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Developing Tools for a Precision Measurement of Newton's Gravitational Constant using Atom Interferometry

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ABSTRACT

We propose a new atom interferometry scheme for making a precision measurement of Newton's Gravitational constant (Big G) using NASA's Cold-Atom Laboratory which is scheduled to be deployed to the International Space Station in 2017. The proposed interferometer consists of splitting a harmonically confined Bose-Einstein condensate into multiple pieces. In a perfect harmonic potential, all of the pieces come to rest at the same time, at which point the harmonic trap is turned off. These initially motionless condensate clouds then accumulate different phases due to the relative velocity they develop caused by the gravitational attraction of a nearby source mass. The trap is then turned back on bringing all of the clouds together at the same time, at which point they are again split to produce a central interference pattern. I have derived equations for the simulation of these schemes using a Lagrangian variational approximation of the solution of the 1D time–dependent Gross–Pitaevskii equation. I have used this method to evaluate different interferometer schemes rapidly and to understand how the resultant interference pattern can be used to obtain Big G.

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1 Introduction

Isaac Newton's universal law of gravitation states that the magnitude of the gravitational force between two point masses, m_1 and m_2 and separated by a distance r is given by

$$F_{gravity} = \frac{Gm_1m_2}{r^2}.$$

That is, the force is directly proportional to the product of the masses and inversely proportional to square of the distance between them. The constant of proportionality between the mass product over the distance squared is called Newton's universal gravitational constant also known as "big G".

The first measurement of big G was made by Cavendish in 1797 using a torsion balance method [15]. Over the past 35 years, measurements of big G have employed torsion balances [3,6,8,10,11,13], beam balances [7], pairs of pendulums suspending optical cavities [5], and most recently atom interferometers [16]. The aggregate of these measurements has not succeeded in appreciably reducing the relative uncertainty, which stands at about 0.015%, in the value of big G [14]. It is important to explore new methods for measuring this constant as it is the most poorly determined of all of the fundamental constants of nature.

Recently the atom interferometry (AI) method has been used to make a precision measurement of big G that is competitive with the traditional methods listed above [2]. AI methods are significant because many of their systematic errors differ from those found in traditional methods. Figure 1 shows a comparison of the results of selected measurements of big G carried out over the past 35 years. The figure also

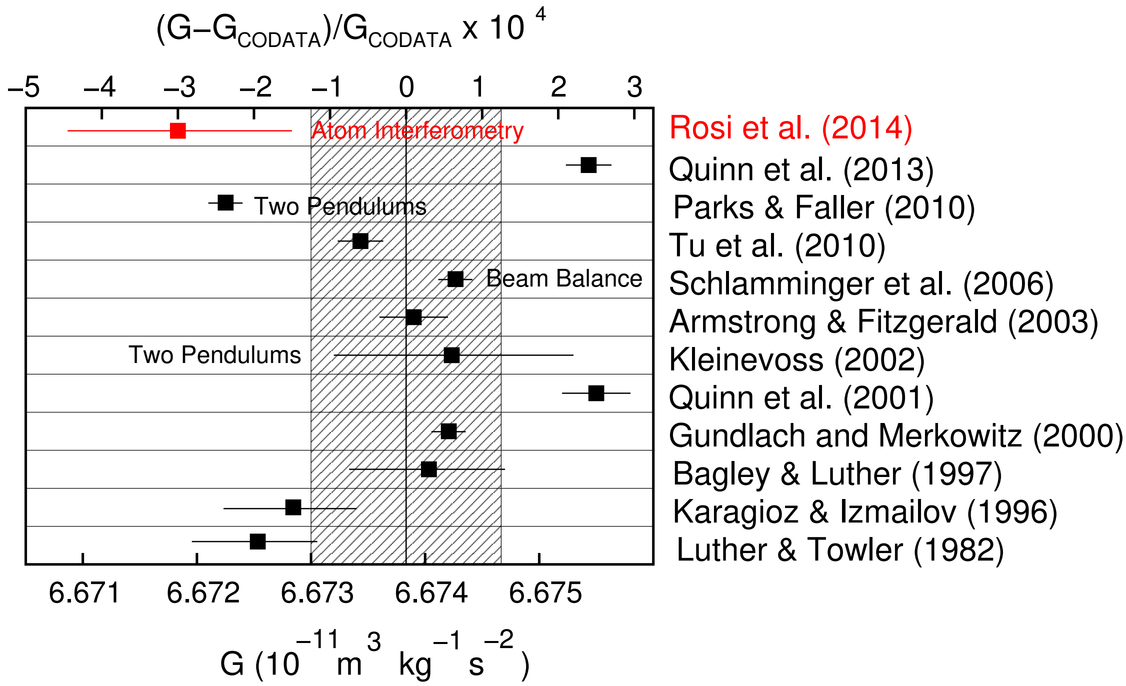

Figure 1: This diagram displays the values of selected measurements of big G from the past 35 years, with the shaded area showing the one-standard-deviation interval around the 2010 CODATA recommended value of $G = 6.67384(80) \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ [1]. The data points are taken from the most recent publications of experimental data from Rosi [2], Quinn [3,4], Parks & Faller [5], Tu [6], Schlamminger [7], Armstrong & Fitzgerald [8], Kleinevoss [9], Gundlach & Merkowitz [10], Bagley & Luther [11], Karagioz & Izmailov [12], and Luther & Towler [13]. This diagram is modeled after Figure 1 of reference [14].

shows how these results compare with the value recommended by the Committee on Data for Science and Technology also known as CODATA.

Figure 1 also shows that a significant number of recent big–G measurements lie outside the one–standard–deviation region around the CODATA recommended value. This deviation is even larger for the recent AI measurement of Tino's group [2] which obtained the value $G = 6.67191(99) \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ [2]. It is clear that this value lies further than one standard deviation away from that of the CODATA value.

This "Big G Crisis" motivated us to study the possibility of designing a new

atom-interferometric measurement of big G. One possibility would be to perform experiments in a micro-gravity environment. The research described in this thesis focuses on the use of micro-gravity environments for precision measurements of big G, with the goal of proposing an experiment that could be carried out on the Cold Atom Laboratory (CAL). The CAL is an ultracold atom experiment user facility that is being built by the National Aeronautics and Space Administration (NASA) and is currently scheduled to be deployed to the International Space Station (ISS) in summer 2017.

2 Atom Interferometry in Micro-Gravity

Atom interferometry (AI) is a measurement process whereby matter-wave systems are split into multiple pieces which then experience different environments. This difference affects the interference pattern produced when the pieces reassemble. AI has been used for many applications [17], such as testing of the Equivalence Principle [18], searching for Dark Matter [19], and now measuring Newton's gravitational constant [2]. AI is the measurement of interference, or a difference in phase, of atomic matter waves. These matter waves produce interference patterns when their phase difference is a minimum of 2π.

When these schemes involve the use of Bose-Einstein condensates (BEC) in a micro-gravity environment for a big G measurement, the micro-gravity allows for extended interrogation times, as well as the elimination of Earth's relatively large gravitational effect on condensates. To understand these processes better, it will

first be necessary to explain what a Bose–Einstein condensate is and give a brief description of quantum mechanics; the theory that describes these systems.

2.1 Bose-Einstein Condensates

A **Bose–Einstein condensate** or BEC is a system of N identical bosonic particles which all share the same single–particle wave function. Bose–Einstein condensates are at the heart of explanations of superfluidity (fluid flow without viscosity) and superconductivity (electric current flow without resistance) [20].

One important system that can be cooled into a Bose–Einstein condensate state is a gas of identical bosonic atoms. A particular isotope of an atom is “bosonic” if it has an even total number of electrons, protons, and neutrons. All that is required of such a gas to form a BEC is that it is cold and dense enough. The general situation for this to happen can be understood using the concept of “matter waves”. The theory of quantum mechanics is based on the idea that matter can have wave–like properties, first introduced in 1924 by Louis Victor deBroglie [21]. De Broglie introduced the concept of matter waves and suggested that the matter wave, or de Broglie wavelength, of a “particle” could be written as $\lambda = 2\pi\hbar/p$ where $\hbar$ is Planck’s constant and p is the magnitude of the particle’s momentum (mass times velocity).

Building upon this idea, Satyendranath Bose [22] suggested that a gas of photons could form an ensemble of identical particles and then Albert Einstein [23] suggested in 1925 that atoms in a gas could be cooled and confined to form a condensate. Cooling the atoms slows them down reducing their momentum which in turn increases

the wavelength of their matter waves. By simultaneously squeezing the atoms into a smaller volume the matter waves of neighboring particles will begin to overlap. The system then falls into the BEC state where the matter–wave shapes of the particles are all the same.

The first gaseous BEC of atoms was formed in 1995 in the lab of Eric Cornell and Carl Weiman at JILA [24], an institute run jointly by the University of Colorado at Boulder and the National Institute of Standards and Technology (NIST). The first BEC was a gas of rubidium atoms (^{87}Rb) and was first trapped and cooled using laser beams and then transferred to a magnetic trap. The gas was further cooled by evaporative cooling and when the gas was cold enough ($T \approx 170$ nK) and dense enough ($\rho \approx 10^{14}$ atoms/cm^3), a BEC of about 2000 atoms was formed. The 2001 Nobel Prize in Physics was awarded for this work and to Wolfgang Ketterle at MIT for his demonstration of important properties of BECs.

BECs magnify quantum properties, usually only observable at the atomic scale, up to the macroscopic scale. Condensates are relatively easy to control and probe making them a good testbeds for studying various fundamental matter–wave interactions. Examples of this include interference between two condensates, studies of superfluidity, slowing of light pulses to very slow speeds, sonic analogs of black holes, and optical lattices produced by counter propagating laser beams [20].

2.1.1 The Basics of Quantum Mechanics

According to the postulates of quantum mechanics (QM), the state of a quantum system is described by a wave function, ψ [25]. Any measurable quantity, $\mathcal{Q}$, is

represented by an operator, $\hat{Q}$, that acts on members of the space of wave functions. These operators must be linear, Hermitian and have a complete set of eigenvectors:

$$\hat{Q}\psi_n = q_n\psi_n, \quad n = 1, 2, \ldots \tag{1}$$

The completeness of the eigenvector set means that any wave function can be written as a linear combination of the eigenvectors.

$$\psi = \sum_n c_n\psi_n. \tag{2}$$

According to QM, the result of a measurement of $\mathcal{Q}$ can only be one of the eigenvalues q_n. Operators that are Hermitian can only have real eigenvalues thus ensuring that measurement predictions are always real (rather than complex) numbers.

Given the wave function of the system at the time of measurement, the theory also predicts the probability of obtaining a particular eigenvalue, say q_m, when $\mathcal{Q}$ is measured.

QM also requires that, immediately after the measurement of the quantity $\mathcal{Q}$ in which the outcome turns out to be eigenvalue q_m, the wave function of the system is the eigenvector, ψ_m, associated with that eigenvalue. This postulate enables the specification of the initial condition for the Schrödinger equation which governs the behavior of the system between measurements.

The time–dependent, many–body Schrödinger equation (TDSE) for a system of

N identical particles is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=\hat{H}_{MB}\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t), \tag{3}$$

where $\hat{H}_{MB}$ is the operator that represents the total energy of the system and is called the Hamiltonian. The Hamiltonian for many–body systems of N identical particles can be written as

$$\hat{H}_{MB}=\sum_{j=1}^{N}\left[-\frac{\hbar^2}{2M}\nabla_j^2+V(\mathbf{r}_j,t)\right]+\sum_{j<k}V_{\text{int}}(\mathbf{r}_j,\mathbf{r}_k). \tag{4}$$

The first term in the Hamiltonian represents the kinetic energies of the individual particles while the second term accounts for potential energies of the particles produced by external fields. The final term represents the energy resulting from binary collisions of atoms.

The wave function solution of the many–body TDSE must also satisfy the normalization condition:

$$\langle\Psi|\Psi\rangle\equiv\int d^3r_1\cdots\int d^3r_N\Psi^*(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=1. \tag{5}$$

Note that the above notation implies integration over all $3N$–dimensional space and that an asterisk superscript denotes complex conjugation in the physics literature. This condition ensures that the sum of the probabilities of all possible outcomes of a particular measurement will be equal to one.

2.1.2 Modeling Bose-Einstein Condensates

The only way to make progress in modeling a gas of N identical bosonic atoms is to resort to an approximation called the Variational Method [25]. In this method a specific form for the wave function solution of the many–body TDSE containing parameters that can be varied is inserted into the energy functional given by

$$E\left[\Psi\right] \equiv \frac{\langle\Psi|\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_{MB}\right)|\Psi\rangle}{\langle\Psi|\Psi\rangle}. \tag{6}$$

The functional is minimized by requiring that the first variation vanish:

$$E[\Psi + \delta\Psi] - E[\Psi] = 0. \tag{7}$$

In order to carry out this procedure the scattering of pairs of atoms in the condensate must be modeled and a trial wave function devised.

The atoms in a Bose–Einstein condensate interact by binary collisions. The atoms participating in such a collision are both cold (i.e., slow–moving) and dilute (i.e., always far apart). Under these circumstances the scattering process that occurs can be modeled by a Dirac delta function of the positions of the scattering atoms. Thus the interaction term in $\hat{H}_{MB}$ takes the form [26]

$$V_{\text{int}}(\mathbf{r}_j, \mathbf{r}_k) = g\,\delta(\mathbf{r}_j - \mathbf{r}_k) \tag{8}$$

where

$$g = \frac{4\pi\hbar^2 a}{M}. \tag{9}$$

Here M is the mass of a condensate atom and a is the scattering length which measures the strength of the atom–atom scattering in dilute, ultracold gases. If $a > 0$, the atoms repel each other on average and, if $a < 0$, atoms attract [20].

With this form of the interaction potential, the many–body Hamiltonian becomes:

$$\hat{H}_{MB} = \sum_{j=1}^{N} \big[-\frac{\hbar^2}{2m}\nabla_j^2 + V(\mathbf{r}_j, t)\big] + g\sum_{j<k} \delta(\mathbf{r}_j - \mathbf{r}_k) \tag{10}$$

A BEC is a gas of identical bosonic atoms all of which share the same single–particle state which we denote by $\phi(\mathbf{r}, t)$. The trial wave function that we will assume for the BEC is the N–fold product of this wave function evaluated at the location of each of the N atoms:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t) = \prod_{i=1}^{N} \phi(\mathbf{r}_i, t). \tag{11}$$

In the variational method, $\phi(\mathbf{r}, t)$ will be allowed to vary arbitrarily except that the many–body wave function must remain normalized. This places the following condition on $\phi(\mathbf{r}, t)$:

$$\int d^3r\, \phi^*(\mathbf{r}, t)\phi(\mathbf{r}, t) = 1. \tag{12}$$

Inserting this trial wave function into the energy functional, computing the first variation and setting it to zero yields the nonlinear Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\phi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{r}, t) + V_{\text{trap}}(\mathbf{r}, t)\phi(\mathbf{r}, t) + g(N-1)\left|\phi(\mathbf{r}, t)\right|^2\phi(\mathbf{r}, t). \tag{13}$$

The details of this can be found in many places in the literature and in textbooks [20, 26].

