $n \equiv 0 \pmod 3$, d divide $\frac{n}{3}$ and $g = \frac{n}{3d}$. Then $\sum_{q\in M_{n,3}^0(d)}\omega_n^{jq} \in \mathbb{Z}$ for $j \in \{0,1,...,n-1\}$.*

Proof. Using Lemma 2.4 and Lemma 2.2, we get $M_{n,3}^0(d) = \bigcup_{h\mid g} G_n(3hd) \bigcup \{0\}$. Therefore

$$\sum_{q\in M_{n,3}^0(d)}\omega_n^{jq} = 1 + \sum_{h\mid g}\sum_{q\in G_n(3hd)}\omega_n^{jq}.$$

By Theorem 2.3, the eigenvalue $\sum_{q\in G_n(3hd)}\omega_n^{jq}$ of the circulant graph $\mathrm{Circ}(\mathbb{Z}_n, G_n(3hd))$ is an integer for each $h \mid g$. Thus $\sum_{q\in M_n^0(d)}\omega_n^{jq} \in \mathbb{Z}$ for each $j \in \{0,1,...,n-1\}$. □

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Lemma 3.5. *Let $n \equiv 0 \pmod 3$, d divide $\frac{n}{3}$ and $g = \frac{n}{3d}$. Then*

$$\sum_{q \in M_{n,3}^2(d)} \omega_3 \omega_n^{jq} + \sum_{q \in M_{n,3}^1(d)} \omega_3^2 \omega_n^{jq} \in \mathbb{Z} \text{ for each } j \in \{0, 1, ..., n-1\}.$$

Proof. It is enough to show that $\sum\limits_{q \in M_{n,3}^2(d)} \omega_3 z^q + \sum\limits_{q \in M_{n,3}^1(d)} \omega_3^2 z^q$ is an integer for all $z \in \mathbb{C}$ satisfying $z^n = 1$. We have

$$\begin{aligned}
z^n - 1 &= (z^d - \omega_3)\left(\sum_{k=1}^{3g} \omega_3^{k-1} z^{n-dk}\right) \\
&= (z^d - \omega_3)\left(\sum_{k=1}^{3g} \omega_3^{k} z^{n-dk}\right)\omega_3^2 \\
&= (z^d - \omega_3)\left(\sum_{\substack{1 \le k \le 3g \\ k \equiv 0 \pmod 3}} z^{n-dk} + \sum_{\substack{1 \le k \le 3g \\ k \equiv 1 \pmod 3}} \omega_3 z^{n-dk} + \sum_{\substack{1 \le k \le 3g \\ k \equiv 2 \pmod 3}} \omega_3^2 z^{n-dk}\right)\omega_3^2 \\
&= (z^d - \omega_3)\left(\sum_{q \in M_{n,3}^0(d)} z^q + \sum_{q \in M_{n,3}^2(d)} \omega_3 z^q + \sum_{q \in M_{n,3}^1(d)} \omega_3^2 z^q\right)\omega_3^2. \qquad (2)
\end{aligned}$$

Let $z \in \mathbb{C}$ such that $z^n = 1$. By Equation (2), we get two possible cases.
Case 1. Assume that $z^d - \omega_3 = 0$.
If $q \in M_{n,3}^1(d)$ then $q = (3y_1 + 1)d$ for some $y_1 \in \mathbb{Z}$. So $z^q = z^{(3y_1+1)d} = \omega_3^{3y_1+1} = \omega_3$.
If $q \in M_{n,3}^2(d)$ then $q = (3y_2 + 2)d$ for some $y_2 \in \mathbb{Z}$. So $z^q = z^{(3y_2+2)d} = \omega_3^{3y_2+2} = \omega_3^2$. Thus

$$\sum_{q \in M_{n,3}^2(d)} \omega_3 z^q + \sum_{q \in M_{n,3}^1(d)} \omega_3^2 z^q = \sum_{q \in M_{n,3}^2(d)} 1 + \sum_{q \in M_{n,3}^1(d)} 1 = |M_{n,3}^1(d) \cup M_{n,3}^2(d)| \in \mathbb{Z}.$$

Case 2. Assume that $z^d - \omega_3 \neq 0$. Then

$$\begin{aligned}
&\sum_{q \in M_{n,3}^0(d)} z^q + \sum_{q \in M_{n,3}^2(d)} \omega_3 z^q + \sum_{q \in M_{n,3}^1(d)} \omega_3^2 z^q = 0 \\
\Rightarrow &\sum_{q \in M_{n,3}^2(d)} \omega_3 z^q + \sum_{q \in M_{n,3}^1(d)} \omega_3^2 z^q = \sum_{q \in M_{n,3}^0(d)} z^q \in \mathbb{Z}, \text{ by Lemma 3.4.}
\end{aligned}$$

□

For $n \equiv 0 \pmod 3$ and $j \in \{0, 1, \ldots, n-1\}$, define

$$Z_n^1(j) = \sum_{q \in G_{n,3}^1(1)} \left(\omega_3 \omega_n^{jq} + \omega_3^2 \omega_n^{-jq}\right) \quad \text{and} \quad Z_n^2(j) = \sum_{q \in G_{n,3}^2(1)} \left(\omega_3 \omega_n^{jq} + \omega_3^2 \omega_n^{-jq}\right).$$

By Lemma 3.3, $Z_n^1(j)$ is an integer if and only if $Z_n^2(j)$ is an integer.

Note that $G_n(d) = dG_{\frac{n}{d}}(1)$, $G^1_{n,3}(d) = dG^1_{\frac{n}{d},3}(1)$ and $G^2_{n,3}(d) = dG^2_{\frac{n}{d},3}(1)$. Therefore, if d is a divisor of $\frac{n}{3}$ then

$$\begin{aligned} Z^1_{\frac{n}{d}}(j) = \sum_{q\in G^1_{\frac{n}{d},3}(1)} \left[\omega_3(\omega_{\frac{n}{d}})^{jq} + \omega_3^2(\omega_{\frac{n}{d}})^{-jq}\right] &= \sum_{q\in G^1_{\frac{n}{d},3}(1)} \left[\omega_3(\omega_n^d)^{jq} + \omega_3^2(\omega_n^d)^{-jq}\right] \\ &= \sum_{q\in dG^1_{\frac{n}{d},3}(1)} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right) \\ &= \sum_{q\in G^1_{n,3}(d)} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right). \end{aligned}$$

Similarly, if d be a divisor of $\frac{n}{3}$ then

$$Z^2_{\frac{n}{d}}(j) = \sum_{q\in G^2_{n,3}(d)} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right).$$

Lemma 3.6. *Let $n \equiv 0 \pmod 3$ and d divide $\frac{n}{3}$. Then $Z^1_{\frac{n}{d}}(j)$ is an integer for each $j \in \{0, 1, \dots, n-1\}$.*

Proof. Let $d_1 = \frac{n}{3}, d_2, ..., d_r = 1$ be the positive divisors of $\frac{n}{3}$ in decreasing order. Apply induction on k to prove that $Z_{\frac{n}{d_k}}(j)$ is an integer for each $j \in \mathbb{Z}$. For $k = 1$, $Z^1_{\frac{n}{d_1}}(j) = Z^1_3(j) = \sum\limits_{q\in G^1_{3,3}(1)} \left(\omega_3\omega_3^{jq} + \omega_3^2\omega_3^{-jq}\right)$ is an integer for each $j \in \{0, 1, \dots, n-1\}$. Assume that $Z^1_{\frac{n}{d_k}}(j)$ is an integer for each $j \in \mathbb{Z}$, where $1 \le k < r$. Let $n = 3d_{k+1}g_{k+1}$ for some $d_{k+1}, g_{k+1} \in \mathbb{Z}$. By Lemma 2.5, we have

$$M^1_{n,3}(d_{k+1}) = \Bigg(\bigcup_{h\in D^1_{g_{k+1},3}} G^1_{n,3}(hd_{k+1})\Bigg) \cup \Bigg(\bigcup_{h\in D^2_{g_{k+1},3}} G^2_{n,3}(hd_{k+1})\Bigg). \tag{3}$$

