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Ramsey Numbers for Connected 2-Colorings of Complete Graphs

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Cover Page Footnote

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Abstract

In 1978, David Sumner introduced a variation of Ramsey numbers by restricting to 2-colorings in which the subgraphs spanned by edges in each color are connected. This paper continues the study of connected Ramsey numbers, including the evaluation of several cases of trees versus complete graphs.

1 Introduction

In this paper, we consider a variation of Ramsey numbers by restricting our attention to connected 2-colorings. Before describing our main results, we review the relevant definitions and background. Recall that a graph G is called *connected* if for every distinct pair of vertices a and b, there exists a path from a to b. If a graph is not connected, it is called *disconnected*. A maximal connected subgraph of a graph G is called a *connected component* of G. The following theorem is usually credited to Erdős and Rado (see Theorem 1.1 of [2]).

Theorem 1.1 (Erdős and Rado)**.** *If a graph G is disconnected, then its complement $\overline{G}$ is connected.*

Denote by K_p the complete graph of order p. It follows that in any 2-coloring of the edges of a complete graph K_p, at most one of the subgraphs spanned by each color is disconnected.

The *degree* of a vertex x in a graph G, denoted $\mathrm{deg_G(x)}$, is the number of edges in G that are incident with x. The *minimum degree* and *maximum degree* of G are given by

$$\delta(G) := \min\{\deg_G(x) \mid x \in V(G)\} \quad \text{and} \quad \Delta(G) := \max\{\deg_G(x) \mid x \in V(G)\},$$

respectively. A subset of vertices whose removal disconnects a connected graph is called a *vertex cut*. The minimum cardinality of a vertex cut of a connected graph G is called the *vertex connectivity of* G, and is denoted $\kappa(G)$. In Theorem 2.6 of [13], Hellwig and Volkmann proved the following theorem, relating the vertex connectivity of a graph to that of its complement, when both graphs are connected.

Theorem 1.2 ([13])**.** *Suppose that G and $\overline{G}$ are both connected. Then*

$$\kappa(G) + \kappa(\overline{G}) \geq \min\{\delta(G), \delta(\overline{G})\} + 1.$$

When a connected graph G satisfies $\kappa(G) = 1$, a vertex whose removal disconnects G is called a *cut-vertex*. If G is connected and has order n, it contains at most $n-2$ cut vertices, and this bound is attained by a path of order n.

If a graph G is connected, then the *edge connectivity* of G, denoted $\lambda(G)$, is the minimum number of edges whose removal disconnects G. If $\lambda(G) = 1$, then an edge whose removal disconnects G is called a *bridge*. Note that if a connected graph contains a bridge, it necessarily contains at least two cut-vertices.

A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required to properly vertex color G is called the *chromatic number* of G, and is denoted $\chi(G)$. An *independent set of vertices* in a graph G is a subset $I \subseteq V(G)$ such that no two vertices in I

are adjacent. In particular, note that each color class of a proper vertex coloring of a graph is an independent set. The maximum cardinality among all independent sets of a graph G is called the *independence number* of G, and is denoted $\alpha(G)$. Observe that if $\alpha(G) = n$, then the complement $\overline{G}$ contains a complete subgraph of order n, but does not contain any complete subgraphs of order $n+1$ (as this would correspond with an independent set with cardinality $n+1$ in G).

A 2-*coloring* of K_p is a map $f : E(K_p) \longrightarrow \{\text{red}, \text{blue}\}$ from the edge set of K_p to a set of two colors (which we assume are red and blue). For a fixed 2-coloring f, we denote by G_R and G_B the subgraph spanned by the red edges and the subgraph spanned by the blue edges, respectively. A 2-coloring of K_p is called *connected* if the subgraphs spanned by each color are both connected. For Ramsey numbers, the restriction to connected 2-colorings is implemented in a similar manner to the way in which one restricts to rainbow-triangle-free colorings when defining Gallai-Ramsey numbers (c.f., [16]).