In most cases $N \gg 1$ so we approximate $N - 1 \approx N$. This yields the **Gross–Pitaevskii (GP) equation** :

$$\boxed{i\hbar\frac{\partial}{\partial t}\phi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{r},t) + V_{\text{trap}}(\mathbf{r},t)\phi(\mathbf{r},t) + gN\left|\phi(\mathbf{r},t)\right|^2\phi(\mathbf{r},t)} \tag{14}$$

This is the basic theoretical tool that is used to analyze and predict the behavior of gaseous Bose–Einstein condensates of atoms. It assumes that there are exactly N atoms in the gas and that its temperature is absolute zero. Most BECs can created at temperatures < 100 nK (nano–Kelvin) which is usually a good approximation.

2.2 Micro-Gravity Environments

2.2.1 Sounding Rockets

Sounding rockets are shot upward into the atmosphere and then allowed to fall freely back to the surface [27,28]. This provides a fairly long interval of free fall interrogation time. However, the rockets crash land after the experiment, and instruments may become damaged in the fall. This means that each run of an experiment may require the condensate equipment to be nearly rebuilt to facilitate further testing, which is not beneficial for a precision measurement requiring many experimental runs.

2.2.2 Drop towers

The next form of experimentation uses drop towers [29], very similar in process to that of sounding rockets to produce micro-gravity. In this case, the interferometer is loaded into a large drum and dropped from a significant height, allowing for time in free fall. This method does however run into similar problems as the sounding rockets, in that there is risk for damage of materials following each experimental run.

2.2.3 The Cold Atom Laboratory

The CAL is our focus for a micro-gravity environment due to the ability to run multiple experiments within a span of time in which the environment for the test will not drastically change. Once deployed to the ISS in June of 2017, the lab set up will be mounted in the station and will begin experimentation with little input or participation from astronauts on board. This provides an environment in which many experimental processes can be completed within same environment of the previous measurements. By using a permanent instrument, the risk of damage within one experimental run is significantly reduced. Creating an experimental scheme for such an environment can lead to improvements in the precision of atom interferometry measurements.

2.3 Framework of our candidate AI scheme

Our goals in this work are (1) to produce a theoretical tool that will enable rapid evaluation of different AI schemes and (2) to apply this tool to the design of a preci-

sion measurement of big G that can be carried out on the CAL. The overview of our proposed AI scheme involves a BEC confined in an external harmonic trap. A harmonic trap keeps the condensate in a parabolic potential energy such that any pieces that would propagate outward from the center would take the same time period to travel back to the initial position. The basic schematic can be seen in Fig. 2(a-g). This figure depicts the motion of condensate cloud pieces and their movement with a source mass present in the system during our candidate AI scheme. A source mass for our theoretical model is assumed to be the single source of gravitational potential energy in the microgravity environment in the CAL.

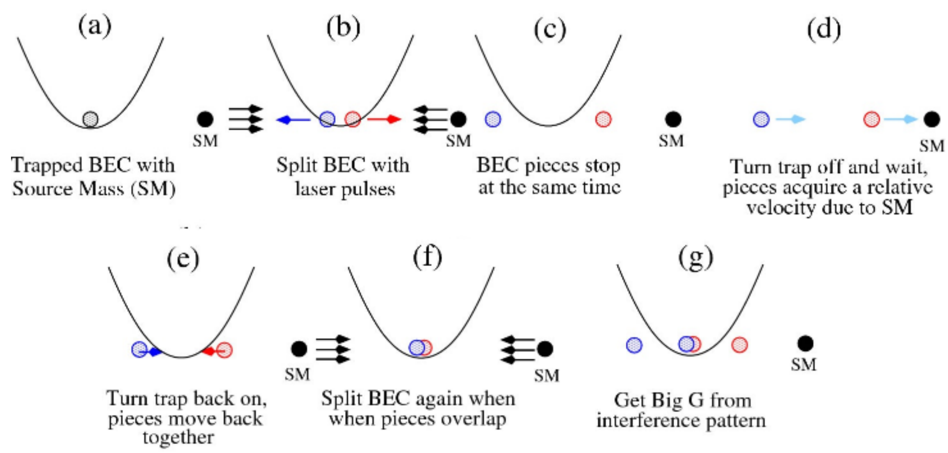

Figure 2: Our scheme for atom interferometry measurements of big G using a mass (SM) for production of gravitational forces on the BEC.

While confined by the harmonic trap, the condensate is pulsed with an optical lattice produced by a pair of counter-propagating laser beams. The optical lattice is a sequence of photons that imparts energy to the BEC, and gives it a momentum "kick". This momentum kick creates two new clouds; a positively "kicked" cloud and a negatively "kicked" cloud (Fig. 2(b)). Now we are left with two separately

moving clouds, traveling to the edge of the harmonic confinement. As the clouds travel outward they lose the initial imparted kinetic energy and come to a stop at the same time. Once this occurs, the harmonic trap is turned off, leaving the two pieces initially motionless in the gravitational environment of the source mass (Fig. 2(d)). The effective source mass potential produced two effects on the atom clouds: (1) it pulls their center of mass toward the source position and(2) it acts as a "tidal force" pushing the two clouds apart from each other. As the clouds travel apart from one another, they acquire different velocities and this relative velocity will be evident in the final interference pattern.

Once a significant amount of wait time has elapsed (this wait time is called the "interrogation time"), the harmonic trap is turned back on and the clouds begin to move back to the center of the trap (Fig. 2(e)). Once they have overlapped there will be an interference pattern produced by the clouds, however, the two clouds will be moving swiftly away from each other. We call this part of the sequence (panels a-e in Fig. 2) the "Initial Split".

Their motion following the re-introduction of the harmonic trap means that an accurate and clearly visible interference pattern will be difficult to obtain. In order to overcome this, in our scheme a second pulsed optical lattice is applied (Fig. 2(f)). This optical lattice imparts the same momentum kicks as the initial split, however now it will produce four clouds from the previous two. For each of the two overlapping initial split cloud pieces, they are then split into two pieces that have an addition of a positive and a negative momentum kick. In the end this leaves two halves the initial clouds nearly stationary in the center of the trap, and two halves traveling with twice

the initial speed towards the edge of the harmonic trap. The clouds within the center of the trap have components of the relative velocity gained when the trap was off, causing an interference pattern. We call this part of the sequence (panels f and g in Fig. 2) the "Final Split". We will use the simulation tools we have developed to assess the effectiveness of this candidate AI scheme for suitability as part of a precision measurment of big G that could be performed on the ISS.

The simulation tools we will develop consist of a set of equations that provide us with an approximate solution of the GP equation. Since the GP equation is assumed to describe condensate behavior, these equations will enable us to simulate condensate behavior rapidly. We now turn to the derivation of these equations of motion.

3 Derivation of the LVM Lagrangian

We need to be able to determine if the AI schemes we create are valid, which takes too much time if using the GP equation. In order to complete these schemes in as little time as possible, we must use an approximation of the GP equation in this case the Lagrangian Variational Method.

3.1 1D Lagrangian Variational Method

The Lagrangian Variational Method (LVM) provides approximations to the solutions of the time–dependent Gross–Pitaevskii equation (GPE), in the form:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2} + V_{\text{ext}}(x,t)\psi(x,t) + gN\left|\psi(x,t)\right|^2\psi(x,t), \qquad (15)$$

where M is the mass of a condensate atom, $\hbar$ is Planck's constant, g measures the strength of atom–atom scattering, N is the number of condensate atoms, and V_{ext} is the potential produced by electric, magnetic, and/or gravitational fields external to the condensate.

The LVM is based on the fact that the GPE can be derived as the Euler–Lagrange equation of motion produced by the following Lagrangian density:

$$\begin{aligned} \mathcal{L}[\psi^*, \psi_x^*, \psi_t^*] &= \frac{i\hbar}{2}\left(\psi\psi_t^* - \psi^*\psi_t\right) + \frac{\hbar^2}{2M}\psi_x^*\psi_x + V_{\text{ext}}(x,t)\psi^*\psi \\ &+ \frac{1}{2}gN(\psi)^2(\psi^*)^2. \end{aligned} \tag{16}$$

This Lagrangian density along with the following Euler–Lagrange equation of motion produces the GPE:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_t^*}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x^*}\right) - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0, \tag{17}$$

where

$$\psi_t^* \equiv \frac{\partial \psi^*}{\partial t} \quad \text{and} \quad \psi_x^* \equiv \frac{\partial \psi^*}{\partial x}. \tag{18}$$

The Lagrangian Variation Method consists of devising a *trial* wave function,

$$\psi_{\text{trial}}(x,t) = \psi_{\text{trial}}(q_1(t), \ldots, q_n(t); x) \tag{19}$$

where the $\{q_i(t)\}$, $i = 1, \ldots, n$ are *variational parameters* that only depend on the time, t. The equations of motion of these variational parameters are derived by

computing the ordinary Lagrangian:

$$L(q_1(t), \ldots, q_n(t)) = \int_{-\infty}^{+\infty} dx \mathcal{L}[\psi^*_{\text{trial}}, \psi^*_{\text{trial},x}, \psi^*_{\text{trial},t}] \tag{20}$$

and then the ordinary Euler–Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1, \ldots, n \tag{21}$$

provides the equation of motion associated with the particular variational parameter.

3.1.1 Scaled units

We can simplify the above method by introducing a set of units appropriate to the problem and a set of scaled variables (both independent and dependent). The scaled variables are defined by first establishing a *length unit*, L_0, and then defining energy, E_0, and time, T_0, units as follows:

$$E_0 \equiv \frac{\hbar^2}{2ML_0^2} \quad \text{and} \quad T_0 \equiv \frac{\hbar}{E_0} = \frac{2ML_0^2}{\hbar}. \tag{22}$$

We then introduce scaled variables which are generally denoted by barred quantities. These consist of scaled space and time coordinates:

$$\bar{x} \equiv \frac{x}{L_0} \quad \text{and} \quad \bar{t} \equiv \frac{t}{T_0}. \tag{23}$$

We also introduce the scaled condensate wave function for the solution of the GPE:

$$\bar{\psi}(\bar{x},\bar{t}) = \psi(x,t) L_0^{1/2}. \tag{24}$$

We can express the original GPE in terms of scaled quantities and this can be done for the Lagrangian density and its associated Euler–Lagrange equation as well.

In terms of scaled quantities, the GPE becomes:

$$i\frac{\partial \bar{\psi}}{\partial \bar{t}} = -\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} + \bar{V}_{\text{ext}}(\bar{x},\bar{t})\psi + \bar{g}N\left|\psi\right|^2\psi. \tag{25}$$

where $\bar{g} \equiv g/(E_0 L_0)$ and $\bar{V}_{\text{ext}}(\bar{x},\bar{t}) = V_{\text{ext}}(x,t)/E_0$. The scaled Lagrangian density becomes

$$\begin{aligned} \bar{\mathcal{L}}\left[\bar{\psi}^*, \bar{\psi}^*_{\bar{x}}, \bar{\psi}^*_{\bar{t}}\right] &= \frac{i}{2}\left(\bar{\psi}\bar{\psi}^*_{\bar{t}} - \bar{\psi}^*\bar{\psi}_{\bar{t}}\right) + \bar{\psi}^*_{\bar{x}}\bar{\psi} + \bar{V}_{\text{ext}}(\bar{x},\bar{t})\bar{\psi}\bar{\psi}^* \\ &+ \frac{1}{2}\bar{g}N\left(\bar{\psi}^*\right)^2\left(\bar{\psi}\right)^2 \end{aligned} \tag{26}$$

and the scaled Euler–Lagrange equation is given by

$$\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \bar{\psi}^*_{\bar{x}}}\right) + \frac{\partial}{\partial \bar{t}}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \bar{\psi}^*_{\bar{t}}}\right) - \frac{\partial \bar{\mathcal{L}}}{\partial \bar{\psi}^*} = 0. \tag{27}$$

Next we turn to the 1D, N_c–gaussian–cloud trial wave function.

3.1.2 The 1D, N_c–gaussian–cloud trial wave function

In the 1D, N_c–gaussian–cloud model we take the trial wave function to be a sum of N_c one–dimensional Gaussian clouds. The j^{th} cloud has its own initial momentum, $\bar{k}_j$, and set of variational parameters. These parameters consist of the cartesian coordinate of the cloud center, $\bar{x}_j$, the cloud width, $\bar{w}_j$, the linear phase coefficient, $\bar{\alpha}_j$, and the quadratic phase coefficient, $\bar{\beta}_j$. The j^{th} cloud also has its own normalization coefficient, A_j, which will be eliminated by fixing the number of atoms in each cloud.

The mathematical form (in scaled units) of the trial wave function is the following:

$$\bar{\psi}(\bar{x}, \bar{t}) = \frac{1}{\sqrt{N_c}} \sum_{j=1}^{N_c} A_j(\bar{t}) e^{f_j(\bar{x},\bar{t}) + i\bar{k}_j\bar{x}} \tag{28}$$

where

$$f_j(\bar{x}, \bar{t}) = -\frac{(\bar{x} - \bar{x}_j(\bar{t}))^2}{2\bar{w}_j^2(\bar{t})} + i\bar{\alpha}_j(\bar{t})\bar{x} + i\bar{\beta}_j(\bar{t})\bar{x}^2 \tag{29}$$

We can calculate the Lagrangian associated with this trial wave function by integrating $\bar{\mathcal{L}}$ over all space:

$$\bar{L}\left(\mathbf{x}, \mathbf{w}, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \int_{-\infty}^{\infty} d\bar{x}\, \bar{\mathcal{L}}\left[\bar{\psi}^*, \bar{\psi}^*_{\bar{x}}, \bar{\psi}^*_{\bar{t}}\right], \tag{30}$$

where our notation for the dependence of $\bar{L}$ means the following

$$
\begin{aligned}
\mathbf{x} &\equiv (\bar{x}_1, \ldots, \bar{x}_{N_c}) \\
\mathbf{w} &\equiv (\bar{w}_1, \ldots, \bar{w}_{N_c}) \\
\boldsymbol{\alpha} &\equiv (\bar{\alpha}_1, \ldots, \bar{\alpha}_{N_c}) \\
\boldsymbol{\beta} &\equiv \left(\bar{\beta}_1, \ldots, \bar{\beta}_{N_c}\right)
\end{aligned}
$$

The equation of motion for a particular variational parameter, $q_j(\bar{t})$, is then given by the ordinary Euler–Lagrange equation:

$$
\frac{\partial}{\partial \bar{t}} \left(\frac{\partial \bar{L}}{\partial \dot{q}_j} \right) - \frac{\partial \bar{L}}{\partial q_j} = 0, \quad j = 1, \ldots, N_c. \tag{31}
$$

With these tools in hand we can now compute the Lagrangian.

The Lagrangian has four large–scale terms and can be written as follows.

$$
\bar{L} = \bar{L}_1 + \bar{L}_2 + \bar{L}_3 + \bar{L}_4 \tag{32}
$$

where

$$
\begin{aligned}
\bar{L}_1 &\equiv \frac{i}{2} \int_{-\infty}^{\infty} d\bar{x} \, \left(\bar{\psi} \bar{\psi}_{\bar{t}}^* - \bar{\psi}^* \bar{\psi}_{\bar{t}} \right) \\
\bar{L}_2 &\equiv \int_{-\infty}^{\infty} d\bar{x} \, \bar{\psi}_{\bar{x}}^* \bar{\psi}_{\bar{x}} \\
\bar{L}_3 &\equiv \int_{-\infty}^{\infty} d\bar{x} \, \bar{V}(\bar{x}, \bar{t}) \bar{\psi} \bar{\psi}^* \\
\bar{L}_4 &\equiv \frac{1}{2} \bar{g} N \int_{-\infty}^{\infty} d\bar{x} \, \left(\bar{\psi}^* \right)^2 \left(\bar{\psi} \right)^2
\end{aligned} \tag{33}
$$

Next we derive, in turn, the value of each of these terms and then put together the full Lagrangian with respect to our trial wave function.

