July 2022

On the Total Set Chromatic Number of Graphs

Mark Anthony C. Tolentino
Ateneo de Manila University, mtolentino@ateneo.edu

Gerone Russel J. Eugenio
Ateneo de Manila University; Central Luzon State University, gerone.eugenio@obf.ateneo.edu

Mari-Jo P. Ruiz
Ateneo de Manila University, mruiz@ateneo.edu

On the Total Set Chromatic Number of Graphs

Cover Page Footnote

Abstract

Given a vertex coloring c of a graph, the *neighborhood color set* of a vertex is defined to be the set of all of its neighbors' colors. The coloring c is called a *set coloring* if any two adjacent vertices have different neighborhood color sets. The *set chromatic number* $\chi_s(G)$ of a graph G is the minimum number of colors required in a set coloring of G. In this work, we investigate a total analog of set colorings; that is, we study set colorings of the total graph of graphs. Given a graph $G = (V, E)$, its *total graph* $T(G)$ is the graph whose vertex set is $V \cup E$ and in which two vertices are adjacent if and only if their corresponding elements in G are adjacent or incident. First, we establish sharp bounds for the set chromatic number of the total graph of a graph. Furthermore, we study the set colorings of the total graph of different families of graphs.

1   Introduction

In graph theory, much research has been devoted to graph colorings that are neighbor-distinguishing. In general, a *neighbor-distinguishing coloring* of a graph is a coloring that induces a vertex labelling for which any two adjacent vertices are assigned distinct labels. The *proper vertex coloring* is a classic example of a neighbor-distinguishing coloring. Other examples can be found in [2, 4, 5, 7, 8, 9, 10].

In this work, we focus on a neighbor-distinguishing coloring called *set coloring*, introduced by Chartrand et al. in [6]. The graphs to be considered in this paper are finite, simple, undirected, and nontrivial. Unless otherwise stated, definitions and notations will follow [3].

1.1   Set coloring

For a graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring, not necessarily proper. For $S \subseteq V(G)$, we denote by $c(S)$ the set of colors assigned to the vertices in S; that is, $c(S) := \{c(v) : v \in S\}$. The *neighborhood color set* $NC(v)$ of a vertex v is defined as $NC(v) = c(N(v))$, where $N(v) := \{u \in V(G) : vu \in E(G)\}$. The coloring c is called a *set coloring* if $NC(u) \neq NC(v)$ for every pair of adjacent vertices u and v of G. Moreover, c is called a set k-coloring if c uses k colors (i.e., $|c(V(G))| = k$). The minimum number of colors required in a set coloring is called the *set chromatic number* of G and is denoted by $\chi_s(G)$.

In [17], it has been established that the *graph set k-colorability problem* (i.e., the decision problem of determinining whether a givem graph is set k-colorable or not) is NP-complete. In [6], it has been shown that a proper k-coloring of any graph G induces a set k-coloring; consequently, $\chi_s(G) \leq \chi(G)$. The following sharp lower bound for the set chromatic number has also been established in [6].

Proposition 1.1 (Chartrand et al., [6]). *For every graph G, $\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil$.*

The following family of graphs has also been studied in [6]. Let n, t be integers such that $n \geq 2$ and $0 \leq t \leq n$. The graph $G_{n,t}$ is the graph whose vertex set may be denoted by $\{v_1, v_2, ..., v_n\} \cup \{u_1, u_2, ..., u_t\}$ and whose edge set is $\{v_i v_j : i \neq j\} \cup \{v_k u_k : k = 1, 2, ..., t\}$. Note that the vertices $v_1, v_2, ..., v_n$ induce a complete subgraph, order n, of $G_{n,t}$. The set chromatic number of $G_{n,t}$ is given by the following proposition.

Proposition 1.2 (Chartrand et al., [6]). *For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.*

1.2 Total set coloring

As has been done in [15] and [20] for equitable colorings and list colorings, respectively, we investigate a "total coloring analog" of set colorings. Recall that a *total coloring* of a graph G is a coloring of the vertices and edges of G such that adjacent or incident elements are not assigned the same colors. The minimum number of colors required for a total coloring of G is called the *total chromatic number* of G and is denoted by $\chi''(G)$. In a similar manner, we introduce the following.

Definition 1.1. *For a graph G, let $c : V(G) \cup E(G) \to \mathbb{N}$ be a coloring of the vertices and edges of G.*

1. *The total neighborhood color set of a vertex $v \in V(G)$ is the set $c(\{u \in V(G) : uv \in E(G)\} \cup \{wv : wv \in E(G)\})$ and is denoted by $TNC(v)$.*

2. *The total neighborhood color set of an edge $e = uv \in E(G)$ is the set $c(\{u, v\} \cup \{xy \in E(G) : x = u \text{ or } y = v, xy \neq uv\})$ and is denoted by $TNC(uv)$.*

3. *The coloring c is a total set coloring if $TNC(x) \neq TNC(y)$ whenever any pair of elements x and y of $V(G) \cup E(G)$ are adjacent or incident. Moreover, c is called a total set k-coloring if c uses k colors (i.e., $|c(V(G) \cup E(G))| = k$).*

4. *The minimum number of colors required for a total set coloring of G is called the total set chromatic number of G.*

Example 1.1. *Consider the cycle $C_7 = v_1 v_2 \cdots v_7 v_1$. Define a coloring c such that $c(y) = 2$ when $y \in \{v_7, v_3 v_4\}$, $c(y) = 3$ when $y \in \{v_4, v_1 v_7\}$, and $c(y) = 1$ otherwise. Then c is a total set 3-coloring of C_7. In fact, for $n \geq 3$, it is clear that $T(C_n) \cong C_{2n}(1, 2)$, which is the circulant graph whose vertex set is the cyclic group $\mathbb{Z}_{2n}$ and whose edge set is $E = \{ij : i - j = \pm 1 \text{ or } \pm 2\}$. By a result in [14], we have $\chi_s(T(C_4)) = 4$ and $\chi_s(T(C_n)) = 3$ for all $n \geq 3$, $n \neq 4$.*

The graph operation called *total graph* can be used to formulate the notions of total coloring [21] and total set coloring. Given a graph G, its total graph $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are connected by an edge if and only if u and v are adjacent (if u, v are both vertices or both edges) or incident (if one of u, v is a vertex while the other is an edge) in G.

