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A New Class of Generalized Dagum Distribution with Applications to Income and Lifetime Data

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Abstract

The generalized beta distribution of the second kind (GB2), McDonald [11], McDonald and Xu [12] is an important distribution with applications in finance and actuarial sciences, as well as economics, where Dagum distribution which is a sub-model of GB2 distribution plays an important role in size distribution of personal income. In this note, a new class of generalized Dagum distribution called gamma-Dagum distribution is presented. The gamma-Dagum (GD) distribution which includes the gamma-Burr III (GB III), gamma-Fisk or gamma-log logistic (GF of GLLog), Zografos and Balakrishnan-Dagum (ZB-D), ZB-Burr III (ZB-B III), ZB-Fisk of ZB-Log logistic (ZB-F or ZB-LLog), Burr III (B III), and Fisk or Log logistic (F or LLog) as special cases is proposed and studied. Some mathematical properties of the new distribution including moments, mean and median deviations, distribution of the order statistics, and Renyi entropy are presented. Maximum likelihood estimation technique is used to estimate the model parameters and applications to real datasets to illustrate the usefulness of the model are presented.

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1 Introduction

Kleiber [9] traced the genesis of Dagum distribution and summarized several statistical properties of this distribution. Domma et al. [5] obtained the maximum likelihood estimates of the parameters of Dagum distribution from censored samples. Dagum [3] distribution is a special case of generalized beta distribution of the second kind (GB2), McDonald [11], McDonald and Xu [12], when the parameter $q = 1$, where the probability density function (pdf) of the GB2 distribution is given by:

$$f_{GB2}(y; a, b, p, q) = \frac{ay^{ap-1}}{b^{ap}B(p, q)[1 + (\frac{y}{b})^a]^{p+q}} \quad \text{for } y > 0. \tag{1}$$

Note that $a > 0, p > 0, q > 0$, are the shape parameters and $b > 0$ is the scale parameter and $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function. Domma and Condino [4] obtained statistical properties of the beta-Dagum distribution. The pdf and cumulative distribution function (cdf) of Dagum distribution are given by:

$$f_D(y; \lambda, \beta, \delta) = \beta\lambda\delta y^{-\delta-1}(1 + \lambda y^{-\delta})^{-\beta-1}, \tag{2}$$

and

$$F_D(y; \lambda, \beta, \delta) = (1 + \lambda y^{-\delta})^{-\beta}, \quad y > 0, \lambda, \beta, \delta > 0, \tag{3}$$

respectively. The hazard and the reverse hazard functions are given by:

$$h_D(y; \lambda, \beta, \delta) = \frac{f_D(y; \lambda, \beta, \delta)}{\overline{F}_D(y; \lambda, \beta, \delta)} = \frac{\beta\lambda\delta y^{-\delta-1}(1 + \lambda y^{-\delta})^{-\beta-1}}{1 - (1 + \lambda y^{-\delta})^{-\beta}}, \tag{4}$$

and

$$\tau_D(y; \lambda, \beta, \delta) = \beta\lambda\delta y^{-\delta-1}(\lambda y^{-\delta} + 1)^{-1}, \tag{5}$$

respectively. The k^{th} raw or non central moments are:

$$E(Y^k) = E(Y^k|\beta, \delta, \lambda) = \beta\lambda^{\frac{k}{\delta}} B\left(\beta + \frac{k}{\delta}, 1 - \frac{k}{\delta}\right), \quad \text{for } \delta > k.$$

Motivated by the various applications of Dagum distribution in finance and actuarial sciences, as well as in economics, where Dagum distribution plays are important role in size distribution of personal income, we construct a new class of Dagum-type distribution called the Gamma-Dagum (GD) distribution and apply the model to real lifetime data.

For any baseline cdf $F(x)$, and $x \in \mathbf{R}$, Zografos and Balakrishnan [23] defined the distribution (when $\theta = 1$) with pdf $g(x)$ and cdf $G(x)$ (for $\alpha > 0$) as follows

$$g(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(\overline{F}(x))]^{\alpha-1}(1-F(x))^{(1/\theta)-1}f(x), \tag{6}$$

and

$$G(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log(\overline{F}(x))} t^{\alpha-1}e^{-t/\theta}dt = \frac{\gamma(-\theta^{-1}\log(\overline{F}(x)),\alpha)}{\Gamma(\alpha)}, \tag{7}$$

respectively, for $\alpha, \theta > 0$, where $g(x) = dG(x)/dx$, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ denotes the gamma function, and $\gamma(z,\alpha) = \int_0^z t^{\alpha-1}e^{-t}dt$ denotes the incomplete gamma function. The corresponding hazard rate function (hrf) is

$$h_G(x) = \frac{[-\log(1-F(x))]^{\alpha-1}f(x)(1-F(x))^{(1/\theta)-1}}{\theta^{\alpha}(\Gamma(\alpha)-\gamma(-\theta^{-1}\log(1-F(x)),\alpha))}. \tag{8}$$

The class of distributions for the special case of $\theta = 1$, is referred to as the ZB-G family of distributions. Also, (when $\theta = 1$), Ristić and Balakrishnan [20] proposed an alternative gamma-generator defined by the cdf and pdf

$$G_2(x) = 1 - \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log F(x)} t^{\alpha-1}e^{-t/\theta}dt, \quad \alpha > 0, \tag{9}$$

and

$$g_2(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(F(x))]^{\alpha-1}(F(x))^{(1/\theta)-1}f(x), \tag{10}$$

respectively. Note that if $\theta = 1$ and $\alpha = n+1$, in equation (6), we obtain the cdf and pdf of the upper record values U given by

$$G_U(u) = \frac{1}{n!}\int_0^{-\log(1-F(u))} y^n e^{-y}dy, \tag{11}$$

and

$$g_U(u) = f(u)[-\log(1-F(u))]^n/n!. \tag{12}$$

Similarly, from equation (10), the pdf of the lower record values is given by

$$g_{_L}(t) = f(t)[-\log(F(t))]^n/n!. \tag{13}$$

In this paper, we consider the generalized family of distributions given in equation (6) via Dagum distribution. Zografos and Balakrishnan [23] motivated the ZB-G model as follows. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be lower record values from a sequence of independent and identically distributed (i.i.d.) random variables from a population with pdf $f(x)$. Then, the pdf of the n^{th} upper record value is given by equation (6), when $\theta = 1$. A logarithmic transformation of the parent distribution F transforms the random variable X with density (6) to a gamma distribution. That is, if X has the density (6), then the random variable $Y = -\log[1 - F(X)]$ has a gamma distribution $GAM(\alpha; 1)$ with density $k(y; \alpha) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}$, $y > 0$. The opposite is also true, if Y has a gamma $GAM(\alpha; 1)$ distribution, then the random variable $X = G^{-1}(1 - e^{-Y})$ has a ZB-G distribution (Zografos and Balakrishnan [23]). In addition to the motivations provided by Zografos and Balakrishnan [23], we are also interested in the generalization of the Dagum distribution via the gamma-generator and establishing the relationship between the distributions in equations (6) and (10), and weighted distributions in general.