Note that hd_{k+1} is also a divisor of $\frac{n}{3}$. If $h > 1$ then $hd_{k+1} = d_s$ for some $s < k+1$, and so by induction hypothesis $Z^1_{\frac{n}{hd_{k+1}}}(j)$ and $Z^2_{\frac{n}{hd_{k+1}}}(j)$ are integers for each $j \in \mathbb{Z}$. Note that the unions in (3) is disjoint. We have

$$\begin{aligned} \sum_{q\in M^1_{n,3}(d_{k+1})} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right) &= \sum_{q\in G^1_{n,3}(d_{k+1})} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right) \\ &+ \sum_{h\in D^1_{g_{k+1},3}, h>1} \Bigg(\sum_{q\in G^1_{n,3}(hd_{k+1})} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right)\Bigg) \\ &+ \sum_{h\in D^2_{g_{k+1},3}} \Bigg(\sum_{q\in G^2_{n,3}(hd_{k+1})} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right)\Bigg). \end{aligned}$$

Therefore

$$\sum_{q\in M^1_{n,3}(d_{k+1})} \left(\omega_3\omega_n^{jq} + \omega_3^2\omega_n^{-jq}\right) = Z^1_{\frac{n}{d_{k+1}}}(j) + \sum_{h\in D^1_{g_{k+1},3}, h>1} Z^1_{\frac{n}{hd_{k+1}}}(j) + \sum_{h\in D^2_{g_{k+1},3}} Z^2_{\frac{n}{hd_{k+1}}}(j),$$

and so

$$Z^1_{\frac{n}{d_{k+1}}}(j) = \sum_{q \in M^1_{n,3}(d_{k+1})} \left(\omega_3 \omega_n^{jq} + \omega_3^2 \omega_n^{-jq}\right) - \sum_{h \in D^1_{g_{k+1},3}, h>1} Z^1_{\frac{n}{hd_{k+1}}}(j) - \sum_{h \in D^2_{g_{k+1},3}} Z^2_{\frac{n}{hd_{k+1}}}(j). \quad (4)$$

By Lemma 3.3 and Lemma 3.5, the first summand in the right of (4) is an integer, and by induction hypothesis the other two summands are also integers. Hence $Z^1_{\frac{n}{d_{k+1}}}(j)$ is an integer for each $j \in \mathbb{Z}$. Thus the proof is complete by induction. □

Corollary 3.7. *Let $n \equiv 0 \pmod 3$ and d a divisor of $\frac{n}{3}$. If $S \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$, then the sum $\sum\limits_{q \in S} (\omega_3 \omega_n^{qj} + \omega_3^2 \omega_n^{-qj})$ is an integer for each $j \in \{0, 1, ..., n-1\}$.*

Proof. The proof follows from Lemma 3.3 and Lemma 3.6. □

Corollary 3.8. *Let $n \equiv 0 \pmod 3$ and d a divisor of $\frac{n}{3}$. If $S \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$, then the sum $\sum\limits_{q \in S} (\omega_6 \omega_n^{qj} + \omega_6^5 \omega_n^{-qj})$ is an integer for each $j \in \{0, 1, ..., n-1\}$.*

Proof. Note that $\omega_3 = -\omega_6^5, \omega_3^2 = -\omega_6$ and $\left(G^1_{n,3}(d)\right)^{-1} = G^2_{n,3}(d)$. Therefore the proof follows from Corollary 3.7. □

In the next result, we get a sufficient condition on the connection set S for which the oriented circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral.

Theorem 3.9. *Let S be a subset of $\mathbb{Z}_n$ such that*

$$S = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \bigcup\limits_{d \in \mathscr{D}} S_n(d) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ and $S_n(d) \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$. Then the oriented circulant graph $Circ(\mathbb{Z}_n, S)$ is HS-integral.

Proof. The proof follows from Theorem 3.2 and Corollary 3.8. □

4 Characterization of HS-integral mixed circulant graphs

In this section, we first characterize HS-integrality of oriented circulant graphs by proving the necessity of Theorem 3.9. After that, we extend this characterization to mixed circulant graphs. Recall that the case $n \not\equiv 0 \pmod 3$ have already been considered in Theorem 3.2. Now let $n \equiv 0 \pmod 3$ and d a divisor of $\frac{n}{3}$. Define $u(d) = [u(d)_1, \ldots, u(d)_n]^T$ to be the n-vector, where

$$u(d)_k = \begin{cases} \omega_3 & \text{if } k \in G^1_{n,3}(d) \\ \omega_3^2 & \text{if } k \in G^2_{n,3}(d) \\ 0 & \text{otherwise.} \end{cases}$$

Let $E = [e_{st}]$ be the $n \times n$ matrix defined by $e_{st} = \omega_n^{st}$. Note that E is an invertible matrix and $EE^* = nI_n$, where E^* is the conjugate transpose of E. Using Theorem 3.9, we get $Eu(d) \in \mathbb{Z}^n$.

Lemma 4.1. *Let $n \equiv 0 \pmod 3$ and $v \in \mathbb{Q}^n(\omega_3)$ such that $Ev \in \mathbb{Q}^n$. Let the coordinates of v be indexed by the elements of $\mathbb{Z}_n$. Then*

(i) $\overline{v}_s = v_{n-s}$ *for all* $1 \leq s \leq n$*;*

(ii) if $d \mid \frac{n}{6}$ *then* $v_s = v_t$ *for all* $s, t \in G^1_{n,3}(d)$*; and*

(iii) if $d \nmid \frac{n}{3}, d \mid n$ *and* $d < n$ *then* $v_r = v_{n-r}$ *for all* $r \in G_n(d)$*.*

Proof. Let $u = Ev \in \mathbb{Q}^n$, where $v \in \mathbb{Q}^n(\omega_3)$. Then $v = \frac{1}{n}E^*u$, and so

$$v_s = \frac{1}{n}\sum_{j=1}^{n} u_j \omega_n^{s(n-j)}. \tag{5}$$

(i) Taking complex conjugate of both sides in Equation 5, we get $\overline{v}_s = v_{n-s}$ for all $1 \leq s \leq n$.

(ii) Assume that $d \mid \frac{n}{3}$ and $s, t \in G^1_{n,3}(d)$. Then $\frac{s}{d}, \frac{t}{d} \in G^1_{\frac{n}{d},3}(1)$, and so ω_n^s, ω_n^t are roots of $\Phi^1_{\frac{n}{d},3}(x)$. Note that $v_s = \frac{1}{n}\sum\limits_{j=1}^{n} u_j \omega_n^{s(n-j)} \in \mathbb{Q}(\omega_3)$. Therefore ω_n^s is a root of $p(x) = \frac{1}{n}\sum\limits_{j=1}^{n} u_j x^{n-j} - v_s \in \mathbb{Q}(\omega_3)[x]$. Hence $p(x)$ is a multiple of the irreducible monic polynomial $\Phi^1_{\frac{n}{d},3}(x)$, and so ω_n^t is also a root of $p(x)$, that is, $v_s = \frac{1}{n}\sum\limits_{j=1}^{n} u_j \omega_n^{t(n-j)} = v_t$.

(iii) Assume that $d \nmid \frac{n}{3}, d \mid n$, $d < n$ and $r \in G_n(d)$. Then $r, n-r \in G_n(d)$, and so ω_n^r, ω_n^{n-r} are roots of $\Phi_{\frac{n}{d}}(x)$. Since $v_r = \frac{1}{n}\sum\limits_{j=1}^{n} u_j \omega_n^{r(n-j)} \in \mathbb{Q}(\omega_3)$, we find that ω_n^r is a root of the polynomial $q(x) = \frac{1}{n}\sum\limits_{j=1}^{n} u_j x^{(n-j)} - v_r \in \mathbb{Q}(\omega_3)[x]$. Therefore $q(x)$ is a multiple of the irreducible monic polynomial $\Phi_{\frac{n}{d}}(x)$. Note that $\Phi_{\frac{n}{d}}(x)$ is irreducible over $\mathbb{Q}(\omega_3)$ as $\frac{n}{d} \not\equiv 0 \pmod 3$. Thus ω_n^{n-r} is also a root of $q(x)$, that is, $v_r = \frac{1}{n}\sum\limits_{j=1}^{n} u_j \omega_n^{(n-r)(n-j)} = v_{n-r}$.

□

In the next theorem, we prove the converse of Theorem 3.9.