3.1.3 Constraints on the trial wave function

Here we make several assumptions about the physical system which have material effects on the values of the variational parameters. These are as follows:

1. We assume that each of the N_c clouds are moving at sufficiently different velocities such that any integral of a quantity containing a factor like $e^{i\left(\bar{k}_j-\bar{k}_{j'}\right)\bar{x}}$, such that $j \neq j'$, can be neglected. If the clouds move with sufficiently different velocities, these factors will be rapidly oscillating and their integrals will integrate to zero.

2. The $A_j(\bar{t})$ are real for all j. This derives from the assumption that the system is a single condensate and has an overall constant phase.

3. The number of atoms in each cloud is fixed. Clouds do not exchange atoms. This plus the normalization condition, fixes a relationship (derived below) between A_j and the widths $\bar{w}_j$.

We can use these assumptions plus the normalization condition on the trial wave function to derive conditions that constrain the values of the A_j.

To find these conditions we require that the full trial wave function be normalized

to unity:

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} d\bar{x}\left|\bar{\psi}(\bar{x},\bar{t})\right|^2 \\
&= \int_{-\infty}^{+\infty} d\bar{x}\left(\frac{1}{\sqrt{N_c}}\sum_{j=1}^{N_c} A_j(\bar{t})e^{f_j(\bar{x},\bar{t})+i\bar{k}_j\bar{x}}\right)^* \\
&\times \left(\frac{1}{\sqrt{N_c}}\sum_{j=1}^{N_c} A_j(\bar{t})e^{f_j(\bar{x},\bar{t})+i\bar{k}_j\bar{x}}\right) \qquad (34)
\end{aligned}$$

We can simplify the above integral by dropping all of the terms in the product $\bar{\psi}^*\bar{\psi}$ that contain rapidly oscillating exponentials such as $e^{i\left(\bar{k}_j-\bar{k}_{j'}\right)\bar{x}}$ where $j \neq j'$. In this case our normalization condition simplifies to

$$\begin{aligned}
1 &= \frac{1}{N_c}\int_{-\infty}^{+\infty} d\bar{x}\left(\sum_{j=1}^{N_c} A_j^2(\bar{t})\exp\left\{f_j(\bar{x},\bar{t})+f_j^*(\bar{x},\bar{t})\right\}\right) \\
&= \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t})\int_{-\infty}^{+\infty} d\bar{x}\exp\left\{-\frac{(\bar{x}-\bar{x}_j)^2}{\bar{w}_j^2}\right\} \\
1 &= \frac{1}{N_c}\sum_{j=1}^{N_c}\left(A_j^2\pi^{1/2}\bar{w}_j(\bar{t})\right) \qquad (35)
\end{aligned}$$

where the value of the integral above is derived in Appendix A Eq. (132).

This last expression is the condition for the trial wave function to be normalized. However, our assumption that the number of atoms in each cloud is fixed adds a further restriction to the above expression. That is that each cloud is individually normalized. This gives, finally,

$$\boxed{A_j^2(\bar{t})\pi^{1/2}\bar{w}_j(\bar{t}) = 1, \quad j = 1, \ldots, N_c.} \qquad (36)$$

These constraints together automatically satisfy Eq. (35) and enable the elimination of all of the A_j in the final Lagrangian.

3.2 Derivation of $\bar{L}_1$

The $\bar{L}_1$ term of the Lagrangian has the form

$$\bar{L}_1 = \int_{-\infty}^{+\infty} d\bar{x}\bar{\mathcal{L}}_1 = \frac{i}{2}\int_{-\infty}^{+\infty} d\bar{x}\left(\psi(\bar{x},\bar{t})\psi^*_{\bar{t}}(\bar{x},\bar{t}) - \psi^*(\bar{x},\bar{t})\psi_{\bar{t}}(\bar{x},\bar{t})\right) \tag{37}$$

In order to compute this integral is convenient to rewrite the integrand as follows:

$$\bar{\mathcal{L}}_1 = \frac{i}{2}\left(\psi(\bar{x},\bar{t})\psi^*_{\bar{t}}(\bar{x},\bar{t}) - \psi^*(\bar{x},\bar{t})\psi_{\bar{t}}(\bar{x},\bar{t})\right) = \mathrm{Im}\left\{\psi^*(\bar{x},\bar{t})\psi_{\bar{t}}(\bar{x},\bar{t})\right\} \tag{38}$$

where $\mathrm{Im}\{z\}$ denotes the imaginary part of the complex number z. Thus we can now write $\bar{L}_1$ in a more convenient form for calculation:

$$\bar{L}_1 = \int d^3\bar{r}\,\mathrm{Im}\left\{\psi^*(\bar{x},\bar{t})\psi_{\bar{t}}(\bar{x},\bar{t})\right\}. \tag{39}$$

To proceed, we next insert the trial wave function into $\psi^*\psi_{\bar{t}}$. It will be efficient first to calculate $\psi_{\bar{t}}$ using the following form of the trial wave function,

$$\psi(\bar{x},\bar{t}) = \frac{1}{\sqrt{N_c}}\sum_{j=1}^{N_c} A_j(\bar{t})e^{f_j(\bar{x},\bar{t})+i\bar{k}_j\bar{x}} \tag{40}$$

It is easy to see that the partial time derivative of this is

$$\psi_{\bar{t}}(\bar{x},\bar{t}) = \frac{1}{\sqrt{N_c}} \sum_{j=1}^{N_c} \left(\dot{A}_j(\bar{t}) + A_j(\bar{t}) \dot{f}_j(\bar{x},\bar{t}) \right) e^{f_j(\bar{x},\bar{t}) + i\bar{k}_j \bar{x}}, \tag{41}$$

where the dot denotes partial differentiation with respect to $\bar{t}$.

Next we multiply the above by ψ^* to get

$$\begin{aligned}
\psi^* \psi_{\bar{t}} &= \left(\frac{1}{\sqrt{N_c}} \sum_{j=1}^{N_c} A_j(\bar{t}) e^{f_j(\bar{x},\bar{t}) + i\bar{k}_j \bar{x}} \right)^* \\
&\times \left(\frac{1}{\sqrt{N_c}} \sum_{j'=1}^{N_c} \left(\dot{A}_{j'}(\bar{t}) + A_{j'}(\bar{t}) \dot{f}_{j'}(\bar{x},\bar{t}) \right) e^{f_{j'}(\bar{x},\bar{t}) + i\bar{k}_{j'} \bar{x}} \right) \\
&\approx \frac{1}{N_c} \sum_{j=1}^{N_c} \left(A_j(\bar{t}) \dot{A}_j(\bar{t}) + A_j^2(\bar{t}) \dot{f}_j(\bar{x},\bar{t}) \right) e^{f_j(\bar{x},\bar{t}) + f_j^*(\bar{x},\bar{t})}
\end{aligned} \tag{42}$$

where in the second equality we have dropped terms that contained rapidly oscillating exponentials (i.e., any terms where $j \neq j'$) as they would be negligible after integration. We have also used the assumption that the A_j are real.

We next take the imaginary part of this last expression. In doing so it is important to note that $f_j + f_j^*$ is real so that the exponential of this will also be real. Thus we can see that, since the term in the above containing the factor $A_j \dot{A}_j$ is real, it will not contribute to the imaginary part. In fact the only factor in the above with an imaginary part is $\dot{f}_j$. Thus, taking the imaginary part of Eq. (42) becomes

$$\begin{aligned}
\mathrm{Im}\left\{ \psi^* \psi_{\bar{t}} \right\} &\approx \frac{1}{N_c} \sum_{j=1}^{N_c} A_j^2(\bar{t}) \mathrm{Im}\left\{ \dot{f}_j(\bar{x},\bar{t}) \right\} e^{f_j(\bar{x},\bar{t}) + f_j^*(\bar{x},\bar{t})} \\
&= \frac{1}{N_c} \sum_{j=1}^{N_c} A_j^2(\bar{t}) \left(\dot{\bar{\alpha}}_j \bar{x} + \dot{\bar{\beta}}_j \bar{x}^2 \right) e^{-(\bar{x} - \bar{x}_j)^2 / \bar{w}_j^2}
\end{aligned} \tag{43}$$

We now insert this expression into the integral for $\bar{L}_1$.

Inserting Eq. (43) into Eq. (39) we have

$$
\begin{aligned}
\bar{L}_1 &= \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t}) \int_{-\infty}^{+\infty} d\bar{x} \left(\dot{\bar{\alpha}}_j \bar{x} + \dot{\bar{\beta}}_j \bar{x}^2\right) e^{-(\bar{x}-\bar{x}_j)^2/\bar{w}_j^2} \\
&= \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t})(\pi^{1/2}\bar{w}_j) \left(\dot{\bar{\alpha}}_j \bar{x}_j + \dot{\bar{\beta}}_j (\bar{x}_j^2 + \tfrac{1}{2}\bar{w}_j^2)\right) \\
&= \frac{1}{N_c}\sum_{j=1}^{N_c} \left(\dot{\bar{\alpha}}_j \bar{x}_j + \dot{\bar{\beta}}_j (\bar{x}_j^2 + \tfrac{1}{2}\bar{w}_j^2)\right).
\end{aligned}
\tag{44}
$$

The value for the integrals appearing above are evaluated in Appendix A and are given in Eqs. (133) and (134). In the last line we have used the normalization conditions to eliminate the $A_j(\bar{t})$ factors.

Thus the final expression for $\bar{L}_1$ is

$$
\boxed{\bar{L}_1 = \frac{1}{N_c}\sum_{j=1}^{N_c} \left(\dot{\bar{\alpha}}_j \bar{x}_j + \dot{\bar{\beta}}_j \left(\bar{x}_j^2 + \frac{1}{2}\bar{w}_j^2\right)\right)}
\tag{45}
$$

Next we turn to the derivation of $\bar{L}_2$.

3.3 Derivation of $\bar{L}_2$

The expression for $\bar{L}_2$ is given by

$$
\bar{L}_2 \equiv \int_{-\infty}^{+\infty} d\bar{x}\, \psi_{\bar{x}}^* \psi_{\bar{x}}
\tag{46}
$$

To proceed we must calculate the space derivative of the trial wave function reproduced here for convenience along with the derivative expression:

$$
\begin{aligned}
\psi(\bar{x},\bar{t}) &= \frac{1}{\sqrt{N_c}}\sum_{j=1}^{N_c} A_j(\bar{t})e^{f_j(\bar{x},\bar{t})+i\bar{k}_j\bar{x}} \\
\psi_{\bar{x}}(\bar{x},\bar{t}) &= \frac{1}{\sqrt{N_c}}\sum_{j=1}^{N_c} A_j(\bar{t})e^{f_j(\bar{x},\bar{t})+i\bar{k}_j\bar{x}}\left(f_{j,\bar{x}}^{(r)}+i\left(f_{j,\bar{x}}^{(i)}+\bar{k}_j\right)\right)
\end{aligned}
\qquad (47)
$$

where we have written f_j in terms of its real and imaginary parts,

$$
f_j(\bar{x},\bar{t}) = -\frac{(\bar{x}-\bar{x}_j)^2}{2\bar{w}_j^2} + i\left(\bar{\alpha}_j\bar{x}+\bar{\beta}_j\bar{x}^2\right) \equiv f_j^{(r)}(\bar{x},\bar{t}) + i\,f_j^{(i)}(\bar{x},\bar{t}), \qquad (48)
$$

and their derivatives with respect to $\bar{x}$ are as follows:

$$
f_{j,\bar{x}}^{(r)} \equiv \frac{\partial f_j^{(r)}}{\partial\bar{x}} = -\frac{(\bar{x}-\bar{x}_j)}{\bar{w}_j^2} \quad \text{and} \quad f_{j,\bar{x}}^{(i)} \equiv \frac{\partial f_j^{(i)}}{\partial\bar{x}} = 2\bar{\beta}_j\bar{x}+\bar{\alpha}_j. \qquad (49)
$$

Before we use these expressions for the derivatives of f_j, we will derive the final version of $\psi_{\bar{x}}^*\psi_{\bar{x}}$ first. If we multiply expression for $\psi_{\bar{x}}$ by its complex conjugate and neglect the rapidly oscillating terms we obtain

$$
\begin{aligned}
\psi_{\bar{x}}^*\psi_{\bar{x}} &= \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t})e^{f_j^*+f_j}\left(\left(f_{j,\bar{x}}^{(r)}\right)^2+\left(f_{j,\bar{x}}^{(i)}+\bar{k}_j\right)^2\right) \\
&= \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t})e^{-(\bar{x}-\bar{x}_j)^2/\bar{w}_j^2}\left(\frac{(\bar{x}-\bar{x}_j)^2}{\bar{w}_j^4}+\left(2\bar{\beta}_j\bar{x}+\bar{\alpha}_j+\bar{k}_j\right)^2\right)
\end{aligned}
\qquad (50)
$$

We now insert this form into Eq. (46):

$$\bar{L}_2 \;=\; \frac{1}{N_c}\sum_{j=1}^{N_c} A_j^2(\bar{t}) \int_{-\infty}^{+\infty} d\bar{x}\, e^{-(\bar{x}-\bar{x}_j)^2/\bar{w}_j^2} \left(\frac{(\bar{x}-\bar{x}_j)^2}{\bar{w}_j^4} + \left(2\bar{\beta}_j\bar{x} + \bar{\alpha}_j + \bar{k}_j\right)^2 \right) \tag{51}$$

While it is possible to use the integrals derived, here it is a little easier to perform the integration directly. To do this we let $x = (\bar{x} - \bar{x}_j)/\bar{w}_j$ and change the integration variable to x in the integral appearing in Eq. (51) and shown below:

$$\begin{aligned}
I \;&\equiv\; \int_{-\infty}^{+\infty} d\bar{x}\, \exp\left\{ -\frac{(\bar{x}-\bar{x}_j)^2}{\bar{w}_j^2} \right\} \left(\frac{(\bar{x}-\bar{x}_j)^2}{\bar{w}_j^4} + \left(2\bar{\beta}_j\bar{x} + \bar{\alpha}_j + \bar{k}_j\right)^2 \right) \\
&=\; \int_{-\infty}^{+\infty} \bar{w}_j\, dx\, e^{-x^2} \left(\frac{x^2}{\bar{w}_j^2} + \left(2\bar{\beta}_j\left(\bar{x}_j + \bar{w}_j x\right) + \bar{\alpha}_j + \bar{k}_j\right)^2 \right) \\
&=\; \int_{-\infty}^{+\infty} \bar{w}_j\, dx\, e^{-x^2} \left(\left(\frac{1}{\bar{w}_j^2} + 4\bar{\beta}_j^2\bar{w}_j^2 \right) x^2 + 4\bar{\beta}_j\bar{w}_j \left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j \right) x \right. \\
&+\; \left. \left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j \right)^2 \right) \\
&=\; \left(\pi^{1/2}\bar{w}_j\right) \left(\frac{1}{2}\left(\frac{1}{\bar{w}_j^2} + 4\bar{\beta}_j^2\bar{w}_j^2 \right) + \left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j \right)^2 \right)
\end{aligned} \tag{52}$$

Inserting this last expression in place of the integral in $\bar{L}_2$ we have the final expression for $\bar{L}_2$:

$$\boxed{\bar{L}_2 = \frac{1}{N_c}\sum_{j=1}^{N_c} \left(\frac{1}{2\bar{w}_j^2} + 2\bar{\beta}_j^2\bar{w}_j^2 + \left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j \right)^2 \right).} \tag{53}$$

Next we turn to the derivation of $\bar{L}_3$.