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. It adjusts the probabilities of actual occurrence of events to arrive at a specification of the probabilities when those events are recorded. Fisher [13] first introduced the concept of weighted distribution, in order to study the effect of ascertainment upon estimation of frequencies. Rao [19] unified concept of weighted distribution and use it to identify various sampling situations. Cox [2] and Zelen [22] introduced weighted distribution to present length biased sampling. Patil [16] used weighted distribution as stochastic models in the study of harvesting and predation. The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can also be seen in Nanda and Jain [14], Gupta and Keating [7], Oluyede [15] and in references therein.

Suppose Y is a non-negative random variable with its natural pdf $f(y; \underline{\theta})$, where $\underline{\theta}$ is a vector of parameters, then the pdf of the weighted random variable

Y^w is given by:

$$f^w(y;\underline{\theta},\underline{\beta}) = \frac{w(y,\underline{\beta})f(y;\underline{\theta})}{\omega}, \tag{14}$$

where the weight function $w(y,\underline{\beta})$ is a non-negative function, that may depend on the vector of parameters $\underline{\beta}$, and $0 < \omega = E(w(Y,\underline{\beta})) < \infty$ is a normalizing constant. In general, consider the weight function $w(y)$ defined as follows:

$$w(y) = y^k e^{ly} F^i(y)\overline{F}^j(y). \tag{15}$$

Setting $k = 0$; $k = j = i = 0$; $l = i = j = 0$; $k = l = 0$; $i \to i-1$; $j = n - i$; $k = l = i = 0$ and $k = l = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

Ristić and Balakrishnan [20], provided motivations for the new family of distributions given in equation (9) when $\theta = 1$, that is for $n \in \mathbf{N}$, equation (9) above is the pdf of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density $f(x)$. Ristić and Balakrishnan [20] used the exponentiated exponential (EE) distribution with cdf $F(x) = (1 - e^{-\beta x})^{\alpha}$, where $\alpha > 0$ and $\beta > 0$, in equation (10) to obtained and study the gamma-exponentiated exponential (GEE) model. See references therein for additional results on the GEE model. Pinho et al. [17] presented the statistical properties of the gamma-exponentiated Weibull distribution. In this note, we obtain a natural extension for Dagum distribution, which we call the gamma-Dagum (GD) distribution.

This paper is organized as follows. In section 2, some basic results, the gamma-Dagum (GD) distribution, series expansion and its sub-models, hazard and reverse hazard functions and the quantile function are presented. The moments and moment generating function, mean and median deviations are given in section 3. Section 4 contains some additional useful results on the distribution of order statistics and Renyi entropy. In section 5, results on the estimation of the parameters of the GD distribution via the method of maximum likelihood are presented. Applications are given in section 6, followed by concluding remarks.

2 GD Distribution, Expansion, Sub-models, Hazard and Reverse Hazard Functions

In this section, the GD distribution, series expansion of its pdf, some sub-models, hazard and reverse hazard functions as well some graphs of these functions are presented. By inserting Dagum distribution in equation (7), the cdf $G_{GD}(x) = G(x)$ of the GD distribution is obtained as follows:

$$\begin{aligned} G_{GD}(x) &= \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{-\log[1-(1+\lambda x^{-\delta})^{-\beta}]} t^{\alpha-1} e^{-t/\theta} dt \\ &= \frac{\gamma(-\theta^{-1}\log[1-(1+\lambda x^{-\delta})^{-\beta}], \alpha)}{\Gamma(\alpha)}, \end{aligned} \tag{16}$$

where $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $\alpha > 0$, $\theta > 0$, and $\gamma(x, \alpha) = \int_0^x x^{\alpha-1} e^{-t} dt$ is the lower incomplete gamma function. The corresponding GD pdf is given by

$$\begin{aligned} g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}} (1+\lambda x^{-\delta})^{-\beta-1} \Big(-\log[1-(1+\lambda x^{-\delta})^{-\beta}] \Big)^{\alpha-1} \\ &\times [1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}. \end{aligned} \tag{17}$$

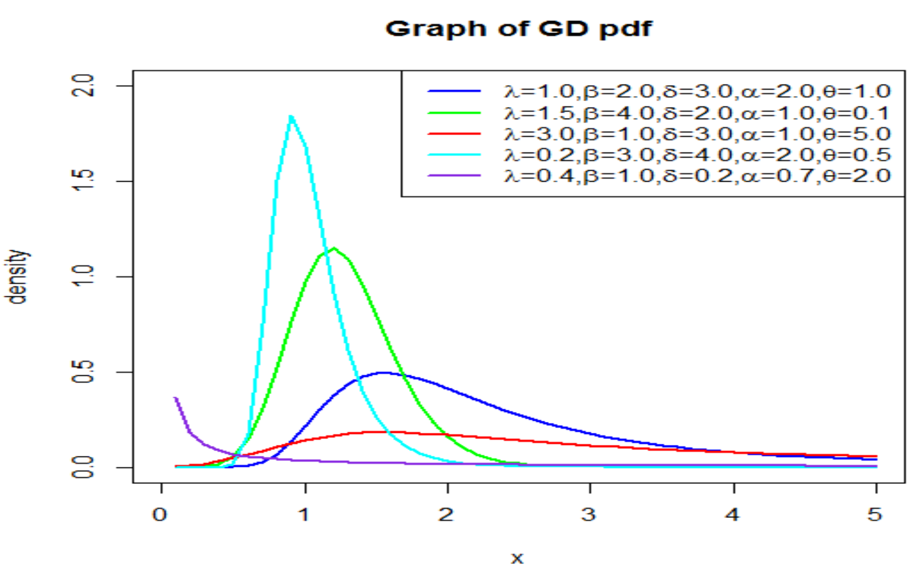

Figure 1: Graph of GD pdf for selected parameters

The graph of pdf for some combinations of values of the model parameters are given in Figure 1. The plots indicate that the GD pdf can be decreasing or right skewed. The GD distribution has a positive asymmetry.

If a random variable X has the GD density, we write $X \sim GD(\lambda, \beta, \delta, \alpha, \theta)$. Let $y = [1 + \lambda x^{-\delta}]^{-\beta}$, and $\psi = 1/\theta$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\alpha-1} = y^{\alpha-1}\Bigg[\sum_{m=0}^{\infty}\binom{\alpha-1}{m} y^m \Big(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\Big)^m\Bigg],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\Big(\sum_{s=0}^{\infty} a_s y^s\Big)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{18}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik [6]), the GD pdf can be written as

$$\begin{aligned}
g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}}{\Gamma(\alpha)\theta^{\alpha}} y^{\alpha-1} \\
&\times \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\alpha-1}{m} b_{s,m} y^{m+s}\sum_{k=0}^{\infty}\binom{\psi-1}{k}(-1)^k y^k \\
&= \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}}{\Gamma(\alpha)\theta^{\alpha}} \\
&\times \sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\alpha-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} y^{\alpha+m+s+k-1} \\
&= \sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\alpha-1}{m}\binom{\psi-1}{k}(-1)^k \frac{b_{s,m}}{(m+s+k+\alpha)\theta^{\alpha}\Gamma(\alpha)} \\
&\times \lambda\beta(m+s+k+\alpha)\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+\alpha)-1},
\end{aligned}$$

where $f(x; \lambda, \beta(m+s+k+\alpha), \delta)$ is the Dagum pdf with parameters λ, $\beta(m+s+k+\alpha)$, and δ. Let $C = \{(m,s,k) \in \mathbf{Z}_+^3\}$, and $\psi = 1/\theta$, then the weights in the GD pdf are

$$w_\nu = (-1)^k\binom{\alpha-1}{m}\binom{\psi-1}{k}\frac{b_{m,s}}{(m+s+k+\alpha)\theta^{\alpha}\Gamma(\alpha)},$$

and

$$g_{GD}(x) = \sum_{\nu\in C} w_v f_D(x; \lambda, \beta(m+s+k+\alpha), \delta). \tag{19}$$