Theorem 4.2. *Let S be a skew-symmetric subset of $\mathbb{Z}_n$. Then the oriented circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral if and only if*

$$S = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \bigcup\limits_{d \in \mathscr{D}} S_n(d) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ and $S_n(d) \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$.

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Proof. We proved the sufficient part in Theorem 3.9. Assume that $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral. If $n \not\equiv 0 \pmod 3$ then by Theorem 3.2, $S = \emptyset$. Let $n \equiv 0 \pmod 3$ and let v be the vector of length n defined by

$$v_k = \begin{cases} -\omega_3^2 & \text{if } k \in S \\ -\omega_3 & \text{if } n-k \in S \\ 0 & \text{otherwise.} \end{cases}$$

Note that each entry of Ev is an HS-eigenvalue of $\text{Circ}(\mathbb{Z}_n, S)$, and so $Ev \in \mathbb{Z}^n$. Thus v satisfies all the conditions of Lemma 4.1.

By conditions (i) and (iii) of Lemma 4.1, if $d \nmid \frac{n}{3}$, $d \mid n$ and $d < n$ then $v_r = v_{n-r} \in \mathbb{R}$ for all $r \in G_n(d)$. Also, by definition of v we have, $v_r \in \mathbb{R}$ if and only if $r \notin S \cup S^{-1}$. Therefore $S \subseteq \bigcup\limits_{d \mid \frac{n}{3}} \left[G_{n,3}^1(d) \cup G_{n,3}^2(d)\right]$.

Further, if $r \in S \cap G_{n,3}^1(d)$ for some $d \mid \frac{n}{3}$, again by condition (ii) of Lemma 4.1, we have $G_{n,3}^1(d) \subseteq S$. Similarly, if $r \in S \cap G_{n,3}^2(d)$ for some $d \mid \frac{n}{3}$ then $G_{n,3}^2(d) \subseteq S$. Thus there exists $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ such that

$$S = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \bigcup\limits_{d \in \mathscr{D}} S_n(d) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $S_n(d) \in \{G_{n,3}^1(d), G_{n,3}^2(d)\}$. □

The following example illustrates Theorem 4.2.

Example 4.3. *Consider* $\Gamma = \mathbb{Z}_{12}$ *and* $S = \{2, 5, 11\}$. *The oriented graph* $\text{Circ}(\mathbb{Z}_{12}, S)$ *is shown in Figure 1a. We see that* $G_{12,3}^2(1) = \{5, 11\}$ *and* $G_{12,3}^1(2) = \{2\}$. *Therefore* $S = G_{12,3}^2(1) \cup G_{12,3}^1(2)$, *and hence* $\text{Circ}(\mathbb{Z}_{12}, S)$ *is HS-integral. Further, using Corollary 2.1, the HS-eigenvalues of* $\text{Circ}(\mathbb{Z}_{12}, S)$ *are obtained as*

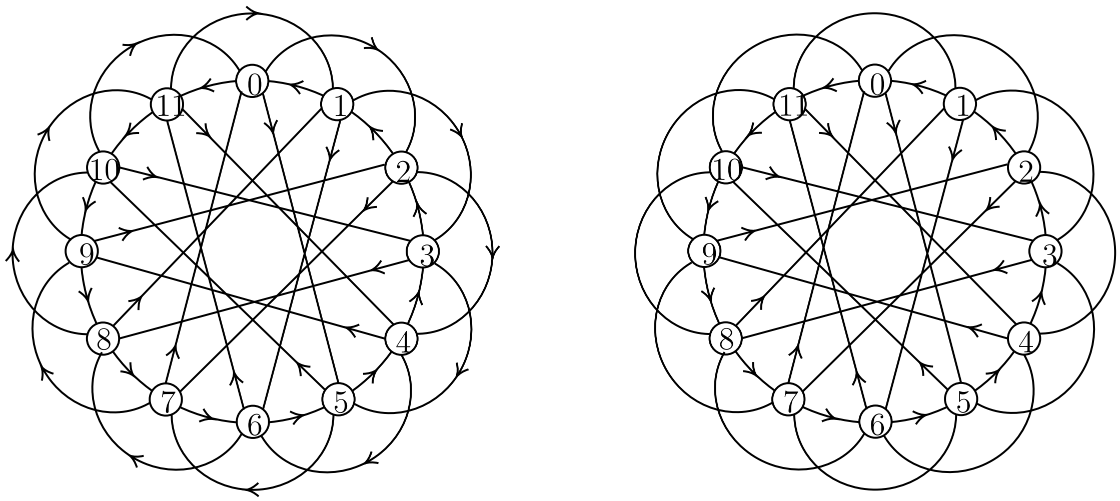
$$\mu_j = \cos\left(\frac{\pi j}{3}\right) + \cos\left(\frac{5\pi j}{6}\right) + \cos\left(\frac{11\pi j}{6}\right) - \sqrt{3}\left[\sin\left(\frac{\pi j}{3}\right) + \sin\left(\frac{5\pi j}{6}\right) + \sin\left(\frac{1\pi j}{6}\right)\right],$$

for each $j \in \mathbb{Z}_{12}$. *We find that* $\mu_0 = 3, \mu_1 = -1, \mu_2 = 2, \mu_3 = -1, \mu_4 = 3, \mu_5 = 2, \mu_6 = -1, \mu_7 = -1, \mu_8 = -6, \mu_9 = -1, \mu_{10} = -1$ *and* $\mu_{11} = 2$. *Thus all the HS-eigenvalues of* $\text{Circ}(\mathbb{Z}_{12}, S)$ *are integers.* □

Lemma 4.4. *Let S be a skew-symmetric subset of $\mathbb{Z}_n$ and $t(\neq 0) \in \mathbb{Q}$. If* $\sum\limits_{k \in S} it\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})$ *is an integer for each* $j \in \{0, ..., n-1\}$ *then*

$$S = \begin{cases} \emptyset & \textit{if } n \not\equiv 0 \pmod 3 \\ \bigcup\limits_{d \in \mathscr{D}} S_n(d) & \textit{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ *and* $S_n(d) \in \{G_{n,3}^1(d), G_{n,3}^2(d)\}$.



(a) $S = \{2, 5, 11\}$ (b) $S = \{2, 5, 10, 11\}$

Figure 1: The graph $\mathrm{Circ}(\mathbb{Z}_{12}, S)$.

Proof. Let $\alpha_j = \sum_{k \in S} it\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})$ for each $j \in \{0, ..., n-1\}$. Assume that $n \not\equiv 0 \pmod 3$, so that $\Phi_n(x)$ is irreducible in $\mathbb{Q}(\omega_3)[x]$. Then ω_n^j is a root of $p(x) = \sum_{k \in S} it\sqrt{3}(x^k - x^{n-k}) - \alpha_j \in \mathbb{Q}(\omega_3)[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi_n(x)$, and so $\omega_n^{-j} = \omega_n^{n-j}$ is also a root of $p(x)$, that is, $\alpha_j = \alpha_{n-j}$. Thus $\alpha_j = -\alpha_{n-j}$ and $\alpha_j = \alpha_{n-j}$ implies that $\alpha_j = \alpha_{n-j} = 0$. By Lemma 3.1, we get $S = \emptyset$.

Assume that $n \equiv 0 \pmod 3$. Let v be the vector of length n defined by

$$v_k = \begin{cases} it\sqrt{3} & \text{if } k \in S \\ -it\sqrt{3} & \text{if } n-k \in S \\ 0 & \text{otherwise.} \end{cases}$$

Since $v \in \mathbb{Q}^n(\omega_3)$ and the j-th coordinate of Ev is α_j, we have $Ev \in \mathbb{Q}^n$. Thus v satisfies all the conditions of Lemma 4.1. Hence

$$S = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \bigcup_{d \in \mathscr{D}} S_n(d) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ and $S_n(d) \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$. □

Lemma 4.5. *Let S be a skew-symmetric subset of $\mathbb{Z}_n$ and $t(\neq 0) \in \mathbb{Q}$. If $\sum_{k \in S} it\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})$ is an integer for each $j \in \{0, ..., n-1\}$ then $\sum_{k \in S \cup S^{-1}} \omega_n^{jk} \in \mathbb{Z}$ for each $j \in \{0, ..., n-1\}$.*

Proof. By Lemma 4.4, there is nothing to prove if $n \not\equiv 0 \pmod 3$. Assume $n \equiv 0 \pmod 3$. Again by Lemma 4.4, $S = \bigcup_{d \in \mathscr{D}} S_n(d)$, where $\mathscr{D} \subseteq \{d : d \mid \frac{n}{3}\}$ and $S_n(d) \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$. Therefore $S \cup S^{-1} = \bigcup_{d \in \mathscr{D}} G_n(d)$. By Theorem 2.3, the graph $\mathrm{Circ}(\mathbb{Z}_n, S \cup S^{-1})$ is integral.