3.4 Derivation of $\bar{L}_3$

3.4.1 General parameter dependence of $\bar{L}_3$

The expression for $\bar{L}_3$ is

$$\bar{L}_3 \equiv \int_{-\infty}^{+\infty} d\bar{x}\, \bar{V}_{\text{ext}}(\bar{x}, \bar{t})\psi^*(\bar{x}, \bar{t})\psi(\bar{x}, \bar{t}). \tag{54}$$

The potential must be specified in order to compute this term. However, as we will show, it is possible to derive a general set of equations of motion due that is valid for *any* external potential. An important component in the derivation of the final equations of motion is that the $\bar{L}_3$ term in the Lagrangian only depend on the centers and width parameters of the N_c clouds due to our choice of trial wave function. The derivation of a general set of equations of motion is possible due to our assumption that the number of atoms in individual clouds is fixed and the fact that the external potential is only a function of position and time.

In this section we derive a general expression for $\bar{L}_3$ using our assumed trial wave function that explicitly depends only on $\mathbf{x}$ and $\mathbf{w}$. To see this we insert the expression

for the trial wave function into Eq. (54). This yields

$$\begin{aligned}
\bar{L}_3\left(\mathbf{x},\mathbf{w}\right) &= \int_{-\infty}^{+\infty} d\bar{x}\,\bar{V}_{\text{ext}}(\bar{x},\bar{t})\left(\frac{1}{\sqrt{N_c}}\sum_{j_1=1}^{N_c} A_{j_1}(\bar{t})e^{f_{j_1}(\bar{x},\bar{t})+i\bar{k}_{j_1}\bar{x}}\right)^* \\
&\times \left(\frac{1}{\sqrt{N_c}}\sum_{j_2=1}^{N_c} A_{j_2}(\bar{t})e^{f_{j_2}(\bar{x},\bar{t})+i\bar{k}_{j_2}\bar{x}}\right) \\
&= \frac{1}{N_c}\sum_{j_1=1}^{N_c} A_{j_1}^2(\bar{t})\int_{-\infty}^{+\infty} d\bar{x}\,e^{f_{j_1}^*(\bar{x},\bar{t})+f_{j_1}(\bar{x},\bar{t})}\bar{V}_{\text{ext}}(\bar{x},\bar{t}) \\
\bar{L}_3\left(\mathbf{x},\mathbf{w}\right) &= \frac{1}{N_c}\sum_{j_1=1}^{N_c} A_{j_1}^2(\bar{t})\int_{-\infty}^{+\infty} d\bar{x}\,\exp\left\{-\frac{\left(\bar{x}-\bar{x}_{j_1}\right)^2}{\bar{w}_{j_1}^2}\right\}\bar{V}_{\text{ext}}(\bar{x},\bar{t}).
\end{aligned} \tag{55}$$

Where we have used constraints 1 and 3 to simplify the integrals. This last expression shows explicitly that $\bar{L}_3$ depends only on the coordinates of the cloud centers and the cloud widths. In Section 3.5, we will find that $\bar{L}_4$ only depends on $\mathbf{x}$ and $\mathbf{w}$ as well. The fact that both $\bar{L}_3$ and $\bar{L}_4$ depend only on $\mathbf{x}$ and $\mathbf{w}$ will enable us to derive equations of motion valid for any external potential.

3.4.2 $\bar{L}_3$ for harmonic trap plus source mass

Next we derive an expression for $\bar{L}_3(\mathbf{x},\mathbf{w})$ for the particular case of a potential consisting of a 1D harmonic trap plus a point source mass. We will assume that the harmonic trap is centered at the origin of coordinates and that a point mass with mass M_{SM} is located at x_{SM} as shown in the figure below.

Thus the exact potential can thus be written (in SI units) as

$$V_{\text{ext}}(\bar{\mathbf{r}},\bar{t}) = \frac{1}{2}M\omega_{T,x}^2 x^2 - \frac{GMM_{SM}}{|x_{SM}-x|} \equiv V_{\text{H}}(x) + V_{\text{G}}(x). \tag{56}$$

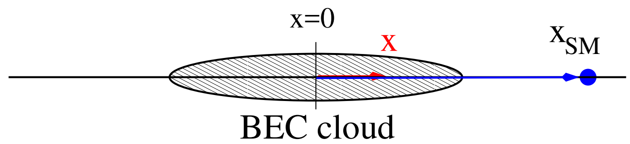

Figure 3: This figure shows schematically the position of the source mass, x_{SM}, relative to locations where condensate atoms, x, are present. We will assume that $|x| \ll |x_{SM}|$.

We want to approximate $V_{\rm G}(\mathbf{r})$ by assuming that the distance of any cloud to the origin is much smaller than the distance of the source mass to the origin. We have chosen the origin to be at the center of the harmonic potential confining the BEC.

First we consider only the gravitational part of the potential:

$$V_{\rm G}(x) = -\frac{GMM_{SM}}{|x_{SM} - x|}. \tag{57}$$

we can approximate this exact expression by making a Taylor expansion about $x = 0$ to second order in x/x_{SM}

$$V_{\rm G}(x) \approx -\frac{GMM_{SM}}{|x_{SM}|^3}\left(x_{SM}^2 + x_{SM}x + x^2\right) \tag{58}$$

This expression is valid only for points x such that $x_{SM} > |x|$ which we take to be the case since we are actually assuming $x_{SM} \gg |x|$. It will be convenient here to introduce the gravitational frequency

$$\omega_{SM} \equiv \left(\frac{GM_{SM}}{|x_{SM}|^3}\right)^{1/2}. \tag{59}$$

Thus we can rewrite the approximate gravitational potential in terms of this quantity

as follows.

$$
\begin{aligned}
V_{\mathrm{G}}(\mathbf{r}) &\approx -M\omega_{SM}^2\left(x_{SM}^2 + x_{SM}x + x^2\right) \\
\bar{V}_{\mathrm{G}}(\mathbf{r}) &\approx -\tfrac{1}{2}\bar{\omega}_{SM}^2\left(\bar{x}_{SM}^2 + \bar{x}_{SM}\bar{x} + \bar{x}^2\right),
\end{aligned} \tag{60}
$$

where, in the second line, we have expressed the gravitational potential in scaled units. Now we are ready to write down the full external potential.

The harmonic potential in scaled units can be written as

$$
\bar{V}_{\mathrm{H}}(\bar{x}) = \tfrac{1}{4}\bar{\omega}_T^2\bar{x}^2 \tag{61}
$$

where $\bar{\omega}_T$ is the frequency of the harmonic trap potential.

Adding this to the gravitation potential yields the full external potential in scaled units

$$
\begin{aligned}
\bar{V}_{\mathrm{ext}}(\bar{x}) &= \tfrac{1}{4}\bar{\omega}_T^2\bar{x}^2 - \tfrac{1}{2}\bar{\omega}_{SM}^2\left(\bar{x}_{SM}^2 + \bar{x}_{SM}\bar{x} + \bar{x}^2\right) \\
&= \tfrac{1}{4}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\bar{x}^2 - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}\bar{x} - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}^2
\end{aligned} \tag{62}
$$

With this expression we are ready to compute $\bar{L}_3$, by inserting the above expression

into Eq. (55) we have

$$
\begin{aligned}
\bar{L}_3\left(\mathbf{x},\mathbf{w}\right) &= \frac{1}{N_c}\sum_{j_1=1}^{N_c} A_{j_1}^2\left(\bar{t}\right)\int_{-\infty}^{+\infty} d\bar{x}\, e^{-\left(\bar{x}-\bar{x}_{j_1}\right)^2/\bar{w}_{j_1}^2} \\
&\times \left(\tfrac{1}{4}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\bar{x}^2 - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}\bar{x} - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}^2\right) \\
&= \frac{1}{N_c}\sum_{j_1=1}^{N_c}\left(\tfrac{1}{4}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\left(\bar{x}_{j_1}^2 + \tfrac{1}{2}\bar{w}_{j_1}^2\right) - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}\bar{x}_{j_1}\right) \\
&- \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}^2
\end{aligned}
\tag{63}
$$

In the above we have used the single–cloud normalization conditions to eliminate the $A_{j_1}^2\left(\bar{t}\right)$ factor.

Thus the final expression for $\bar{L}_3$ for the case of an external harmonic trap plus the gravitational potential produced by a point mass far away from the condensate is given by

$$
\boxed{\bar{L}_3\left(\mathbf{x},\mathbf{w}\right) = \frac{1}{N_c}\sum_{j=1}^{N_c}\left(\tfrac{1}{4}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\left(\bar{x}_j^2 + \tfrac{1}{2}\bar{w}_j^2\right) - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}\bar{x}_j\right) - \tfrac{1}{2}\bar{\omega}_{SM}^2\bar{x}_{SM}^2} \tag{64}
$$

We note again that this a particular form for $\bar{L}_3$ and that the general equations of motion are valid as long as $\bar{L}_3$ only depends on $\mathbf{x}$ and $\mathbf{w}$.

3.5 Derivation of $\bar{L}_4$

The expression for $\bar{L}_4$ is

$$
\bar{L}_4\left(\mathbf{x},\mathbf{w}\right) \equiv \frac{1}{2}\bar{g}N\int_{-\infty}^{\infty} d\bar{x}\,\left|\psi\right|^4 \tag{65}
$$

In order to perform this integral we must first calculate $|\psi|^4$. We can write this quantity as follows:

$$
\begin{aligned}
|\psi|^4 &= \left[\left(\frac{1}{\sqrt{N_c}} \sum_{j_1=1}^{N_c} A_{j_1}(\bar{t}) \exp\left\{ \left(f_{j_1}(\bar{x},\bar{t}) + i\bar{k}_{j_1}\bar{x} \right) \right\} \right)^* \right. \\
&\times \left. \left(\frac{1}{\sqrt{N_c}} \sum_{j_2=1}^{N_c} A_{j_2}(\bar{t}) \exp\left\{ \left(f_{j_2}(\bar{x},\bar{t}) + i\bar{k}_{j_2}\bar{x} \right) \right\} \right) \right]^2 \\
&= \frac{1}{N_c^2} \left[\sum_{j_1,j_2=1}^{N_c} A_{j_1} A_{j_2} \exp\left\{ f_{j_1}^* + f_{j_2} + i\left(\bar{k}_{j_2} - \bar{k}_{j_1} \right) \bar{x} \right\} \right]^2 \\
&= \frac{1}{N_c^2} \left[\sum_{j_1=1}^{N_c} A_{j_1}^2 e^{f_{j_1}^* + f_{j_1}} + \sum_{\substack{j_1,j_2=1 \\ j_1 \neq j_2}}^{N_c} A_{j_1} A_{j_2} e^{f_{j_1}^* + f_{j_2} + i\left(\bar{k}_{j_2} - \bar{k}_{j_1} \right)\bar{x}} \right]^2 \\
&\equiv \frac{1}{N_c^2} \left[T_1 + T_2 \right]^2 = \frac{1}{N_c^2} \left[T_1^2 + 2T_1T_2 + T_2^2 \right] \approx \frac{1}{N_c^2} \left[T_1^2 + T_2^2 \right]
\end{aligned}
\tag{66}
$$

In the above we have divided the double sum of the previous line into a term, T_1, where $j_1 = j_2$ and a term, T_2, where $j_1 \neq j_2$. The term T_1 therefore definitely does not oscillate rapidly with respect to $\bar{x}$ while T_2 definitely does oscillate. Thus when the square is carried out the term T_1^2 definitely does not oscillate while the $2T_1T_2$ term definitely does oscillate. The term T_2^2 has parts that oscillate and other parts that do not. Since oscillating terms will be neglected after integration we have left $2T_1T_2$ out of the final result above.

Write the above expression in terms of the sum has the following form:

$$|\psi|^4 \approx \frac{1}{N_c^2}\left[\sum_{j_1,j_2=1}^{N_c} A_{j_1}^2 A_{j_2}^2 e^{f_{j_1}^*+f_{j_1}+f_{j_2}^*+f_{j_2}} + \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} \sum_{\substack{j_1',j_2'=1\\ j_1'\neq j_2'}} A_{j_1}A_{j_2}A_{j_1'}A_{j_2'} e^{f_{j_1}^*+f_{j_2}+f_{j_1'}^*+f_{j_2'}+i\left(\bar{k}_{j_2}-\bar{k}_{j_1}+\bar{k}_{j_2'}-\bar{k}_{j_1'}\right)\bar{x}}\right] \tag{67}$$

The second term above still has oscillating terms. We only want to keep the non–oscillating terms. The terms that don't oscillate are those where $j_1' = j_2$ and $j_2' = j_1$ (since $j_1 = j_2$ and $j_1' = j_2'$ are excluded already). Thus we can evaluate the primed sums keeping only those terms where $j_1' = j_2$ and $j_2' = j_1$:

$$|\psi|^4 \approx \frac{1}{N_c^2}\left[\sum_{j_1,j_2=1}^{N_c} A_{j_1}^2 A_{j_2}^2 e^{f_{j_1}^*+f_{j_1}+f_{j_2}^*+f_{j_2}} + \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 e^{f_{j_1}^*+f_{j_2}+f_{j_2}^*+f_{j_1}}\right] \tag{68}$$

Now note that, in the last expression, the first double sum is the same as the second double sum except that it includes the term where $j_1 = j_2$. We can therefore write the first double sum as this special term plus another instance of the second term. Carrying out this procedure gives.

$$|\psi|^4 \approx \frac{1}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4 e^{2f_{j_1}^*+2f_{j_1}} + 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 e^{f_{j_1}^*+f_{j_1}+f_{j_2}^*+f_{j_2}}\right] \tag{69}$$

This expression now contains all of the non–oscillating terms found in $|\psi|^4$.