For a graph G, it is clear that $\chi''(G) = \chi(T(G))$. Similarly, G has a total set k-coloring if and only if $T(G)$ has a set k-coloring. Hence, the total set chromatic number of G is equal to $\chi_s(T(G))$. Moreover, it follows that $\chi_s(T(G)) \leq \chi''(G)$; that is, the total chromatic number is an upper bound for the total set chromatic number. Additionally, it immediately follows from the NP-completeness of the graph set k-colorability problem that the total set k-colorability problem is also NP-complete.

For the rest of the paper, we will adopt the above total graph formulation of total set colorings; that is, we study set colorings in relation to the operation total graph [11]. Previous studies have also focused on set colorings in relation to other graph operations such as join [23, 13], corona and vertex/edge deletions [6], comb product [13], and middle graph [12]. Other neighbor-distinguishing colorings have also been studied in relation to graph operations such as those in [1, 16, 18, 19, 22, 24].
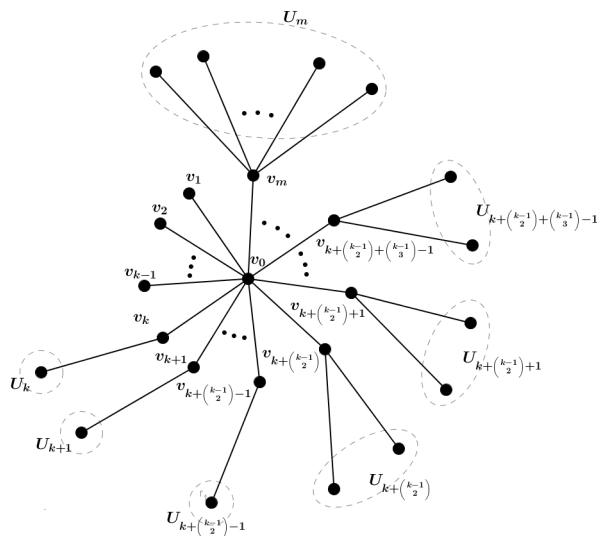
Figure 1: The graph H_m

We will use the following notations for the rest of the paper: For a positive integer k, we denote by $\mathbb{N}_k$ the set $\{1, 2, ..., k\}$. Given a vertex v in a graph G, we denote by $S_G(v)$ the set of all pendant neighbors, in G, of v.

2   Lower Bounds for the Total Set Chromatic Number

Given a graph G, it is clear that $\omega(T(G)) \geq \Delta(G) + 1$. Thus, Proposition 1.1 also provides a lower bound for $\chi_s(T(G))$. We can also prove the sharpness of this lower bound.

Theorem 2.1. *For any graph G,*

$$\chi_s(T(G)) \geq 1 + \lceil \log_2(\Delta(G) + 1) \rceil. \tag{1}$$

Moreover, for every integer $m \geq 4$, there is a graph H_m with $\Delta(H_m) = m$ and $\chi_s(T(H_m)) = 1 + \lceil \log_2(\Delta(H_m) + 1) \rceil$.

Proof. We prove the sharpness of (1) by construction. Let $m \geq 4$ be an integer and set $k := 1 + \lceil \log_2(m+1) \rceil$. Denote by $S_1, S_2, ..., S_{2^{k-1}-1}$ the nonempty subsets of the set $\mathbb{N}_k \smallsetminus \{1\}$, where $S_1, S_2, ..., S_{k-1}$ are the 1-subsets, $S_k, S_{k+1}, ..., S_{k+\binom{k-1}{2}-1}$ are the 2-subsets, and so on. Now, take the star graph $K_{1,m}$ and label its vertices as $v_0, v_1, ..., v_m$, where $\deg(v_0) = m$ and $\deg(v_i) = 1$ for $i \neq 0$. Since $2^{k-1} - 1 \geq m$, for each $i \in \{1, 2, ..., m\}$, we can have a set U_i containing $|S_i| - 1$ vertices such that $U_1, U_2, ..., U_m$, and $V(K_{1,m})$ are mutually disjoint. Note that $U_1, U_2, ..., U_{k-1}$ are empty.

We now construct H_m as follows: set $V(H_m) = V(K_{1,m}) \cup \bigcup_{i=1}^{m} U_i$ and $E(H_m) = E(K_{1,m}) \cup \bigcup_{i=1}^{m} T_i$, where $T_i := \{v_i u : u \in U_i\}$. Clearly, $\Delta(H_m) = m$. Since $\chi_s(T(H_m)) \geq k$ by (1), it remains to show that $T(H_m)$ is set k-colorable.

Recall that $V(T(H_m)) = V(H_m) \cup E(H_m)$. We consider two cases.
Case 1. Suppose $m < 2^{k-1} - 1$. We define a k-coloring $c_1 : V(T(H_m)) \to \mathbb{N}_k$ as follows:

1. We set $c_1(v) = 1$ for $v \in \{v_0, v_0v_1, v_0v_2, ..., v_0v_m\} \cup \bigcup_{i=1}^{m} U_i$.

2. For $i = 1, 2, ..., m$, we set $c_1(\{v_i\} \cup T_i) = S_i$ such that $c_1(v_i) = \min S_i$.

Note that the vertices in each clique induced by the set $\{v_0v_i, v_i\} \cup T_i$ receive different colors. We now obtain the neighborhood color sets of the vertices in $T(H_m)$.

- $NC(v_0) = \mathbb{N}_k$

For $i \in \{1, 2, ..., m\}$:

- $NC(v_0v_i) = \{1\} \cup S_i$

- $NC(v_i) = (\{1\} \cup S_i) - \{c_1(v_i)\}$

- $NC(t) = (\{1\} \cup S_i) - \{c_1(t)\}$ for $t \in T_i$

- $NC(u) = c_1(\{v_i, t\})$, for $t \in T_i$ and $u \in U_i$ such that $tu \in E(T(H_m))$

Note that since $m < 2^{k-1} - 1$, we have $S_i \subsetneq \mathbb{N}_k - \{1\}$, and consequently $\{1\} \cup S_i \subsetneq \mathbb{N}_k$, for all $i = 1, 2, ..., m$. From here, it is easy to verify that c_1 is a set k-coloring; hence, $\chi_s(T(H_m)) = k$ when $m < 2^{k-1} - 1$.

Case 2. Suppose $m = 2^{k-1} - 1$. We define a k-coloring $c_2 : V(T(H_m)) \to \mathbb{N}_k$ as follows:

1. We set $c_2(v) = 1$ for $v \in \{v_0, v_m, v_0v_1, v_0v_2, ..., v_0v_m\} \cup T_m \cup \left(\bigcup_{i=1}^{m-1} U_i \right)$.