Note that $\sum\limits_{k\in S\cup S^{-1}} \omega_n^{jk}$ is an eigenvalue of $\text{Circ}(\mathbb{Z}_n, S\cup S^{-1})$ for each $j\in\{0,...,n-1\}$. Hence $\sum\limits_{k\in S\cup S^{-1}} \omega_n^{jk}\in\mathbb{Z}$ for each $j\in\{0,...,n-1\}$. □

Lemma 4.6. *Let $S\subseteq\mathbb{Z}_n$ such that $0\notin S$. Then the mixed circulant graph $\text{Circ}(\mathbb{Z}_n,S)$ is HS-integral if and only if both $\text{Circ}(\mathbb{Z}_n,S\setminus\overline{S})$ and $\text{Circ}(\mathbb{Z}_n,\overline{S})$ are HS-integral.*

Proof. Assume that the mixed circulant graph $\text{Circ}(\mathbb{Z}_n,S)$ is HS-integral. Let the HS-spectrum of $\text{Circ}(\mathbb{Z}_n,S)$ be $\{\gamma_0,\gamma_1,...,\gamma_{n-1}\}$, where $\gamma_j=\lambda_j+\mu_j$,

$$\lambda_j=\sum_{k\in S\setminus\overline{S}}\omega_n^{jk} \text{ and } \mu_j=\sum_{k\in\overline{S}}(\omega_6\omega_n^{jk}+\omega_6^5\omega_n^{-jk}) \text{ for each } j\in\{0,...,n-1\}.$$

By assumption $\gamma_j\in\mathbb{Z}$, and so $\gamma_j-\gamma_{n-j}=\sum\limits_{k\in\overline{S}} i\sqrt{3}(\omega_n^{jk}-\omega_n^{-jk})\in\mathbb{Z}$ for each $j\in\{0,...,n-1\}$. By Lemma 4.5, we get $\sum\limits_{k\in\overline{S}\cup\overline{S}^{-1}}\omega_n^{jk}\in\mathbb{Z}$ for each $j\in\{0,...,n-1\}$. Since

$$\mu_j=\frac{1}{2}\sum_{k\in\overline{S}\cup\overline{S}^{-1}}\omega_n^{jk}+\frac{1}{2}\sum_{k\in\overline{S}} i\sqrt{3}(\omega_n^{jk}-\omega_n^{-jk}),$$

the number μ_j is a rational algebraic integer for each $j\in\{0,...,n-1\}$. Thus $\text{Circ}(\mathbb{Z}_n,\overline{S})$ is HS-integral. Now we have $\gamma_j,\mu_j\in\mathbb{Z}$ for each $j\in\{0,...,n-1\}$, and so $\lambda_j=\gamma_j-\mu_j\in\mathbb{Z}$ for each $j\in\{0,...,n-1\}$. Thus $\text{Circ}(\mathbb{Z}_n,S\setminus\overline{S})$ is also HS-integral.

Conversely, assume that both $\text{Circ}(\mathbb{Z}_n,S\setminus\overline{S})$ and $\text{Circ}(\mathbb{Z}_n,\overline{S})$ are HS-integral. Then Lemma 2.1 implies that $\text{Circ}(\mathbb{Z}_n,S)$ is HS-integral. □

Theorem 4.7. *Let $S\subseteq\mathbb{Z}_n$ such that $0\notin S$. Then the mixed circulant graph $\text{Circ}(\mathbb{Z}_n,S)$ is HS-integral if and only if $S\setminus\overline{S}=\bigcup\limits_{d\in\mathscr{D}_1}G_n(d)$ and*

$$\overline{S}=\begin{cases}\emptyset & \text{if } n\not\equiv 0 \pmod 3\\ \bigcup\limits_{d\in\mathscr{D}_2}S_n(d) & \text{if } n\equiv 0 \pmod 3,\end{cases}$$

where $\mathscr{D}_1\subseteq\{d:d\mid n\}$, $\mathscr{D}_2\subseteq\{d:d\mid\frac{n}{3}\}$, $\mathscr{D}_1\cap\mathscr{D}_2=\emptyset$ and $S_n(d)\in\{G_{n,3}^1(d),G_{n,3}^2(d)\}$.

Proof. By Lemma 4.6, $\text{Circ}(\mathbb{Z}_n,S)$ is HS-integral if and only if $\text{Circ}(\mathbb{Z}_n,S\setminus\overline{S})$ and $\text{Circ}(\mathbb{Z}_n,\overline{S})$ are HS-integral. Therefore the result follows from Theorem 2.3 and Theorem 4.2. □

The following example illustrates Theorem 4.7.

Example 4.8. *Consider $\Gamma=\mathbb{Z}_{12}$ and $S=\{2,5,10,11\}$. The mixed graph $\text{Circ}(\mathbb{Z}_{12},S)$ is shown in Figure 1b. We see that $G_{12,3}^2(1)=\{5,11\}$ and $G_{12}(2)=\{2,10\}$. Therefore $S=G_{12,3}^2(1)\cup G_{12}(2)$, and hence $\text{Circ}(\mathbb{Z}_{12},S)$ is HS-integral. Further, using Corollary 2.1, the HS-eigenvalues of $\text{Circ}(\mathbb{Z}_{12},S)$ are obtained as*

$$\gamma_j=2\cos\left(\frac{\pi j}{3}\right)+\cos\left(\frac{5\pi j}{6}\right)+\cos\left(\frac{11\pi j}{6}\right)-\sqrt{3}\left[\sin\left(\frac{5\pi j}{6}\right)+\sin\left(\frac{11\pi j}{6}\right)\right],$$

for each $j\in\mathbb{Z}_{12}$. One can see that $\gamma_0=4,\gamma_1=1,\gamma_2=3,\gamma_3=-2,\gamma_4=1,\gamma_5=1,\gamma_6=0,\gamma_7=1$, $\gamma_8=-5,\gamma_9=-2,\gamma_{10}=-3$ and $\gamma_{11}=1$. Thus all the HS-eigenvalues of $\text{Circ}(\mathbb{Z}_{12},S)$ are integers. □

5 Characterization of Eisenstein integral mixed circulant graphs

In 1918, Ramanujan [12] introduced the sum, known as Ramanujan sum, defined by

$$C_n(q)=\sum_{a\in G_n(1)}\cos\left(\frac{2\pi aq}{n}\right)\quad\text{for each } n,q\in\mathbb{N}. \tag{6}$$

It is well known that $C_n(q)$ is an integer for all $n,q\in\mathbb{Z}$. For $n\equiv 0 \pmod 3$, define

$$T_n(q)=\sum_{a\in G_{n,3}^1(1)} i\sqrt{3}(\omega_n^{aq}-\omega_n^{-aq})=\sum_{a\in G_{n,3}^1(1)} -2\sqrt{3}\sin\left(\frac{2\pi aq}{n}\right).$$

Lemma 5.1. *Let $n\equiv 0 \pmod 3$. Then $T_n(q)\in\mathbb{Z}$ for all $n,q\in\mathbb{Z}$.*

Proof. We have

$$Z_n^1(q)=\sum_{a\in G_{n,3}^1(1)}\left(\omega_3\omega_n^{aq}+\omega_3^2\omega_n^{-aq}\right)=-\frac{1}{2}\sum_{q\in G_n(1)}\omega_n^{jq}+\frac{i\sqrt{3}}{2}\sum_{a\in G_{n,3}^1(1)}(\omega_n^{aq}-\omega_n^{-aq})=-\frac{C_n(q)}{2}+\frac{T_n(q)}{2}.$$

Thus $T_n(q)=2Z_n^1(q)+C_n(q)\in\mathbb{Z}$ for all $n,q\in\mathbb{Z}$. □

Lemma 5.2. *Let $n=3^tm$ with $m\not\equiv 0 \pmod 3$. Then the following statements hold.*

(i) If $t=1$ then $G_n(1)=\left(m+3G_{\frac{n}{3}}(1)\right)\cup\left(2m+3G_{\frac{n}{3}}(1)\right)$.