We are now in a position to write $|\psi|^4$ in terms of coordinates. The result is

$$\begin{aligned} |\Psi|^4 \quad &\approx \quad \frac{1}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4 e^{-2\left(\bar{x}-\bar{x}_{j_1}\right)^2/\bar{w}_{j_1}^2}\right. \\ &+ \quad 2\left.\sum_{\substack{j_1,j_2=1 \\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 e^{-\left(\bar{x}-\bar{x}_{j_1}\right)^2/\bar{w}_{j_1}^2-\left(\bar{x}-\bar{x}_{j_2}\right)^2/\bar{w}_{j_2}^2}\right] \end{aligned} \tag{70}$$

We now consider the exponent of the exponential appearing in the second term above. Consider the quantity

$$S\left(\bar{x}_{j_1}, \bar{w}_{j_1}, \bar{x}_{j_2}, \bar{w}_{j_2}; \bar{x}\right) \equiv \frac{\left(\bar{x}-\bar{x}_{j_1}\right)^2}{\bar{w}_{j_1}^2} + \frac{\left(\bar{x}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_2}^2}. \tag{71}$$

We can rewrite this as

$$S\left(\bar{x}_{j_1}, \bar{w}_{j_1}, \bar{x}_{j_2}, \bar{w}_{j_2}; \bar{x}\right) = \frac{\left(\bar{x}-\bar{x}_{j_1 j_2}\right)^2}{\bar{w}_{j_1 j_2}^2} + \frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2} \tag{72}$$

where

$$\bar{x}_{j_1 j_2} \equiv \frac{\bar{w}_{j_1}^2 \bar{x}_{j_2} + \bar{w}_{j_2}^2 \bar{x}_{j_1}}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2} \quad \text{and} \quad \bar{w}_{j_1 j_2} \equiv \frac{\bar{w}_{j_1}\bar{w}_{j_2}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}} \tag{73}$$

We can show this by multiplying out the squares in the definition of S and completing the square:

$$
\begin{aligned}
S\left(\bar{x}\right) &= \frac{\left(\bar{x}-\bar{x}_{j_1}\right)^2}{\bar{w}_{j_1}^2}+\frac{\left(\bar{x}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_2}^2} \\
&= \frac{\bar{x}^2-2\bar{x}_{j_1}\bar{x}+\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2}+\frac{\bar{x}^2-2\bar{x}_{j_2}\bar{x}+\bar{x}_{j_2}^2}{\bar{w}_{j_2}^2} \\
&= \left(\frac{1}{\bar{w}_{j_1}^2}+\frac{1}{\bar{w}_{j_2}^2}\right)\bar{x}^2-2\left(\frac{\bar{x}_{j_1}}{\bar{w}_{j_1}^2}+\frac{\bar{x}_{j_2}}{\bar{w}_{j_2}^2}\right)\bar{x}+\left(\frac{\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2}+\frac{\bar{x}_{j_2}^2}{\bar{w}_{j_2}^2}\right) \\
&= \left(\frac{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}\right)\bar{x}^2-2\left(\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}\right)\bar{x}+\left(\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}^2+\bar{w}_{j_2}^2\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}\right) \\
&\equiv a\bar{x}^2-2b\bar{x}+c=a\left(\bar{x}-\frac{b}{a}\right)^2+c-\frac{b^2}{a} \qquad (74)
\end{aligned}
$$

where we defined a, b, and c as

$$
a\equiv\frac{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2},\quad b\equiv\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}\quad c\equiv\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}^2+\bar{w}_{j_2}^2\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}. \qquad (75)
$$

It is important to note that, in terms of our new notation, we have

$$
\frac{b}{a}=\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}=\bar{x}_{j_1j_2}\quad\text{and}\quad a=\frac{1}{\bar{w}_{j_1j_2}^2}. \qquad (76)
$$

These will be immediately useful.

With these definitions it is easy to see how the square is completed. Now we can rewrite S by replacing the a, b, and c with their definitions:

$$
\begin{aligned}
S(\eta) &= a\left(\bar{x}-\frac{b}{a}\right)^2 + c - \frac{b^2}{a} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\bar{w}_{j_1}^2\bar{x}_{j_2}^2+\bar{w}_{j_2}^2\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2} - \left(\frac{\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}\right)^2 \\
&\times \left(\frac{\bar{w}_{j_1}^2\bar{w}_{j_2}^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right) \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\left(\bar{w}_{j_1}^2\bar{x}_{j_2}^2+\bar{w}_{j_2}^2\bar{x}_{j_1}^2\right)}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2} - \frac{\left(\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}\right)^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\left(\bar{w}_{j_1}^2\bar{x}_{j_2}^2+\bar{w}_{j_2}^2\bar{x}_{j_1}^2\right)\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)-\left(\bar{w}_{j_1}^2\bar{x}_{j_2}+\bar{w}_{j_2}^2\bar{x}_{j_1}\right)^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\bar{w}_{j_1}^4\bar{x}_{j_2}^2+\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_2}^2+\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_1}^2+\bar{w}_{j_2}^4\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&- \frac{\bar{w}_{j_1}^4\bar{x}_{j_2}^2+2\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_1}\bar{x}_{j_2}+\bar{w}_{j_2}^4\bar{x}_{j_1}^2}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_2}^2+\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_1}^2-2\bar{w}_{j_1}^2\bar{w}_{j_2}^2\bar{x}_{j_1}\bar{x}_{j_2}}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{x}_{j_2}^2+\bar{x}_{j_1}^2-2\bar{x}_{j_1}\bar{x}_{j_2}\right)}{\bar{w}_{j_1}^2\bar{w}_{j_2}^2\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)} \\
&= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\left(\bar{x}_{j_1}^2-2\bar{x}_{j_1}\bar{x}_{j_2}+\bar{x}_{j_2}^2\right)}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2} \\
S\left(\bar{x}\right) &= \frac{\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2}{\bar{w}_{j_1j_2}^2} + \frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2} \qquad (77)
\end{aligned}
$$

Thus Eq. (77) is identical to Eq. (72). This this form may be used to rewrite $|\psi|^4$ so that the integrals will be easy.

We have

$$
\begin{aligned}
|\psi|^4 \;\approx\; & \frac{1}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4\, e^{-2\left(\bar{x}-\bar{x}_{j_1}\right)^2/\bar{w}_{j_1}^2}\right. \\
+ \; & \left. 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 e^{-\left(\bar{x}-\bar{x}_{j_1}\right)^2/\bar{w}_{j_1}^2-\left(\bar{x}-\bar{x}_{j_2}\right)^2/\bar{w}_{j_2}^2}\right] \\
= \; & \frac{1}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4\, e^{-\left(\bar{x}-\bar{x}_{j_1}\right)^2/\left(\bar{w}_{j_1}/\sqrt{2}\right)^2}\right. \\
+ \; & \left. 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 e^{-\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2/\bar{w}_{j_1j_2}^2-\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2/\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)}\right]
\end{aligned}
\tag{78}
$$

Now we can insert this form (Eq. (78)) into $\bar{L}_4$; giving

$$
\begin{aligned}
\bar{L}_4 \;=\; & \frac{\left(\frac{1}{2}\bar{g}N\right)}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4 \int_{-\infty}^{+\infty} d\bar{x}\, e^{-\left(\bar{x}-\bar{x}_{j_1}\right)^2/\left(\bar{w}_{j_1}/\sqrt{2}\right)^2}\right. \\
+ \; & \left. 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 \int_{-\infty}^{+\infty} d\bar{x}\, e^{-\left(\bar{x}-\bar{x}_{j_1j_2}\right)^2/\bar{w}_{j_1j_2}^2-\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2/\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)}\right] \\
= \; & \frac{\left(\frac{1}{2}\bar{g}N\right)}{N_c^2}\left[\sum_{j_1=1}^{N_c} A_{j_1}^4 \left(\pi^{1/2}\bar{w}_{j_1}/\sqrt{2}\right)\right. \\
+ \; & \left. 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} A_{j_1}^2 A_{j_2}^2 \left(\pi^{1/2}\bar{w}_{j_1j_2}\right) e^{-\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2/\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)}\right] \\
= \; & \frac{\left(\frac{1}{2}\bar{g}N\right)}{N_c^2}\left[\sum_{j_1=1}^{N_c} \left(\frac{1}{\pi^{1/2}\bar{w}_{j_1}}\right)^2 \left(\pi^{1/2}\bar{w}_{j_1}/\sqrt{2}\right)\right. \\
+ \; & \left. 2\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}} \left(\frac{1}{\pi^{1/2}\bar{w}_{j_1}}\right)\left(\frac{1}{\pi^{1/2}\bar{w}_{j_2}}\right) \left(\pi^{1/2}\bar{w}_{j_1j_2}\right) e^{-\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2/\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)}\right]
\end{aligned}
\tag{79}
$$

Thus the expression for $\bar{L}_4$ becomes

$$\boxed{\bar{L}_4(\mathbf{x},\mathbf{w}) = \frac{\left(\frac{1}{2}\bar{g}N\right)}{(2\pi)^{1/2}N_c^2}\left[\sum_{j_1=1}^{N_c}\frac{1}{\bar{w}_{j_1}} + 2^{3/2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right]}$$

where we have used the expression for $\bar{w}_{j_1 j_2}$ defined earlier. This equation can be written more compactly as follows:

$$\bar{L}_4(\mathbf{x},\mathbf{w}) = \frac{\left(\frac{1}{2}\bar{g}N\right)}{\pi^{1/2}N_c^2}\sum_{j_1=1}^{N_c}\sum_{j_2=1}^{N_c}\frac{(2-\delta_{j_1 j_2})\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}. \tag{80}$$

It is relevant to note here that $\bar{L}_4$ depends only on $\mathbf{x}$ and $\mathbf{w}$, just as seen in $\bar{L}_3$.

3.6 The final Lagrangian

Now that we have derived all four pieces of $\bar{L}$ it is time to put it all together. The result is

$$\begin{aligned}\bar{L} &= \frac{1}{N_c}\sum_{j=1}^{N_c}\left(\dot{\bar{\alpha}}_j\bar{x}_j + \dot{\bar{\beta}}_j\left(\bar{x}_j^2+\frac{1}{2}\bar{w}_j^2\right) + \frac{1}{2\bar{w}_j^2} + 2\bar{\beta}_j^2\bar{w}_j^2 + \left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right)^2\right)\\ &+ \bar{L}_3(\mathbf{x},\mathbf{w}) + \bar{L}_4(\mathbf{x},\mathbf{w})\end{aligned} \tag{81}$$

To derive general equations of motion it will only be necessary to know that $\bar{L}_3$ and $\bar{L}_4$ depend on $\mathbf{x}$ and $\mathbf{w}$ alone. It will turn out that the final equations of motion can be cast in terms of derivatives of $\bar{L}_3$ and $\bar{L}_4$. We now turn to this derivation.

4 Derivation of the 1D, N–gaussian–cloud equations of motion

The final set of equations of motion for the parameters of the N_c–cloud, gaussian trial wave function divides naturally into two parts: (1) an equation of motion for the coordinate of the center, $\bar{x}_j$ of each cloud, and (2) an equation of motion for the width, $\bar{w}_j$ of each cloud. As we shall see, while each variational parameter, q_j, obeys an equation of motion set by the ordinary Euler–Lagrange equation:

$$\frac{d}{d\bar{t}}\left(\frac{\partial \bar{L}}{\partial \dot{q}_j}\right) - \frac{\partial \bar{L}}{\partial q_j} = 0, \tag{82}$$

it will be possible to derive second–order differential equations for both the centers and widths reminiscent of Newton's Laws of Motion. In the full set of equations of motion, only the centers and widths and their time derivatives need to be solved for. All other parameters can be written in terms of these. We begin with the derivation of the equations for the center coordinate of the Gaussian clouds.

4.1 Equations of motion for the cloud centers

4.1.1 $\bar{\alpha}_j$ Euler–Lagrange equations

To obtain the equations of motion (EOMs) for the cloud–center coordinates we need the EOMs for the $\bar{\alpha}_j$ and the $\bar{x}_j$. We begin with the EOMs for the $\bar{\alpha}_j$. The Euler–

Lagrange equations for these are

$$\frac{d}{d\bar{t}}\left(\frac{\partial \bar{L}}{\partial \dot{\bar{\alpha}}_j}\right) - \frac{\partial \bar{L}}{\partial \bar{\alpha}_j} = 0, \quad j = 1, \ldots, N_c \tag{83}$$

We can compute the derivatives of $\bar{L}$ using Eq. (81).

The derivative of $\bar{L}$ with respect to $\dot{\bar{\alpha}}_j$ is

$$\frac{\partial \bar{L}}{\partial \dot{\bar{\alpha}}_j} = \frac{1}{N_c}\bar{x}_j, \quad j = 1, \ldots, N_c. \tag{84}$$

The derivative of $\bar{L}$ with respect to $\bar{\alpha}_j$ is

$$\frac{\partial \bar{L}}{\partial \bar{\alpha}_j} = \frac{1}{N_c}\left[4\bar{\beta}_j\bar{x}_j + 2\bar{\alpha}_j + 2\bar{k}_j\right], \quad j = 1, \ldots, N_c \tag{85}$$

Thus the EOM for the $\bar{\alpha}_j$ is

$$\dot{\bar{x}}_j = 4\bar{\beta}_j\bar{x}_j + 2\bar{\alpha}_j + 2\bar{k}_j, \quad j = 1, \ldots, N_c. \tag{86}$$

4.1.2 $\bar{x}_j$ Euler–Lagrange equations

To complete the derivation of equations of motion for the cloud center coordinates we need the EOMs associated with the $\bar{x}_j$.

The $\bar{x}_j$ Euler–Lagrange EOM reads

$$\frac{d}{d\bar{t}}\left(\frac{\partial \bar{L}}{\partial \dot{\bar{x}}_j}\right) - \frac{\partial \bar{L}}{\partial \bar{x}_j} = 0, \quad j = 1, \ldots, N_c. \tag{87}$$

We can easily calculate the derivatives appearing above

$$\frac{\partial \bar{L}}{\partial \dot{\bar{x}}_j} = 0, \quad j = 1, \ldots, N_c. \tag{88}$$

This result simplifies the Euler–Lagrange EOM to

$$\frac{\partial \bar{L}}{\partial \bar{x}_j} = 0, \quad j = 1, \ldots, N_c. \tag{89}$$

Thus the derivative of $\bar{L}$ with respect to $\bar{x}_j$ will be set to zero and this has the form:

$$\begin{aligned}\frac{\partial \bar{L}}{\partial \bar{x}_j} &= \frac{1}{N_c}\Big[\dot{\alpha}_j + 2\dot{\bar{\beta}}_j \bar{x}_j + 2\left(2\bar{\beta}_j \bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right)\left(2\bar{\beta}_{jx}\right) \\ &+ N_c\left(\frac{\partial \bar{L}_3}{\partial \bar{x}_j} + \frac{\partial \bar{L}_4}{\partial \bar{x}_j}\right)\Big] = 0 \quad j = 1, \ldots, N_c.\end{aligned} \tag{90}$$

This is the Euler–Lagrange equation for $\bar{x}_j$.

4.1.3 Cloud–center equations of motion

We can derive equations of motion for the cloud–center coordinates that contain only centers and widths by differentiating the Euler–Lagrange (E–L) equation for $\bar{\alpha}_j$ with respect to time and combining these with the E–L equation for $\bar{x}_j$.