If G_1 and G_2 are two graphs, then the *Ramsey number* $r(G_1, G_2)$ is the least natural number p such that every 2-coloring of K_p contains a red copy of G_1 (i.e., a subgraph isomorphic to G_1 in which all of its edges are red) or a blue copy of G_2. The existence of $r(G_1, G_2)$ follows from Frank Ramsey's ubiquitous theorem [21]. A 2-coloring of $K_{r(G_1,G_2)-1}$ that avoids a red copy of G_1 and a blue copy of G_2 is called a *critical coloring* for $r(G_1, G_2)$.

The *connected Ramsey number* $r_c(G_1, G_2)$ is the least natural number p such that every connected 2-coloring of K_p contains a red copy of G_1 or a blue copy of G_2. A connected 2-coloring of $K_{r_c(G_1,G_2)-1}$ that avoids a red copy of G_1 and a blue copy of G_2 is called a *connected critical coloring* for $r_c(G_1, G_2)$. Since every connected 2-coloring of a complete graph is a 2-coloring, it follows that $r_c(G_1, G_2) \leq r(G_1, G_2)$, with equality holding whenever a critical coloring for $r(G_1, G_2)$ is also a connected critical coloring for $r_c(G_1, G_2)$. When this occurs, we say that (G_1, G_2) is *Ramsey connected.*

In Section 2, we focus on proving that connected Ramsey numbers have a useful property that Ramsey numbers possess. Specifically, in Corollary 2.2, it is shown that if $r_c(G_1, G_2) = p$, then of all $n \geq p$, every connected 2-coloring of K_n contains a red copy of G_1 or a blue copy of G_2. Our attention then turns to the evaluation of $r_c(G_1, G_2)$ for various graphs, with a focus on connected graphs G_1 that are n-good.

Section 3 focuses on connected Ramsey numbers for trees versus complete graphs. Letting P_m denote a path of order m and $K_{1,m}$ denote a star of order $m+1$, Theorems 3.1, 3.3, and 3.5 include the following evaluations:

$$r_c(P_m, K_3) = m, \quad \text{for all } m \geq 4,$$
$$r_c(P_5, K_n) = n + 2, \quad \text{for all } n \geq 3,$$
$$r_c(K_{1,3}, K_n) = 2n, \quad \text{for all } n \geq 3.$$

There is an interest in proving results involving the maximum degree and independence numbers in graphs (e.g., see [10], [11], [12], and [17]). Some of our results can be interpreted as giving such relationships for a special collection of graphs: those that are connected and have connected complements. We will refer to such graphs as being *totally connected.* After Theorems 3.1, 3.3, and 3.5, we give corollaries of these results restated as properties of totally connected graphs. Finally, in Section 4, some interesting directions for future research concerning connected Ramsey numbers are discussed.

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2 The Connected Ramsey Number

As with Ramsey numbers, if H is a subgraph of G_1, then the connected Ramsey number satisfies

$$r_c(H, G_2) \leq r_c(G_1, G_2)$$

since the existence of a red G_1 implies the existence of a red H. One property that is not immediately obvious with connected Ramsey numbers is whether or not $r_c(G_1, G_2) = p$ implies that every connected 2-coloring of K_n contains a red copy of G_1 or a blue copy of G_2 when $n > p$. Sumner [22] noted that this is indeed true, as it follows from Theorem 11 of [3]. We give a self-contained proof of this fact in Theorem 2.1 and Corollary 2.2.

