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Counting Power Domination Sets in Complete m-ary Trees

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Cover Page Footnote

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Abstract

Motivated by the question of computing the probability of successful power domination by placing k monitors uniformly at random, in this paper we give a recursive formula to count the number of power domination sets of size k in a labeled complete m-ary tree. As a corollary we show that the desired probability can be computed in exponential with linear exponent time.

1 Introduction

The study of power domination sets arises from the monitoring of electrical network using Phase Measurement Units (PMUs or monitors). This problem was first studied in terms of graphs in [4] in 2002 and has been a topic of much interest since then (see e.g. [1–3, 6, 7]). A PMU placed at a network node measures the voltage at the node and all current phasors at the node [1], and subsequently measures the voltage at some neighboring nodes using the propagation rules described in Definition 1. Since PMUs are expensive, it is desirable to find the minimum number of PMUs needed to monitor a network. This problem is known to be NP-complete even for planar bipartite graphs ([3]). Since the cost of technology typically decreases but the cost of employment increases, it is feasible that the cost of placing extra PMUs is preferred to the cost of determining the minimum number of PMUs and an optimal placement. Thus, in this paper, we begin to investigate how probable it is that a randomly placed set of k PMUs will monitor a network. Our main result is an exact formula for the number of power dominating sets of size k for the complete m-ary tree of height h. As a consequence we can compute the probability that placing PMUs on $k \geq 0$ network nodes chosen uniformly at random will monitor a network shaped as a complete m-ary tree of height h.

1.1 Terminology

Let T be a tree. A *rooted tree* is a tree in which one vertex has been designated the *root*, and denoted $r(T)$, or simply r when T is clear from context. In a rooted tree, the *parent* of a vertex v is the vertex adjacent to v on the path to the root. Since every vertex has a unique path to the root, every vertex other than the root has a unique parent. The root has no parent. A *child* of a vertex v is any vertex w for which v is the parent of w. A *descendant* of a vertex v is any vertex which is either the child of v or is, recursively, the descendant of any of the children of v.

The length of a path corresponds to the number of edges in the path. The *height* of a vertex, v, in a rooted tree is the length of the longest descending path to a leaf from that vertex, denoted $h(v)$. The height of the tree is the height of the root.

The *complete m-ary tree of height h* is the tree of height h satisfying that each internal vertex has m children. Throughout the literature this tree is also referred to as a full m-ary tree or a perfect m-ary tree. We denote by $T_{m,h}$ the complete m-ary of height h rooted at the center-most vertex, r.

For the purpose of counting power domination sets in a complete m-ary tree we will introduce a new concept. The *m-ary extended tree of height h*, denoted $T^+_{m,h}$, is formed by adding an additional vertex, r', and edge $\{r, r'\}$ to the root to the tree $T_{m,h}$. That is, $T^+_{m,h}$ has vertex set $V = V(T_{m,h}) \cup \{r'\}$ and edge set $E = E(T_{m,h}) \cup \{\{r, r'\}\}$. We refer to the added vertex r' as the *stem* of the tree.

Suppose G is a complete m-ary tree or an m-ary extended tree. If r_i is a child of the root r we let V_i be the descendants of r_i and let G_i be the induced subgraph of G on $V_i \cup \{r_i, r\}$, that is $G_i = G[V_i \cup \{r_i, r\}]$. Observe that G_i is an m-ary extended tree of height $h-1$ with root r_i and stem r. In our proofs we will label the children of r by $r_1, r_2, \dots, r_m$ and refer to G_i as the *i^{th} extended subtree of* G.

For any positive integer n, we will use $[n]$ to denote the set $\{1, 2, \dots, n\}$, and for any non-negative integers q, n with $q \leq n$, we will use $[q, n]$ to denote the set $\{q, q+1, \dots, n\}$.

1.2 Power Domination

Definition 1 (Graph Power Domination)**.** Let $G = (V, E)$ and $S \subseteq V$. Set $\mathcal{P}^0(S) = N[S]$ (the closed neighborhood of S) and for $k \geq 1$ we let $\mathcal{P}^k(S) = \mathcal{P}^{k-1}(S) \cup N^*\left(\mathcal{P}^{k-1}(S)\right)$ where

$$N^*\left(\mathcal{P}^{k-1}(S)\right) = \bigcup_{v \in \mathcal{P}^{k-1}(S)} \left\{x \in V : N_G(v) \setminus \mathcal{P}^{k-1}(S) = \{x\}\right\}$$

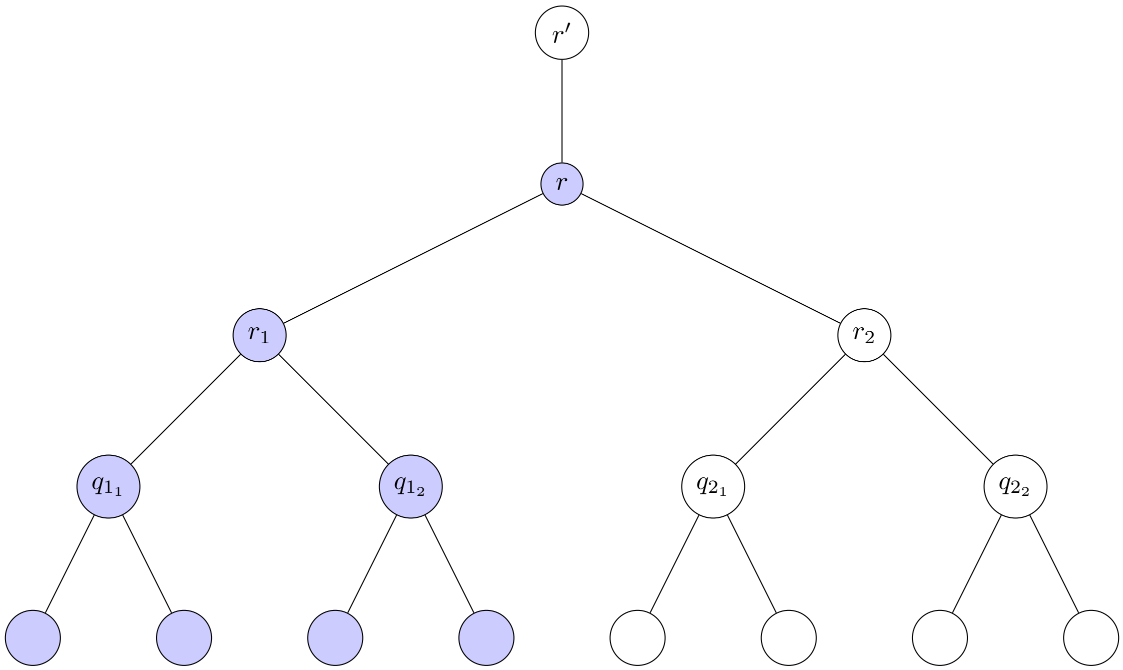


Figure 1: $T^+_{2,3}$: Binary extended tree of height 3 with highlighted 1st extended subtree of $T_{2,3}$

That is, $x \in N^*(A)$ if there exists some $a \in A$ such that x is the only neighbor of a not in A. For a finite graph we see that eventually $\mathcal{P}^k(S) = \mathcal{P}^{k+1}(S)$ and we denote this by $\mathcal{P}^\infty(S)$. If $\mathcal{P}^\infty(S) = V$ then we say that S is a *power dominating set* for G. The minimum cardinality of a power dominating set for G is referred to as the *power dominating number* of G and is denoted by $\gamma_P(G)$.