2. For $i = 1, 2, ..., m - 1$, we set $c_2(\{v_i\} \cup T_i) = S_i$ such that $c_2(v_i) = \min S_i$.

3. We set $c_2(U_m) = S_{m-1}$.

To show that c_2 is a set coloring, we will just show that

(a) $NC(w_m) \neq NC(w_i)$ for $i = 1, 2, ..., m - 1$, and that

(b) $NC(u) \neq NC(v)$ for adjacent vertices $u, v \in \{v_0, v_0v_m, v_m\} \cup T_m \cup U_m$.

The verification for the remaining adjacencies can be done easily, as in Case 1. To prove (a) and (b), we need to consider only the following neighborhood color sets: $NC(v_0) = \mathbb{N}_k$; $NC(v_0v_i) = \{1\} \cup S_i$ for $i = 1, 2, ..., m - 1$; $NC(v_0v_m) = \{1\}$; $NC(v_m) = \{1\} \cup S_{m-1}$; $NC(t) = \{1, c_2(u)\}$ for $t \in T_m, u \in U_m$ such that $tu \in E(T(H_m))$; and $NC(u) = \{1\}$ for $u \in U_m$.

For $i \in \{1, 2, ..., m - 1\}$, we have $S_i \neq \emptyset$, which implies that $NC(w_m) \neq NC(w_i)$. We now verify (b). Suppose $t \in T_m$ and $u \in U_m$ such that $tu \in E(T(H_m))$. First, it is clear that $NC(v_0), NC(v_0v_m)$, and $NC(v_m)$ are pairwise distinct. Since $c_2(u) \neq 1$, we have $NC(v_0v_m) \neq NC(t)$. Moreover, since $m = 2^{k-1} - 1$, we have $m \geq 7$ and $k \geq 4$, which imply that $|S_{m-1}| = k - 2 \geq 2$. Thus, $NC(v_m) \neq NC(t)$. Finally, $NC(v_m) \neq NC(t)$ and $NC(t) \neq NC(u)$ follow easily from the cardinalities of these sets. This proves that c_2 is a set k-coloring; hence, $\chi_s(T(H_m)) = k$ when $m = 2^{k-1} - 1$.

Therefore, for $m \geq 4$, H_m is a graph satisfying $\Delta(H_m) = m$ and $\chi_s(T(H_m)) = 1 + \lceil \log_2(\Delta(H_m) + 1) \rceil$. $\square$

We now establish a second lower bound for the total set chromatic number of graphs that have a vertex with at least one pendant neighbor.

Theorem 2.2. *Let G be any graph and let W be the set of all vertices of G that have at least one pendant neighbor. If $W \neq \emptyset$, then*

$$\chi_s(T(G)) \geq \max\{|S_G(v)| : v \in W\}. \tag{2}$$

Proof. Let $v \in W$. We set $Q :=$ the set of all nonpendant neighbors, in G, of v, $T := \{vq \in E(G) : q \in Q\}$, $S := S_G(v)$, and $R := \{vs \in E(G) : s \in S\}$. Note that T and R can also be viewed as subsets of $V(T(G))$.

Let c be a set k-coloring of $T(G)$. Suppose, for now, that $Q \neq \emptyset$ and let $t \in T$. Let H be the clique of $T(G)$ formed by v, t, and all the vertices in R. Then $H \cong K_{|S|+2}$. Permuting colors if necessary, we have $c(V(H)) = \mathbb{N}_\ell$ for some $\ell \leq k$. Let X be the maximal subset of $V(H)$ such that for all $x \in X$, there exists $y \in \{X\} - x$ for which $c(x) = c(y)$. Since the remaining vertices in $V(H) \smallsetminus X$ receive unique colors, we must have $|V(H)| - |X| + 1 \leq \ell$ or $|X| \geq |V(H)| - \ell + 1$. The neighborhood color sets of vertices in X are given as follows.

1. If $v \in X$, then $NC(v) = \mathbb{N}_\ell \cup c(S) \cup c(Q) \cup c(T \smallsetminus \{t\})$.

2. If $t \in X$, then $NC(t) = \mathbb{N}_\ell \cup c[N_{T(G)}(t) \smallsetminus (V(H) \smallsetminus \{t\})]$.

3. Let $r \in R \cap X$ and $s \in S \cap N(r)$.

   (a) If $c(s) \notin \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell \cup \{c(s)\} \cup c(T \smallsetminus \{t\})$.
   (b) If $c(s) \in \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell \cup c(T \smallsetminus \{t\})$.

At best, all the neighborhood color sets in (1)-(3) above are all distinct from each other. Since there are $k - \ell$ colors not in $\mathbb{N}_\ell$, (3a) provides for $k - \ell$ distinct neighborhood color sets. Hence, the maximum number of neighborhood color sets available for vertices in X is $k - \ell + 3$. So we must have $k - \ell + 3 \geq |X|$. Then $k - \ell + 3 \geq |V(H)| - \ell + 1$. With $|V(H)| = |S| + 2$, we have $k \geq |S|$ and the conclusion follows.

Now, suppose $Q = \emptyset$. (Note that, in this case, v and the vertices in S induce a star component of G.) Then we can take H to be the clique formed by v and the vertices in R. As before, we assume that $c(V(H)) = \mathbb{N}_\ell$ for some $\ell \leq k$ and we can construct the set X as before. Then the neighborhood color sets of vertices in X are as follows.

(1') If $v \in X$, then $NC(v) = \mathbb{N}_\ell \cup c(S)$.

(3') Let $r \in R \cap X$ and $s \in S \cap N(r)$.

   (a') If $c(s) \neq \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell \cup \{c(s)\}$.
   (b') If $c(s) \in \mathbb{N}_\ell$, then $NC(r) = \mathbb{N}_\ell$.

Then the maximum number of neighborhood color sets for vertices in X is $k - \ell + 2$. We now have $k - \ell + 2 \geq |X| \geq |V(H)| - \ell + 1 = |S| + 1 - \ell + 1$. Therefore, we also obtain $k \geq |S|$ and the conclusion also follows in this case. $\square$

When (2) is an equality, we can establish some properties of optimal total set colorings of G. Such properties are useful, for instance, for constructing optimal total set colorings. For convenience, we restrict our attention to connected graphs that are not stars, the total set chromatic number of which will be discussed in the next section. In the following, R is defined as in the proof of Theorem 2.2.