Theorem 2.1. *Let $n \geq 5$ and let c be a connected 2-coloring of K_n. Then there exists some vertex whose removal results in a connected 2-coloring of K_{n-1}.*

Proof. This result is equivalent to proving that every connected 2-coloring of K_n contains a vertex that is not a cut-vertex in G_R or G_B. From Theorem 1.1, we know that each vertex is a cut-vertex in at most one of G_R and G_B. Consider a connected 2-coloring of K_n, where $n \geq 5$ and suppose that every vertex is a cut-vertex in G_R or G_B. Since every connected graph of order n has at most $n-2$ cut-vertices, it follows that G_R and G_B each have at least two cut-vertices. Thus, from Theorem 1.2, it follows that $1 \geq \min\{\delta(G_R), \delta(G_B)\}$. Without loss of generality, assume that $\delta(G_R) = 1$ and $\deg_{G_R}(x) = 1$. Let y be the neighbor of x in G_R. It follows that x is a cut-vertex in G_B and y is a cut-vertex in G_R. Let $z_1, z_2, \ldots, z_{n-2}$ be the remaining vertices and note that xz_i is blue for all $1 \leq i \leq n-2$. Since G_R and G_B are both connected, y must be incident with at least one red edge and at least one blue edge to the set $\{z_1, z_2, \ldots, z_{n-2}\}$. Without loss of generality, assume that yz_1 is blue and yz_{n-2} is red (see the first image in Figure 1). If any other edge yz_i is blue, for $2 \leq i \leq n-3$, then

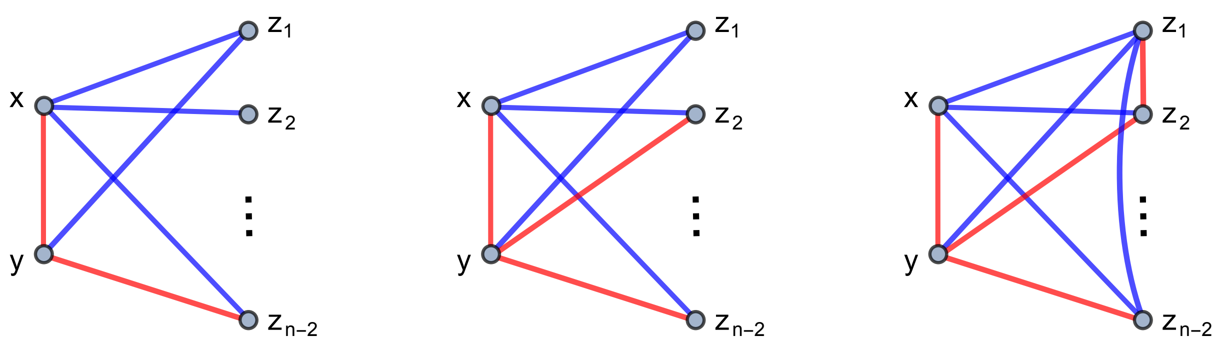

Figure 1: A connected 2-coloring of K_n, where $n \geq 5$.

all of $z_1, z_2, \ldots, z_{n-2}$ are cut-vertices in G_R. Along with y, this results in $n-1$ cut-vertices in G_R, which is a contradiction since every connected graph of order n has at most $n-2$ cut-vertices. It follows that $yz_2, yz_3, \ldots, yz_{n-2}$ are all red and z_1 is a cut-vertex in G_B since its removal disconnects y (see the second image in Figure 1). Observe that removing any one of $z_2, z_3, \ldots, z_{n-2}$ does not disconnect G_B. So, they must all be cut-vertices in G_R.

Since G_R is connected, z_1 must be incident with some red edge. Without loss of generality, suppose that z_1z_2 is red. If any of $z_1z_3, \ldots, z_1z_{n-2}$ are red, then z_2 is not a cut-vertex in G_R, which is a contradiction. So, assume that $z_1z_3, \ldots, z_1z_{n-2}$ are all blue (see the third image in Figure 1). At this point, we see that z_{n-2} is not a cut-vertex in G_R or G_B, contradicting our assumption. Thus, every connected 2-coloring of K_n, where $n \geq 5$, contains some vertex that is not a cut vertex in G_R or G_B. □

In a connected 2-coloring of K_n, we say that a vertex is *removable* if it is not a cut-vertex in G_R and it is not a cut-vertex in G_B. Theorem 2.1 then states that every connected 2-coloring of K_n, where $n \geq 5$, has a removable vertex. This fact will be useful in proving bounds for connected Ramsey numbers using induction.