Definition 2 (Forcing Pairs)**.** Let $G = (V, E)$ and $S \subseteq V$ be given. A pair (x, y) is a *forcing pair* for S in G if for some $k \geq 0$, $x \in \mathcal{P}^k(S)$ and $N_G(x) \setminus \mathcal{P}^k(S) = \{y\}$. We denote this by $x \xrightarrow{G,S} y$, or simply by $x \to y$ when G and S are clear from context. We also say x *forces* y. The ordered set $(x_1, x_2, \dots, x_k)$ is a *forcing chain* for S in G if $x_i \xrightarrow{G,S} x_{i+1}$ for all $i \in [k-1]$. This may be denoted by $x_1 \xrightarrow{G,S} x_2 \xrightarrow{G,S} \cdots \xrightarrow{G,S} x_k$, dropping the superscripts when clear by context.

Remark 1. Note that it is possible for a vertex to be forced by more than one vertex. Let G be the graph with three vertices, x, x', and y, and two edges, namely $\{x, y\}$ and $\{x', y\}$. If $S = \{x, x'\}$, then we have both $x \xrightarrow{G,S} y$ and $x' \xrightarrow{G,S} y$.

2 Power Domination in Complete m-ary Trees

The following terminology will be central to our counting argument.

Definition 3. Let $m \geq 2$, $h \geq 0$ and $G = T^+_{m,h}$. We say $S \subseteq V(G) \setminus \{r'\}$ is

- Type I if S is a power dominating set for G;
- Type II if S is not a power dominating set for G, but $S \cup \{r'\}$ is a power dominating set for G;
- Type 0 otherwise.

We let $F^k_{m,h}$ denote the number of Type I sets of size k that exist for $T^+_{m,h}$ and $H^k_{m,h}$ denote the number of Type II sets of size k that exist for $T^+_{m,h}$.

The letter H (with super-/sub-scripts) is used to remind us that the set in question needs *help* from r' to successfully power-dominate and the letter F (with super-/sub-scripts) is used to remind us that the set in question will successfully power-dominate $G - r'$ then exit to *force* or assist in forcing r'.

Notation 1. *Let G be an m-ary extended tree, with root r, stem r', and extended subtrees $\{G_i\}_{i\in[d]}$, and let $S \subseteq V(G) \setminus \{r'\}$. We let $\textsc{Obs}(G,S)$ denote the set of vertices observed in the (attempted) power domination of G by S, that is $\textsc{Obs}(G,S) = \mathcal{P}^\infty(S)$ in the graph G. We note that this notation is needed as we will later make arguments about $\textsc{Obs}(G,S)$ by first appealing to $\textsc{Obs}(G',S)$ for some subgraph G' of G.*

Proposition 1. Let G be an m-ary extended tree and let $S \subseteq V(G) \setminus \{r'\}$. If $S^+ = S \cup \{r'\}$ is a power dominating set for G then $S_i^+ = (S \cap V(G_i)) \cup \{r\}$ is a power dominating set for G_i for all $i \in [m]$.

Proof. Choose and fix $S \subseteq V(G) \setminus \{r'\}$ and $i \in [m]$. Since the neighborhoods of the sets $\{r\}$ and $\{r, r'\}$ are the same, it follows that $\textsc{Obs}(G, S \cup \{r, r'\}) = \textsc{Obs}(G, S \cup \{r\})$.

Assume, by way of contradiction, that $\textsc{Obs}(G, S^+) = V(G)$ but $\textsc{Obs}(G_i, S_i^+) \neq V(G_i)$. Choose and fix $x \in V(G_i) \setminus \textsc{Obs}(G_i, S_i^+)$ such that the distance from x to r is minimized. In particular the internal vertices of the path from r to x are all contained within $\textsc{Obs}(G_i, S_i^+)$. Since $x \in \textsc{Obs}(G, S^+)$ there is some $y \in V(G)$ such that $y \xrightarrow{G,S^+} x$. Now $x \notin \{r, r_i\}$ since $r \in S_i^+$ and $r_i \in N_{G_i}(r)$. It follows that $y \in V(G_i) \setminus \{r\}$.

It must be the case that y is either a child or the parent of x. We will show that y is the parent of x. Assume to the contrary that y is a child of x. Note that in G_i, with initial set S_i^+, the vertex y doesn't force x. However, we do have that $y \xrightarrow{G,S^+} x$, and so there is a forcing chain $C = (v_1, v_2, \dots, v_j, y)$ that starts outside of $V(G_i)$ and ends at y that allows $y \xrightarrow{G,S^+} x$. Since each of these paths must go through x we have that $v_i = x$ for some $i \leq j$ and therefore $x \xrightarrow{G,S^+} y$. This would contradict that $y \xrightarrow{G,S^+} x$ so we must proceed under the assumption that y is the parent of x.

By assumption, all of the ancestors of x in G_i are contained within $\textsc{Obs}(G_i, S_i^+)$. Since $x \notin \textsc{Obs}(G_i, S_i^+)$, it follows that y must have a second child, x', satisfying $x' \in V(G_i) \setminus \textsc{Obs}(G_i, S_i^+)$. Hence $y \xrightarrow{G,S^+} x$ only after x' is observed. The only way this can occur is if a child of x' forces x'; however, the argument that x must be forced by a parent and not a child could also be made for x'. This is a contradiction. □

Lemma 1. Let $h, \ell, k, m \in \mathbb{Z}$, where $m \geq 2$, $h \geq 0$, and $G = T^+_{m,h+1}$. Then

$$\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_\ell \\ i_{\ell+1} \leq i_{\ell+2} \leq \cdots \leq i_m \\ i_1 + i_2 + \cdots + i_m = k}} \binom{m}{\ell} \binom{\ell}{s_0, s_1, \dots, s_k} \binom{m-\ell}{t_0, t_1, \dots, t_k} \left(\prod_{1 \leq j \leq \ell} H^{i_j}_{m,h} \right) \left(\prod_{\ell < j \leq m} F^{i_j}_{m,h} \right)$$

where $s_\alpha = |\{j \in [\ell] : i_j = \alpha\}|$ and $t_\beta = |\{j \in [\ell+1, m] : i_j = \beta\}|$ counts the number of ways to select a set $S \subseteq V(G) \setminus \{r, r'\}$ of size k so that $S \cap G_i$ is Type II for exactly ℓ of the indices $i \in [m]$ and $S \cap G_i$ is Type I for the other $m - \ell$ the indices.

Proof. Let $h, \ell, k, m \in \mathbb{Z}$, where $m \geq 2$, $h \geq 0$, and $G = T^+_{m,h+1}$. Observe that when $\ell \notin [0, m]$ the sum will equal zero, leaving nothing to be shown. We will proceed assuming $0 \leq \ell \leq m$. Choose and fix non-negative integers $i_1, i_2, \dots, i_m$ so that we have $i_1 \leq i_2 \leq \cdots \leq i_\ell$, $i_{\ell+1} \leq i_{\ell+2} \cdots \leq i_m$, and $\sum_{j \in [m]} i_j = k$.