Lemma 2.3. *Let G be any connected graph that is not a star. If there is a vertex $v \in V(G)$ such that $\chi_s(T(G)) = |S_G(v)|$, then any total set $\chi_s(T(G))$-coloring of G satisfies the following:*

1. *Let q be a nonpendant neighbor, in G, of v. If $y \in \{v, vq\}$, then there is a $y' \in [R \cup \{vq, v\}] \setminus \{y\}$ such that $c(y) = c(y')$.*

2. $|c(S_G(v) \cup \{vs \in E(G) : s \in S_G(v)\})| = |S_G(v)|$

Proof. The assumptions imply that v has at least one nonpendant neighbor and at least one pendant neighbor. We set $Q :=$ the set of all nonpendant neighbors, in G, of v, $T := \{vq \in E(G) : q \in Q\}$, $S := S_G(v)$, and $R := \{vs \in E(G) : s \in S\}$. Let c be a set $|S|$-coloring of $T(G)$.

First, we prove (1). Suppose $|c(R \cup \{v, vq\})| = m$ for some $m \leq |S|$. Moreover, let X be the set of vertices x in $R \cup \{v, vq\}$ for which $c(x) = c(x')$ for some $x' \in [R \cup \{v, vq\}] \setminus \{x\}$. Let $\gamma = |\{v, vz\} \setminus X|$. Following a similar argument as in the proof of Theorem 2.2, we have $|S| - l + 3 - \gamma \geq (|S| + 2) - l + 1$. Thus, $\gamma = 0$ and the conclusion follows.

Now, let us prove (2). First, suppose $|c(R)| = l$ for some $l \leq |S|$. Let $P \subseteq R$ be the set of all p in R for which $c(p) = c(y)$ for some $y \in R \setminus \{p\}$. The conclusion follows immediately when $P = \emptyset$; thus, let us assume that $P \neq \emptyset$. Then, for all $p \in P$, we have $NC(p) = c(R) \cup c(T \cup \{v\}) \cup \{c(s)\}$, where $s \in S \cap N_{T(G)}(p)$. Hence, there are at most $|S| - l + 1$ possible neighborhood color sets for vertices in P; consequently, we must have $|P| \leq |S| - l + 1$. On the other hand, $|P| = |R| - |R \setminus P| = |S| - (l - |c(P)|)$. Therefore, we must have $|c(P)| = 1$ and $|P| = |S| - l + 1$.

Now, only one of the vertices in P may have neighborhood color set equal to $c(R) \cup c(T \cup \{v\})$; thus, we must have at least $|S| - l$ colors that are in $c(S)$ but not in $c(R) \cup c(T \cup \{v\})$. Thus, $|c(S) \setminus c(R)| \geq |c(S) \setminus [c(R) \cup c(T \cup \{v\})]| \geq |S| - l$. Finally, we have $c(R \cup S)| = |c(R)| + |c(S) \setminus c(R)| \geq l + |S| - l = |S|$ and the conclusion follows. □

Note, in particular, that Lemma 2.3(2) implies that $NC(v)$ must contain all the colors used by an optimal set coloring of $T(G)$. The usefulness of Theorem 2.2 and Lemma 2.3 will be evident in the next section, where we determine the total set chromatic number of different families of trees. The sharpness of (2) also follows from some of these results.

3 Total Set Chromatic Number of Some Tree Families

We now determine the total set chromatic number of different families of trees. We begin with the following proposition.

Proposition 3.1. *For $n \geq 3$, $\chi_s(T(P_n)) = 3$.*

Proof. Suppose $P_n = v_1 v_2 \cdots v_n$. Clearly, $\chi_s(T(P_n)) \geq 3$. On the other hand, it is easy to verify that the coloring $c : V(T(P_n)) \to \mathbb{N}_3$ defined below is a set 3-coloring of $T(P_n)$.

$$c(y) = \begin{cases} 3, & \text{if } y = v_i, \text{ where } i \equiv 0 \pmod{3}, \\ & \text{or if } y = v_j v_{j+1}, \text{ where } j \equiv 1 \pmod{3} \text{ and } j \geq 4, \\ 2, & \text{if } y = v_i, \text{ where } i \equiv 1 \pmod{3}, \\ & \text{or if } y = v_j v_{j+1}, \text{ where } j \equiv 2 \pmod{3}, \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

$\square$

We now consider the total set chromatic number of stars. We have the following result.

Proposition 3.2. *For $m \geq 1$,*

$$\chi_s(T(K_{1,m})) = \begin{cases} 3, & \text{if } m \leq 2, \\ 4, & \text{if } m = 3, \\ m, & \text{if } m \geq 4. \end{cases} \tag{4}$$

Proof. The cases when $m \leq 3$ can be easily verified. Now, suppose $m \geq 4$. Theorem 2.2 implies that $\chi_s(T(K_{1,m})) \geq m$.

Let $V(K_{1,m}) = \{v_i : i = 0, 1, ..., m\}$ and $E(K_{1,m}) = \{v_0 v_i : i = 1, 2, ..., m\}$. We now a construct a coloring $c : V(T(K_{1,m})) \to \mathbb{N}_m$ as follows:

$$c(y) = \begin{cases} 1, & \text{if } y \in \{v_0, v_1, v_m\} \cup \{v_0 v_2, v_0 v_3, ..., v_0 v_m\}, \\ 2, & \text{if } y = v_0 v_1, \\ 2 - i + m, & \text{if } y = v_i \text{ for } i \in \{2, 3, ..., m-1\}. \end{cases} \tag{5}$$

It can easily be checked that c is a set m-coloring of $T(K_{1,m})$ and the desired conclusion follows. $\square$

Let us now consider *double-stars*, by which we mean trees that have exactly two nonpendant vertices. We denote by $S_{m,n}$ the double-star in which one nonpendant vertex has exactly m pendant neighbors while the other nonpendant vertex has exactly n pendant neighbors. Our result is as follows.

