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On the uniqueness of continuation of a partially defined metric

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Abstract

The problem of continuation of a partially defined metric can be efficiently studied using graph theory. Let $G = G(V, E)$ be an undirected graph with the set of vertices V and the set of edges E. A necessary and sufficient condition under which the weight $w\colon E \to \mathbb{R}^+$ on the graph G has a unique continuation to a metric $d\colon V \times V \to \mathbb{R}^+$ is found.

1 Introduction

The problem of continuation of a weight $w\colon E \to \mathbb{R}^+$ defined on the set of edges $E(G)$ of the graph $G = (V, E)$ to a pseudometric was considered in [1]. In particular, it was found a set of necessary and sufficient conditions under which the weight w can be extended to a pseudometric $d\colon V \times V \to \mathbb{R}^+$, see Theorem 1.1. The set $\mathfrak{M}_w$ of all such extensions can be partially ordered, see Definition 1.1. It was shown that the shortest-path pseudometric d_w is the greatest element of $\mathfrak{M}_w$ and that the least continuation exist if and only if G is complete k-partite. The analogous problem of continuation of a weight to an ultrametric was considered in [2]. Moreover, in [2] the question of uniqueness of such continuation was studied. The aim of this paper is to fill this gap in [1], i.e., to find a uniqueness criterion for continuation of a weight to a metric (pseudometric).

Note that a problem of continuation of a weight to a metric can be reformulated as a problem of continuation of a partially defined metric. In such setting, some cases of this problem were considered earlier. For example, the free amalgamation property for finite metric spaces states that there always exists a metric on the union $X \cup Y$ of finite metric spaces (X, d_1), (Y, d_2) which agrees with d_1 on X and d_2 on Y, if $d_1 = d_2$ for elements of Z where $Z = X \cap Y$, see [3, 4]. A review of results related to extensions of continuous and uniformly continuous pseudometrics can be found in [5].

Recall the basic definitions from the graph theory, see, for example, [6]. A *graph* is a pair (V, E) consisting of a nonempty set V and a (probably empty) set E elements of which are unordered pairs of different points from V. For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called *the set of vertices* and *the set of edges*, respectively. A *path* in a graph G is a subgraph P of G whose vertices can be numbered so that

$$V(P) = \{x_0, x_1, ..., x_k\}, \quad E(P) = \{\{x_0, x_1\}, ..., \{x_{k-1}, x_k\}\},$$

where all x_i are distinct. A graph G is *connected* if any two distinct vertices of G can be joined by a path. A finite graph C is a *cycle* if $|V(C)| \geq 3$ and there exists an enumeration $(v_1, \dots, v_n)$ of its vertices such that

$$(\{v_i, v_j\} \in E(C)) \Leftrightarrow (|i - j| = 1 \text{ or } |i - j| = n - 1).$$

The edge $e = \{u, v\}$ is said to *join* u and v, and the vertices u and v are called *adjacent* in G. The graph G is *empty* if no two vertices are adjacent, i.e., if $E(G) = \varnothing$.

A *weighted graph* (G, w) is a graph $G = (V, E)$ together with a weight $w\colon E \to \mathbb{R}^+$ where $\mathbb{R}^+ = [0, \infty)$. If (G, w) is a weighted graph, then for each subgraph F of the graph G we define the weight of F as

$$w(F) = \sum_{e \in E(F)} w(e).$$

Recall also that a *pseudometric* d on the set X is a function $d : X \times X \to \mathbb{R}^+$ such that $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. The pseudometric d on X is a *metric* if, in addition, $((d(x,y) = 0) \Rightarrow (x = y))$ for all $x, y \in X$.

Let (G, w) be a weighted graph. If there exists a pseudometric d on the set $V(G)$ such that the equality

$$w(\{u,v\}) = d(u,v)$$

holds for all edges $\{u,v\} \in E(G)$, then we say that the weight w is *pseudometrizable*.

Let (G, w) be a connected weighted graph and let $u, v \in V(G)$. Denote by $\mathcal{P}_{u,v} = \mathcal{P}_{u,v}(G)$ the set of all paths joining u and v in G. Write

$$d_w(u,v) = \inf\{w(P) : P \in \mathcal{P}_{u,v}\}, \tag{1}$$

where $w(P)$ is the weight of the path P. The function d_w is known as *weighted shortest-path pseudometric*.