Let $\mathcal{M} = \{G_1, G_2, \dots, G_m\}$ be the set of extended subtrees of G. Choose a subset of $\mathcal{A} \subseteq \mathcal{M}$ so that $|\mathcal{A}| = \ell$ and let $\mathcal{B} = \mathcal{M} \setminus \mathcal{A}$ and observe that $|\mathcal{B}| = m - \ell$. This selection can be done in $\binom{m}{\ell}$ ways. Make a function f from $\mathcal{A} \cup \mathcal{B}$ into multiset $\{i_1, i_2, \dots, i_m\}$ so that f maps $\mathcal{A}$ into the multisets $\{i_1, \dots, i_\ell\}$ and $\mathcal{B}$ into the multiset $\{i_{\ell+1}, \dots, i_m\}$, this function can be created in

$$\binom{\ell}{s_0, s_1, \dots, s_k} \cdot \binom{m-\ell}{t_0, t_1, \dots, t_k}$$

ways where $s_\alpha = |\{j \in [\ell] : i_j = \alpha\}|$ and $t_\beta = |\{j \in [\ell+1, m] : i_j = \beta\}|$. Let $\{A_1, A_2, \dots, A_\ell\} = \mathcal{A}$ and $\{B_{\ell+1}, \dots, B_m\} = \mathcal{B}$ so that $f(A_j) = i_j$ and $f(B_{j'}) = i_{j'}$ for each $j \in [\ell]$ and $j' \in [\ell+1, m]$. Then $H^{i_j}_{m,h}$ counts the number of ways to select i_j vertices from A_j so that the i_j vertices are of Type II in A_j. Similarly,

$F^{i_j}_{m,h}$ counts the number of ways to select i_j vertices from B_j so that the i_j vertices are of Type I in B_j. Hence, for the fixed selection of $\mathcal{A}$ and $\mathcal{B}$, and for the fixed function f,

$$\left(\prod_{1\le j\le \ell} H^{i_j}_{m,h}\right)\left(\prod_{\ell<j\le m} F^{i_j}_{m,h}\right)$$

counts the number of ways that a set S of k vertices can be selected from $V(G)\setminus\{r,r'\}$ so that $S\cap V(A_j)$ is Type II and $S\cap V(B_{j'})$ is Type I for each $j\in[\ell]$ and $j'\in[\ell+1,m]$. Hence, when we account for the number of options we had for selecting $\mathcal{A}$ (and hence $\mathcal{B}$), and for the possible assignments of function f, the number of choices is

$$\binom{m}{\ell}\binom{\ell}{s_0,s_1,\dots,s_k}\binom{m-\ell}{t_0,t_1,\dots,t_k}\left(\prod_{1\le j\le \ell} H^{i_j}_{m,h}\right)\left(\prod_{\ell<j\le m} F^{i_j}_{m,h}\right).$$

Thus by summing over all choices of $(i_1,i_2,\dots,i_m)$ restricted to $i_1\le i_2\le\cdots\le i_\ell$, $i_{\ell+1}\le i_{\ell+2}\cdots\le i_m$, and $\sum_{j\in[m]} i_j=k$ we count the number of ways to select a set $S\subseteq V(G)\setminus\{r,r'\}$ of size k so that $S\cap G_i$ is Type II for exactly ℓ of the indices $i\in[m]$ and $S\cap G_i$ is Type I for the other $m-\ell$ the indices. This yields the desired result. □

Notation 2. *We use the expression from Lemma 1 repeated throughout the remainder of the paper so we set* $\langle \mathcal{H}^\ell\oplus\mathcal{F}^{m-\ell}\rangle^k_{m,h}=$

$$\sum_{\substack{i_1\le i_2\le\cdots\le i_\ell\\ i_{\ell+1}\le i_{\ell+2}\le\cdots\le i_m\\ i_1+i_2+\cdots+i_m=k}}\binom{m}{\ell}\binom{\ell}{s_0,s_1,\dots,s_k}\binom{m-\ell}{t_0,t_1,\dots,t_k}\left(\prod_{1\le j\le \ell} H^{i_j}_{m,h}\right)\left(\prod_{\ell<j\le m} F^{i_j}_{m,h}\right)$$

where $s_\alpha=|\{j\in[\ell]: i_j=\alpha\}|$ *and* $t_\beta=|\{j\in[\ell+1,m]: i_j=\beta\}|$.

Observation 1. For all $m\ge 2$:

$$H^k_{m,0}=\begin{cases}1, & k=0\\ 0, & \text{otherwise}\end{cases}\quad\text{and}\quad F^k_{m,0}=\begin{cases}1, & k=1\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let $G=T^+_{m,0}$. Since $V(G)\setminus\{r'\}=\{r\}$ we only have two sets to consider: $S_1=\{r\}$ and $S_2=\emptyset$. For S_1, we have $k=1$, and $\textsc{Obs}(G,S_1)=V(G)$, yielding the values $F^1_{m,0}=1$ and $H^1_{m,0}=0$. For S_2, we have $k=0$, $\textsc{Obs}(G,S_2)=\emptyset$, and $\textsc{Obs}(G,S_2\cup\{r'\})=V(G)$, yielding the values $H^0_{m,0}=1$ and $F^0_{m,0}=0$. □

Lemma 2. For all $m\ge 2$:

$$H^k_{m,1}=\begin{cases}m, & k=m-1\\ 0, & \text{otherwise}\end{cases}\quad\text{and}\quad F^k_{m,1}=\begin{cases}m+1, & k=m\\ \binom{m}{k-1}, & \text{otherwise.}\end{cases}$$

Proof. Let $G=T^+_{m,1}$. Label the vertices so that the parent of r is r' and the children of r are $\{r_1,\dots,r_m\}$. Then $G=(\{r',r,r_1,\dots,r_m\},\{rr',rr_1,\dots,rr_m\})$. Fix $k\in\mathbb{Z}$. Observe that if $k\notin[0,m+1]$ then $H^k_{m,1}=F^k_{m,1}=0$ since it would be an impossible selection. As this agrees with our claim, we will proceed under the assumption that $k\in[0,m+1]$. Let $S\subseteq V(G)\setminus\{r'\}$ of size k. Assume first that $r\in S$; there are $\binom{m}{k-1}$ such sets. Since $N[S]\supseteq N[\{r\}]=V(G)$ we may conclude that S is a Type I set. If instead, we have $r\notin S$, then necessarily $k\in[0,m]$. If $0\le k\le m-2$ then at least two children of r are not in S; call them r_i and r_j. Then $N[S]\subseteq N[S\cup\{r'\}]\subseteq V(G)\setminus\{r_i,r_j\}$ and therefore r cannot force either vertex even if $\{r'\}$ is added to S. So in this case S is Type 0. If $k=m-1$ then $S=\{r_1,r_2,\dots,r_m\}\setminus\{r_i\}$ for some i; there are $\binom{m}{m-1}=m$ such sets. Here, $N[S]=V(G)\setminus\{r',r_i\}$ and therefore r cannot force either vertex. However, $N[S\cup\{r'\}]=V(G)\setminus\{r_i\}$ so $r\xrightarrow{G,S\cup\{r'\}} r_i$ and therefore S is Type II. Finally, if $k=m$ (and $r\notin S$) then there is only $\binom{m}{m}=1$ choice for S, namely $S=V(G)\setminus\{r',r\}$. In this case $N[S]=V(G)\setminus\{r'\}$ and r forces r' so it is a Type I set. The result follows. □