If x is a vertex in a graph G, denote by $G - x$ the graph formed by deleting x (and all edges incident with x). We note that regardless of whether or not x is a removable vertex in K_n, one cannot predict whether or not a vertex $y \neq x$ is removable in $K_n - x$. For example, Figure 2 shows all four possibilities when $n = 6$.

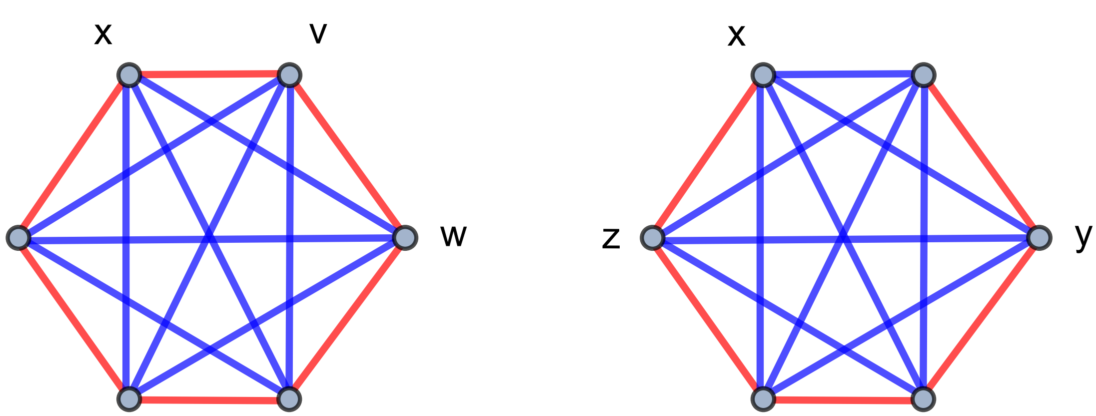

Figure 2: In both connected 2-colorings of K_6, x is a removable vertex. In the first graph, v and w are both removable, while in the second graph y and z are not removable. Also, v and z are both removable in their respective colorings of $K_6 - x$ and neither w nor y are removable in their respective colorings of $K_6 - x$.

For a pair of graphs G_1 and G_2, suppose that $r_c(G_1, G_2) = p$. Then if we consider a connected 2-coloring of K_{p+1}, we can remove a vertex that is not a cut vertex in G_R or G_B. The result is a connected 2-coloring of K_p, which necessarily contains a red G_1 or blue G_2. This observation can be applied inductively to obtain the following corollary.

Corollary 2.2. *If G_1 and G_2 are graphs such that $r_c(G_1, G_2) = p$, then for $n \geq p$, every connected 2-coloring of K_n contains a red copy of G_1 or a blue copy of G_2.*

For the Ramsey numbers $r(K_3, K_3) = 6$, $r(K_3, K_4) = 9$, $r(K_3, K_5) = 14$, and $r(K_4, K_4) = 18$, the lower bounds all correspond with known critical colorings in which the graph spanned by each color is a connected circulant graph (see [9] and [18]). This observation implies that (K_3, K_3), (K_3, K_4), (K_3, K_5), and (K_4, K_4) are all Ramsey-connected. In fact, it is known that for all $m, n \geq 4$, (K_m, K_n) is Ramsey-connected, as this follows from the following theorem due to Sumner (see Theorem 2.1 of [22]).

Theorem 2.3 ([22])**.** *Let G_1 and G_2 be graphs of order at least 4 and having edge connectivity at least 2. Then (G_1, G_2) is Ramsey-connected.*

From this theorem, it makes sense to restrict our attention to $r_c(G_1, G_2)$ when at least one of G_1 and G_2 has a bridge (i.e., when $\lambda(G_1) = 1$ or $\lambda(G_2) = 1$).