We need the following for the proof of the main result of this paper.

Theorem 1.1 ([1]). *Let (G, w) be a weighted graph. The following statements are equivalent.*

(i) *The weight w is pseudometrizable.*

(ii) *The equality*

$$w(\{u,v\}) = d_w(u,v)$$

holds for all $\{u,v\} \in E(G)$.

(iii) *For every cycle $C \subseteq G$ we have the inequality*

$$2 \max_{e \in E(C)} w(e) \leq w(C).$$

Definition 1.1. *Let G be a graph and let w be a pseudometrizable weight on $E(G)$. Denote by $\mathfrak{M}_w$ is the set of all pseudometrics ρ on $V(G)$ satisfying the equality*

$$\rho(u,v) = w(\{u,v\})$$

for each $\{u,v\} \in E(G)$. Let us introduce a partial order $\leqslant$ on the set $\mathfrak{M}_w$ as:

$$(\rho_1 \leqslant \rho_2) \quad \text{if and only if} \quad (\rho_1(u,v) \leqslant \rho_2(u,v)) \text{ for all } u,v \in V(G).$$

Proposition 1.1 ([1]). *Let (G, w) be a nonempty weighted graph with a pseudometrizable weight w. If G is connected, then the shortest-path pseudometric d_w belongs to $\mathfrak{M}_w$ and this pseudometric is the greatest element of the poset $(\mathfrak{M}_w, \leqslant)$, i.e., the inequality*

$$\rho \leqslant d_w$$

holds for each $\rho \in \mathfrak{M}_w$. Conversely, if the poset $(\mathfrak{M}_w, \leqslant)$ contains the greatest element, then G is connected.

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2 Uniqueness of continuation

Clearly, if a weighted graph (G, w) is not connected, then the continuation of the weight w is not unique. Thus, it has sense to consider only connected graphs G. For the convergent sequence (a_n) of reals we write $a_n \to a - 0$ if $a_n \to a$ and $a_n \leqslant a$ for all $n \in \mathbb{N}$. If, for a weighted graph (G, w), the shortest-path pseudometric d_w is a metric, then we say that w is *metrizable*.

Theorem 2.1. *Let (G, w) be a connected weighted graph with the metrizable weight w. The set $\mathfrak{M}_w$ consists only of one element ρ if and only if for every two non-adjacent vertices $u, v \in V(G)$ there exists a sequence (P_n) of paths $P_n \in \mathcal{P}_{u,v}$ (possibly stationary) such that*

$$2 \max_{e \in P_n} w(e) - w(P_n) \to d_w(u, v) - 0, \quad \text{as} \quad n \to \infty. \tag{2}$$

Moreover, if condition (2) holds and w is metrizable weight, then $\rho(u, v) = d_w(u, v)$.

Proof. Let condition (2) hold and let w be a metrizable weight. Definition of metrizability of w means that $d_w(u, v) > 0$ for all $u, v,\ u \neq v$. Suppose there exists $\rho' \in \mathfrak{M}_w$ such that $\rho' \neq d_w$. Then by Proposition 1.1 the inequality $\rho'(u, v) = b < d_w(u, v)$ holds for some non-adjacent $u, v \in V(G)$.

According to (2) for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$2 \max_{e \in P_{n_0}} w(e) - w(P_{n_0}) > d_w(u, v) - \varepsilon\,. \tag{3}$$

Let ε be such that

$$d_w(u, v) > b + \varepsilon\,. \tag{4}$$

It follows from (3) and (4) that

$$2 \max_{e \in P_{n_0}} w(e) - w(P_{n_0}) > b.$$

Transform this inequality to the following

$$\max_{e \in P_{n_0}} w(e) > w(P_{n_0}) - \max_{e \in P_{n_0}} w(e) + b. \tag{5}$$

Consider the weighted cycle (C_{n_0}, w') such that $E(C_{n_0}) = E(P_{n_0}) \cup \{u, v\}$, $w'(e) = w(e)$, $e \in P_{n_0}$ and $w'\{u, v\} = b$. The value in the right side of inequality (5) is the sum of all weights of edges of the cycle C_{n_0} except the maximal weight. Hence, $\max\limits_{e \in P_{n_0}} w(e)$ is the maximal weight of the cycle C_{n_0}. Inequality (5) implies

$$2 \max_{e \in Cn_0} w'(e) > w'(C_{n_0})$$

and according to condition (iii) of Theorem 1.1 this contradicts to the fact that ρ' is a pseudometric.