Corollary 1. Let $m \geq 2$ and $G = T^+_{m,1}$. If S is a Type II set then

$$N_G[r] \setminus \textsc{Obs}(G,S) = \{r', r_i\} \text{ for some } i \in [m].$$

Furthermore,

$$H^k_{m,1} = \left\langle \mathcal{H}^1 \oplus \mathcal{F}^{m-1} \right\rangle^k_{m,0}$$

and

$$F^k_{m,1} = \left\langle \mathcal{H}^0 \oplus \mathcal{F}^{m} \right\rangle^k_{m,0} + \sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,0}.$$

Proof. The only case where S is Type II is when $S = \{r_1, r_2, \ldots, r_m\} \setminus \{r_i\}$ for some $i \in [m]$. It follows from the proof that if S is a Type II set then

$$N_G[r] \setminus \textsc{Obs}(G,S) = \{r', r_i\} \text{ for some } i \in [m].$$

Observe that $V(G) \setminus \{r, r'\} = \{r_1, r_2, \ldots, r_m\}$. Now $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^X_{m,0}$ counts the number of ways to select a set $S \subseteq V(G) \setminus \{r, r'\}$ of size X so that $S \cap G_i$ is Type II for exactly ℓ of the indices $i \in [m]$ and $S \cap G_i$ is Type I for the other $m - \ell$ the indices.

In this context $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^X_{m,0}$ counts the number of ways to exactly ℓ of the children of r to be in the set S and therefore $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^X_{m,0} = 0$ unless $X = m - \ell$ in which case $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^X_{m,0} = \binom{m}{m-\ell}$.

It follows that

$$\left\langle \mathcal{H}^1 \oplus \mathcal{F}^{m-1} \right\rangle^k_{m,0} = \begin{cases} \binom{m}{m-1}, & k = m-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\left\langle \mathcal{H}^0 \oplus \mathcal{F}^{m} \right\rangle^k_{m,0} = \begin{cases} \binom{m}{m}, & k = m \\ 0, & \text{otherwise} \end{cases}$$

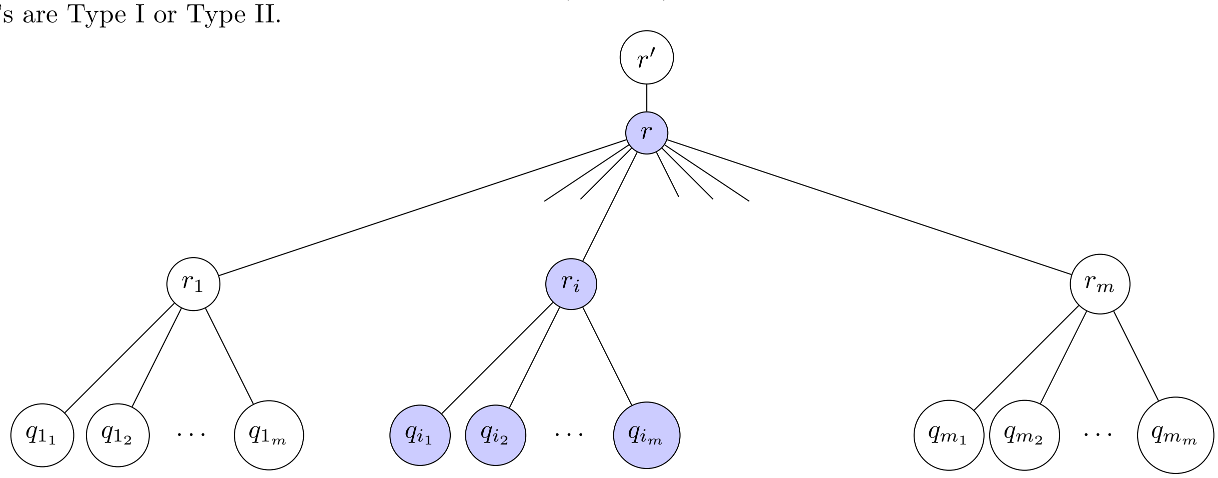
$$\sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,0} = \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{k-1} \right\rangle^{k-1}_{m,0} = \binom{m}{k-1}.$$

□

Lemma 3. For all $m \geq 2$,

$$H^k_{m,2} = \left\langle \mathcal{H}^m \oplus \mathcal{F}^0 \right\rangle^k_{m,1} \quad \text{and} \quad F^k_{m,2} = \sum_{\ell=0}^{m-1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^k_{m,1} + \sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,1}.$$

Proof. Let $G = T^+_{m,2}$ with root r and stem r'. Label the children of r by $r_1, \ldots, r_m$ and for $i \in [m]$ label the children of r_i by $q_{i_1}, \ldots, q_{i_m}$. The i^{th} extended subtree of G is G_i with vertex set $\{r, r_i, q_{i_1}, \ldots, q_{i_m}\}$ (see figure below). Let $S \subseteq V(G) \setminus \{r'\}$ and for each $i \in [m]$ let $S_i = (S \cap V(G_i)) \setminus \{r\}$. By Proposition 1 if S_i is Type 0 for any i then S does not contribute to $H^k_{m,2}$ or $F^k_{m,2}$, thus we only consider the cases where all the S_i's are Type I or Type II.



Case 1: Suppose $r \notin S$ and S_i is Type II for G_i for each $i \in [m]$. By Lemma 1, with $\ell = m$ and $h = 1$, there are $\left\langle \mathcal{H}^m \oplus \mathcal{F}^0 \right\rangle_{m,1}^k$ ways to choose S. It follows from the proof of Lemma 2 that for each $i \in [m]$, $S_i = \{q_{i_1}, q_{i_2}, \ldots, q_{i_m}\} \setminus \{q_{i_{j(i)}}\}$ for some $j(i) \in [m]$. Therefore

$$S = \bigcup_{i \in [m]} \left(\{q_{i_1}, q_{i_2}, \ldots, q_{i_m}\} \setminus \{q_{i_{j(i)}}\} \right)$$

which in turn implies that $r \notin \textsc{Obs}(G, S)$ since for any $i \in [m]$, r_i cannot force both q_{i_j} and r. On the other hand, since $\{r, r'\} \subseteq N_G[r] = \{r', r, r_1, \ldots, r_m\}$ we have that $r_i \xrightarrow{G, S \cup \{r\}} q_{i_{j(i)}}$ for each $i \in [m]$ so that $S \cup \{r\}$ is a dominating set for G. We conclude that whenever $r \notin S$ and S_i is Type II for S_i for each $i \in [m]$, then S is Type II. Moreover, there are $\langle \mathcal{H}^m \oplus \mathcal{F}^0 \rangle_{m,1}^k$ such sets, S.