In 1972, Chvátal and Harary [6] proved that if G is any graph and m is the order of a largest connected component in G, then

$$r(G, K_n) \geq (m-1)(n-1) + 1.$$

In the case where T_m is a tree of order m, Chvátal [5] proved in 1977 that

$$r(T_m, K_n) = (m-1)(n-1) + 1. \tag{1}$$

In 1983, this led Burr and Erdős [4] to call a graph G *n-good* if it satisfies

$$r(G, K_n) = (m-1)(n-1) + 1,$$

where m is the order of a largest connected component in G.

When evaluating the star-critical Ramsey number for trees versus complete graphs in 2010, Hook and Isaak [14] proved that the unique critical coloring for $r(T_m, K_n)$ is the 2-coloring of $K_{(m-1)(n-1)}$ formed by replacing the vertices in a blue K_{n-1} with red copies of K_{m-1}. As this is the only critical coloring and its red subgraph is disconnected, we find that

$$r_c(T_m, K_n) < r(T_m, K_n),$$

implying that (T_m, K_n) is not Ramsey-connected. This raises an interesting question. If a graph G is n-good, does it follow that G is not Ramsey-connected?

To answer this question, denote by C_m a cycle of order m, where $m \geq 3$. It is a well-known conjecture of Erdős, Faudree, Rousseau, and Schelp [7] that C_m is n-good for all $m \geq n \geq 3$, except for $(m, n) = (3, 3)$. While the conjecture is still open in general, it has been verified in many cases. In all such cases with $m, n \geq 4$, Theorem 2.3 implies that (C_m, K_n) is Ramsey-connected since $\lambda(C_m) = 2$ and $\lambda(K_n) \geq 2$.

3 Trees Versus Complete Graphs

In this section, we consider the case of trees versus complete graphs, focusing on paths and stars. The *path* P_m of order m is a sequence $x_1 x_2 \cdots x_m$ of distinct vertices such that $x_i x_{i+1}$ is an edge for each $1 \leq i \leq m-1$. For $p \geq 4$, every connected 2-coloring of K_p contains a red P_4 since the only connected 2-coloring of K_4 is the union of a red P_4 and a blue P_4. It follows that $r_c(P_4, K_n) = 4$ for all $n \geq 3$. We now consider the connected Ramsey number $r_c(P_m, K_3)$.

Theorem 3.1. *For all $m \geq 4$, $r_c(P_m, K_3) = m$.*

Proof. The connected 2-coloring of K_{m-1} in which G_B is a blue P_{m-1} and all other edges are colored red avoids a red P_m and a blue K_3. Hence, $r_c(P_m, K_3) \geq m$. To prove the reverse inequality, we use induction on $m \geq 4$, with $r_c(P_4, K_3) = 4$ serving as the base case. Assume that $r_c(P_{m-1}, K_3) = m - 1$ a consider a connected 2-coloring of K_m. By Theorem 2.1, there exists a removable vertex, resulting in a connected 2-coloring of K_{m-1}, which necessarily contains a red P_{m-1} or a blue K_3. In the latter case, we are done, so assume there is a red P_{m-1}. Suppose that this red path is given by $x_1 x_2 \cdots x_{m-1}$ and the other vertex is labeled y. If either yx_1 or yx_{m-1} are red, then the addition of such an edge to the existing red P_{m-1} will produce a red P_m. So, suppose that both yx_1 and yx_{m-1} are blue. Since this 2-coloring is connected, there exists an edge yx_i that is red for some $2 \leq i \leq m-2$ (see Figure 3). If $x_1 x_{m-1}$ is red, then $yx_i x_{i+1} \cdots x_{m-1} x_1 x_2 \cdots x_{i-1}$ is a red P_m. If $x_1 x_{m-1}$ is