To find the EOM for $\bar{x}_j$ we differentiate Eq. (86) with respect to $\bar{t}$. Resulting in the following:

$$\ddot{\bar{x}}_j = 4\dot{\bar{\beta}}_j \bar{x}_j + 4\bar{\beta}_j \dot{\bar{x}}_j + 2\dot{\bar{\alpha}}_j. \tag{91}$$

We can eliminate the $\dot{\bar{x}}_j$ appearing on the right–hand–side of this equation by using Eq. (86):

$$\begin{aligned}\ddot{\bar{x}}_j &= 4\dot{\bar{\beta}}_j\bar{x}_j + 4\bar{\beta}_j\left(4\bar{\beta}_j\bar{x}_j + 2\bar{\alpha}_j + 2\bar{k}_j\right) + 2\dot{\bar{\alpha}}_j \\ &= 2\left[\dot{\bar{\alpha}}_j + 2\dot{\bar{\beta}}_j\bar{x}_j + 2\left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right)\left(2\bar{\beta}_j\right)\right]\end{aligned} \tag{92}$$

Now, we can write Eq. (90) as

$$\dot{\bar{\alpha}}_j + 2\dot{\bar{\beta}}_j\bar{x}_j + 2\left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right)\left(2\bar{\beta}_j\right) = -N_c\left(\frac{\partial\bar{L}_3}{\partial\bar{x}_j} + \frac{\partial\bar{L}_4}{\partial\bar{x}_j}\right) \tag{93}$$

Now note that the left–hand–side of the above equation (Eq. (93)) is identical to the term in square brackets appearing in the previous equation (Eq. (92)). Hence we can rewrite Eq. (92) in the very simple form:

$$\ddot{\bar{x}}_j = -2N_c\left(\frac{\partial\bar{L}_3}{\partial\bar{x}_j} + \frac{\partial\bar{L}_4}{\partial\bar{x}_j}\right) \tag{94}$$

Finally we introduce a "variational potential"

$$\boxed{\bar{U}(\mathbf{x},\mathbf{w}) \equiv 2N_c\bar{L}_3(\mathbf{x},\mathbf{w}) + 2N_c\bar{L}_4(\mathbf{x},\mathbf{w}) \equiv \bar{U}_{\mathrm{ext}}(\mathbf{x},\mathbf{w}) + \bar{U}_{\mathrm{int}}(\mathbf{x},\mathbf{w}).} \tag{95}$$

in terms of which we can write the equation of motion for $\bar{x}_j$. The final result is a compactly written equation valid for any potential:

$$\boxed{\ddot{\bar{x}}_j = -\frac{\partial\bar{U}}{\partial\bar{x}_j}, \quad j = 1, \ldots, N_c.} \tag{96}$$

We note that only cloud–center and cloud–width variational parameters and their time derivatives are present in these equations.

4.2 Equations of motion for the cloud widths

Next we turn to the EOMs for the cloud widths. We can also derive a second–order EOM for the widths similar to that for the centers.

4.2.1 $\bar{\beta}_j$ Euler–Lagrange equations

To derive this equation we first consider the EOMs associated with the $\bar{\beta}_j$. The Euler–Lagrange equation for this quantity is

$$\frac{d}{d\bar{t}}\left(\frac{\partial \bar{L}}{\partial \dot{\bar{\beta}}_j}\right) - \frac{\partial \bar{L}}{\partial \bar{\beta}_j} = 0, \quad j = 1, \ldots, N_c. \tag{97}$$

First we must compute the derivative of $\bar{L}$ with respect to $\dot{\bar{\beta}}_j$. The result is

$$\frac{\partial \bar{L}}{\partial \dot{\bar{\beta}}_j} = \frac{1}{N_c}\left[\bar{x}_j^2 + \tfrac{1}{2}\bar{w}_j^2\right]. \tag{98}$$

The derivative of $\bar{L}$ with respect to the $\bar{\beta}_j$ is

$$\frac{\partial \bar{L}}{\partial \bar{\beta}_j} = \frac{1}{N_c}\left[4\bar{\beta}_j\bar{w}_j^2 + 2\left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right)(2\bar{x}_j)\right]. \tag{99}$$

With these derivatives, the EOM associated with the $\bar{\beta}_j$ is

$$2\bar{x}_j\dot{\bar{x}}_j + \bar{w}_j\dot{\bar{w}}_j = 4\bar{\beta}_j\bar{w}_j^2 + (4\bar{x}_j)\left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right) \tag{100}$$

where again this equation holds for $j = 1, \ldots, N_c$.

We can simplify Eq. (100) by substituting the expression for $\dot{\bar{x}}_j$ from Eq. (86):

$$2\bar{x}_j \left(4\bar{\beta}_j\bar{x}_j + 2\bar{\alpha}_j + 2\bar{k}_j\right) + \bar{w}_j\dot{\bar{w}}_j = (4\bar{x}_j)\left(2\bar{\beta}_j\bar{x}_j + \bar{\alpha}_j + \bar{k}_j\right) + 4\bar{\beta}_j\bar{w}_j^2. \tag{101}$$

Note that all but one of the terms on each side cancel. This leaves us with a particularly simple relationship between $\bar{\beta}_j$ and $\bar{w}_j$ and $\dot{\bar{w}}_j$ which can be expressed as:

$$\dot{\bar{w}}_j = 4\bar{\beta}_j\bar{w}_j, \tag{102}$$

and we note that this equation holds for $j = 1, \ldots, N_c$. This equation will be key in deriving the equation for the widths. Note also that, if the widths and their derivatives are solved for, then the values of the $\bar{\beta}_j$ can be immediately calculated.

4.2.2 $\bar{w}_j$ Euler–Lagrange equations

Next we need the Euler–Lagrange equations of motion associated with the $\bar{w}_j$. Once we have derived these EOMs we will be ready to derive final equations of motion for the cloud widths similar to those for the cloud centers.

The Euler–Lagrange equations for the $\bar{w}_j$ are

$$\frac{d}{d\bar{t}}\left(\frac{\partial \bar{L}}{\partial \dot{\bar{w}}_j}\right) - \frac{\partial \bar{L}}{\partial \bar{w}_j} = 0. \tag{103}$$

The derivative of $\bar{L}$ with respect to the $\dot{\bar{w}}_j$ is

$$\frac{\partial \bar{L}}{\partial \dot{\bar{w}}_j} = 0. \tag{104}$$

The derivative of $\bar{L}$ with respect to the $\bar{w}_j$ is

$$\frac{\partial \bar{L}}{\partial \bar{w}_j} = \frac{1}{N_c}\left[\dot{\bar{\beta}}_j \bar{w}_j - \frac{1}{\bar{w}_j^3} + 4\bar{\beta}_j^2 \bar{w}_j + N_c \frac{\partial \bar{L}_3}{\partial \bar{w}_j} + N_c \frac{\partial \bar{L}_4}{\partial \bar{w}_j}\right]. \tag{105}$$

Thus the Euler–Lagrange equations can be written in this form

$$\dot{\bar{\beta}}_j \bar{w}_j + 4\bar{\beta}_j^2 \bar{w}_j = \frac{1}{\bar{w}_j^3} - N_c \frac{\partial \bar{L}_3}{\partial \bar{w}_j} - N_c \frac{\partial \bar{L}_4}{\partial \bar{w}_j} \tag{106a}$$

$$4\dot{\bar{\beta}}_j \bar{w}_j + 16\bar{\beta}_j^2 \bar{w}_j = \frac{4}{\bar{w}_j^3} - 2\frac{\partial \bar{U}(\mathbf{x}, \mathbf{w})}{\partial \bar{w}_j}, \tag{106b}$$

where we have written the equations in the second line in a form that will be convenient below in the derivation of the final cloud–width equations of motion. We have highlighted the left–hand–side of the second line in red for reference in the derivation below.

To obtain equations for the cloud widths analogous to those for the cloud centers, we differentiate both sides of Eq. (102) we obtain

$$\ddot{\bar{w}}_j = 4\dot{\bar{\beta}}_j \bar{w}_j + 4\bar{\beta}_j \dot{\bar{w}}_j = 4\dot{\bar{\beta}}_j \bar{w}_j + 16\bar{\beta}_j^2 \bar{w}_j \tag{107}$$

where we have used Eq. (102) to replace $\dot{\bar{w}}_j$ appearing in the first line of the above equation.

Note now that the right-hand-side of the above is the same as the red–highlighted part of Eq. (107). Thus the red terms here can be replaced with right–hand–side of Eq. (107). This yields the final evolution equations for the widths:

$$\boxed{\ddot{\bar{w}}_j = \frac{4}{\bar{w}_j^3} - 2\frac{\partial \bar{U}}{\partial \bar{w}_j}, \quad j = 1, \ldots, N_c.} \tag{108}$$

The derivatives of $\bar{U}(\mathbf{x}, \mathbf{w})$ will with respect to $\bar{x}_j$ and $\bar{w}_j$ will be given in Appendix B.

4.3 Full set of equations of motion

4.3.1 The final general equations of motion

Now we are finally able to set out the full equations of motion for all of the variational parameters. They consist of a pair of second–order ordinary differential equations for the cloud centers and widths as well as expressions for the $\bar{\beta}_{j\eta}$ and the $\bar{\alpha}_{j\eta}$ in terms of the centers, widths and their time derivatives:

$$\ddot{\bar{x}}_j = -\frac{\partial \bar{U}}{\partial \bar{x}_j}, \tag{109a}$$

$$\ddot{\bar{w}}_j = \frac{4}{\bar{w}_j^3} - 2\frac{\partial \bar{U}}{\partial \bar{w}_j}, \tag{109b}$$

$$\bar{\beta}_j = \frac{\dot{\bar{w}}_j}{4\bar{w}_j}, \tag{109c}$$

$$\bar{\alpha}_j = \tfrac{1}{2}\dot{\bar{x}}_j - 2\bar{\beta}_j\bar{x}_j - \bar{k}_j, \tag{109d}$$

$$j = 1, \ldots, N_c$$

The equations for the cloud centers and cloud widths (Eqs. (109a) and (109b)) form a closed set that contain only the $\bar{x}_j$, $\dot{\bar{x}}_j$,$\bar{w}_j$, and $\dot{\bar{w}}_j$. Once these quantities are obtained, all of the other variational parameters can be calculated. We note one more time that these equations hold for any external potential, $\bar{V}_{\text{ext}}$.

4.3.2 The final equations of motion for harmonic trap plus source mass

The equations of motion for the case when an harmonic trap with frequency ω_T is present along with a source mass of mass M_{SM} and located at x_{SM} can be found by computing the derivatives found in Eqs. (109a) and (109b). The potential for this case is derived in Appendix B and is given by Eqs. (138) and (137). The final results are the following:

$$\ddot{\bar{x}}_j + \left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\bar{x}_j = -\left(\frac{8\bar{g}N}{\pi^{1/2}N_c}\right)\sum_{j_1=1}^{N_c}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right)$$

$$\times\left(\frac{\bar{x}_{j_1}-\bar{x}_j}{\bar{w}_{j_1}^2+\bar{w}_{j\eta}^2}\right) - \bar{\omega}_{SM}^2\bar{x}_{SM}, \tag{110a}$$

$$\ddot{\bar{w}}_j + \left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\bar{w}_j = \frac{4}{\bar{w}_j^3} - \frac{(4\bar{g}N)}{(\pi)^{1/2}N_c}\sum_{j_1=1}^{N_c}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right)$$

$$\times\left(\frac{\left[2\left(\bar{x}_{j_1}-\bar{x}_j\right)^2-\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)\right]\bar{w}_j}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)\left(2-\delta_{jj_1}\right) \tag{110b}$$

$$j = 1, \ldots, N_c$$

4.3.3 Initial conditions for the equations of motion

The basic idea for establishing the initial conditions for the EOMs for the variational parameters is that a condensate is formed at $t = 0$ in a static potential. If this potential does not change, then the condensate will remain stationary thereafter. Thus the initial values of the variational parameters should correspond to a stationary solution of the EOMs. Below we describe the equations that this solution must satisfy.

Recall that the $\bar{k}_j$ are assumed known and correspond to the initial wave vectors of the N_c clouds in the system. The EOMs can't be solved without specifying initial conditions. We assume that these equations describe approximately the evolution of the condensate for the case where a condensate is formed and then split into N_c *equal–size* clouds with known wave vectors. In this case the initial positions of the cloud centers are all the same and correspond to the location of the original condensate. The initial velocities of the cloud centers are determined from the given wave vectors so that (in scaled units) we must have

$$\dot{\bar{x}}_j(0) = 2\bar{k}_j. \tag{111}$$

Now, since, from the above EOM for $\bar{\alpha}_j$ we see that

$$\dot{\bar{x}}_j(0) = 2\bar{\alpha}_j(0) + 2\bar{k}_j + 4\bar{\beta}_j(0)\bar{x}_j(0). \tag{112}$$

Now we have

$$\bar{\beta}_j(0) = \frac{\dot{\bar{w}}_j(0)}{\bar{w}_j(0)} = 0 \tag{113}$$

since we assume that, for a initially stationary condensate we will have $\dot{\bar{w}}(0) = 0$. Thus we must also have $\bar{\alpha}_j(0) = 0$.

The guiding principle is that the initial conditions for the variational parameter, at the time the condensate is initially formed, are such that the variational equations will be stationary (i.e. none of the parameters will vary in time unless conditions change). Thus, if we assume that the initial widths are denoted by $\bar{w}_j(0) = \bar{w}_j^{(0)}$, then the initial conditions for the variational parameters at the time a condensate is newly formed are the following:

$$\bar{x}_j(0) = \bar{x}_j^{(0)} \quad \dot{\bar{x}}_j(0) = 2\bar{k}_j \quad \bar{w}_j(0) = \bar{w}_j^{(0)} \quad \dot{\bar{w}}_j(0) = 0 \tag{114}$$

These initial conditions determine all of the others.

One final note regarding the initial widths. Since the variational widths should remain stationary for an initially formed condensate, the initial widths, $\bar{w}_j^{(0)}$ must satisfy the width EOM with the time derivative term set to zero. That is

$$0 = \frac{4}{\left(\bar{w}_j^{(0)}\right)^3} - 2\frac{\partial \bar{U}}{\partial \bar{w}_j}(\mathbf{x}^{(0)}, \mathbf{w}^{(0)}) \tag{115}$$

for all j.

5 Solutions to the 1-D LVM

In order to model then the behaviors of a BEC we must use the equations of motion from the LVM, Eqs. (110a) and (110b). To do so for both the initial split and the final

split, calculations must be done to find their respective final and initial conditions. The evolution through the initial split is done using the equations of motion and their initial conditions until the predetermined time t_2 of the overlap and final split. In the initial split at t_2 there are two clouds each with individual parameters that are found in the solutions to the equations of motion.

The t_2 solutions are then set equal to the initial solutions for the final split sequence at $t_2 = t_3$. This equivalence connects the two parts of the sequence, after which the final pieces also evolve as described by the equations of motion and the final split's new initial parameters. The final split is then evolved from t_3 to the end of the interrogation time. This enables us to calculate the interference pattern created by the two center clouds of the final split.

5.1 Extracting big G from the LVM solutions

We found that through varying the interrogation time and size of the source mass, we could cause a shift in the center of mass for the final overlapping condensate, as well as a change in the number of interference fringes. Through calculations of the solutions for the LVM equations of motion, we could then calculate the value of big G from these shifts and interference fringes. Using the notation and positions of BEC clouds shown in Fig. 4, we can derive an approximate expression for the center of mass shift based on Newton's 2nd Law, enabling us to extract the value of big G.