It follows therefore that the GD density is a linear combination of the Dagum pdfs. The statistical and mathematical properties can be readily obtained from those of the Dagum distribution. Note that $g_{GD}(x)$ is a weighted pdf with the weight function

$$w(x) = [-\log(1 - F(x))]^{\alpha-1}[1 - F(x)]^{\frac{1}{\theta}-1}, \tag{20}$$

that is,

$$\begin{aligned} g_{GD}(x) &= \frac{[-\log(1 - F(x))]^{\alpha-1}[1 - F(x)]^{\frac{1}{\theta}-1}}{\theta^\alpha\Gamma(\alpha)} f(x) \\ &= \frac{w(x)f(x)}{E_F(w(X))}, \end{aligned} \tag{21}$$

where $0 < E_F[[-\log(1 - F(x))]^{\alpha-1}[1 - F(x)]^{\frac{1}{\theta}-1}] = \theta^\alpha\Gamma(\alpha) < \infty$, is the normalizing constant. Similarly,

$$g_2(x) = \frac{[-\log(F(X))]^{\alpha-1}[F(X)]^{\frac{1}{\theta}-1}}{\theta^\alpha\Gamma(\alpha)} f(x) = \frac{w(x)f(x)}{E_F(w(X))}, \tag{22}$$

where $0 < E_F(w(X)) = E_F([-\log(F(X))]^{\alpha-1}[F(X)]^{\frac{1}{\theta}-1}) = \theta^\alpha\Gamma(\alpha) < \infty$.

2.1 Some Sub-models of the GD Distribution

Some of the sub-models of the GD distribution are listed below:

- If $\theta = 1$, we obtain the gamma-Dagum distribution via the ZB-D (ZB-Dagum) distribution.
- When $\lambda = \theta = 1$, we have the ZB-Burr III (ZB-B III) distribution.
- When $\beta = \theta = 1$, we obtain the ZB-Fisk or ZB-Log logistic (ZB-F or ZB-LLog) distribution.
- If $\alpha = 1$, we get the exponentiated Dagum (ED) distribution, which is also a Dagum distribution.
- When $\beta = 1$, we have the gamma-Fisk or gamma-Log logistic (GF or GLLog) distribution.
- If $\lambda = 1$, we obtain the gamma-Burr III (GB III) distribution.

- If $\theta = 1$, and $\alpha = 1$ we have Dagum (D) distribution.
- When $\lambda = \alpha = \theta = 1$, we have Burr III (B III) distribution.
- When $\lambda = \alpha = 1$ we have exponentiated Burr III (EB III) distribution.
- When $\beta = \alpha = 1$, we obtain Fisk or Log logistic (F or LLog) distribution.

2.2 Hazard and Reverse Hazard Functions

Let X be a continuous random variable with distribution function F, and probability density function (pdf) f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar [21]). The hazard and reverse hazard functions of the GD distribution are

$$h_G(x) = \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(-\log(1-[1+\lambda x^{-\delta}]^{-\beta}))^{\alpha-1}[1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}}{\theta^\alpha(\Gamma(\alpha)-\gamma(-\theta^{-1}\log(1-(1+\lambda x^{-\delta})^{-\beta}),\alpha))}, \tag{23}$$

and

$$\tau_G(x) = \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(-\log(1-[1+\lambda x^{-\delta}]^{-\beta}))^{\alpha-1}[1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}}{\theta^\alpha(\gamma(-\theta^{-1}\log(1-(1+\lambda x^{-\delta})^{-\beta}),\alpha))}, \tag{24}$$

for $x \geq 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $\alpha > 0$, and $\theta > 0$, respectively.

The graph of hazard function for selected parameters are given in Figure 2. The plots show various shapes including monotonically decreasing, monotonically increasing, and bathtub followed by upside down bathtub shapes for five combinations of values of the parameters. This very attractive flexibility makes the GD hazard rate function useful and suitable for monotonic and non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

2.3 GD Quantile Function

The quantile function of GD distribution is obtained by solving the equation

$$G(Q(y)) = y, \quad 0 < y < 1. \tag{25}$$

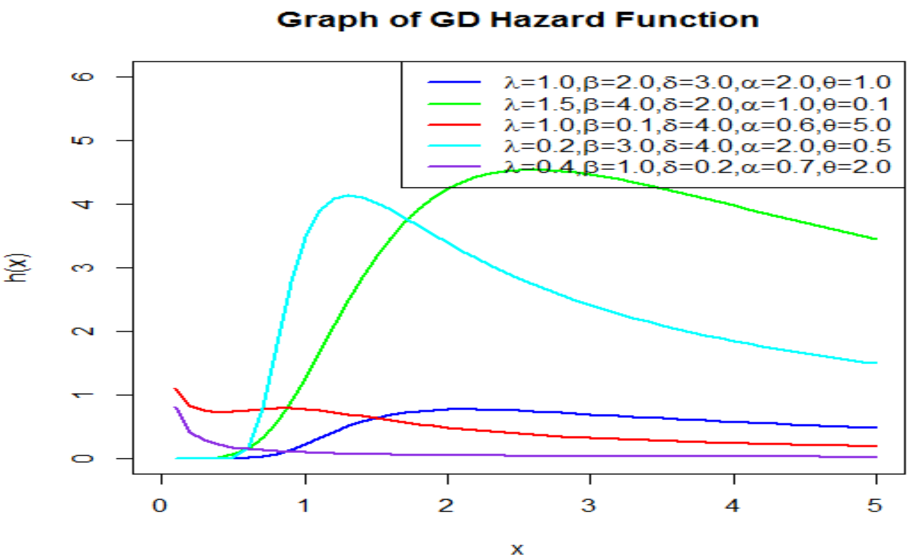

Figure 2: Graph of hazard function for selected parameters

Note that the inverse or quantile function of Dagum distribution, $F_D(x) = [1+\lambda x^{-\delta}]^{-\beta}$ is given by $Q_D(.)$, that is

$$Q_D(y) = \lambda^{\frac{1}{\delta}} \left(y^{\frac{-1}{\beta}} - 1 \right)^{\frac{-1}{\delta}}. \tag{26}$$

The quantile function of the GD distribution is obtained by inverting equation (16) to obtain

$$Q_{GD}(y) = \lambda^{\frac{1}{\delta}} \left[\left(1 - e^{-\theta u} \right)^{\frac{-1}{\beta}} - 1 \right]^{\frac{-1}{\delta}}, \tag{27}$$

where $u = \gamma^{-1}(y\Gamma(\alpha), \alpha)$.

3 Moments, Moment Generating Function, Mean and Median Deviations

In this section, we present the moments, moment generating function, mean and median deviations for the GD distribution. These measures can be readily obtained for the sub-models given in section 2.