Proposition 3.3. *For $m \geq n \geq 5$,*

$$\chi_s(T(S_{m,n})) = \begin{cases} m, & \text{if } n < m, \\ m+1, & \text{if } n = m. \end{cases} \tag{6}$$

Proof. Suppose $V(S_{m,n}) = \{v_i : i = 0, 1, ..., m\} \cup \{w_i : i = 0, 1, ..., n\}$ and $E(S_{m,n}) = \{v_0 w_0\} \cup \{v_0 v_i : i = 1, 2, ..., m\} \cup \{w_0 w_i : i = 1, 2, ..., n\}$. Theorem 2.2 implies that $\chi_s(T(S_{m,n})) \geq m$. If $n < m$, we construct the coloring $c_1 : V(T(S_{m,n})) \to \mathbb{N}_m$ as follows:

$$c_1(y) = \begin{cases} i, & \text{if } y = v_i \text{ for } i \in \{1, 2, ..., m\}, \\ 2, & \text{if } y = v_0 v_1, \\ m, & \text{if } y = w_0 w_1, \\ n, & \text{if } y \in \{w_3, w_0 w_2\}, \\ i - 1, & \text{if } y = w_i \text{ for } i \in \{4, 5, ..., n\}, \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

On the other hand, if $n = m$, we construct the coloring $c_2 : V(T(S_{m,m})) \to \mathbb{N}_{m+1}$ as follows:

$$c_2(y) = \begin{cases} i, & \text{if } y = v_i \text{ for } i \in \{1, 2, ..., m\}, \\ 2, & \text{if } y = v_0 v_1, \\ m+1, & \text{if } y = w_0 w_1, \\ m, & \text{if } y = w_0 w_2, \\ i, & \text{if } y = w_i \text{ for } i \in \{3, 5, ..., m\}, \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

As in the previous propositions, it can be verified that c_1 is a set m-coloring of $T(S_{m,n})$, where $5 \leq n < m$, and that c_2 is a set $(m+1)$-coloring of $T(S_{m,m})$, where $m \geq 5$.

To complete the proof, we need to show that $\chi_s(T(S_{m,m})) \geq m+1$ for $m \geq 5$. If $\chi_s(T(S_{m,m})) = m$ and c is a set m-coloring of $T(S_{m,m})$, then Lemma 2.3(2) implies that $|NC(v_0)| = |NC(w_0)| = m$, which means that v_0 and w_0 must have the same neighborhood color sets. This is a contradiction and we must have $\chi_s(T(S_{m,m})) \geq m+1$ for $m \geq 5$. $\qquad \square$

We now consider a more general family of trees with height 2, for which we will use the following notations: The root vertex will be denoted by v_0. The children of v_0 will be denoted by $v_1, v_2, ..., v_{\deg v_0}$. For each $i \in \{1, 2, ..., \deg v_0\}$ for which $\deg v_i \geq 2$, the children of v_i will be denoted by $v_{i,1}, v_{i,2}, ..., v_{i,(\deg v_i)-1}$. Let us begin with the following lemma, which involves a total set coloring algorithm.

Lemma 3.1. *Let G be a tree of height 2 with root v_0. If there is an internal vertex w such that $\Delta(G) = \deg w \geq 2 + \deg v_0 \geq 7$, then G is total set $(\deg w - 1)$-colorable.*

Proof. Without loss of generality, we assume that $w = v_1$ and we set $m = |S_G(v_1)| = \deg v_1 - 1$. We construct a coloring $c : V(G) \cup E(G) \to \mathbb{N}_m$ using Algorithm 1.

In Algorithm 1, for $i \in \{3, ..., \deg v_0\}$, the color $i+1$ is assigned to the vertex v_i (line 30) or to the edge $v_i v_{i+1}$ (line 39). Since we want c to use only m colors, we must have $1 + \deg v_0 \leq m$, which follows from our assumption that $\deg w \geq 2 + \deg v_0$. Moreover, it can be easily verified that the sets $\{\alpha_1, ..., \alpha_{m-2}\}$, $\{\beta_1, ..., \beta_{m-2}\}$, $\{\delta_1, ..., \delta_{m-2}\}$, $\{\gamma_1, ..., \gamma_{m-2}\}$, and $\{\epsilon_1, ..., \epsilon_{m-1}\}$ contain sufficient colors to color the vertices of $v_{i,j}$ (see lines 5, 13, 22, 32, and 40 of Algorithm 1). Thus, Algorithm 1 can always construct a coloring $c : V(G) \cup E(G) \to \mathbb{N}_m$.

We now prove that c is a set coloring of $T(G)$. First, note that $2 \notin NC(v_0)$ while $2 \in NC(v_0 v_i)$ and $2 \in NC(v_i)$ for all $i \in \mathbb{N}_{\deg v_0}$. Therefore, $NC(v_0) \neq NC(v_0 v_i)$ and $NC(v_0) \neq NC(v_i)$ for all $i \in \mathbb{N}_{\deg v_0}$. We now compute the neighborhood color sets of all the vertices of $T(G)$ except for v_0.

1. Neighborhood color set of vertices of the form $v_0 v_i$:

$$NC(v_0 v_1) = \mathbb{N}_5$$

$$NC(v_0 v_2) = \begin{cases} \{1, 2, 4\} & \text{if } |S_G(v_2)| \leq m-2 \\ \mathbb{N}_3 \cup \{5, 6\} & \text{if } |S_G(v_2)| \in \{m-1, m\} \end{cases}$$

$$i \geq 3 : \quad NC(v_0 v_i) = \mathbb{N}_3 \cup \{i+1\}$$

2. Neighborhood color set of vertices of the form v_i:

$$NC(v_1) = \mathbb{N}_m$$

$$NC(v_2) = \begin{cases} \{2,3\} & \text{if } |S_G(v_2)| = 0 \\ \{2,3\} \cup \{\beta_1, \beta_2, ..., \beta_{|S_G(v_2)|}\} & \text{if } 0 < |S_G(v_2)| \leq m - 2 \\ \mathbb{N}_m & \text{if } |S_G(v_2)| \in \{m-1, m\} \end{cases}$$

$$i \geq 3: \ NC(v_i) = \begin{cases} \{1,2\} & \text{if } |S_G(v_i)| = 0 \\ \{1,2\} \cup \{\gamma_1, \gamma_2, ..., \gamma_{|S_G(v_i)|}\} & \text{if } 0 < |S_G(v_i)| \leq m - 2 \\ \mathbb{N}_m & \text{if } |S_G(v_i)| \in \{m-1, m\} \end{cases}$$

3. Neighborhood color set of vertices of the form $v_i v_{i,j}$:

   (a) $i = 1$: $NC(v_1 v_{1,1}) = \{1,5\}$; $NC(v_1 v_{1,2}) = \{1,4\}$;
       for $j \geq 3$: $NC(v_1 v_{1,j}) = \{1,4,5,\alpha_{j-2}\}$