(ii) If $t=1$ then

$$G_{n,3}^1(1)=\begin{cases} m+3G_{\frac{n}{3}}(1) & \text{if } m\equiv 1 \pmod 3\\ 2m+3G_{\frac{n}{3}}(1) & \text{if } m\equiv 2 \pmod 3.\end{cases}$$

(iii) If $t\geq 2$ then

$$G_{n,3}^1(1)=\begin{cases} \left(m+3G_{\frac{n}{3}}(1)\right)\cup\left(4m+3G_{\frac{n}{3},3}^2(1)\right) & \text{if } m\equiv 1 \pmod 3\\ \left(2m+3G_{\frac{n}{3}}(1)\right)\cup\left(5m+3G_{\frac{n}{3},3}^1(1)\right) & \text{if } m\equiv 2 \pmod 3.\end{cases}$$

(iv) If $t\geq 2$ then

$$G_{n,3}^1(1)=\begin{cases} \left(7m+3G_{\frac{n}{3}}(1)\right)\cup\left(4m+3G_{\frac{n}{3},3}^1(1)\right) & \text{if } m\equiv 1 \pmod 3\\ \left(8m+3G_{\frac{n}{3}}(1)\right)\cup\left(5m+3G_{\frac{n}{3},3}^2(1)\right) & \text{if } m\equiv 2 \pmod 3.\end{cases}$$

(v) If $t\geq 2$ then

$$G_{n,3}^1(1)=\begin{cases} \left(m+3G_{\frac{n}{3}}(1)\right)\cup\left(4m+3G_{\frac{n}{3}}(1)\right)\cup\left(7m+3G_{\frac{n}{3}}(1)\right) & \text{if } m\equiv 1 \pmod 3\\ \left(2m+3G_{\frac{n}{3}}(1)\right)\cup\left(5m+3G_{\frac{n}{3}}(1)\right)\cup\left(8m+3G_{\frac{n}{3}}(1)\right) & \text{if } m\equiv 2 \pmod 3.\end{cases}$$

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Proof. (i) Assume that $t=1$. Let $k \in m+3G_{\frac{n}{3}}(1)$. We get $k=m+3r$ for some $r \in G_{\frac{n}{3}}(1)$. Then $\gcd(r,\frac{n}{3})=1$ suggests that $\gcd(m+3r,n)=1$. Therefore $m+3G_{\frac{n}{3}}(1) \subseteq G_n(1)$. Similarly, $2m+3G_{\frac{n}{3}}(1) \subseteq G_n(1)$. Conversely, the size of both $\big(m+3G_{\frac{n}{3}}(1)\big) \cup \big(2m+3G_{\frac{n}{3}}(1)\big)$ and $G_n(1)$ are same, and hence both are equal.

(ii) Assume that $t=1$ and $m \equiv 1 \pmod 3$. Using Part (i), we have $m+3G_{\frac{n}{3}}(1) \subseteq G_{n,3}^1(1)$, and each element of $m+3G_{\frac{n}{3}}(1)$ is congruent to 1 modulo 3. Hence $G_{n,3}^1(1)=m+3G_{\frac{n}{3}}(1)$. Similarly, if $t=1$ and $m \equiv 2 \pmod 3$ then $G_{n,3}^1(1)=2m+3G_{\frac{n}{3}}(1)$.

(iii) The proof is similar to Part (i).

(iv) The proof is similar to Part (i).

(v) Use Part (iii) and Part (iv).

□

Let $\Im(z)$ denote the imaginary part of the complex number z.

Lemma 5.3. *Let $n=3m$ with $m \not\equiv 0 \pmod 3$. Then*

$$T_n(q)=\begin{cases} -2\sqrt{3}\Im(\omega_3^q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_3^{2q})C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3.\end{cases}$$

Moreover, $\frac{T_n(q)}{3}$ is an integer for all $q \in \mathbb{Z}$.

Proof. We have

$$\begin{aligned} T_n(q) &= \sum_{a \in G_{n,3}^1(1)} i\sqrt{3}(\omega_n^{aq}-\omega_n^{-aq})\\ &= \begin{cases} \sum\limits_{a \in G_{\frac{n}{3}}(1)} i\sqrt{3}(\omega_n^{mq}\omega_n^{3aq}-\omega_n^{-mq}\omega_n^{-3aq}) & \text{if } m \equiv 1 \pmod 3\\ \sum\limits_{a \in G_{\frac{n}{3}}(1)} i\sqrt{3}(\omega_n^{2mq}\omega_n^{3aq}-\omega_n^{-2mq}\omega_n^{-3aq}) & \text{if } m \equiv 2 \pmod 3\end{cases}\\ &= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}) \sum\limits_{a \in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m \equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}) \sum\limits_{a \in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m \equiv 2 \pmod 3\end{cases}\\ &= \begin{cases} -2\sqrt{3}\Im(\omega_3^q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_3^{2q})C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3.\end{cases} \end{aligned}$$

Here the second equality follows from Part (ii) of Lemma 5.2. Since $2\Im(\omega_3)=\sqrt{3}$, therefore $\frac{T_n(q)}{3}$ is an integer for all $q \in \mathbb{Z}$. □

Lemma 5.4. *Let $n=3^t m$ with $m \not\equiv 0 \pmod 3$ and $t \geq 2$. Then*

$$2T_n(q)=\begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3.\end{cases}$$

Moreover, $\frac{T_n(q)}{3}$ is an integer for all $q \in \mathbb{Z}$.

Proof. We use the fact that $G^1_{n,3}(1)$ can be written as disjoint unions in two different ways using Part (iii) and Part (iv) of Lemma 5.2. We have

$$\begin{aligned}
&2T_n(q)\\
&= \sum_{a\in G^1_{n,3}(1)} i\sqrt{3}(\omega_n^{aq}-\omega_n^{-aq}) + \sum_{a\in G^1_{n,3}(1)} i\sqrt{3}(\omega_n^{aq}-\omega_n^{-aq})\\
&= \begin{cases} \sum\limits_{a\in G_{\frac{n}{3}}(1)} i\sqrt{3}((\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})\omega_n^{3aq}-(\omega_n^{-mq}+\omega_n^{-4mq}+\omega_n^{-7mq})\omega_n^{-3aq}) & \text{if } m\equiv 1 \pmod 3\\ \sum\limits_{a\in G_{\frac{n}{3}}(1)} i\sqrt{3}((\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})\omega_n^{3aq}-(\omega_n^{-2mq}+\omega_n^{-5mq}+\omega_n^{-8mq})\omega_n^{-3aq}) & \text{if } m\equiv 2 \pmod 3\end{cases}\\
&= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})\sum\limits_{a\in G_{\frac{n}{3}}(1)}\omega_{\frac{n}{3}}^{aq} & \text{if } m\equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})\sum\limits_{a\in G_{\frac{n}{3}}(1)}\omega_{\frac{n}{3}}^{aq} & \text{if } m\equiv 2 \pmod 3\end{cases}\\
&= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})C_{\frac{n}{3}}(q) & \text{if } m\equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})C_{\frac{n}{3}}(q) & \text{if } m\equiv 2 \pmod 3.\end{cases}
\end{aligned}$$

Here the second equality follows from Part (v) of Lemma 5.2. If $t=2$, then

$$2T_n(q) = \begin{cases} C_9(q)C_{\frac{n}{3}}(q) & \text{if } m\equiv 1 \pmod 3\\ C_9(q)C_{\frac{n}{3}}(q) & \text{if } m\equiv 2 \pmod 3.\end{cases}$$

Thus $\frac{T_n(q)}{3}$ is an integer for all $q\in\mathbb{Z}$. Assume that $t\geq 3$. If $C_{\frac{n}{3}}(q)\neq 0$ then both $-2\sqrt{3}\Im(\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})$ and $-2\sqrt{3}\Im(\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})$ are rational algebraic integers, and hence both are integers for each $q\in\mathbb{Z}$. As $C_{\frac{n}{3}}(q)$ is an integer multiple of 3, we find that $\frac{T_n(q)}{3}$ is an integer for all $q\in\mathbb{Z}$. □