3.1 Moments and Moment Generating Function

Let $\beta^* = \beta(m+s+k+\alpha)$, and $Y \sim D(\lambda, \beta^*, \delta)$. Note that from $Y \sim D(\alpha, \beta^*, \delta)$, the r^{th} moment of the random variable Y is

$$E(Y^r) = \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \tag{28}$$

$r < \delta$, so that the r^{th} raw moment of GD distribution is given by:

$$E(X^r) = \sum_{\nu \in C} w_\nu E(Y^r) = \sum_{\nu \in C} w_\nu \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \tag{29}$$

$r < \delta$. The moment generating function (MGF) , for $|t| < 1$, is given by:

$$\begin{aligned} M_X(t) &= \sum_{\nu \in C} w_\nu M_Y(t) \\ &= \sum_{\nu \in C} \sum_{i=0}^{\infty} w_\nu \frac{t^i}{i!} \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \end{aligned} \tag{30}$$

for $r < \delta$.

Theorem 3.1.

$$E\{[-\log(1 - F(X))]^r [(1 - F(X))^s]\} = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{(s\theta+1)^\alpha \theta^\alpha \Gamma(\alpha)}. \tag{31}$$

Proof:

$$\begin{aligned} & E\{[-\log(1 - F(X))]^r [(1 - F(X))^s]\} \\ = & \int_0^\infty \frac{f(x)}{\theta^\alpha \Gamma(\alpha)} [-\log(1 - F(x))]^{r+\alpha-1} [1 - F(x)]^{s+(1/\theta)-1} dx \\ = & \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{(s\theta+1)^\alpha \theta^\alpha \Gamma(\alpha)}. \end{aligned} \tag{32}$$

Corollary 3.2. *If* $s = 0$, *we have* $E[-\log(1 - F(X))^r] = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{\theta^\alpha \Gamma(\alpha)}$, *and if* $r = 0$, $E[(1 - F(X))^s] = [s\theta + 1]^{-\alpha}$.

Proof: Let $s = 0$ in equation (31) or equation (32), then

$$E[-\log(1 - F(X))^r] = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{\theta^\alpha \Gamma(\alpha)}. \tag{33}$$

Let $\theta^* = s + \frac{1}{\theta}$, then with $r = 0$ in equation (31) or equation(32), we obtain

$$E[(1 - F(X))^s] = [s\theta + 1]^{-\alpha}. \tag{34}$$

3.2 Mean and Median Deviations

If X has the GD distribution, we can derive the mean deviation about the mean μ by

$$\delta_1 = \int_0^\infty |x-\mu| g_{GD}(x)dx = 2\mu G_{GD}(\mu) - 2\mu + 2T(\mu), \tag{35}$$

and the median deviation about the median M by

$$\delta_2 = \int_0^\infty |x-M| g_{GD}(x)dx = 2T(M) - \mu, \tag{36}$$

where $\mu = E(X)$ is given in equation (29), $M = Q_{GD}(0.5)$ in equation (27) and $T(a) = \int_a^\infty x \cdot g_{GD}(x)dx$. Let $\beta^* = \beta(m+s+k+\alpha)$, then

$$\begin{aligned} T(a) &= \sum_{\nu\in C} w_\nu T_{D(\lambda,\beta^*,\delta)}(a) \\ &= \sum_{\nu\in C} w_\nu \beta^* \lambda^{\frac{1}{\delta}} \left[B\left(\beta^* + \frac{1}{\delta}, 1-\frac{1}{\delta}\right) - B\left(t(a); \beta^* + \frac{1}{\delta}, 1-\frac{1}{\delta}\right)\right], \end{aligned} \tag{37}$$

where $t(a) = (1+\lambda a^{-\delta})^{-1}$, and $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$.

4 Order Statistics and Renyi Entropy

Order Statistics play an important role in probability and statistics. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present the distribution of the order statistics, and Renyi entropy for the GD distribution.

4.1 Order Statistics

The pdf of the i^{th} order statistics from the GD pdf $g(x)$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\frac{\gamma(-\theta^{-1}\log(\overline{F}(x),\alpha))}{\Gamma(\alpha)}\right]^{i+j-1}, \end{aligned}$$

where $\overline{F}(x) = 1 - F(x)$. Using the fact that

$$\gamma(x,\alpha) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\alpha}}{(m+\alpha)m!}, \tag{38}$$

and setting $c_m = (-1)^m/((m+\alpha)m!)$, we have

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\alpha)]^{i+j-1}}[-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)} \\ &\times \left[\sum_{m=0}^{\infty}\frac{(-1)^m(-\theta^{-1}\log(\overline{F}(x)))^m}{(m+\alpha)m!}\right]^{i+j-1} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\alpha)]^{i+j-1}}[-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)} \\ &\times \sum_{m=0}^{\infty}d_{m,i+j-1}(-\theta^{-1}\log(\overline{F}(x)))^m, \end{aligned}$$

where $d_0 = c_0^{(i+j-1)}$, $d_{m,i+j-1} = (mc_0)^{-1} \sum_{l=1}^{m} [(i+j-1)l - m + l] c_l d_{m-l,i+j-1}$. Now,

$$\begin{aligned}
g_{i:n}(x) &= \frac{n! g(x)}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\alpha)]^{i+j-1}} [-\theta^{-1} \log(\overline{F}(x))]^{\alpha(i+j-1)+m} \\
&= \frac{n! [-\log(\overline{F}(x))]^{\alpha-1} [\overline{F}(x)]^{\psi-1} f(x)}{(i-1)!(n-i)! \Gamma(\alpha) \theta^{\alpha}} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j-1}} \\
&\times [-\theta^{-1} \log(\overline{F}(x))]^{\alpha(i+j-1)+m} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\
&\times \frac{[\log(\overline{F}(x))]^{\alpha(i+j)+m-1}}{\theta^{\alpha(i+j)+m}} [\overline{F}(x)]^{\psi-1} f(x) \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\
&\times \frac{\Gamma(\alpha(i+j)+m)}{\theta^{\alpha(i+j)+m}} \frac{[-\log(\overline{F}(x))]^{\alpha(i+j)+m-1}}{\Gamma(\alpha(i+j)+m)} [\overline{F}(x)]^{\psi-1} f(x).
\end{aligned}$$

That is,

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j d_{m,i+j-1} \Gamma(\alpha(i+j)+m)}{[\Gamma(\alpha)]^{i+j}} \\
&\times g(x; \alpha(i+j)+m, \beta, \lambda, \delta, \theta),
\end{aligned}$$

where $g(x; \alpha(i+j)+m, \beta, \lambda, \delta, \theta)$ is the GD pdf with parameters $\lambda, \beta, \delta, \theta$, and shape parameter $\alpha^* = \alpha(i+j)+m$. It follows therefore that

$$\begin{aligned}
E(X_{i:n}^j) &= \frac{n!}{(i-1)!(n-i)!} \sum_{\nu \in C} \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j w_\nu d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\
&\times \Gamma(\alpha(i+j)+m) \beta^* \lambda^{j/\delta} B\left(\beta^* + \frac{j}{\delta}, 1 - \frac{j}{\delta}\right),
\end{aligned}$$

for $j < \delta$, where $B(.,.)$ is the beta function. These moments are often used in several areas including reliability, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

4.2 Renyi Entropy

Renyi entropy is an extension of Shannon entropy. Renyi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [g(x;\lambda,\beta,\delta,\alpha,\theta)]^v dx\right), v \neq 1, v > 0. \tag{39}$$