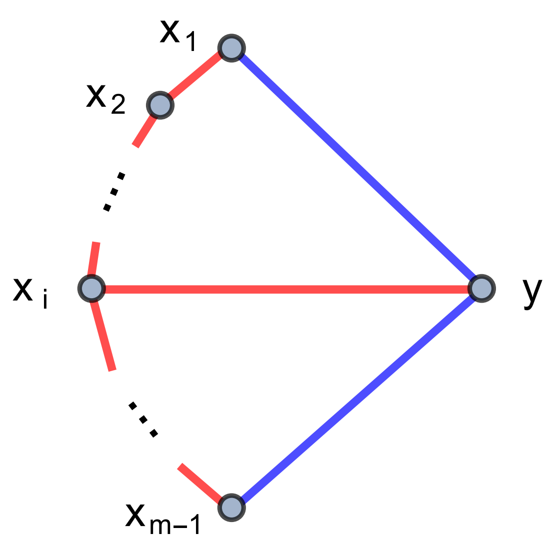

Figure 3: A connected 2-coloring of K_m that contains a red P_{m-1}.

blue, then the subgraph induced by $\{y, x_1, x_{m-1}\}$ is a blue K_3, completing the proof that $r_c(P_m, K_3) = m$. □

From Equation (1), it follows that $r(P_m, K_3) = 2m - 1$. So, Theorem 3.1 implies that (P_m, K_3) is not Ramsey-connected whenever $m \geq 4$. The following corollary is also a consequence of Theorem 3.1.

Corollary 3.2. *Let $m \geq 4$. If a totally connected graph G satisfies $\alpha(G) \leq 2$ and $|V(G)| \geq m$, then G contains a subgraph isomorphic to P_m.*

The *broom* $B_{k,\ell}$ is the tree formed by adding a single edge between a leaf (a vertex with degree 1) in $P_{\ell-1}$ and the center vertex of $K_{1,k}$, where $k \geq 1$ and $\ell \geq 2$. This definition coincides with the definition of a broom given in [8]. The order of $B_{k,\ell}$ is $k + \ell$ and $\Delta(B_{k,\ell}) = k + 1$. Brooms play an important role in the lower bound construction for the connected Ramsey number $r_c(P_5, K_n)$.

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Theorem 3.3. *For all $n \geq 3$, $r_c(P_5, K_n) = n + 2$.*

Proof. The case when $n = 3$ follows from Theorem 3.1, so assume $n \geq 4$. To obtain the lower bound, we consider the connected 2-coloring of K_{n+1} formed by starting with a red $B_{n-2,3}$ and coloring all additional edges blue (e.g., Figure 4 shows the $n = 6$ case.). The

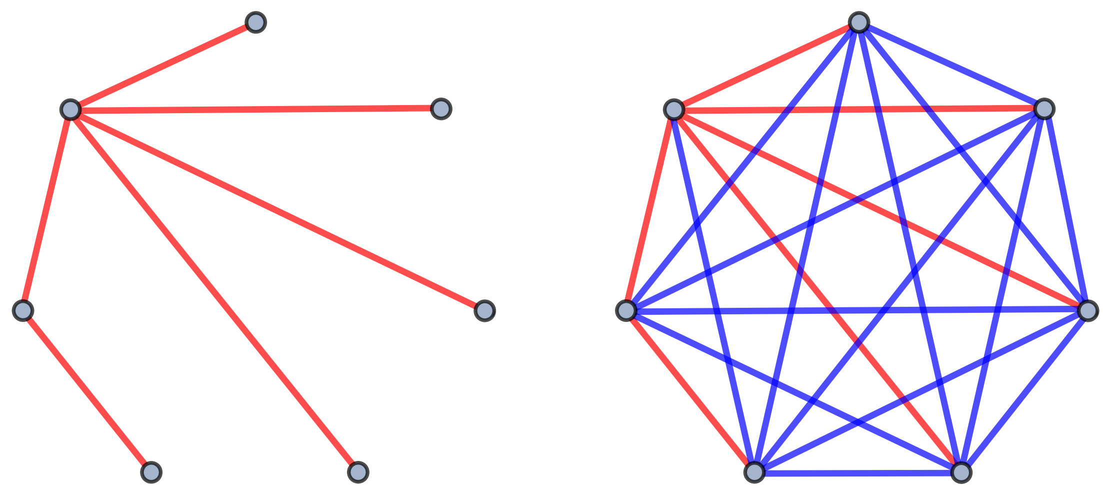

Figure 4: The broom $B_{4,3}$ and the corresponding construction of a connected 2-coloring of K_7 that avoids a red P_5 and a blue K_6.