   (b) $i = 2$

       i. If $|S_G(v_2)| = 1$: $NC(v_2 v_{2,1}) = \{3,4,\beta_1\}$
       ii. If $2 \leq |S_G(v_2)| \leq m - 2$: $NC(v_2 v_{2,j}) = \{2,3,4,\beta_j\}$
       iii. If $|S_G(v_2)| \in \{m-1, m\}$: $NC(v_2 v_{2,1}) = \{3,6\}$; $NC(v_2 v_{2,2}) = \{3,5\}$;
            for $j \geq 3$: $NC(v_2 v_{2,j}) = \{3,5,6,\delta_{j-2}\}$

   (c) $i \geq 3$:

       i. If $0 < |S_G(v_i)| \leq m - 2$: $NC(v_i v_{i,j}) = \{1, i+1, \gamma_j\}$
       ii. If $|S_G(v_i)| \in \{m-1, m\}$: $NC(v_i v_{i,1}) = \{1\}$;
           for $j \geq 2$: $NC(v_i v_{i,j}) = \{1, i+1, \epsilon_{j-1}\}$

4. Neighborhood color set of vertices of the form $v_{i,j}$:

   (a) $i = 1$: $NC(v_{1,1}) = \{1,4\}$; $NC(v_{1,2}) = \{1,5\}$; for $j \geq 3$: $NC(v_{1,j}) = \{1\}$
   (b) $i = 2$:

       i. If $0 < |S_G(v_2)| \leq m - 2$: $NC(v_{2,j}) = \{2,4\}$
       ii. If $|S_G(v_2)| \in \{m-1, m\}$: $NC(v_{2,1}) = \{3,5\}$; $NC(v_{2,2}) = \{3,6\}$;
           for $j \geq 3$: $NC(v_{2,j}) = \{3\}$

   (c) $i \geq 3$:

       i. If $0 < |S_G(v_i)| \leq m - 2$: $NC(v_{i,j}) = \{1, i+1\}$
       ii. If $|S_G(v_i)| \in \{m-1, m\}$: $NC(v_{i,1}) = \{1, i+1\}$; for $j \geq 2$: $NC(v_{i,j}) = \{1\}$

To complete the proof that c is indeed a total set m-coloring of G, we use the computed total neighborhood color sets above to verify the total set coloring condition for the remaining adjacencies. For distinct $i, \ell \in \mathbb{N}_{\deg v_0}$, it is easy to see that $NC(v_0 v_i) \neq NC(v_0 v_\ell)$. Moreover, it is clear that $NC(v_0 v_1) \neq NC(v_1) = \mathbb{N}_m$ and that $NC(v_0 v_i) \neq NC(v_i) = \mathbb{N}_m$ when $i \geq 2$ and $|S_G(v_i)| \in \{m-1, m\}$. On the other hand, when $|S_G(v_2)| \leq m - 2$, we have 4 in $NC(v_0 v_2)$ but not in $NC(v_2)$, which implies that $NC(v_0 v_2) \neq NC(v_2)$. Similarly, when $i \geq 3$ and $|S_G(v_i)| \leq m - 2$, we have $i + 1$ in $NC(v_0 v_i)$ but not in $NC(v_i)$.

Algorithm 1 Constructing a total set m-coloring c of a graph G as in Lemma 3.1

1: $c(v_0) \leftarrow 2$

2: $c(v_0 v_1) \leftarrow 1, \quad c(v_1) \leftarrow 1$ ▷ Start: coloring first branch

3: $c(v_1 v_{1,1}) \leftarrow 4, \quad c(v_{1,1}) \leftarrow 1$

4: $c(v_1 v_{1,2}) \leftarrow 5, \quad c(v_{1,2}) \leftarrow 1$

5: Suppose $\mathbb{N}_m \smallsetminus \{4, 5\} = \{\alpha_1, \alpha_2, ..., \alpha_{m-2}\}$.

6: **for** $j \leftarrow 3$ to m **do**

7:     $c(v_1 v_{1,j}) \leftarrow 1, \quad c(v_{1,j}) \leftarrow \alpha_{j-2}$

8: **end for** ▷ End: coloring first branch

9: $c(v_0 v_2) \leftarrow 3$ ▷ Start: coloring second branch

10: **if** $|S_G(v_2)| \leq m - 2$ **then**

11:     $c(v_2) \leftarrow 4$

12:     **if** $|S_G(v_2)| > 0$ **then**

13:         Suppose $\mathbb{N}_m \smallsetminus \{3, 4\} = \{\beta_1, \beta_2, ..., \beta_{m-2}\}$.

14:         **for** $j \leftarrow 1$ to $|S_G(v_2)|$ **do**

15:             $c(v_2 v_{2,j}) \leftarrow 2, \quad c(v_{2,j}) \leftarrow \beta_j$

16:         **end for**

17:     **end if**

18: **else** ▷ i.e. $|S_G(v_2)| \in \{m-1, m\}$

19:     $c(v_2) \leftarrow 3$

20:     $c(v_2 v_{2,1}) \leftarrow 5, \quad c(v_{2,1}) \leftarrow 3$

21:     $c(v_2 v_{2,2}) \leftarrow 6, \quad c(v_{2,2}) \leftarrow 3$

22:     Suppose $\mathbb{N}_m \smallsetminus \{5, 6\} = \{\delta_1, \delta_2, ..., \delta_{m-2}\}$ such that $\delta_{m-2} = 3$.

23:     **for** $j \leftarrow 3$ to $|S_G(v_2)|$ **do**

24:         $c(v_2 v_{2,j}) \leftarrow 3, \quad c(v_{2,j}) \leftarrow \delta_{j-2}$

25:     **end for**

26: **end if** ▷ End: coloring second branch

27: **for** $i \leftarrow 3$ to $\deg v_0$ **do** ▷ Start: coloring remaining branches

28:     $c(v_0 v_i) \leftarrow 1$

29:     **if** $|S_G(v_i)| \leq m - 2$ **then**

30:         $c(v_i) \leftarrow i + 1$

31:         **if** $|S_G(v_i)| > 0$ **then**

32:             Suppose $\mathbb{N}_m \smallsetminus \{1, i + 1\} = \{\gamma_1, \gamma_2, ..., \gamma_{m-2}\}$.