Case 2: Suppose $r \notin S$ and for some $\ell \in [0, m-1]$, we have S_i is Type II for the corresponding G_i for ℓ of the sets G_i and the remaining $m - \ell$ sets S_i are Type I for their corresponding G_i. Observe that by Lemma 1 there are $\sum_{\ell=0}^{m-1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,1}^k$ such ways to choose S. The restriction of S to the extended subtrees of G results in ℓ Type II sets and $m - \ell > 0$ Type I sets. Without loss of generality, we may assume S_i is Type II for G_i for all $i \in [\ell]$ and S_j is Type I for G_j for all $j \in [\ell+1, m]$. Appealing to the proof of Lemma 2 (as we did in Case 1) we may conclude that

$$\begin{aligned}
\textsc{Obs}(G, S) &\supseteq \bigcup_{i \in [m]} \textsc{Obs}(G, S_i) \\
&= \left(\bigcup_{i \in [\ell]} \textsc{Obs}(G, S_i) \right) \cup \left(\bigcup_{i \in [\ell+1, m]} \textsc{Obs}(G, S_i) \right) \\
&= \left(\bigcup_{i \in [\ell]} \{r_i, q_{i_1}, q_{i_2}, \ldots, q_{i_m}\} \setminus \{q_{i_{j(i)}}\} \right) \cup \left(\bigcup_{i \in [\ell+1, m]} V(G_i) \right).
\end{aligned}$$

It then follows that $r_m \xrightarrow{G,S} r \xrightarrow{G,S} r'$ and then for $i \in [\ell]$, $r_i \xrightarrow{G,S} q_{i_{j(i)}}$. Thus, these $\sum_{\ell=0}^{m-1} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle_{m,1}^k$ sets are all Type I.

Case 3: Suppose $r \in S$ and for some $\ell \in [0, m]$, we have S_i is Type II for G_i for ℓ of the sets G_i and the remaining $m - \ell$ sets S_i are Type I for their corresponding G_i. Without loss of generality, we may assume S_i is Type II for G_i for all $i \in [\ell]$ and S_j is Type I for G_j for all $j \in [\ell+1, m]$. By Lemma 1, and noting that there are $k - 1$ vertices in S other than r, there are $\sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,1}^{k-1}$ ways to choose S. Again by appealing to the proof of Lemma 2 we have that

$$\begin{aligned}
\textsc{Obs}(G, S) &\supseteq N_G[r] \cup \left(\bigcup_{i \in [m]} \textsc{Obs}(G, S_i) \right) \\
&\supseteq N_G[r] \cup \left(\bigcup_{i \in [m]} \{q_{i_1}, q_{i_2}, \ldots, q_{i_m}\} \setminus \{q_{i_{j(i)}}\} \right) \\
&= V(G) \setminus \left(\bigcup_{i \in [m]} \{q_{i_{j(i)}}\} \right).
\end{aligned}$$

It then follows that $r_i \xrightarrow{G,S} q_{i_{j(i)}}$ for each $i \in [m]$. We conclude that these $\sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,1}^{k-1}$ sets are all Type I.

□

Corollary 2. Let $m \geq 2$ and $G = T_{m,2}^+$. If S is a Type II set then $N_G[r] \setminus \textsc{Obs}(G, S) = \{r', r\}$.

In the proof of Lemma 3, we twice claimed that *without loss of generality* we may assume that if ℓ of the subtrees of T^+ are Type II then we may assume they are the first ℓ trees. We will use this claim in the rest paper; in particular in the proofs of Lemma 4 and Theorem 1.

Lemma 4. Let $m \geq 2$, $h \geq 1$, and $G = T^+_{m,h}$. If S is a Type II set then

$$N_G[r] \setminus \textsc{Obs}(G,S) = \begin{cases} \{r', r_i\} \text{ for some } i \in [m], & \text{if } h \text{ is odd} \\ \{r', r\}, & \text{if } h \text{ is even} \end{cases}.$$

Furthermore,

$$H^k_{m,h} = \begin{cases} \left\langle \mathcal{H}^1 \oplus \mathcal{F}^{m-1} \right\rangle^k_{m,h-1}, & \text{if } h \text{ is odd} \\ \left\langle \mathcal{H}^m \oplus \mathcal{F}^0 \right\rangle^k_{m,h-1}, & \text{if } h \text{ is even} \end{cases}$$

and

$$F^k_{m,h} = \begin{cases} \left\langle \mathcal{H}^0 \oplus \mathcal{F}^m \right\rangle^k_{m,h-1} + \sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,h-1}, & \text{if } h \text{ is odd} \\ \sum_{\ell=0}^{m-1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^k_{m,h-1} + \sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,h-1}, & \text{if } h \text{ is even} \end{cases}.$$

Proof. The statement is true when $h \in \{1, 2\}$ by the corollaries to Lemma 2 and 3. We proceed by strong induction on h by assuming that the claim is true for $h \in [n]$. Let $G = T^+_{m,h}$ and let $S \subseteq V(G) \setminus \{r'\}$ be a Type II set.

Case 1: Suppose $h = n+1$ is odd. By Proposition 1, if for any $i \in [m]$ it is the case that S_i is Type 0 for G_i, then S is Type 0 for G. Since S is presumed to be a Type II set, we may proceed under the assumption S_i is Type II for the first ℓ extended subtrees and Type I for the latter $m - \ell$ extended subtrees. By inductive hypothesis $N_G[r_i] \setminus \textsc{Obs}(G_i, S_i) = \{r, r_i\}$ for each $i \in [\ell]$.

Subcase 1: Let $\ell = 0$ and suppose $r \notin S$. Since S_i is Type I for each $i \in [m]$ we have

$$\textsc{Obs}(G,S) \supseteq \bigcup_{i \in [m]} \textsc{Obs}(G_i, S_i) = \bigcup_{i \in [m]} V(G_i) = V(G) \setminus \{r'\}.$$

However $r \xrightarrow{G,S} r'$ so the $\left\langle \mathcal{H}^0 \oplus \mathcal{F}^m \right\rangle^k_{m,h-1}$ sets in question are Type I. This gives the first summand for $F^k_{m,h}$ in the case that h is odd.