Renyi entropy tends to Shannon entropy as $v \to 1$. Let $y = [1+\lambda x^{-\delta}]^{-\beta}$. Note that for $\alpha > 1$ and ν/θ a natural number,

$$\begin{aligned} g_{GD}^v(x) &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k}x^{-v\delta-v} \\ &\times [1+\lambda x^{-\delta}]^{-v\beta-v}y^{m+s+v\alpha-v+k} \\ &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k} \\ &\times x^{-v\delta-v}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^\infty g_{GD}^v(x)dx &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k} \\ &\times \int_0^\infty x^{-v\delta-v}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}dx. \end{aligned}$$

Let $t = [1+\lambda x^{-\delta}]^{-1}$, then

$$\begin{aligned} &\int_0^\infty x^{-v\delta-v}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}dx \\ &= \frac{\lambda^{-v-\frac{v}{\delta}+\delta}}{\delta}\int_0^1 t^{\beta(m+s+k+v\alpha)-\frac{v}{\delta}+\delta-1}(1-t)^{v+\frac{v}{\delta}-\delta-1}dt \\ &= \frac{\lambda^{-v-\frac{v}{\delta}+\delta}}{\delta}B\left(\beta(m+s+k+v\alpha)+\delta-\frac{v}{\delta}, v+\frac{v}{\delta}-\delta\right), \end{aligned}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function. Consequently, Renyi entropy for GD distribution is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\left[\frac{\lambda^{v-\frac{v}{\delta}}\beta^v\delta^{v-1}}{\theta^{v\alpha}(\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{(v/\theta)-1}{k}b_{s,m}\right. \\ &\times \left. B\left(\beta(m+s+k+v\alpha)+\delta-\frac{v}{\delta}, v+\frac{v}{\delta}-\delta\right)\right], \end{aligned}$$

for $v > 0$, $v \neq 1$.

5 Maximum Likelihood Estimation

Consider a random sample $x_1, x_2, ..., x_n$ from the gamma-Dagum distribution. The likelihood function is given by

$$\begin{aligned} L(\lambda,\beta,\delta,\theta,\alpha) &= \frac{(\lambda\beta\delta)^n}{[\theta^\alpha\Gamma(\alpha)]^n}\prod_{i=1}^{n}\Bigg\{x_i^{-\delta-1}[1+\lambda x_i^{-\delta}]^{-\beta-1} \\ &\times \left[-\log\left(1-(1+\lambda x_i^{-\delta})^{-\beta}\right)\right]^{\alpha-1}\left[1-(1+\lambda x_i^{-\delta})^{-\beta}\right]^{(1/\theta)-1}\Bigg\}. \end{aligned}$$

Now, the log-likelihood function denoted by ℓ is

$$\begin{aligned} \ell &= \log[L(\lambda,\beta,\delta,\theta,\alpha)] \\ &= n\log(\lambda)+n\log(\beta)+n\log(\delta)+(-\delta-1)\sum_{i=1}^{n}\log(x_i) \\ &+ (-\beta-1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})+(\alpha-1)\sum_{i=1}^{n}\log\left[-\log\left(1-(1+\lambda x_i^{-\delta})\right)\right] \\ &+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\log\left[1-(1+\lambda x_i^{-\delta})\right]-n\alpha\log(\theta)-n\log(\Gamma(\alpha)). \end{aligned} \tag{40}$$

The entries of the score function are given by

$$\begin{aligned} \frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda}+(-\beta-1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{1+\lambda x_i^{-\delta}} \\ &+ (\alpha-1)\sum_{i=1}^{n}\frac{\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}}{(1-(1+\lambda x_i^{-\delta})^{-\beta})\log(1-(1+\lambda x_i^{-\delta})^{-\beta})} \\ &+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}}{\log(1-(1+\lambda x_i^{-\delta})^{-\beta})}, \end{aligned} \tag{41}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}-\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta}) \\ &+ (\alpha-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})(-1)}{(1-(1+\lambda x_i^{-\delta})^{-\beta})(\log(1-(1+\lambda x_i^{-\delta})^{-\beta}))} \\ &+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-\beta}]}, \end{aligned} \tag{42}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n} \log(x_i) + (-\beta - 1) \sum_{i=1}^{n} \frac{\lambda x_i^{-\delta} \log(x_i^{-\delta})(-1)}{1 + \lambda x_i^{-\delta}} \\ &+ (\alpha - 1) \sum_{i=1}^{n} \frac{\lambda x_i^{-\delta} \log(\lambda x_i^{-\delta})}{(1 - (1 + \lambda x_i^{-\delta})^{-\beta})(\log(1 - (1 + \lambda x_i^{-\delta})^{-\beta}))}, \end{aligned} \quad (43)$$

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - n\log(\alpha) + \sum_{i=1}^{n} \log\left(-\log\left(1 - (1 + \lambda x_i^{-\delta})^{-\beta}\right)\right), \quad (44)$$

and

$$\frac{\partial \ell}{\partial \theta} = -\frac{n\alpha}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log\left(-\log\left(1 - (1 + \lambda x_i^{-\delta})^{-\beta}\right)\right). \quad (45)$$

The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters $\lambda, \beta, \delta, \alpha, \theta$ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{5X5} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$, can be numerically obtained by MATHLAB or MAPLE software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j}\bigg|_{\mathbf{\Delta} = \hat{\mathbf{\Delta}}} \right]_{5X5}, \quad i, j = 1, 2, 3, 4, 5. \quad (46)$$

For a given set of observations, the matrix given in equation (46) is obtained after the convergence of the Newton-Raphson procedure in MATHLAB software.

5.1 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GD distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Delta}} = (\hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\alpha}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (\lambda, \beta, \delta, \alpha, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\eta)\%$ confidence intervals for $\lambda, \beta, \delta, \theta$ and α are:

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{I^{-1}_{\beta\beta}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I^{-1}_{\delta\delta}(\widehat{\mathbf{\Delta}})}$$

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I^{-1}_{\alpha\alpha}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{I^{-1}_{\theta\theta}(\widehat{\mathbf{\Delta}})},$$

respectively, where $I^{-1}_{\lambda\lambda}(\hat{\mathbf{\Delta}})$, $I^{-1}_{\beta\beta}(\hat{\mathbf{\Delta}})$, $I^{-1}_{\delta\delta}(\hat{\mathbf{\Delta}})$, $I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Delta}})$ and $I^{-1}_{\theta\theta}(\hat{\mathbf{\Delta}})$ are the diagonal elements of $I^{-1}_{n}(\hat{\mathbf{\Delta}})$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the GD distribution with its sub-models for a given data set. For example, to test $\theta = \alpha = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\alpha}, \hat{\theta})) - \ln(L(\tilde{\lambda}, \tilde{\beta}, \tilde{\delta}, 1, 1))]$, where $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$, $\hat{\alpha}$ and $\hat{\theta}$, are the unrestricted estimates, and $\tilde{\lambda}$, $\tilde{\beta}$, and $\tilde{\delta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_{\epsilon}$, where χ^2_{ϵ} denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

6 Applications

In this section, we present examples to illustrate the flexibility of the GD distribution and its sub-models for data modeling. We also compare the four parameter GD sub-model to the gamma-exponentiated Weibull (GEW) distribution [17]. The pdf of GEW is given by

$$g_{GEW}(x) = \frac{k\alpha^{\delta}}{\lambda\Gamma(\delta)} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k} \left[1 - e^{-(\frac{x}{\lambda})^k}\right]^{\alpha-1} \left[-\log\left(1 - e^{-(\frac{x}{\lambda})^k}\right)\right]^{\delta-1}, \quad (49)$$

for $\alpha, \delta, \lambda, k > 0$.