33:             **for** $j \leftarrow 1$ to $|S_G(v_i)|$ **do**

34:                 $c(v_i v_{i,j}) \leftarrow 1, \quad c(v_{i,j}) \leftarrow \gamma_j$

35:             **end for**

36:         **end if**

37:     **else** ▷ i.e. $|S_G(v_i)| \in \{m-1, m\}$

38:         $c(v_i) \leftarrow 1$

39:         $c(v_i v_{i,1}) \leftarrow i + 1, \quad c(v_{i,1}) \leftarrow 1$

40:         Suppose $\mathbb{N}_m \smallsetminus \{i + 1\} = \{\epsilon_1, \epsilon_2, ..., \epsilon_{m-1}\}$ such that $\epsilon_{m-1} = 1$.

41:         **for** $j \leftarrow 2$ to $|S_G(v_i)|$ **do**

42:             $c(v_i v_{i,j}) \leftarrow 1, \quad c(v_{i,j}) \leftarrow \epsilon_{j-1}$

43:         **end for**

44:     **end if**

45: **end for** ▷ End: coloring remaining branches

Now, when $|S_G(v_2)| = 1$, we have $NC(v_0v_2) = \{1, 2, 4\} \neq \{3, 4, \beta_1\} = NC(v_2v_{2,1})$. Aside from this case, we have $NC(v_0v_i) \neq NC(v_iv_{i,j})$, where $i \in \mathbb{N}_{\deg v_0}$ and $j \in \mathbb{N}_{|S_G(v_i)|}$, because their cardinalities do not match.

Next, it is clear that $\mathbb{N}_m = NC(v_1) \neq NC(v_1v_{1,j})$ for any $j \in \mathbb{N}_m$. Moreover, when $i \geq 2$ and $|S_G(v_i)| \in \{m - 1, m\}$, it is also clear that $\mathbb{N}_m = NC(v_i) \neq NC(v_iv_{i,j})$ for any $j \in \mathbb{N}_{|S_G(v_i)|}$. On the other hand, when $|S_G(v_2)| \leq m - 2$, we have 4 in $NC(v_2v_{2,j})$, for any $j \in \mathbb{N}_{|S_G(v_2)|}$, but not in $NC(v_2)$. Similarly, when $i \geq 3$ and $|S_G(v_i)| \leq m - 2$, we have $i + 1$ in $NC(v_iv_{i,j})$, for any $j \in \mathbb{N}_{|S_G(v_i)|}$, but not in $NC(v_i)$.

Due to mismatch of cardinalities, we can also conclude that $NC(v_i) \neq NC(v_{i,j})$ for any $i \in \mathbb{N}_{\deg v_0}$ and $j \in \mathbb{N}_{|S_G(v_i)|}$. Now, note that when $\lambda \in \{\alpha, \beta, \gamma, \delta, \epsilon\}$ and $j \neq h$, then $\lambda_j \neq \lambda_h$. Then it is also evident in the total neighborhood color sets that for each $i \in \mathbb{N}_{\deg v_0}$, we have $NC(v_iv_{i,j}) \neq NC(v_iv_{i,h})$ for any distinct $j, h \in \mathbb{N}_{|S_G(v_i)|}$.

Finally, it can be easily verified that $NC(v_iv_{i,j}) \neq NC(v_{i,j})$ for any $i \in \mathbb{N}_{\deg v_0}$ and $j \in \mathbb{N}_{|S_G(v_i)|}$. For example, when $i \geq 3$ and $0 < |S_G(v_i)| \leq m - 2$, we have $NC(v_iv_{i,j}) = \{1, i + 1, \gamma_j\} \neq \{1, i + 1\} = NC(v_{i,j})$ because $\gamma_j \notin \{1, i + 1\}$ for any $j \in \mathbb{N}_{|S_G(v_i)|}$. This concludes the proof that the coloring c constructed by Algorithm 1 is a total set m-coloring of G. □

The following is an extension of Lemma 3.1 to the case where the degree of the root vertex of the height-2 tree G is equal to $\Delta(G) - 1$; that is, the root vertex and the internal vertex with maximal degree have the same number of children.

Corollary 3.2. *Let G be a tree of height 2 with root v_0. If there is an internal vertex w such that $\Delta(G) = \deg w = 1 + \deg v_0 \geq 7$, then G is total set $(\deg w - 1)$-colorable.*

Proof. Without loss of generality, we assume that $w = v_1$ and we set $m = |S_G(v_1)| = \deg v_0 = -1 + \deg v_1$. If G is a double-star (i.e. only v_0 and v_1 have children), then G is isomorphic to $S_{m,m-1}$. The desired conclusion then follows from Proposition 3.3.

We now assume that G is not a double-star. Then there must be at least one $i \in \mathbb{N}_m \setminus \{1\}$ for which $|S_G(v_i)| \geq 1$. So we can assume that $|S_G(v_m)| \geq 1$.

Let H be the subgraph of G induced by the vertices $v_0, v_1, v_2, ..., v_{m-1}$, and the vertices of the form $v_{i,j}$, where $i \in \mathbb{N}_{m-1}$ and $j \in \mathbb{N}_{|S_G(v_i)|}$. Then H satisfies the assumptions of Lemma 3.1; thus, we have a total set m-coloring of H.

We now extend the coloring c of H to a coloring c' of G. We define c' using Algorithm 2.

It is evident from Algorithm 2 that the coloring c of H and the coloring c' of G induce the same total neighborhood color sets for the elements in H, except possibly for v_0. From here, is is straightforward to check that c' is indeed a total set m-coloring of G. □

In Figure 2, a tree satisfying the assumptions of Corollary 3.2 is shown together with a total set 7-coloring generated by Algorithm 2. Note that the coloring of the subgraph induced by the vertices $v_0, v_1, v_2, ..., v_6$, and the children of $v_1, v_2, ..., v_6$ was, as indicated in Algorithm 2, generated using Algorithm 1.