Lemma 5.5. *Let $n\equiv 0 \pmod 3$. Then $C_n(q)$ and $\frac{T_n(q)}{3}$ are integers of the same parity for all $q\in\mathbb{Z}$.*

Proof. Since $T_n(q)-C_n(q)=2Z^1_n(q)$ is an even integer, $C_n(q)$ and $T_n(q)$ are integers of the same parity for each $q\in\mathbb{Z}$. By Lemma 5.3 and Lemma 5.4, $\frac{T_n(q)}{3}$ is an integer for all $q\in\mathbb{Z}$. Hence $C_n(q)$ and $\frac{T_n(q)}{3}$ are integers of the same parity for all $q\in\mathbb{Z}$. □

Let S be a subset of $\mathbb{Z}_n$ and $j\in\{0,1,...,n-1\}$. Define

$$\alpha_j(S)=\sum_{k\in S\setminus\overline{S}}\omega_n^{jk} \quad\text{and}\quad \beta_j(S)=\sum_{k\in\overline{S}}(\omega\omega_n^{jk}+\overline{\omega}\omega_n^{-jk}),$$

where $\omega=\frac{1}{2}-\frac{i\sqrt{3}}{6}$. It is clear that $\alpha_j(S)$ and $\beta_j(S)$ are real numbers. We have

$$\sum_{k\in S}\omega_n^{jk}=\alpha_j(S)+\beta_j(S)+\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right)(\beta_j(S)-\beta_{n-j}(S)).$$

Note that $\alpha_j(S)=\alpha_{n-j}(S)$ for each j. Therefore if $\alpha_j(S)+\beta_j(S)\in\mathbb{Z}$ for each $j\in\{0,1,...,n-1\}$ then $\beta_j(S)-\beta_{n-j}(S)=[\alpha_j(S)+\beta_j(S)]-[\alpha_{n-j}(S)+\beta_{n-j}(S)]$ is also an integer for each $j\in\{0,1,...,n-1\}$. Hence the mixed circulant graph $\text{Circ}(\mathbb{Z}_n,S)$ is Eisenstein integral if and only if $\alpha_j(S)+\beta_j(S)$ is an integer for each $j\in\{0,1,...,n-1\}$.

Lemma 5.6. *Let $S \subseteq \mathbb{Z}_n$ such that $0 \notin S$. Then the mixed circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is Eisenstein integral if and only if $2\alpha_j(S)$ and $2\beta_j(S)$ are integers of the same parity for each $j \in \{0, 1, ..., n-1\}$.*

Proof. Let the mixed circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ be Eisenstein integral and $j \in \{0, 1, ..., n-1\}$. Then $\alpha_j(S) + \beta_j(S)$ and $\beta_j(S) - \beta_{n-j}(S) = \sum\limits_{k \in \overline{S}} \frac{-i\sqrt{3}}{3}(\omega_n^{jk} - \omega_n^{-jk})$ are integers. By Lemma 4.5, $\sum\limits_{k \in \overline{S} \cup \overline{S}^{-1}} \omega_n^{jk} \in \mathbb{Z}$. Since

$$2\beta_j(S) = \sum_{k \in \overline{S} \cup \overline{S}^{-1}} \omega_n^{jk} - \sum_{k \in \overline{S}} \frac{i\sqrt{3}}{3}(\omega_n^{jk} - \omega_n^{-jk}),$$

we find that $2\beta_j(S)$ is an integer. Therefore $2\alpha_j(S) = 2(\alpha_j(S) + \beta_j(S)) - 2\beta_j(S)$ is also an integer of the same parity with $2\beta_j(S)$.

Conversely, assume that $2\alpha_j(S)$ and $2\beta_j(S)$ are integers of the same parity for each $j \in \{0, 1, ..., n-1\}$. Then $\alpha_j(S) + \beta_j(S)$ is an integer for each $j \in \{0, 1, ..., n-1\}$. Hence the mixed circulant graph $\text{Circ}(\mathbb{Z}_n, S)$ is Eisenstein integral. □

Lemma 5.7. *Let $S \subseteq \mathbb{Z}_n$ such that $0 \notin S$. Then the mixed circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is Eisenstein integral if and only if $\alpha_j(S)$ and $\beta_j(S)$ are integers for each $j \in \{0, 1, ..., n-1\}$.*

Proof. By Lemma 5.6, it is enough to show that $2\alpha_j(S)$ and $2\beta_j(S)$ are integers of the same parity if and only if $\alpha_j(S)$ and $\beta_j(S)$ are integers. If $\alpha_j(S)$ and $\beta_j(S)$ are integers, then clearly $2\alpha_j(S)$ and $2\beta_j(S)$ are even integers. Conversely, assume that $2\alpha_j(S)$ and $2\beta_j(S)$ are integers of the same parity. Since $\alpha_j(S)$ is an algebraic integer, the integrality of $2\alpha_j(S)$ implies that $\alpha_j(S)$ is an integer. Thus $2\alpha_j(S)$ is even, and so by the assumption $2\beta_j(S)$ is also an even integer. Hence $\beta_j(S)$ is an integer. □

Theorem 5.8. *Let $S \subseteq \mathbb{Z}_n$ such that $0 \notin S$. Then the mixed circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is Eisenstein integral if and only if* $\text{Circ}(\mathbb{Z}_n, S)$ *is HS-integral.*

Proof. By Lemma 5.7, it is enough to show that $\alpha_j(S)$ and $\beta_j(S)$ are integers for each $j \in \{0, 1, ..., n-1\}$ if and only if $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral. Note that $\alpha_j(S)$ is an eigenvalue of the circulant graph $\text{Circ}(\mathbb{Z}_n, S \setminus \overline{S})$. By Theorem 2.3, $\alpha_j(S)$ is an integer for each $j \in \{0, 1, ..., n-1\}$ if and only if $S \setminus \overline{S} = \bigcup\limits_{d \in \mathscr{D}_1} G_n(d)$ for some $\mathscr{D}_1 \subseteq \{d : d \mid n\}$. Assume that $\alpha_j(S)$ and $\beta_j(S)$ are integers for each j. Then $-\frac{i\sqrt{3}}{3} \sum\limits_{k \in \overline{S}} (\omega_n^{jk} - \omega_n^{-jk}) = \beta_j(S) - \beta_{n-j}(S)$ is also an integer for each j. Using Theorem 2.3 and Lemma 4.4, we see that $S \setminus \overline{S}$ and $\overline{S}$ satisfy the conditions of Theorem 4.7. Hence $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral.

Conversely, assume that $\text{Circ}(\mathbb{Z}_n, S)$ is HS-integral. Then $\text{Circ}(\mathbb{Z}_n, S \setminus \overline{S})$ is integral, and hence $\alpha_j(S)$ is an integer for each $j \in \{0, 1, ..., n-1\}$. By Theorem 4.7, we have

$$\overline{S} = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \bigcup\limits_{d \in \mathscr{D}_2} S_n(d) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D}_2 \subseteq \{d : d \mid \frac{n}{3}\}$ and $S_n(d) \in \{G^1_{n,3}(d), G^2_{n,3}(d)\}$. Then

$$\begin{aligned}\beta_j(S) &= \frac{1}{2}\sum_{k\in\overline{S}\cup\overline{S}^{-1}} \omega_n^{jk} - \frac{1}{6}\sum_{k\in\overline{S}} i\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})\\ &= \frac{1}{2}\sum_{d\in\mathcal{D}_2}\sum_{k\in G_n(d)} \omega_n^{jk} + \frac{1}{6}\sum_{d\in\mathcal{D}_2}\sum_{k\in S_n(d)} i\sqrt{3}(\omega_n^{jk} - \omega_n^{-jk})\\ &= \frac{1}{2}\sum_{d\in\mathcal{D}_2} C_{\frac{n}{d}}(j) \pm \frac{1}{6}\sum_{d\in\mathcal{D}_2} T_{\frac{n}{d}}(j)\\ &= \frac{1}{2}\sum_{d\in\mathcal{D}_2}\left(C_{\frac{n}{d}}(j) \pm \frac{1}{3}T_{\frac{n}{d}}(j)\right).\end{aligned}$$

By Lemma 5.5, $C_{\frac{n}{d}}(j) \pm \frac{1}{3}T_{\frac{n}{d}}(j)$ are even integers for all $d \in \mathcal{D}_2$. Hence $\beta_j(S)$ is an integer for each $j \in \{0, 1, ..., n-1\}$. □

The following example illustrates Theorem 5.8.