broom $B_{n-2,3}$ does not contain a red P_5 as a subgraph. The blue subgraph does not contain a K_n since not all of the red edges are incident with any single vertex. So, the removal of any single vertex results in a 2-colored K_n that contains at least one red edge. From this coloring, it follows that $r_c(P_5, K_n) \geq n + 2$.

In order to prove the reverse inequality, consider a connected 2-coloring of K_{n+2}. Since $r_c(P_4, K_n) = 4$ for all $n \geq 4$, it follows that there exists a red P_4 or a blue K_n. In the latter case, we are done, so assume there is a red P_4 given by $x_1x_2x_3x_4$. Label the other vertices $y_1, y_2, \ldots, y_{n-2}$. If an edge x_1y_i is red for some $1 \leq i \leq n-2$, then $y_ix_1x_2x_3x_4$ is a red P_5. If an edge x_4y_i is red for some $1 \leq i \leq n-2$, then $y_ix_4x_3x_2x_1$ is a red P_5. So assume that all edges of the forms x_1y_i and x_4y_i are blue for all $1 \leq i \leq n-2$. Figure 5 shows our current connected 2-coloring of K_{n+2}. Since we are considering a connected 2-coloring of K_{n+2}, at least one of x_2 and x_3 must join to $\{y_1, y_2, \ldots, y_{n-2}\}$ via a red edge. Without loss of generality, assume that x_2y_1 is red. If x_3y_1 is also red, then $x_1x_2y_1x_3x_4$ forms a red P_5. So assume that x_3y_1 is blue. Now consider the edges joining y_1 to $\{y_2, y_3, \ldots, y_{n-2}\}$. If such an edge is red, say y_1y_2, then $y_2y_1x_2x_3x_4$ is a red P_5. So, assume that all such edges are blue. Now, we proceed by induction, assuming that exactly one of x_2 and x_3 joins to each of $y_1, y_2, \ldots, y_i$ via a red edge, the subgraph induced by $\{y_1, y_2, \ldots, y_i\}$ is a blue K_i, and y_i joins to $\{y_{i+1}, y_{i+2}, \ldots, y_{n-2}\}$ via blue edges. In order for the 2-coloring of K_{n+2} to be connected, at least one of x_2 and x_3 must join to $\{y_{i+1}, y_{i+2}, \ldots, y_{n-2}\}$ via a red edge (and the other joins via a blue edge if a red P_5 is avoided). If any edge joining y_{i+1} to $\{y_{i+2}, y_{i+3}, \ldots, y_{n-2}\}$ is red, then a red P_5 is formed. So, all such edges must be blue. The final result of this inductive process is that avoiding a red P_5 forces the subgraph induced by $\{y_1, y_2, \ldots, y_{n-2}\}$ to be a blue K_{n-2}. Then $\{x_1, x_4, y_1, y_2, \ldots, y_{n-2}\}$ forms a blue K_n. It follows that $r_c(P_5, K_n) \leq n + 2$. □

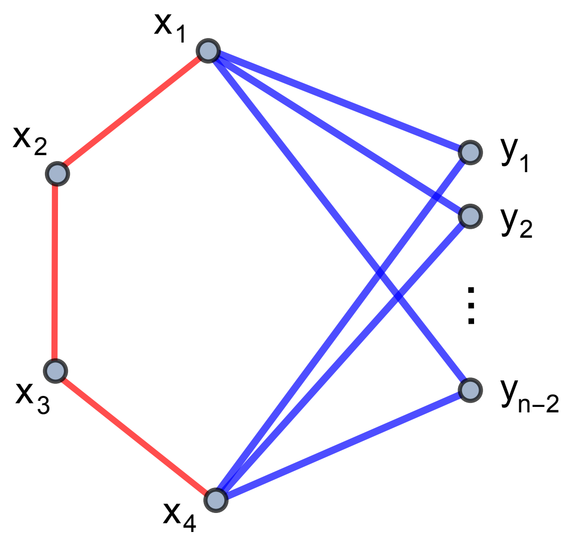

Figure 5: A connected 2-coloring of K_{n+2}, where $n \geq 3$.