The maximum likelihood estimates (MLEs) of the GD parameters λ, β, δ, α, and θ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard

Table 1: Descriptive Statistics

Data	N	Mean	Median	Mode	SD	Variance	Skewness	Kurtosis	Min.	Max.
Air Conditioning System	188	92.07	54.00	14.00	107.92	11646.00	2.16	5.19	1.0	603.0
Baseball Player Salary	818	3.260059	1.151	0.4	4.36406	19.04505	2.09955602	5.12663081	0.4	33
Bladder	128	9.365625	6.395	2.02	10.50833	110.42497	3.32566967	16.1537298	0.08	79.05

error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 3 and 4. Also, presented are values of the Kolmogorov-Smirnov statistic, $KS = max_{1\leq i\leq n}\{G_{GD}(x_i) - \frac{i-1}{n}, \frac{i}{n} - G_{GD}(x_i)\}$, and the sum of squares $SS = \sum_{j=1}^{n}\left[G_{GD}(x_{(j)}; \hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2$. These statistics are used to compare the distributions presented in these tables. Plots of the fitted densities and the histogram of the data are given in Figures 4 and 5. Probability plots (Chambers et al. [1]) are also presented in Figures 4 and 5. For the probability plot, we plotted $G_{GD}(x_{(j)}; \hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. Table 1 gives the descriptive statistics for the data sets.

The first example consists of the salaries of 818 professional baseball players for the year 2009 (USA TODAY). Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC,KS and SS for baseball player salaries data are give in Table 2.

Table 2: Estimates of Models for Baseball Player Salaries Data Set

	Estimates					Statistics				
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS
$GD(\lambda,\beta,\delta,\alpha,\theta)$	0.000024	99.9919	7.4586	0.525	16.2026	2884.5	2894.5	2894.6	2918.0	8.4755
	(0.000004361)	(0.1127)	(0.04404)	(0.05415)	(1.4405)					
$ZB-D(\lambda,\beta,\delta,\alpha,1)$	0.000816	98.7384	1.5113	3.1337	1	3201.7	3209.7	3209.8	3228.6	6.5759
	(0.000473)	(0.000063)	(0.1338)	(0.598)						
$ZB-BurrIII(1,\beta,\delta,\alpha,1)$	1	1.7986	1.1777	0.8104	1	3366.3	3372.3	3372.3	3386.4	5.9665
		(0.4834)	(0.05465)	(0.1796)						
$ZB-Fisk(\lambda,1,\delta,\alpha,1)$	0.008731	1	1.9669	5.4912	1	3221.5	3227.5	3227.5	3241.6	6.3701
	(0.005115)		(0.1193)	(0.621)						

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the baseball player salaries data are given in Figure 3.

The LR test statistic of the hypothesis H_0: GD against H_a: ZB-D, H_0: GD against H_a: ZB-BurrIII and H_0: GD against H_a: ZB-Fisk are 317.2 (p-value < 0.0001), 481.8 (p-value < 0.0001) and 337 (p-value < 0.0001). We can conclude that there is a significant difference between GD and ZB-D, ZB-BurrIII and ZB-Fisk distributions. The values of the statistics AIC, AICC and BIC shows that GD is the best fit for baseball salary data. Note that ZB-BurrIII gives the smallest SS value.

Table 3: Estimates of Models for Air Conditioning System Data Set

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	2.1816	31.0783	0.538	0.1856	0.05384	2065.1	2075.1	2075.4	2091.2	0.0334	0.0401
	(0.8567)	(7.1966)	(0.05068)	(0.0188)	(0.034)						
ZB-D	10.611	14.8939	0.9507	0.1885	1	2084.7	2092.7	2092.9	2105.7	0.5144	0.0982
	(1.9869)	(1.0488)	(0.0375)	(0.0194)	-						
ZB-BurrIII	1	51.7658	0.7498	0.2579	1	2101.5	2107.5	2107.6	2117.2	0.3876	0.0786
	-	(1.0861)	(0.0327)	(0.02906)	-						
ZB-Fisk	102.03	1	1.2902	1.2013	1	2078.5	2084.5	2084.6	2094.2	0.0797	0.0467
	(51.2069)	-	(0.07275)	(0.1991)	-						
D	118.02	1.1792	1.2873	1	1	2077.4	2083.4	2083.5	2093.1	0.0687	0.0421
	(71.3654)	(0.2375)	(0.09666)								
	k	α	δ	λ							
GEW	0.2651	1.3363	0.05339	0.0007		2338.4	2346.4	2346.6	2359.3	8.4196	0.3863
	(0.001118)	(0.4333)	(0.003965)	(0.000001897)							

The second example consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes (Proschan [18]). Initial value for GD model in SAS code are $\lambda = 1.2, \beta = 14, \delta = 1, \alpha = 0.9, \theta = 0.01$. Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, KS and SS for air conditioning system data are give in Table 3. The asymptotic covariance matrix of the MLEs of the GD model parameters, which is the inverse of the observed Fisher information matrix $\mathbf{I}_n^{-1}(\hat{\boldsymbol{\Delta}})$ is given by:

$$\begin{pmatrix} 0.7339 & -6.1407 & 0.02873 & -0.00537 & 0.006472 \\ -6.1407 & 51.7908 & -0.2362 & 0.04324 & -0.05223 \\ 0.02873 & -0.2362 & 0.002568 & 0.00014 & 0.001453 \\ -0.00537 & 0.04324 & 0.00014 & 0.000353 & 0.000194 \\ 0.006472 & -0.05223 & 0.001453 & 0.000194 & 0.001156 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by $\lambda \in (2.1816 \pm 1.6791)$, $\beta \in (31.0783 \pm 14.1053)$, $\delta \in (0.5380 \pm 0.0993)$, $\alpha \in (0.1856 \pm 0.03685)$, and $\theta \in (0.05384 \pm 0.06664)$, repectively.

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the air conditioning system data are given in Figure 4.

The LR test statistic of the hypothesis H_0: ZB-D against H_a: GD, H_0: ZB-BurrIII against H_a: GD and H_0: ZB-F against H_a: GD are 19.6 (p-value < 0.0001), 36.4 (p-value < 0.0001) and 13.4 (p-value $= 0.0012$). We can conclude that there is a significant difference between GD and ZB-D, ZB-BurrIII and ZB-Fisk distributions. Considering the statistics ($-2\log(L)$, AIC, BIC, KS and the values of SS given in Table 2, we observe that the GD distribution gives a better fit than the ZB-D, ZB-BurrIII, ZB-F, D, and GEW distributions. The value of the KS statistics (smaller is better) confirms that the GD model yields a better fit than the other models in Table 3.