Example 5.9. *Consider the HS-integral graph Circ*$(\mathbb{Z}_{12}, S)$ *of Example 4.8. By Theorem 5.8, the graph Circ*$(\mathbb{Z}_{12}, S)$ *is Eisenstein integral. Indeed, the eigenvalues of Circ*$(\mathbb{Z}_{12}, S)$ *are obtained as*

$$\begin{aligned}\gamma_j =& 2\cos\left(\frac{\pi j}{3}\right) + \cos\left(\frac{5\pi j}{6}\right) + \cos\left(\frac{11\pi j}{6}\right) + \frac{1}{\sqrt{3}}\left[\sin\left(\frac{5\pi j}{6}\right) + \sin\left(\frac{11\pi j}{6}\right)\right]\\ &+ \omega_3\frac{2}{\sqrt{3}}\left[\sin\left(\frac{5\pi j}{6}\right) + \sin\left(\frac{11\pi j}{6}\right)\right] \quad \textit{for each } j \in \mathbb{Z}_{12}.\end{aligned}$$

One can see that $\gamma_0 = 4, \gamma_1 = 1, \gamma_2 = -1-2\omega_3, \gamma_3 = -2, \gamma_4 = -3-2\omega_3, \gamma_5 = 1, \gamma_6 = 0, \gamma_7 = 1, \gamma_8 = -1+2\omega_3, \gamma_9 = -2, \gamma_{10} = 1+2\omega_3$ *and* $\gamma_{11} = 1$. *Thus* γ_j *is an Eisenstein integer for each* $j \in \mathbb{Z}_{12}$. □

6 Eigenvalues and HS-eigenvalues of unitary oriented circulant graphs in terms of generalized Möbius function

Let $n \equiv 0 \pmod 3$. The underlying graph of $\text{Circ}(\mathbb{Z}_n, G^1_{n,3}(1))$ is known as an unitary simple circulant graph. The graph $\text{Circ}(\mathbb{Z}_n, G^1_{n,3}(1))$ is called an *unitary oriented circulant graph* (*UOCG*). Using Theorem 4.7 and Theorem 5.8, UOCG is an HS-integral as well as Eisenstein integral graph. The eigenvalues and the HS-eigenvalues of UOCG are $\frac{C_n(j)}{2} + \frac{T_n(j)}{6} + \omega_3^2\frac{T_n(j)}{3}$ and $\frac{C_n(j)}{2} + \frac{T_n(j)}{2}$, respectively, for each $j \in \{0, 1, \ldots, n-1\}$. Note that $C_n(j)$ is an eigenvalue of the underlying graph of UOCG for each $j \in \{0, 1, \ldots, n-1\}$. Using Lemma 5.3 and Lemma 5.4, we can express $T_n(j)$ in terms of $C_{\frac{n}{3}}(j)$. This, in turn, express the eigenvalues and the HS-eigenvalues of UOCG in terms of Ramanujan sums. The Ramanujan sum $C_n(j)$

is well known [11] in terms of Möbius function. In particular, we have

$$C_n(j) = \sum_{d|\delta} d\mu(n/d) = \frac{\mu(n/\delta)\varphi(n)}{\varphi(n/\delta)},$$

where $\delta = \gcd(n, j)$. We attempt to obtain similar expression for $T_n(j)$ in terms of generalized Möbius function. That, in turn, will express the eigenvalues and the HS-eigenvalues of UOCG in terms of generalized Möbius function.

The classical Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Let δ be the indicator function defined by

$$\delta(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Theorem 6.1. *[11]* $\sum\limits_{d|n} \mu(d) = \delta(n)$.

E. Cohen [4] introduced a generalized Möbius inversion formula of arbitrary direct factor sets. Let P and Q be two non-empty subsets of $\mathbb{N}$ such that if $n_1, n_2 \in \mathbb{N}$ with $\gcd(n_1, n_2) = 1$, then $n_1 n_2 \in P$ (resp. $n_1 n_2 \in Q$) if and only if $n_1, n_2 \in P$ (resp. $n_1, n_2 \in Q$). If each integer $n \in \mathbb{N}$ possesses a unique factorization of the form $n = ab$ with $a \in P, b \in Q$, then the sets P and Q are called direct factor sets of $\mathbb{N}$. In what follows, P will denote such a direct factor set with (conjugate) factor set Q. The Möbius function can be generalized to an arbitrary direct factor set P by setting

$$\mu_P(n) = \sum_{d|n, d \in P} \mu\left(\frac{n}{d}\right),$$

where μ is the classical Möbius function. For example, $\mu(n) = \mu_{\{1\}}(n)$ and $\mu_{\mathbb{N}}(n) = \delta(n)$.

Theorem 6.2. *[4]* $\sum\limits_{d|n, d \in Q} \mu_P\left(\frac{n}{d}\right) = \delta(n)$.

Theorem 6.3. *[4] If $f(n)$ and $g(n)$ are arithmetic functions then*

$$f(n) = \sum_{d|n, d \in Q} g\left(\frac{n}{d}\right) \text{ if and only if } g(n) = \sum_{d|n} f(d)\mu_P\left(\frac{n}{d}\right).$$

For the remaining part of this section, we consider the direct factors $P = \{2^k : k \geq 0\}$, and Q, the set of all odd natural numbers.

Lemma 6.4. *Let $P = \{2^k : k \geq 0\}$. Then*

$$\mu_P(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mu(n) & \text{if } n \text{ is odd.} \end{cases}$$

Note that if $n \equiv 0 \pmod 3$ then $D^2_{n,3} = D^2_{\frac{n}{3},3}$.

Theorem 6.5. *Let $n = 3m$ with m an odd integer and $D^2_{n,3} = \emptyset$. Then*

$$T_n(j) = \sum_{\substack{d \mid \frac{n}{3} \\ \frac{j}{d} \equiv 1 \pmod 3}} -3d\mu\left(\frac{n}{3d}\right) + \sum_{\substack{d \mid \frac{n}{3} \\ \frac{j}{d} \equiv 2 \pmod 3}} 3d\mu\left(\frac{n}{3d}\right).$$

Proof. Let

$$f_n(j) = \sum_{a \in M^1_{n,3}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) = \begin{cases} -n & \text{if } j \equiv \frac{n}{3} \pmod n \\ n & \text{if } j \equiv \frac{2n}{3} \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 2.5, we have

$$\begin{aligned} f_n(j) = \sum_{a \in M^1_{n,3}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) &= \sum_{d \in D^1_{n,3}} \sum_{a \in G^1_{n,3}(d)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) \\ &= \sum_{d \in D^1_{n,3}} \sum_{a \in dG^1_{\frac{n}{d},3}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) \\ &= \sum_{d \in D^1_{n,3}} \sum_{a \in G^1_{\frac{n}{d},3}(1)} i\sqrt{3}[(\omega_n^d)^{aj} - (\omega_n^d)^{-aj}] = \sum_{d \in D^1_{n,3}} T_{\frac{n}{d}}(j). \end{aligned}$$

In the last equation, we have used the fact that $\omega_n^d = \exp(\frac{2\pi i}{n/d})$ is a primitive $\frac{n}{d}$-th root of unity. Since $D^1_{n,3} \subseteq Q$, from Theorem 6.3 we have

$$\begin{aligned} T_n(j) &= \sum_{\substack{d \mid n \\ j \equiv \frac{d}{3} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d \mid n \\ j \equiv \frac{2d}{3} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) \\ &= \sum_{\substack{d \mid n \\ j \equiv \frac{d}{3} \pmod d}} -d\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d \mid n \\ j \equiv \frac{2d}{3} \pmod d}} d\mu_P\left(\frac{n}{d}\right) \\ &= \sum_{\substack{3d \mid n \\ j \equiv d \pmod{3d}}} -3d\mu\left(\frac{n}{3d}\right) + \sum_{\substack{3d \mid n \\ j \equiv 2d \pmod{3d}}} 3d\mu\left(\frac{n}{3d}\right) \\ &= \sum_{\substack{d \mid \frac{n}{3} \\ \frac{j}{d} \equiv 1 \pmod 3}} -3d\mu\left(\frac{n}{3d}\right) + \sum_{\substack{d \mid \frac{n}{3} \\ \frac{j}{d} \equiv 2 \pmod 3}} 3d\mu\left(\frac{n}{3d}\right). \end{aligned}$$