From Equation (1), it follows that $r(P_5, K_n) = 4n - 3$. So, Theorem 3.3 implies that (P_5, K_n) is not Ramsey-connected whenever $n \geq 3$. Since the independence number of a graph corresponds with the largest complete subgraph in its complement, Theorem 3.3 also implies the following corollary.

Corollary 3.4. *Let $n \geq 5$. If a totally connected graph G satisfies $\alpha(G) \leq n - 3$ and $|V(G)| \geq n$, then G contains a subgraph isomorphic to P_5.*

For cycles, it is easily confirmed that $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$. This follows from considering a proper vertex coloring of C_n using 2 colors when n is even and 3 colors when n is odd. The cardinality of the largest color class is $\lfloor \frac{n}{2} \rfloor$ and corresponds with an independent set of vertices. It can similarly be argued that $\alpha(P_n) = \lceil \frac{n}{2} \rceil$ (for example, see Lemma 4 of [15]).

Theorem 3.5. *For all $n \geq 3$, $r_c(K_{1,3}, K_n) = 2n$.*

Proof. For the lower bound, consider the connected 2-coloring of K_{2n-1} formed by taking C_{2n-1} for the red graph and color all other edges blue. Since $\alpha(C_{2n-1}) = \lfloor \frac{2n-1}{2} \rfloor = n - 1$, the graph G_B does not contains a subgraph isomorphic to K_n. It follows that $r_c(K_{1,3}, K_n) \geq 2n$.

To prove the reverse inequality, consider a connected 2-coloring of K_{2n} that lacks a red $K_{1,3}$. Then the maximum red degree of any vertex is 2. The only connected graphs of order $2n$ with maximum vertex degree 2 are C_{2n} and P_{2n}. In this case, we have that $\alpha(C_{2n}) = \alpha(P_{2n}) = n$, from which it follows that G_B must contain a subgraph isomorphic to K_n. It follows that $r_c(K_{1,3}, K_n) \leq 2n$. □

From Equation (1), it follows that $r(K_{1,3}, K_n) = 3n - 2$. So, Theorem 3.5 implies that (P_m, K_3) is not Ramsey-connected whenever $n \geq 3$. Theorem 3.5 also implies the following corollary.

Corollary 3.6. *Let $n \geq 3$. If a totally connected graph G satisfies $\Delta(G) \leq 2$ and $|V(G)| \geq n$, then $\alpha(G) \geq \lfloor \frac{n}{2} \rfloor$.*

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4 Conclusion

Unfortunately, there does not appear to be a single algebraic expression that describes the value of $r_c(T_m, K_n)$ in terms of m and n, when T_m is a tree of order m. Based on Theorems 3.1 and 3.3, we offer the following conjecture for the general values of $r_c(P_m, K_n)$.

Conjecture 4.1. *For all* $m \geq 5$ *and* $n \geq 3$, $r_c(P_m, K_n) = m + n - 3$.

Besides the values of connected Ramsey numbers for trees versus complete graphs, one can consider other graphs. Other than certain cycles, what other n-good graphs G exist such that (G, K_n) is Ramsey-connected?

In 2015, Rahadjeng, Baskoro, and Assiyatun [19] introduced a connected version of the size Ramsey number. The connected size Ramsey number was further studied in [1], [20], [23], and [24]. At the present time, no other generalizations/variations of Ramsey numbers have had connected versions studied in the literature. Connected analogues of multicolor Ramsey numbers, anti-Ramsey numbers, hypergraph Ramsey numbers, and star-critical Ramsey numbers are all unexplored territory.

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