Table 4: Estimates of Models for Remission Times Data Set

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	36.5904	4.6432	1.6783	0.1763	0.827	818.9	828.9	829.4	843.2	0.0153	0.0345
	(20.0777)	(0.6152)	(0.2154)	(0.02128)	(0.3848)						
ZB-D	14.4218	5.6739	1.4787	0.1932	1	821.4	829.4	829.7	840.8	0.0422	0.0457
	(4.9839)	(0.5272)	(0.1038)	(0.02313)	-						
ZB-BurrIII	1	14.0701	0.955	0.3086	1	846.8	852.8	853.0	861.4	0.3641	0.0945
	-	(0.8515)	(0.054)	(0.04013)	-						
ZB-Fisk	78.8617	1	1.9597	0.6595	1	819.6	825.6	825.8	834.1	0.0205	0.0396
	(47.7597)	-	(0.1721)	(0.1125)	-						
D	0.04426	56.2014	0.7713	1	1	884.2	890.2	890.4	898.8	0.9490	0.1382
	(0.02063)	(25.1544)	(0.04672)								
	k	α	δ	λ							
GEW	0.9718	1.8349	0.03244	1.9013		1154.7	1162.7	1163.0	1174.1	25.4662	0.6689
	(0.008578)	(0.9136)	(0.002901)	(0.002509)							

The third example represents the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, [10]). Initial value for GD model in SAS code are $\lambda = 7, \beta = 1, \delta = 0.84, \alpha = 0.21, \theta = 0.3$. Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC and SS for remission times data are give in Table 4. The asymptotic covariance matrix of the MLEs of the GD model parameters, which is the inverse of the observed Fisher information matrix $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}})$ is given by:

$$\begin{pmatrix} 403.11 & -11.7265 & 3.5502 & -0.1583 & 2.8776 \\ -11.7265 & 0.3785 & -0.1238 & 0.002601 & -0.1345 \\ 3.5502 & -0.1238 & 0.0464 & -0.00018 & 0.06253 \\ -0.1583 & 0.002601 & -0.00018 & 0.000453 & 0.000548 \\ 2.8776 & -0.1345 & 0.06253 & 0.000548 & 0.148 \end{pmatrix}$$

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the remission times data are given in Figure 5.

The LR test statistic of the hypothesis H_0: ZB-D against H_a: GD and H_0: ZB-BurrIII against H_a: GD are 2.5 (p-value = 0.1138) and 27.9 (p-value < 0.0001). We can conclude that there is a significant difference between GD and ZB-BurrIII distributions. Note that the GD distribution gives smaller AIC, AICC, BIC, SS and the value of KS statistic shows that this model yields a better fit than its sub-models and the GEW distribution.

7 Concluding Remarks

A new class of generalized Dagum distribution called the gamma-Dagum distribution is proposed and studied. The GD distribution has the GB, GF, ED and D distributions as special cases. The density of this new class of distributions can be expressed as a linear combination of Dagum density functions. The GD distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, the GD model is fitted to real data sets to illustrate the usefulness of the distribution.

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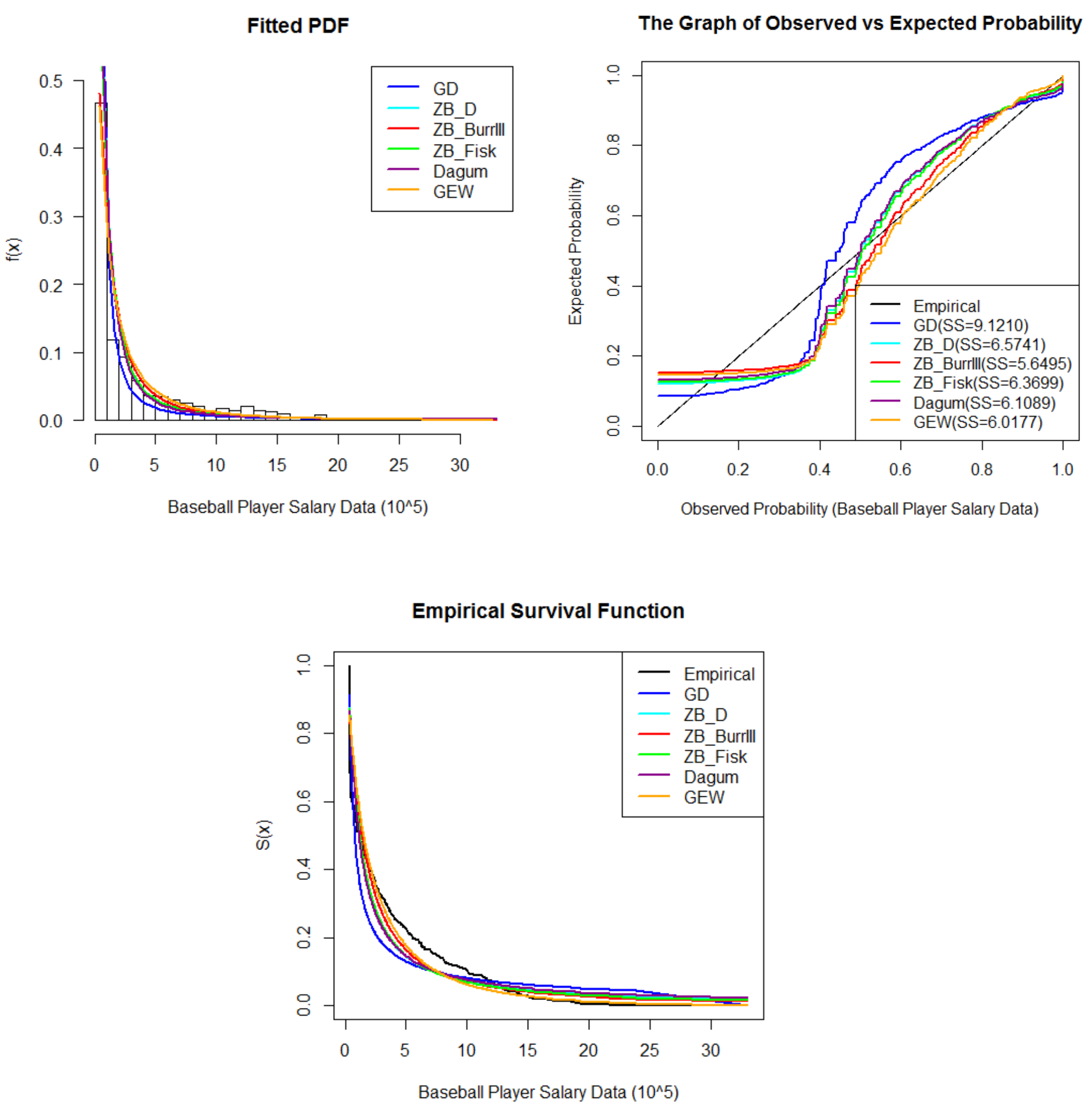

Figure 3: Fitted densities, probability plots of the baseball player salaries data