This expression begins with the use of the gravitational forces, F_+ and F_+

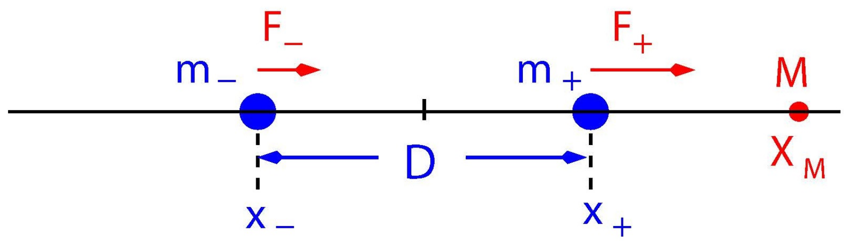

Figure 4: This figure depicts the situation where the two condensate halves, shown in blue, have separated and are experiencing different gravitational forces due to the source mass, shown in red.

$$F_+ = \frac{GMm_+}{(x_M - |x_+|)^2} \quad , \quad F_- = \frac{GMm_-}{(x_M + |x_-|)^2}, \tag{116}$$

with which we can write the accelerations for each cloud using Newton's 2nd Law of motion. The result is:

$$\ddot{x}_+ = \frac{GM}{(x_M - |x_+|)^2} \qquad and \qquad \ddot{x}_- = \frac{GM}{(x_M + |x_-|)^2}. \tag{117}$$

We can then make a Taylor expansion of the right-hand-sides above, assuming that $|x_\pm|/x_M \ll 1$, and write in terms of $\delta x = x_+ - x_-$. We obtain,

$$\ddot{x}_+ \approx \frac{GM}{{x_M}^2}\big(1 + \frac{2x_+}{x_M}\big) \qquad and \qquad \ddot{x}_- \approx \frac{GM}{{x_M}^2}\big(1 + \frac{2x_-}{x_M}\big) \tag{118}$$

Subtracting the x_- equation from the x_+ equation gives

$$\delta\ddot{x} = \frac{2GM}{{x_M}^3}\delta x \tag{119}$$

We assume this equation of motion holds true when the harmonic trap is turned off

(which we call $t = 0$). In this case we have

$$\begin{aligned} \delta x(0) &= D \\ \delta \dot{x}(0) &= 0 \end{aligned} \tag{120}$$

Using these equations, we can find a solution for δx in terms of the separation distance D, the wait time T, and the gravitational potential frequency, $\omega^2 = \frac{2GM}{{x_M}^3}$. The center of mass shift is found to be

$$\delta x(T) = D\cosh(\omega T) \tag{121}$$

This overall shift in the center of mass can be seen within the overlapping cloud solutions of the LVM as shown in Fig. 5 below. This graph provides the value of $\delta x(T)$ from which we can get the value of G.

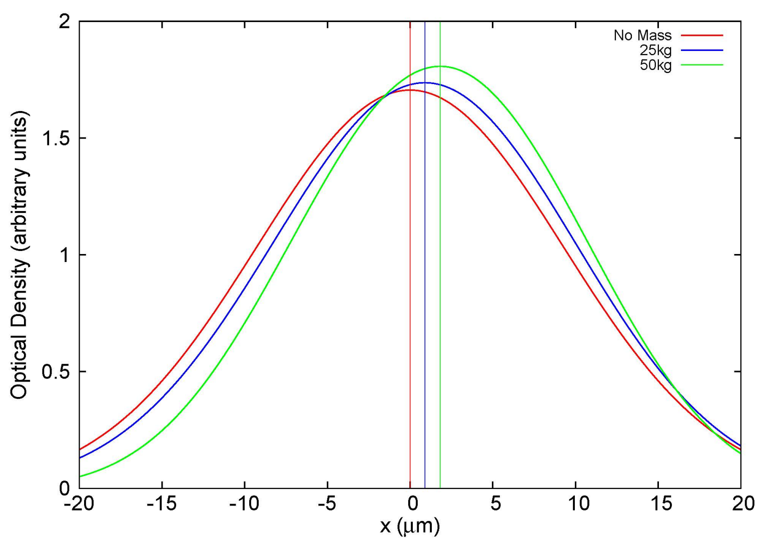

Figure 5: This figure shows the change in the center of mass for the overlapping condensate clouds as the source mass size is increased.

These variables D, T, and M also enable us to determine a threshold in which fringes from the interferometry scheme will be produced. This threshold is determined by the gradient of the phase for the overlapping clouds, $\frac{\delta\phi}{w(T)}$, and the relative velocity of each cloud, $\delta\nu$. In order to produce a minimum of one single fringe, the phase difference between the clouds must be at least 2π, and is determined using the equation:

$$\delta\nu(T, D) = \frac{\hbar}{M}\Big(\frac{\delta\phi}{w(T)}\Big). \tag{122}$$

In order to determine the relative velocity of the cloud pieces, we can examine the gravitational force of the source mass on the clouds. In order to equate $\delta\nu$ to known variables, we redefine the total relative velocity as

$$\delta\nu \equiv \frac{d}{dt}(x_+ - x_-) = D\omega\sinh(\omega T), \tag{123}$$

the time derivative of δx.

Combining now Eq. (122) and Eq. (123), we obtain the following:

$$\Big(\frac{\delta\phi}{w(T)}\Big)\frac{\hbar}{M} = D\omega\sinh(\omega T). \tag{124}$$

Using that equation, we are able to estimate values for wait time and separation distances that will not only be feasible on the CAL, but will also produce interference patterns. These equations also allow for the calculation of big G using the measured values for phase, time, and positions.

5.2 Comparison of LVM and exact GPE Solutions

In addition to determining whether or not the scheme would be possible with specific interrogation times and sizes of the condensate scheme, we also compared the approximation of the GPE solution. Shown in Fig. 6, we follow the progression of our scheme using identical parameters in both the LVM and the GPE simulations. The BEC is created within the same harmonic trap, is given an identical initial momentum kick, and has the same source mass potential during the wait time where the harmonic trap is off. We found that the LVM model matches well to the initial split of the condensate in position, and relatively close in the widths due to the assumption that the cloud takes on a Gaussian distribution.

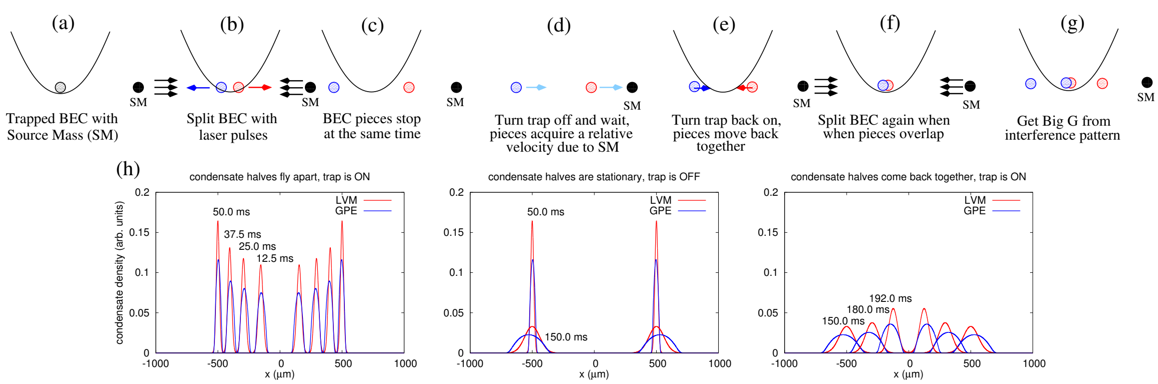

Figure 6: This figure shows the density of condensate clouds with respect to position over the time of the initial split for the LVM in red and the GPE in blue. The first graph within (h) shows the expansion of the clouds once split, the middle shows their relaxation and shift without confinement, and the final graph shows their re-confinement and movement towards the center. All figures are labeled with time stamps stating the time for the given positions of the clouds.

6 Conclusion

Through this research we have completed our two goals, producing tools for rapid evaluation of AI schemes and applying these tools to the design of a precision measurement of big G. These tools developed show that our LVM solutions are a valid approximation to the GPE, as well as an evaluating tool for proposed AI schemes on the CAL.

The evaluation tools we describe give us the ability to determine the range in parameter values necessary for the production of interference patterns, through which we can extract big G. With these solutions, we are also able to determine those parameters specific to the environment of the CAL. These specifications included the potential traps and number of condensate atoms created with the use of an atom chip, condensate interrogation times of up to twenty seconds, and the shaking of the system that occurs on the ISS. Our tools allow for the creation of schemes that take these factors into account and can provide solutions for extracting the value of big G.

References

[1] Peter J. Mohr, Barry N. Taylor, and David B. Newell. Codata recommended values of the fundamental physical constants: 2010. *Rev. Mod. Phys.*, 84:1527–1605, Nov 2012.

[2] G. Rosi, F. Sorrentino, L. Cacciapuoti, M. Prevedelli, and G. M. Tino. Precision measurement of the newtonian gravitational constant using cold atoms. *Nature*, 510:518–521, Jun 2014.

[3] T. J. Quinn, C. C. Speake, S. J. Richman, R. S. Davis, and A. Picard. A new determination of G using two methods. *Phys. Rev. Lett.*, 87:111101, Aug 2001.

[4] T. J. Quinn, C. C. Speake, S. J. Richman, R. S. Davis, and A. Picard. A new determination of G using two methods. *Phys. Rev. Lett.*, 87:111101, Aug 2001.

[5] Harold V. Parks and James E. Faller. Simple pendulum determination of the gravitational constant. *Phys. Rev. Lett.*, 105:110801, Sep 2010.

[6] Liang-Cheng Tu, Qing Li, Qing-Lan Wang, Cheng-Gang Shao, Shan-Qing Yang, Lin-Xia Liu, Qi Liu, and Jun Luo. New determination of the gravitational constant g with time-of-swing method. *Phys. Rev. D*, 82:022001, Jul 2010.

[7] St. Schlamminger, E. Holzschuh, W. Kündig, F. Nolting, R. E. Pixley, J. Schurr, and U. Straumann. Measurement of newton's gravitational constant. *Phys. Rev. D*, 74:082001, Oct 2006.

[8] T. R. Armstrong and M. P. Fitzgerald. New measurements of g using the measurement standards laboratory torsion balance. *Phys. Rev. Lett.*, 91:201101, Nov 2003.

[9] U. Kleinevoss. Bestimmung der newtonschen gravitationskonstanten g. *PhD Thesis*, 2002.

[10] Jens H. Gundlach and Stephen M. Merkowitz. Measurement of newton's constant using a torsion balance with angular acceleration feedback. *Phys. Rev. Lett.*, 85:2869–2872, Oct 2000.

[11] Charles H. Bagley and Gabriel G. Luther. Preliminary results of a determination of the newtonian constant of gravitation: A test of the kuroda hypothesis. *Phys. Rev. Lett.*, 78:3047–3050, Apr 1997.

[12] O. V. Karagioz and V. P. Izmailov. Measurement of the gravitational constant with a torsion balance. *Measurement Techniques*, 39(10):979–987, 1996.

[13] Gabriel G. Luther and William R. Towler. Redetermination of the newtonian gravitational constant g. *Phys. Rev. Lett.*, 48:121–123, Jan 1982.

[14] S. Schlamminger. Fundamental constants: A cool way to measure big g. *Nature*, 510:478–480, June 2014.

[15] W.J. Kossler. A simple cavendish experimental apparatus. *American Journal of Physics*, 84(3):234, 2016.

[16] J. M. McGuirk, G. T. Foster, J. B. Fixler, M. J. Snadden, and M. A. Kasevich. Sensitive absolute-gravity gradiometry using atom interferometry. *Phys. Rev. A*, 65(3):033608, Feb 2002.

[17] Alexander D. Cronin, Jörg Schmiedmayer, and David E. Pritchard. Optics and interferometry with atoms and molecules. *Rev. Mod. Phys.*, 81(3):1051–1129, Jul 2009.

[18] Savas Dimopoulos, Peter W. Graham, Jason M. Hogan, and Mark A. Kasevich. Testing general relativity with atom interferometry. *Phys. Rev. Lett.*, 98:111102, Mar 2007.

[19] Clare Brrage, Edmund J. Copeland, and E.A. Hinds. Probing dark energy with atom interferometry. *Journal of Cosmology and Astroparticle Physics*, 2015:042, Mar 2015.

[20] C. Pethick and H. Smith. *Bose-Einstein condensation in dilute gases*. Cambridge University Press, 2002.

[21] Louis de Broglie. Xxxv. a tentative theory of light quanta. *Philosophical Magazine Series 6*, 47(278):446–458, 1924.

[22] S.N. Bose. Plancks gesetz und lichtquantenhypothese. *Zeitschrift für Physik*, 26:178–181, 1924.

[23] A. Einstein. Quantentheorie des einatomigen idealen gases. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 3:1, 1925.

[24] MH Anderson, JR Ensher, MR Matthews, CE Wieman, and EA Cornell. Observation of bose–einstein condensation in a dilute atomic vapor. *Science*, 269(5221):198–201, JUL 14 1995.

[25] Albert Messiah. *Quantum Mechanics Volume II*. Elsevier Science B.V., 1961.

[26] L.P. Pitaevskii and S. Stringari. *Bose-Einstein Condensation*. International Series of Monographs on Physics. Clarendon Press, 2003.

[27] H. Müntinga, H. Ahlers, M. Krutzik, A. Wenzlawski, S. Arnold, D. Becker, K. Bongs, H. Dittus, O. Hellmig, W. Herr, S. Herrmann, E. Kajari, S. Kleinert, C. Lämmerzahl, W. Lewoczko-Adamczyk, J. Malcolm, N. Meyer, R. Nolte, A. Peters, M. Popp, J. Reichel, A. Roura, J. Rudolph, M. Schiemangk, M. Schneider, S.T. Seidel, K. Sengstock, V. Tamma, T. Valensuela, A. Vogel, R. Walser, T. Wendrich, P. Windpassinger, W. Zeller, T. van Zoest, W. Ertmer, W.P. Schleich, and E.M. Rasel. Interferometry with bose-einstein condensates in microgravity. *Phys. Rev. Lett.*, 110(9):093602, Feb 2013.

[28] S. T. Seidel, D. Becker, M. D. Lachmann, W. Herr, E. M. Rasel, and Quantus Collaboration. Atom Interferometry on Sounding Rockets with Bose-Einstein Condensates. In *APS Division of Atomic, Molecular and Optical Physics Meeting Abstracts*, May 2016.

[29] T. van Zoest, N. Gaaloul, Y. Singh, H. Ahlers, W. Herr, S. T. Seidel, W. Ertmer, E. Rasel, M. Eckart, E. Kajari, S. Arnold, G. Nandi, W. P. Schleich, R. Walser, A. Vogel, K. Sengstock, K. Bongs, W. Lewoczko-Adamczyk, M. Schiemangk,

T. Schuldt, A. Peters, T. Könemann, H. Müntinga, C. Lämmerzahl, H. Dittus, T. Steinmetz, T. W. Hänsch, and J. Reichel. Bose-einstein condensation in microgravity. *Science*, 328:1540–1543, Jun 2010.