□

The case that $n = 3m$ with m an even integer is not covered in Theorem 6.5. Now assume that $n = 3m$ with m an even integer, so that $n \equiv 0 \pmod 6$. For a divisor d of $\frac{n}{6}$, $r \in \{1, 5\}$

and $g \in \mathbb{Z}$, define the following sets:

$$M^r_{n,6}(d) = \{dk : 0 \leq dk < n, k \equiv r \pmod 6\};$$
$$G^r_{n,6}(d) = \{dk : 1 \leq dk < n, k \equiv r \pmod 6, \gcd(dk, n) = d\}; \quad \text{and}$$
$$D^r_{g,6} = \{k : k \text{ divides } g, k \equiv r \pmod 6\}.$$

Lemma 6.6. *Let $n \equiv 0 \pmod 6$, d divides $\frac{n}{6}$ and $g = \frac{n}{6d}$. Then the following hold:*

(i) $G^1_{n,6}(d) \cap G^5_{n,6}(d) = \emptyset$;

(ii) $G_n(d) = G^1_{n,6}(d) \cup G^5_{n,6}(d)$;

(iii) $M^1_{n,6}(d) = \left(\bigcup\limits_{h \in D^1_{g,6}} G^1_{n,6}(hd) \right) \cup \left(\bigcup\limits_{h \in D^5_{g,6}} G^5_{n,6}(hd) \right)$;

(iv) $M^5_{n,6}(d) = \left(\bigcup\limits_{h \in D^1_{g,6}} G^5_{n,6}(hd) \right) \cup \left(\bigcup\limits_{h \in D^5_{g,6}} G^1_{n,6}(hd) \right)$.

Proof. The proof is similar to the proof of Lemma 2.5. □

Note that $D^5_{n,6} = D^5_{\frac{n}{6},6}$. In the next result, we calculate $T_n(j)$ for the values of n not covered in Theorem 6.5.

Theorem 6.7. *Let $n \equiv 0 \pmod 6$ and $D^5_{n,6} = \emptyset$. Then*

$$T_n(j) = \sum_{\substack{d \mid \frac{n}{6} \\ \frac{j}{d} \equiv 1 \text{ or } 2 \pmod 6}} -3d\mu_P\left(\frac{n}{6d}\right) + \sum_{\substack{d \mid \frac{n}{6} \\ \frac{j}{d} \equiv 4 \text{ or } 5 \pmod 6}} 3d\mu_P\left(\frac{n}{6d}\right),$$

Proof. Let

$$f_n(j) = \sum_{a \in M^1_{n,6}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) = \begin{cases} -\frac{n}{2} & \text{if } j \equiv \frac{n}{6} \pmod n \\ -\frac{n}{2} & \text{if } j \equiv \frac{2n}{6} \pmod n \\ 0 & \text{if } j \equiv \frac{3n}{6} \pmod n \\ \frac{n}{2} & \text{if } j \equiv \frac{4n}{6} \pmod n \\ \frac{n}{2} & \text{if } j \equiv \frac{5n}{6} \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 6.6 we get

$$\begin{aligned} f_n(j) = \sum_{a \in M^1_{n,6}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) &= \sum_{d \in D^1_{n,6}} \sum_{a \in G^1_{n,6}(d)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) \\ &= \sum_{d \in D^1_{n,6}} \sum_{a \in dG^1_{\frac{n}{d},6}(1)} i\sqrt{3}(\omega_n^{aj} - \omega_n^{-aj}) \\ &= \sum_{d \in D^1_{n,6}} \sum_{a \in G^1_{\frac{n}{d},6}(1)} i\sqrt{3}[(\omega_n^{d})^{aj} - (\omega_n^{d})^{-aj}] \\ &= \sum_{d \in D^1_{n,6}} T_{\frac{n}{d}}(j). \end{aligned}$$

In the last equation, we have used the fact that ω_n^d is a primitive $\frac{n}{d}$-th root of unity and $G^1_{\frac{n}{d},6}(1) = G^1_{\frac{n}{d},3}(1)$ for all $d \in D^1_{n,6}$. Since $D^1_{n,6} \subset Q$, by Theorem 6.3 we get

$$\begin{aligned}
T_n(j) = & \sum_{\substack{d|n \\ j\equiv \frac{d}{6} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ j\equiv \frac{2d}{6} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) \\
& + \sum_{\substack{d|n \\ j\equiv \frac{4d}{6} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ j\equiv \frac{5d}{6} \pmod d}} f_d(j)\mu_P\left(\frac{n}{d}\right) \\
= & \sum_{\substack{d|n \\ j\equiv \frac{d}{6} \text{ or } \frac{2d}{6} \pmod d}} -\frac{d}{2}\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ j\equiv \frac{4d}{6} \text{ or } \frac{5d}{6} \pmod d}} \frac{d}{2}\mu_P\left(\frac{n}{d}\right) \\
= & \sum_{\substack{6d|n \\ j\equiv d \text{ or } 2d \pmod{6d}}} -3d\mu_P\left(\frac{n}{6d}\right) + \sum_{\substack{6d|n \\ j\equiv 4d \text{ or } 5d \pmod{6d}}} 3d\mu_P\left(\frac{n}{6d}\right) \\
= & \sum_{\substack{d|\frac{n}{6} \\ \frac{j}{d}\equiv 1 \text{ or } 2 \pmod 6}} -3d\mu_P\left(\frac{n}{6d}\right) + \sum_{\substack{d|\frac{n}{6} \\ \frac{j}{d}\equiv 4 \text{ or } 5 \pmod 6}} 3d\mu_P\left(\frac{n}{6d}\right).
\end{aligned}$$

□

References

[1] O. Ahmadi, N. Alon, I.F. Blake and I.E. Shparlinski. Graphs with integral spectrum. *Linear Algebra and its Applications* 430(1) (2009), 547–552.

[2] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić and D. Stevanović. A survey on integral graphs. *Publikacije Elektrotehničkog fakulteta. Serija Matematika* (2002), 42–65.

[3] R.B. Bapat, D. Kalita, and S. Pati. On weighted directed graphs. *Linear Algebra and its Applications*, 436(1) (2012), 99–111.

[4] E. Cohen. A class of residue systems (mod r) and related arithmetical functions. i. a generalization of möbius inversion. *Pacific Journal of Mathematics* 9(1) (1959), 13–23.

[5] P. Csikvári. Integral trees of arbitrarily large diameters. *Journal of Algebraic Combinatorics* 32(3) (2010), 371–377.

[6] F. Harary and A.J. Schwenk. Which graphs have integral spectra? In *Graphs and Combinatorics*. Springer (1974), 45–51.

[7] K. Ireland and M. Rosen. *A classical introduction to modern number theory.* Springer Verlag (1984).

[8] M. Kadyan and B. Bhattacharjya. HS-integral and Eisenstein integral mixed Cayley graphs over abelian groups. *Linear Algebra and its Applications* 645 (2022), 68-90.

[9] M. Kadyan and B. Bhattacharjya. Integral mixed circulant graphs. *Discrete Mathematics* 346(1) (2023), 113142.

[10] B. Mohar. A new kind of Hermitian matrices for digraphs. *Linear Algebra and its Applications* 584 (2020), 343–352.

[11] M.R. Murty. *Problems in analytic number theory*. Volume 206, Springer Science & Business Media (2008).

[12] S. Ramanujan. On certain trigonometrical sums and their applications in the theory of numbers. *Transactions of the Cambridge Philosophical Society* 22(13) (1918), 259–276.

[13] W. So. Integral circulant graphs. *Discrete Mathematics* 306(1) (2006), 153–158.

[14] M. Watanabe. Note on integral trees. *Mathematics Reports* 2 (1979), 95–100.

[15] M. Watanabe and A.J. Schwenk. Integral starlike trees. *Journal of the Australian Mathematical Society* 28(1) (1979), 120–128.