Subcase 2: Let $\ell = 1$ and suppose $r \notin S$. By inductive hypothesis $r, r_1 \notin \textsc{Obs}(G_1, S_1)$ and therefore $r', r_1 \notin \bigcup_{i \in [m]} \textsc{Obs}(G_i, S_i)$. Since r cannot force both r' and r_1, it follows that S is not Type I. Observe that $N_G[r] \setminus \textsc{Obs}(G,S) = \{r', r_1\}$ which agrees with the first claim in the case where h is odd. We show that the sets S in this case are Type II. Assume, by way of contradiction, that there is some vertex $v \in V(G)$ with $v \notin \textsc{Obs}(G, S \cup \{r'\})$. Since $S_2, \ldots, S_m$ are Type I then $v \in V(G) \setminus \bigcup_{i \in [2,m]} V(G_i)$. Clearly $r' \in \textsc{Obs}(G, S \cup \{r'\})$ so $v \in V(G_1)$. Now $r' \xrightarrow{G, S \cup \{r'\}} r \xrightarrow{G, S \cup \{r'\}} r_1$ so $v \in V(G_1) \setminus \{r, r_1\}$. However, S_1 is a Type II set so $r \xrightarrow{G_1, S_1} r_1 \xrightarrow{G_1, S_1} \cdots \xrightarrow{G_1, S_1} v$. But then $r \xrightarrow{G, S \cup \{r'\}} r_1 \xrightarrow{G, S \cup \{r'\}} \cdots \xrightarrow{G, S \cup \{r'\}} v$ as well. It follows that the $\left\langle \mathcal{H}^1 \oplus \mathcal{F}^{m-1} \right\rangle^k_{m,h-1}$ sets in question are Type II. This gives the result for $H^k_{m,h}$ when h is odd.

Subcase 3: Let $2 \leq \ell \leq m$ and suppose $r \notin S$. By inductive hypothesis $r_i \notin \text{Obs}(G_i, S_i)$ for all $i \in [\ell]$ and, since $\ell \geq 2$, r cannot force all of the vertices r_i with $i \in [\ell]$. Thus $\textsc{Obs}(G,S) \subseteq V(G) \setminus \{r', r, r_1, \ldots, r_\ell\}$ so S is not Type I. Assume, by way of contradiction, that S is Type II. Then for each $v \in V(G)$ there is a chain

$$r' \xrightarrow{G, S \cup \{r'\}} r \xrightarrow{G, S \cup \{r'\}} r_i \xrightarrow{G, S \cup \{r'\}} \cdots \xrightarrow{G, S \cup \{r'\}} v.$$

However, this is impossible since r can only force a child once all of the other children have been observed. The sets in question are Type 0.

Subcase 4: Let $0 \leq \ell \leq m$ and suppose $r \in S$. Note that

$$\textsc{Obs}(G,S) \supseteq N_G[r] \cup \bigcup_{i \in [m]} \textsc{Obs}(G_i, S_i \cup \{r\}) = N_G[r] \cup \bigcup_{i \in [m]} V(G_i) = V(G).$$

It follows that the $\sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle^{k-1}_{m,h-1}$ sets in question are Type I. This gives the second summand for $F^k_{m,h}$ in the case where h is odd.

Case 2: Suppose $h = n+1$ is even. By Proposition 1, if for any $i \in [m]$ it is the case that S_i is Type 0 (for G_i), then S is Type 0 for G. Assuming this is not the case, we may proceed under the assumption S_i is Type II for the first ℓ extended subtrees and Type I for the latter $m-\ell$ extended subtrees. By inductive hypothesis, for each $i \in [\ell]$, there is some $j(i) \in [m]$ such that $N_G[r_i] \setminus \textsc{Obs}(G_i, S_i) = \{r, q_{i_{j(i)}}\}$. Without loss of generality we will assume that $N_G[r_i] \setminus \textsc{Obs}(G_i, S_i) = \{r, q_{i_1}\}$ for each $i \in [\ell]$.

Subcase 1: Let $\ell = m$ and suppose $r \notin S$. Since S_i is Type II for each $i \in [m]$ we have $r, q_{i_1} \notin \textsc{Obs}(G_i, S_i)$. Observe that this implies that $r \notin \textsc{Obs}(G, S)$ since otherwise $r_i \xrightarrow{G,S} r$ implies $r_i \xrightarrow{G_i,S_i} r$ for some $i \in [m]$. Thus the sets in question are not Type I. Observe that $N_G[r] \setminus \textsc{Obs}(G, S) = \{r, r'\}$ which agrees with the first claim in the case where h is even. For each $i \in [m]$, $\textsc{Obs}(G_i, S_i \cup \{r\}) = V(G_i)$; however, $N_G[r] \setminus (\cup_{i\in[m]} \textsc{Obs}(G_i, S_i)) = \{r', r\}$ since by inductive hypothesis $N_{G_i}[r] \setminus \textsc{Obs}(G_i, S_i) = \{r\}$. It follows that $V(G_i) = \textsc{Obs}(G_i, S_i \cup \{r\}) = \textsc{Obs}(G_i, S_i \cup \{r'\})$ and therefore

$$\begin{aligned}\textsc{Obs}(G, S \cup \{r'\}) &\supseteq N_G[r'] \cup \left(\bigcup_{i\in[m]} \textsc{Obs}(G_i, S_i \cup \{r'\}) \right) \\ &= N_G[r] \cup \left(\bigcup_{i\in[m]} V(G_i) \right) \\ &= V(G).\end{aligned}$$

It follows that the $\langle \mathcal{H}^m \oplus \mathcal{F}^0 \rangle^k_{m,h-1}$ sets in question are Type II. This gives the result for $H^k_{m,h}$ in the case that h is even.

Subcase 2: Let $0 \le \ell \le m-1$ and suppose $r \notin S$. By inductive hypothesis $N_{G_i}(r_i) \setminus \textsc{Obs}(G_i, S_i) = \{r, q_{i_1}\}$ for each $i \in [\ell]$. Importantly $\{r_1, r_2, \ldots, r_m\} \subset \textsc{Obs}(G, S)$ and since S_m is Type I, then $r_m \xrightarrow{G_m,S_m} r$. Therefore $r_m \xrightarrow{G,S} r \xrightarrow{G,S} r'$ and subsequently, for each $i \in [\ell]$, $r_i \xrightarrow{G,S} q_{i_1}$ initiates a forcing chain equivalent to the one in Subcase 1. It follows that the $\sum_{\ell=0}^{m-1} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^k_{m,h-1}$ sets in question are Type I. This gives the first summand for $F^k_{m,h}$ in the case that h is even.

Subcase 3: Let $0 \le \ell \le m$ and suppose $r \in S$. For each $i \in [\ell]$, S_i is Type II for G_i and therefore $\textsc{Obs}(G_i, S_i \cup \{r\}) = V(G_i)$. It then follows that

$$\begin{aligned}\textsc{Obs}(G, S) &= \textsc{Obs}(G, S \cup \{r\}) \\ &= N_G[r] \cup \left(\bigcup_{i\in[m]} \textsc{Obs}(G_i, S_i \cup \{r\}) \right) \\ &= N_G[r] \cup \left(\bigcup_{i\in[m]} V(G_i) \right) \\ &= V(G).\end{aligned}$$

It follows that the $\sum_{\ell=0}^{m} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^{k-1}_{m,h-1}$ sets in question are Type I. This gives the second summand for $F^k_{m,h}$ in the case that h is even.