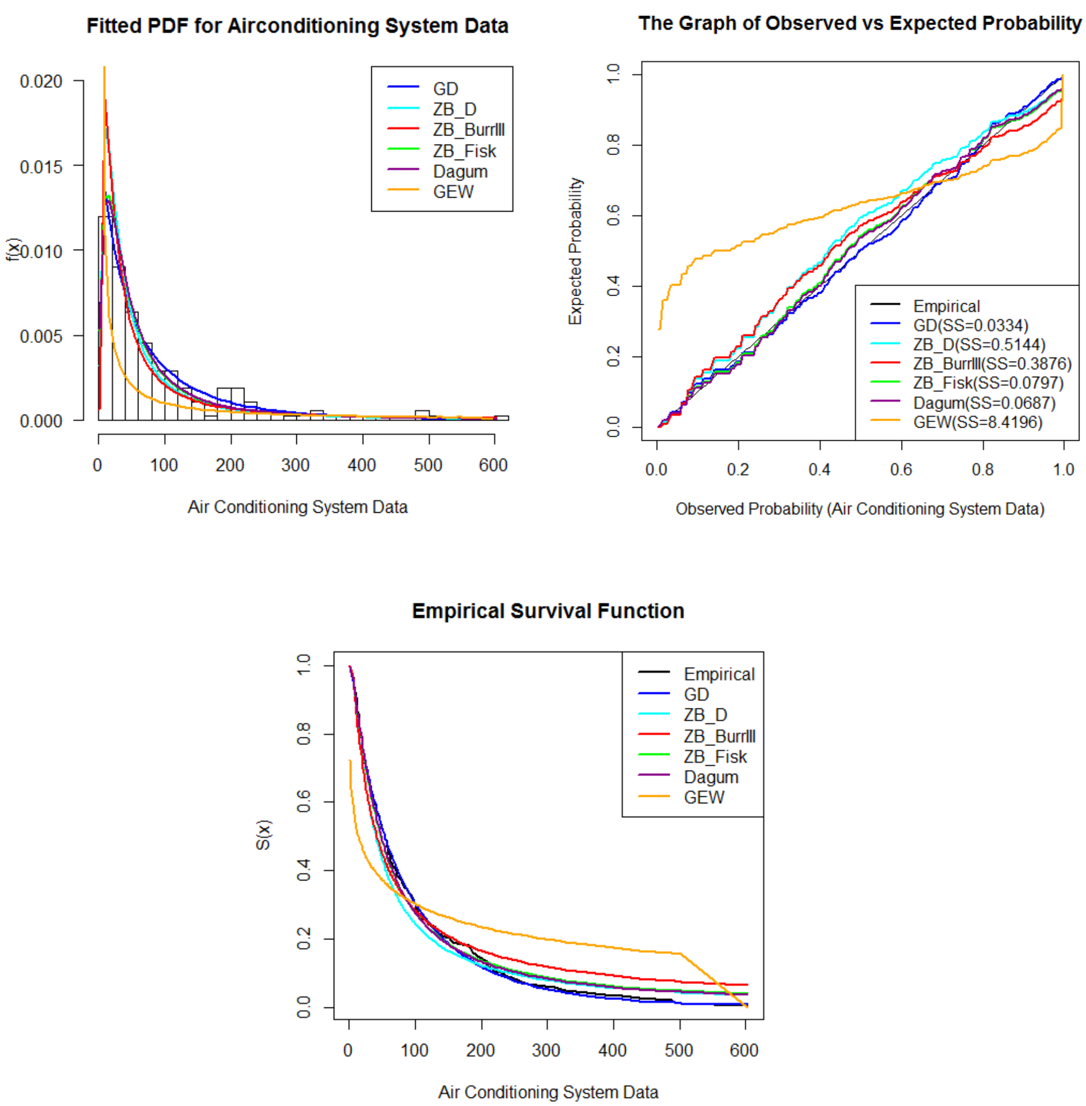

Figure 4: Fitted densities, probability plots of the air conditioning system data

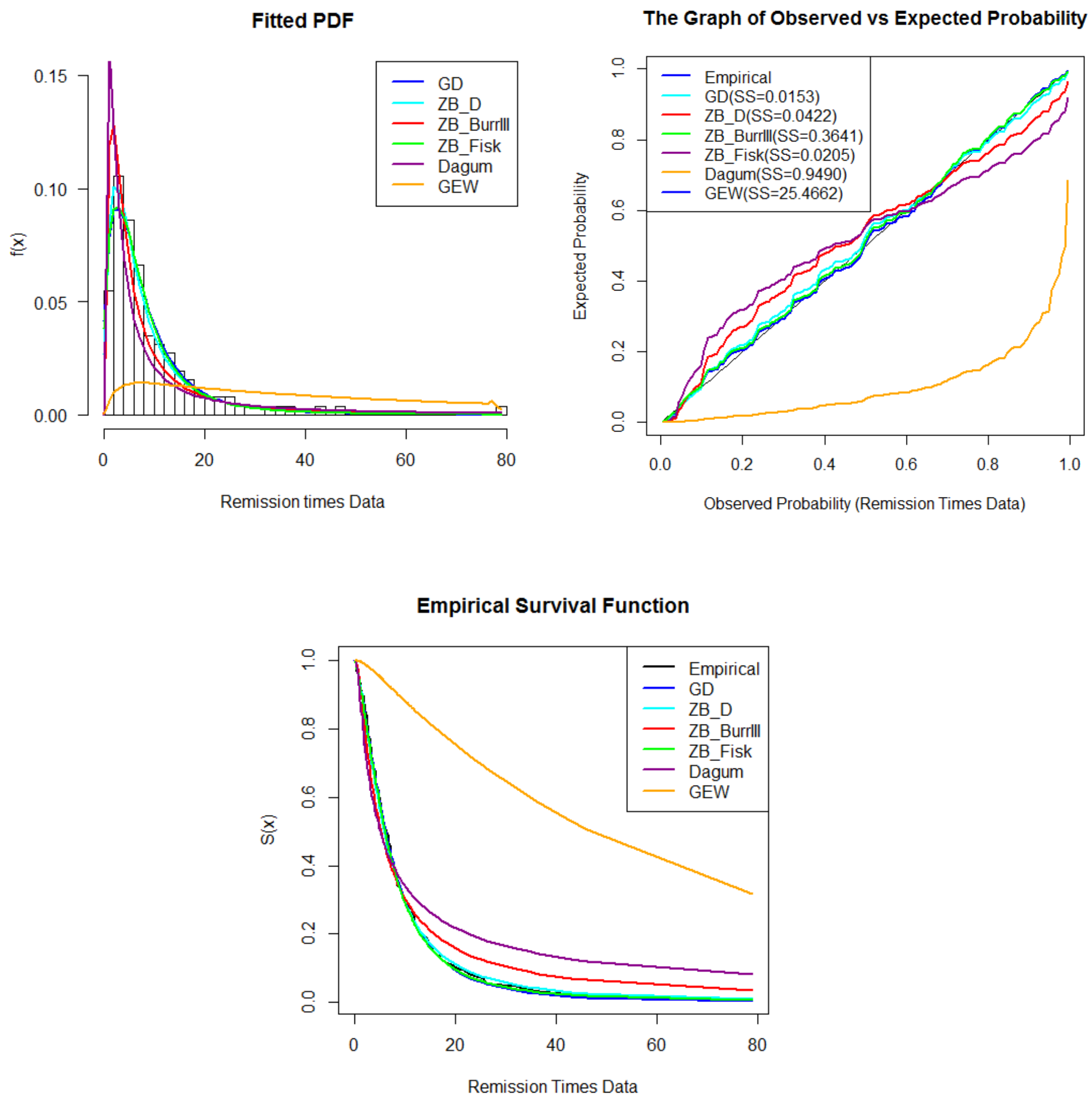

Figure 5: Fitted densities, probability plots of remission times data