Appendix A Some Useful Gaussian Integrals

This appendix derives some Gaussian integrals useful in the derivation presented in the main body of the text. Consider the following class of Gaussian integrals:

$$J_k(\eta_0, w_0) \equiv \int_{-\infty}^{\infty} \eta^k e^{-(\eta-\eta_0)^2/w_0^2} \, d\eta, \quad k = 0, 1, 2, \ldots \tag{125}$$

we can evaluate this class of integrals by changing the variable of integration:

$$x \equiv \frac{\eta - \eta_0}{w_0}, \quad \eta = \eta_0 + w_0 x, \quad d\eta = w_0 dx. \tag{126}$$

Expressed in terms of this new integration variable, the integral now has the form

$$J_k(\eta_0, w_0) = \int_{-\infty}^{\infty} (\eta_0 + w_0 x)^k \, e^{-x^2} \, w_0 dx. \tag{127}$$

We now use the binomial theorem to express the factor $(\eta_0 + w_0 x)^k$ as a series of powers of x:

$$(\eta_0 + w_0 x)^k = \sum_{s=0}^{k} \binom{k}{s} \eta_0^{k-s} \, (w_0 x)^s \,. \tag{128}$$

Inserting this into the integral in Eq. (127) gives

$$J_k(\eta_0, w_0) = w_0 \sum_{s=0}^{k} \binom{k}{s} \eta_0^{k-s} w_0^s \int_{-\infty}^{\infty} x^s e^{-x^2} \, dx. \tag{129}$$

The integral now appearing in the sum above is well–known (after all, integration is the art of transforming the integral until you can look it up!). We have

$$\int_{-\infty}^{\infty} x^s e^{-x^2}\, dx = \left\{ \begin{array}{ll} 0 & s = \text{ odd integer} \\ \left(\frac{s!}{(s/2)!}\right) \frac{\pi^{1/2}}{2^s} & s = \text{ even integer} \end{array} \right\} \tag{130}$$

Using this result we can write a final expression for the integrals:

$$J_k\left(\eta_0, w_0\right) = \left(w_0 \pi^{1/2}\right) \sum_{m=0}^{[k/2]} \binom{k}{2m} \eta_0^{k-2m} \left(\frac{w_0}{2}\right)^{2m} \frac{(2m)!}{m!}. \tag{131}$$

Where the upper limit of the sum, $[k/2]$, is the greatest integer less than or equal to $k/2$.

In the derivation of the final Lagrangian in the main body of this document, integrals of this type are found repeatedly. However, only for $k = 0, 1, 2$. Thus we present the values of these integrals for convenient reference:

$$J_0\left(\eta_0, w_0\right) = \int_{-\infty}^{\infty} e^{-(\eta-\eta_0)^2/w_0^2}\, d\eta = \left(w_0 \pi^{1/2}\right) \tag{132}$$

$$J_1\left(\eta_0, w_0\right) = \int_{-\infty}^{\infty} \eta\, e^{-(\eta-\eta_0)^2/w_0^2}\, d\eta = \left(w_0 \pi^{1/2}\right) \eta_0 \tag{133}$$

$$J_2\left(\eta_0, w_0\right) = \int_{-\infty}^{\infty} \eta^2 e^{-(\eta-\eta_0)^2/w_0^2}\, d\eta = \left(w_0 \pi^{1/2}\right) \left(\eta_0^2 + \frac{1}{2} w_0^2\right) \tag{134}$$

Appendix B Derivatives of $\bar{U}(\mathbf{x}, \mathbf{w})$

In this appendix we present the derivation of the formulas resulting from differentiating the variational potential, $\bar{U}(\mathbf{x}, \mathbf{w})$. In the first section we write down the explicit expressions for the external and interaction parts of the potential. Subsequent sections contain derivations of the center and width derivatives of these potentials needed in the final equations of motion.

B.1 The external and interaction potentials

The variational potential is composed of a part due to the external fields acting on the condensate atoms (external variational potential) and a part due to the binary scattering of condensate atoms (interaction variational potential). The general form of the varitional potential is thus

$$\bar{U}(\mathbf{x}, \mathbf{w}) = \bar{U}_{\text{ext}}(\mathbf{x}, \mathbf{w}) + \bar{U}_{\text{int}}(\mathbf{x}, \mathbf{w}) \tag{135}$$

where

$$\bar{U}_{\text{ext}}(\mathbf{x}, \mathbf{w}) \equiv 2N_c\bar{L}_3(\mathbf{x}, \mathbf{w}) \quad \text{and} \quad \bar{U}_{\text{int}}(\mathbf{x}, \mathbf{w}) \equiv 2N_c\bar{L}_4(\mathbf{x}, \mathbf{w}). \tag{136}$$

Here we will assume that the external potential consists of a harmonic trap plus a point source mass located at $\mathbf{r}_{SM}$ with mass M_{SM}.

The expression for $\bar{L}_3(\mathbf{x}, \mathbf{w})$ was derived in Section 3.4.2. Thus we can write the

form of $\bar{U}_{\text{ext}}$ using Eq. (64):

$$\bar{U}_{\text{ext}}(\mathbf{x}, \mathbf{w}) = \sum_{j_1=1}^{N_c} \left(\tfrac{1}{2} \left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2 \right) \left(\bar{x}_{j_1}^2 + \tfrac{1}{2} \bar{w}_{j_1}^2 \right) - \bar{\omega}_{SM}^2 \bar{x}_{SM} \bar{x}_{j_1} \right) - \bar{\omega}_{SM}^2 \bar{x}_{SM}^2 \quad (137)$$

The expression for $\bar{L}_4(\mathbf{x}, \mathbf{w})$ was derived in Section 3.5 and is given in Eq. (80). Using that equation we can immediately write down both forms of the variational interaction potential:

$$\begin{aligned} \bar{U}_{\text{int}}(\mathbf{x}, \mathbf{w}) &= \frac{(\bar{g}N)}{(2\pi)^{1/2} N_c} \left[\sum_{j_1=1}^{N_c} \frac{1}{\bar{w}_{j_1}} + 2^{3/2} \sum_{\substack{j_1, j_2 = 1 \\ j_1 \neq j_2}}^{N_c} \frac{\exp\left\{ -\frac{\left(\bar{x}_{j_1} - \bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2 + \bar{w}_{j_2}^2} \right\}}{\left(\bar{w}_{j_1}^2 + \bar{w}_{j_2}^2 \right)^{1/2}} \right] \\ &= \frac{(\bar{g}N)}{\pi^{1/2} N_c} \sum_{j_1=1}^{N_c} \sum_{j_2=1}^{N_c} \frac{(2 - \delta_{j_1 j_2}) \exp\left\{ -\frac{\left(\bar{x}_{j_1} - \bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2 + \bar{w}_{j_2}^2} \right\}}{\left(\bar{w}_{j_1}^2 + \bar{w}_{j_2}^2 \right)^{1/2}}. \end{aligned} \quad (138)$$

Next we compute the derivatives of these potentials.

B.2 Derivatives of the external potential

The cloud– and width–center derivatives of $\bar{U}_{\text{ext}}(\mathbf{x}, \mathbf{w})$ are derived as follows. Using Eq. (137) we have

$$\begin{aligned} \frac{\partial \bar{U}_{\text{ext}}}{\partial \bar{x}_j} &= \frac{\partial}{\partial \bar{x}_j} \left\{ \sum_{j_1=1}^{N_c} \left(\tfrac{1}{2} \left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2 \right) \left(\bar{x}_{j_1}^2 + \tfrac{1}{2} \bar{w}_{j_1}^2 \right) - \bar{\omega}_{SM}^2 \bar{x}_{SM} \bar{x}_{j_1} \right) - \bar{\omega}_{SM}^2 \bar{x}_{SM}^2 \right\} \\ &= \left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2 \right) \bar{x}_j - \bar{\omega}_{SM}^2 \bar{x}_{SM}, \quad j = 1, \ldots, N_c. \end{aligned} \quad (139)$$

The derivative of $\bar{L}_3$ with respect to the cloud width is given by

$$\begin{aligned}\frac{\partial \bar{U}_{\rm ext}}{\partial \bar{w}_j} &= \frac{\partial}{\partial \bar{w}_j}\left\{\sum_{j_1=1}^{N_c}\left(\tfrac{1}{2}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\left(\bar{x}_{j_1}^2 + \tfrac{1}{2}\bar{w}_{j_1}^2\right) - \bar{\omega}_{SM}^2 \bar{x}_{SM}\bar{x}_{j_1}\right) - \bar{\omega}_{SM}^2\bar{x}_{SM}^2\right\} \\ &= \tfrac{1}{2}\left(\bar{\omega}_T^2 - 2\bar{\omega}_{SM}^2\right)\bar{w}_j, \quad j = 1,\ldots,N_c. \end{aligned} \tag{140}$$

B.3 Derivatives of the interaction potential

Next we turn to the derivatives of the interaction potential. We begin with the derivative of $\bar{U}_{\rm int}(\mathbf{x},\mathbf{w})$ with respect to the cloud center coordinates, $\bar{x}_j$. For this derivation, it will be convenient to use the long form of $\bar{U}_{\rm int}(\mathbf{x},\mathbf{w})$:

$$\bar{U}_{\rm int}(\mathbf{x},\mathbf{w}) = \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\left[\sum_{j_1=1}^{N_c}\frac{1}{\bar{w}_{j_1}} + 2^{3/2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right] \tag{141}$$

The cloud–center derivative of the above equation is therefore

$$\begin{aligned}\frac{\partial \bar{U}_{\rm int}}{\partial \bar{x}_j} &= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\frac{\partial}{\partial \bar{x}_j}\left[\sum_{j_1=1}^{N_c}\frac{1}{\bar{w}_{j_1}} + 2^{3/2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right] \\ &= \frac{(2\bar{g}N)}{(\pi)^{1/2}N_c}\frac{\partial}{\partial \bar{x}_j}\left[\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right], \end{aligned} \tag{142}$$

In the last line we can see that there is a double sum over j_1 and j_2. Now, since $j_1 \neq j_2$ in the double sum and because the derivative only acts on j term, then only terms where either $j_1 = j$ OR $j_2 = j$ will have a non–vanishing derivative. We can

thus remove all of the terms for which j_1 and j_2 are not equal to j. This gives

$$\begin{aligned}\frac{\partial \bar{U}_{\text{int}}}{\partial \bar{x}_j} &= \frac{(2\bar{g}N)}{(\pi)^{1/2}N_c}\frac{\partial}{\partial \bar{x}_j}\left[\sum_{\substack{j_2=1\\ j_2\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_j-\bar{x}_{j_2}\right)^2}{\bar{w}_j^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_j^2+\bar{w}_{j_2}^2\right)^{1/2}}\right.\\ &+ \left.\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right],\\ &= \frac{(4\bar{g}N)}{(\pi)^{1/2}N_c}\frac{\partial}{\partial \bar{x}_j}\left[\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right].\end{aligned} \tag{143}$$

In the second equality we have simplified the first equality by noting that the two sums appearing there are identical. Note that the prefactor has been multiplied by two. Finally, having removed all of the terms whose derivatives are zero, we can carry out the final steps of the differentiation.

$$\begin{aligned}\frac{\partial \bar{U}_{\text{int}}}{\partial \bar{x}_j} &= \frac{(4\bar{g}N)}{(\pi)^{1/2}N_c}\left[\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\frac{\partial}{\partial \bar{x}_j}\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\right],\\ &= \frac{(8\bar{g}N)}{(\pi)^{1/2}N_c}\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\left\{\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)}{\bar{w}_{j_1}^2+\bar{w}_{j\eta}^2}\right\}.\end{aligned} \tag{144}$$

The final form for this derivative is

$$\frac{\partial \bar{U}_{\text{int}}}{\partial \bar{x}_j} = \left(\frac{8\bar{g}N}{\pi^{1/2}N_c}\right)\sum_{j_1=1}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\left(\frac{\bar{x}_{j_1}-\bar{x}_j}{\bar{w}_{j_1}^2+\bar{w}_{j\eta}^2}\right). \tag{145}$$

where we have included the $j_1 = j$ term because it is zero anyway.

Finally we turn to the derivative of the interaction variational potential with respect to the cloud–width parameters. We have

$$
\begin{aligned}
\frac{\partial \bar{U}_{\text{int}}}{\partial \bar{w}_j} &= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\frac{\partial}{\partial \bar{w}_j}\left[\sum_{j_1=1}^{N_c}\frac{1}{\bar{w}_{j_1}} + 2^{3/2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right], \\
&= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\left[\frac{-1}{\bar{w}_j^2} + 2^{3/2}\frac{\partial}{\partial \bar{w}_j}\left(\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j_2}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j_2}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j_2}^2\right)^{1/2}}\right)\right], \\
&= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\left[\frac{-1}{\bar{w}_j^2} + 2^{5/2}\frac{\partial}{\partial \bar{w}_j}\left(\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j}^2\right)^{1/2}}\right)\right], \\
&= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\left[\frac{-1}{\bar{w}_j^2} + 2^{5/2}\sum_{\substack{j_1=1\\ j_1\neq j}}^{N_c}\frac{\partial}{\partial \bar{w}_j}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_{j}\right)^2}{\bar{w}_{j_1}^2+\bar{w}_{j}^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_{j}^2\right)^{1/2}}\right)\right],
\end{aligned}
\tag{146}
$$

Now, separately, consider the factor in the above to be differentiated recalling that

$j_1 \neq j$:

$$\begin{aligned}
\frac{\partial}{\partial \bar{w}_j}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right) &= \frac{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}\frac{\partial}{\partial \bar{w}_j}\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)} \\
&- \frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\frac{\partial}{\partial \bar{w}_j}\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)}
\end{aligned} \tag{147}$$

Furthermore we can separately consider these inner derivatives:

$$\begin{aligned}
\frac{\partial}{\partial \bar{w}_j}\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\} &= \exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\frac{\partial}{\partial \bar{w}_j}\left(-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right) \\
&= \exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\left(\frac{2\bar{w}_j\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)
\end{aligned} \tag{148}$$

and also consider

$$\begin{aligned}
\frac{\partial}{\partial \bar{w}_j}\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2} &= \bar{w}_j\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{-1/2} \\
&= \left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}\left(\frac{\bar{w}_j\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)
\end{aligned} \tag{149}$$

Now insert Eqs. (148) and (149) into Eq. (147):

$$
\begin{aligned}
\frac{\partial}{\partial \bar{w}_j}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right) &= \frac{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\left(\frac{2\bar{w}_j\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)} \\
&- \frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}\left(\frac{\bar{w}_j\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)} \\
&= \frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\bar{w}_j\left(2\left(\bar{x}_{j_1}-\bar{x}_j\right)^2-\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2} \\
&= \frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\frac{\bar{w}_j\left(2\left(\bar{x}_{j_1}-\bar{x}_j\right)^2-\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)\right)}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}
\end{aligned} \tag{150}
$$

Note that we have written the result of differentiating the original function as a product of the original function times an extra factor.

Now we are ready to write the final result in two equivalent forms:

$$
\begin{aligned}
\frac{\partial \bar{U}_{\text{int}}}{\partial \bar{w}_j} &= \frac{(\bar{g}N)}{(2\pi)^{1/2}N_c}\left[\frac{-1}{\bar{w}_j^2}+2^{5/2}\sum_{\substack{j_1=1 \\ j_1\neq j}}^{N_c}\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right)\right. \\
&\times \left.\left(\frac{\bar{w}_j\left[2\left(\bar{x}_{j_1}-\bar{x}_j\right)^2-\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)\right]}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)\right], \\
&= \frac{(2\bar{g}N)}{(\pi)^{1/2}N_c}\sum_{j_1=1}^{N_c}\left(2-\delta_{j_1 j}\right)\left(\frac{\exp\left\{-\frac{\left(\bar{x}_{j_1}-\bar{x}_j\right)^2}{\bar{w}_{j_1}^2+\bar{w}_j^2}\right\}\bar{w}_j}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^{1/2}}\right) \\
&\times \left(\frac{\left[2\left(\bar{x}_{j_1}-\bar{x}_j\right)^2-\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)\right]\bar{w}_j}{\left(\bar{w}_{j_1}^2+\bar{w}_j^2\right)^2}\right)
\end{aligned} \tag{151}
$$