□

Theorem 1. *Let $m \ge 2$ and $h \ge 2$. The number of power dominating sets of size k for $G = T_{m,h}$ is:*

$$N(m,h,k) = \begin{cases} \sum_{\ell=0}^{m-1} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^k_{m,h-1} + \sum_{\ell=0}^{m} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^{k-1}_{m,h-1}, & h \text{ is even} \\ \sum_{\ell=0}^{1} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^k_{m,h-1} + \sum_{\ell=0}^{m} \langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \rangle^{k-1}_{m,h-1}, & h \text{ is odd.} \end{cases}$$

Proof. Let $G = T_{m,h}$ with root r and root children $r_1, r_2, \ldots, r_m$. For each $i \in [m]$, let G_i be the extended subtree with stem r, root r_i and containing all the descendants of r_i. For each $i \in [r]$ let the children of r_i be $\{q_{i_j}\}_{j\in[m]}$. Choose $S \subseteq V(G)$ and let $S_i = S \cap (V(G_i) \setminus \{r\})$. By Proposition 1 we need only consider cases where, for each $i \in [m]$, S_i is a Type I or Type II set for G_i. Thus we will assume, without loss of generality, that the first ℓ sets are Type II and the remaining $m - \ell$ sets are Type I.

Case 1: Suppose $r \in S$. Note that $\textsc{Obs}(G, S) \supseteq \bigcup_{i\in[m]} \textsc{Obs}(G_i, S_i \cup \{r\}) = V(G)$. It follows that for any $0 \le \ell \le m$, the set S will be a power dominating set and the number of sets in question is $\sum_{\ell=0}^{m} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h-1}^{k-1}$. This gives the second summand in both instances: where h is even or h is odd.

Case 2: Suppose h is even and $r \notin S$. Observe that for each $i \in [\ell]$, we have S_i is a Type II set for G_i, and consequently $r, q_{i_j} \notin \textsc{Obs}(G_i, S_i)$ for some $j \in [m]$. Thus, if all the sets are Type II then no child of r can force r and S is not a power dominating set. On the other hand, if $\ell < m$ then $r_m \xrightarrow{G,S} r$ and for each $i \in [\ell]$, $r_i \xrightarrow{G,S} q_{i_j}$ initiates the same forcing chain that begins with $r_i \xrightarrow{G_i, S_i \cup \{r\}} q_{i_j}$. Hence, the $\sum_{\ell=0}^{m-1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h-1}^{k}$ sets in question are power dominating sets.

Case 3: Suppose h is odd and $r \notin S$. Observe that for each Type II set $r, r_i \notin \textsc{Obs}(G_i, S_i)$. Thus, if $\ell \ge 2$, the vertex r cannot force both r_1 and r_2 and S is not a power dominating set. On the other hand, if $0 \le \ell \le 1$ then $r_m \xrightarrow{G,S} r \xrightarrow{G,S} r_i$ for $i \in [\ell]$, initiating the same forcing chain that begins with $r \xrightarrow{G_i, S_i \cup \{r\}} r_i$. Hence, the $\sum_{\ell=0}^{1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h-1}^{k}$ sets in question are power dominating sets.

□

Corollary 3. Let $G = T_{m,h}$ with $m, h \ge 2$. The probability that a uniformly at random selected subset of $V(G)$ of size k is a power dominating set is

$$p(m, h, k) = \frac{N(m, h, k)}{\binom{|V(G)|}{k}}$$

where $|V(G)| = \frac{m^{h+1}-1}{m-1}$.

Example 1. We will show that $N(2, 2, 4) = 33$. Using Theorem 1 yields

$$\begin{aligned} N(2,2,4) &= \sum_{\ell=0}^{1} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{2-\ell} \right\rangle_{2,2-1}^{4} + \sum_{\ell=0}^{2} \left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{2-\ell} \right\rangle_{2,2-1}^{3} \\ &= \left\langle \mathcal{H}^0 \oplus \mathcal{F}^{2-0} \right\rangle_{2,2-1}^{4} + \left\langle \mathcal{H}^1 \oplus \mathcal{F}^{2-1} \right\rangle_{2,2-1}^{4} + \left\langle \mathcal{H}^0 \oplus \mathcal{F}^{2-0} \right\rangle_{2,2-1}^{3} + \left\langle \mathcal{H}^1 \oplus \mathcal{F}^{2-1} \right\rangle_{2,2-1}^{3} + \left\langle \mathcal{H}^2 \oplus \mathcal{F}^{2-2} \right\rangle_{2,2-1}^{3}. \end{aligned}$$

Computing the first summand is done as follows:

$$\begin{aligned} \left\langle \mathcal{H}^0 \oplus \mathcal{F}^2 \right\rangle_{2,1}^{4} &= \sum_{\substack{i_1 \le i_2 \\ i_1 + i_2 = 4}} \binom{2}{0} \binom{0}{s_0, s_1, \ldots, s_4} \binom{2}{t_0, t_1, \ldots, t_4} \left(\prod_{1 \le j \le 0} H_{2,1}^{i_j} \right) \left(\prod_{0 < j \le 2} F_{2,1}^{i_j} \right) \\ &= \sum_{\substack{i_1 \le i_2 \\ i_1 + i_2 = 4}} \binom{2}{t_0, t_1, \ldots, t_4} \left(\prod_{0 < j \le 2} F_{2,1}^{i_j} \right) \\ &= \binom{2}{1,0,0,0,1} \left(\prod_{0 < j \le 2} F_{2,1}^{i_j} \right) + \binom{2}{0,1,0,1,0} \left(\prod_{0 < j \le 2} F_{2,1}^{i_j} \right) + \binom{2}{0,0,2,0,0} \left(\prod_{0 < j \le 2} F_{2,1}^{i_j} \right) \\ &= 2\left(F_{2,1}^0 F_{2,1}^4\right) + 2\left(F_{2,1}^1 F_{2,1}^3\right) + 1\left(F_{2,1}^2 F_{2,1}^2\right) = 2 \cdot 0 \cdot 0 + 2 \cdot 1 \cdot 1 + 1 \cdot 3 \cdot 3 = 11. \end{aligned}$$

By similar computations we obtain the other four summands:

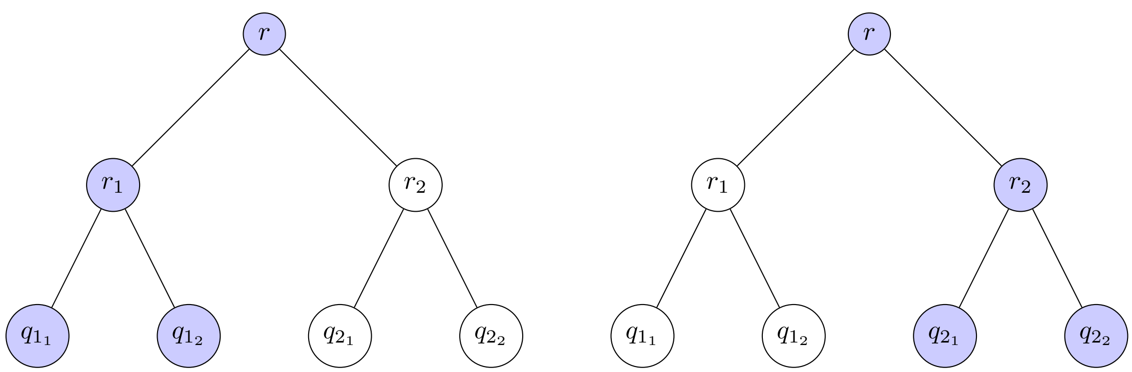
$$\begin{aligned}
\left\langle \mathcal{H}^1 \oplus \mathcal{F}^1 \right\rangle_{2,1}^4 &= 2\left(H_{2,1}^0\right)\left(F_{2,1}^4\right) + 2\left(H_{2,1}^1\right)\left(F_{2,1}^3\right) + 2\left(H_{2,1}^2\right)\left(F_{2,1}^2\right) + 2\left(H_{2,1}^3\right)\left(F_{2,1}^1\right) + 2\left(H_{2,1}^4\right)\left(F_{2,1}^0\right) \\
&= 2\cdot 0\cdot 0 + 2\cdot 2\cdot 1 + 2\cdot 0\cdot 3 + 2\cdot 0\cdot 1 + 2\cdot 0\cdot 0 = 4, \\
\left\langle \mathcal{H}^0 \oplus \mathcal{F}^2 \right\rangle_{2,1}^3 &= 2\left(F_{2,1}^0 F_{2,1}^3\right) + 2\left(F_{2,1}^1 F_{2,1}^2\right) = 2\cdot 0\cdot 1 + 2\cdot 1\cdot 3 = 6, \\
\left\langle \mathcal{H}^1 \oplus \mathcal{F}^1 \right\rangle_{2,1}^3 &= 2\left(H_{2,1}^0\right)\left(F_{2,1}^3\right) + 2\left(H_{2,1}^1\right)\left(F_{2,1}^2\right) + 2\left(H_{2,1}^2\right)\left(F_{2,1}^1\right) + 2\left(H_{2,1}^3\right)\left(F_{2,1}^0\right) \\
&= 2\cdot 0\cdot 1 + 2\cdot 2\cdot 3 + 2\cdot 0\cdot 1 + 2\cdot 0\cdot 0 = 12, \text{ and} \\
\left\langle \mathcal{H}^2 \oplus \mathcal{F}^0 \right\rangle_{2,1}^3 &= 2\left(H_{2,1}^0 H_{2,1}^3\right) + 2\left(H_{2,1}^1 H_{2,1}^2\right) = 2\cdot 0\cdot 0 + 2\cdot 2\cdot 0 = 0.
\end{aligned}$$

Hence,

$$N(2,2,4) = 11 + 4 + 6 + 12 + 0 = 33.$$

Since the inputs are relatively small, we verified $N(2,2,4) = 33$ using Theorem 1; however, it is simpler in this case to observe that $\binom{7}{4} = 35$ and exactly two subsets of size 4 fail to be power dominating sets, as illustrated in Figure 2.

Figure 2: These two subsets of size 4 are the only ones that fail to be power dominating sets.



The previous computation can be a bit cumbersome when done by hand and therefore it is desirable to use a computer. Assuming constant time for arithmetic operations,

$$\binom{m}{\ell}\binom{\ell}{s_0, s_1, \ldots, s_k}\binom{m-\ell}{t_0, t_1, \ldots, t_k}\left(\prod_{1\le j\le \ell} H_{m,h}^{i_j}\right)\left(\prod_{\ell<j\le m} F_{m,h}^{i_j}\right)$$

is computed in linear time relative to k (since $m < k$). The complexity of computing $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h}^k$ is increased by the complexity of listing the partitions of k and keeping the ones that are the concatenation of two non-increasing sequences. This can be accomplished by first constructing a tree of all non-decreasing partitions of $k-i$ ($0 \le i \le k$) into $j \le m$ parts by using the constructive recurrence $p(k,m) = p(k-1, m-1) + p(k-m, m)$ which uses less than 2^{k+1} iterations since the complete binary tree has $2^{k+1} - 1$ vertices.

It follows that the computational complexity to determine $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h}^k$ is $O(m) \cdot (2^{k+1}) = 2^{O(k)}$. Since $N(m,h,k)$ requires only $2m+1$ computations equivalent to $\left\langle \mathcal{H}^\ell \oplus \mathcal{F}^{m-\ell} \right\rangle_{m,h}^k$ (and some constant time arithmetic operations) we have that $N(m,h,k)$ is computed in $2^{O(k)}$ time (exponential with linear exponent).

3 Future Directions

Power domination in graphs arose from the phase measurement unit problem where it is desirable to place the least number of monitors in an electrical power system. The desire for minimality of monitors in this problem is motivated by the expense of purchasing and placing such monitors in the grid. The work on this paper was motivated by considering the additional time cost of determining the minimum number of monitors needed as well as an optimal placement of those monitors. This problem is known to be NP-complete even for planar bipartite graphs ([3]). This inspires the following questions.

Question 1. For fixed $m, h \geq 2$, what is the minimum value of k_α such that $p(m, h, k_\alpha) \geq \alpha$? For fixed $m \geq 2$ do the following limits exist

$$\lim_{h\to\infty} \left\{ \min_{k\geq 1} \left\{ \frac{k}{n} \right\} \middle| \, p(m,h,k) = 1 \right\}$$

where n is the number of vertices in the m-ary tree?

Question 2. Is it possible to compute $N(m, h, k)$ in polynomial time? If so, is there a pattern to optimal select the $N(m, h, k)$ vertices in the power-dominating set?

Determining a closed-form formula for $N(m, h, k)$ (or at least its asymptotic behaviors) would be beneficial in answering the previous questions. The sequence $\left\{ \sum_k N(2, h, k) \right\}_{h\geq 1} = \{1, 7, 94, 19192, \ldots\}$ does not appear in the Online Encyclopedia of Integer Sequences ([5]).

Question 3. Does $\left\{ \sum_k N(m, h, k) \right\}_{h\geq 1}$ count anything other than power domination sets of any size for a complete m-ary tree of height h (when $m \geq 2$)?

References

[1] Baldwin TL, Mili L, Boisen MB, Adapa R. *Power system observability with minimal phasor measurement placement.* IEEE Transactions on Power systems. 1993; 8(2): 707-15.

[2] Benson K, Ferrero D, Flagg M, Hogben L, Furst V, Vasilevska V, Wissman B. *Zero forcing and power domination for graph products.* Australasian Journal of Combinatorics. 2018; 70(2): 221-235.

[3] Brueni DJ, Heath LS. *The PMU Placement Problem.* SIAM Journal on Discrete Mathematics. 2005; 19(3): 744-761.

[4] Haynes TW, Hedetniemi SM, Hedetniemi ST, Henning MA. *Domination in graphs applied to electric power networks.* SIAM Journal on Discrete Mathematics. 2002; 15(4): 519-529.

[5] OEIS Foundation Inc. (2021), *The On-Line Encyclopedia of Integer Sequences.* Published electronically at http://oeis.org

[6] Ferrero D, Varghese S, Vijayakumar A. *Power domination in honeycomb networks.* Journal of Discrete Mathematical Sciences and Cryptography. 2011; 14(6): 521-529.

[7] Zhao M, Kang L, Chang GJ. *Power domination in graphs.* Discrete Mathematics. 2006; 306(15): 1812-1816.

[8] Brimkov B, Carlson J, Hicks IV, Patel R, Smith L. *Power domination throttling.* Theoretical Computer Science. 2019; 795: 142-153.