Algorithm 2 Constructing a total set m-coloring c for a graph G as in Corollary 3.2

1: **for** $x \in V(H) \cup E(H)$ **do**
2:    $c'(x) \leftarrow c(x)$ ▷ c as constructed by Algorithm 1
3: **end for**
4: $c'(v_0 v_m) \leftarrow 1$
5: **if** $|S_G(v_m)| \geq 2$ **then**
6:    $c'(v_m) \leftarrow 1$
7:    $c'(v_m v_{m,1}) \leftarrow 4$, $c'(v_{m,1}) \leftarrow 1$
8:    $c'(v_m v_{m,2}) \leftarrow 6$, $c'(v_{m,2}) \leftarrow 1$
9:    Suppose $\mathbb{N}_m \setminus \{4,6\} = \{\alpha_1, \alpha_2, ..., \alpha_{m-2}\}$ such that $\alpha_1 = 5$.
10:    **for** $j \leftarrow 3$ to $|S_G(v_m)|$ **do**
11:       $c'(v_m v_{m,j}) \leftarrow 1$, $c'(v_{m,j}) \leftarrow \alpha_{j-2}$
12:    **end for**
13: **else**   ▷ i.e. $|S_G(v_m)| = 1$
14:    $c'(v_m) \leftarrow 4$, $c'(v_m v_{m,1}) \leftarrow 6$, $c'(v_{m,1}) \leftarrow 1$
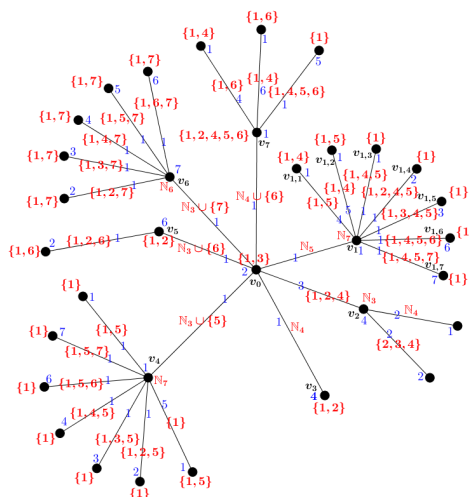15: **end if**

Figure 2: A tree of height 2 with a total set 7-coloring generated by Algorithm 2
.

Finally, with Theorem 2.2, Lemma 3.1, and Corollary 3.2, we obtain the following result.

Theorem 3.3. *Let G be a tree of height 2 with root v_0. If there is an internal vertex w such that $\Delta(G) = \deg w \geq 1 + \deg v_0 \geq 7$, then $\chi_s(T(G)) = \Delta(G) - 1$.*

Given the results discussed in this paper, the authors suggest the following problems:

1. Can we establish other lower bounds, similar to (2), that are applicable to graphs with no pendant vertices?

2. Are $\Delta(G)$ colors always sufficient to construct a total set coloring of any tree G?

References

[1] J. Aruldoss and S. Mary. Harmonious coloring of middle and central graph of some special graphs. *Int. J. Math. Appl.*, 4:187–191, 2017.

[2] S. Arumugam, K. Premalatha, M. Bača, and Semaničová-Feňovčíková. Local antimagic vertex coloring of a graph. *Graphs Combin.*, 33:275–285, 2017.

[3] J. A. Bondy and R. Murty. *Graph Theory with Applications.* Elsevier Science Publishing Co., Inc., 1976.

[4] G. Chartrand, S. English, and P. Zhang. Kaleidoscopic colorings of graphs. *Discuss. Math. Graph Theory*, 37(3):711–727, 2017.

[5] G. Chartrand, J. Hallas, and P. Zhang. Proper mean colorings of graphs. *Discrete Math. Lett.*, 3:1–8, 2020.

[6] G. Chartrand, F. Okamoto, C. W. Rasmussen, and P. Zhang. The set chromatic number of a graph. *Discuss. Math. Graph Theory*, 29(3):545–561, 2009.

[7] G. Chartrand, F. Okamoto, and P. Zhang. The metric chromatic number of a graph. *Australas. J. Combin.*, 44:273–286, 2009.

[8] G. Chartrand, F. Okamoto, and P. Zhang. Neighbor-distinguishing vertex colorings of graphs. *J. Combin. Math. Combin. Comput.*, 74:223–251, 2010.

[9] G. Chartrand, F. Okamoto, and P. Zhang. The sigma chromatic number of a graph. *Graphs Combin.*, 26:755–773, 2010.

[10] G. Chartrand and P. Zhang. *Chromatic Graph Theory.* Boca Raton: Chapman & Hall/CRC, 2009.

[11] G. R. E. Eugenio. The set chromatic numbers of the middle and total graphs of a graph. Master's thesis, Ateneo de Manila University, 2020.

[12] G. R. J. Eugenio, M. P. Ruiz, and M. A. C. Tolentino. The set chromatic numbers of the middle graph of graphs. *J. Phys. Conf. Ser.*, 1836:1–10, 2021.

[13] B. C. Felipe, A. D. Garciano, and M. A. C. Tolentino. On the set chromatic number of the join and comb product of graphs. *J. Phys. Conf. Ser.*, 1538:1–10, 2019.

[14] B. C. L. Felipe. The set chromatic numbers of some families of graphs. Master's thesis, Ateneo de Manila University, 2018.

[15] H. Fu. Some results on equalized total coloring. *Congr. Numer.*, 102:111–119, 01 1994.

[16] A. D. Garciano, R. M. Marcelo, M. P. Ruiz, and M. A. C. Tolentino. Sigma coloring and edge deletion. *J. Inf. Process.*, 28:859–864, 2020.

[17] R. Gera, F. Okamoto, C. Rasmussen, and P. Zhang. Set colorings in perfect graphs. *Mathematica Bohemica*, 136(1):61–68, 2011.

[18] L. Guangrong and Z. Limin. Total chromatic number of one kind of join graphs. *Discrete Math.*, 306:1895–1905, 2006.

[19] F. Hanna. Equitable coloring of graph products. *Opuscula Math.*, 26(1):31–44, 2006.

[20] M. Juvan, B. Mohar, and R. Škrekovski. List total colourings of graphs. *Combin. Probab. Comput.*, 7:181–188, 06 1998.

[21] H. Kaul, J. A. Mudrock, and M. J. Pelsmajer. Total equitable list coloring. *Graphs Combin.*, 34:1637–1649, 2018.

[22] A. Noga and M. Bojan. The chromatic number of graph powers. *Combin. Probab. Comput.*, 11:1–10, 1993.

[23] F. Okamoto, C. W. Rasmussen, and P. Zhang. Set vertex colorings and joins of graphs. *Czechoslov. Math. J.*, 59(4):929–941, 2009.

[24] J. Vivin, M. Venkatachalam, and N. Mohanapriya. On b-chromatic number of some line, middle and total graph families. *International J.Math. Combin.*, 1:116–125, 2016.