Conversely, let the set $\mathfrak{M}_w$ consist only of one element. By Proposition 1.1 this element is d_w. Suppose that for some non-adjacent $u, v \in V(G)$ and for all sequences (P_n) of paths

$P_n \in \mathcal{P}_{u,v}$ condition (2) does not hold. Let us show that there exists another continuation $\tilde{\rho} \in \mathfrak{M}_w$ of the weight w such that $\tilde{\rho}(u,v) < d_w(u,v)$.

By supposition, there exists $\varepsilon > 0$ such that for all paths $P \in \mathcal{P}_{u,v}$ the relation

$$q(P) \in (-\infty, d_w(u,v) - \varepsilon) \cup (d_w(u,v), +\infty)$$

holds, where

$$q(P) = 2 \max_{e \in P} w(e) - w(P). \tag{6}$$

Suppose that for some P the relation $q(P) \in (d_w(u,v), +\infty)$ holds. Consequently, (6) implies

$$2 \max_{e \in P} w(e) > d_w(u,v) + w(P). \tag{7}$$

Consider the weighted cycle (C, w') defined as $E(C) = E(P) \cup \{u,v\}$, $w'(e) = w(e)$, $e \in E(P)$, $w'(\{u,v\}) = d_w(u,v)$. Inequality (7) implies

$$2 \max_{e \in C} w'(e) > w'(C).$$

Hence, according to condition (iii) of Theorem 1.1 the function d_w is not a metric, which is false by the supposition of the theorem. Consequently, the relation $q(P) \in (d_w(u,v), +\infty)$ is impossible

Suppose now that for every P we have $q(P) \in (-\infty, d_w(u,v) - \varepsilon)$. Which means

$$-\infty < 2 \max_{e \in P} w(e) - w(P) < d_w(u,v) - \varepsilon\,.$$

Hence, we have

$$2 \max_{e \in P} w(e) < w(P) + d_w(u,v) - \varepsilon\,. \tag{8}$$

Let r be any real number from the interval $(d_w(u,v) - \varepsilon, d_w(u,v))$ and let a weighted cycle $(C, \tilde{w})$ be such that $E(C) = E(P) \cup \{u,v\}$, $\tilde{w}(e) = w(e)$ for all $e \in E(P)$ and $\tilde{w}(\{u,v\}) = r$.

Consider first the case where $\max_{e \in P} w(e) > r$. Hence, $\max_{e \in P} w(e)$ is a maximal weight in the cycle C. Inequality (8) implies

$$2 \max_{e \in C} \tilde{w}(e) = 2 \max_{e \in P} w(e) < w(P) + r = \tilde{w}(C).$$

By condition (iii) of Theorem 1.1 this means that the cycle $(C, \tilde{w})$ is metrizable.

Under the supposition $\max_{e \in P} w(e) \leqslant r$ we see that $\{u,v\}$ is an edge of maximal weight in the cycle C. Hence,

$$2\tilde{w}(\{u,v\}) = 2r < r + d_w(u,v) \leqslant r + w(P) = \tilde{w}(C).$$

And the cycle C is also metrizable.

Thus, the cycle $(C, \tilde{w})$ is metrizable for every $P \in \mathcal{P}_{u,v}$.

Consider a weighted graph (G', w') such that $V(G') = V(G)$, $E(G') = E(G)$, $w'(e) = w(e)$ for every $e \in E(G)$ and $w'(\{u,v\}) = r$. By condition (iii) of Theorem 1.1 the weight w' is metrizable since all the cycles in G' are metrizable. The shortest-path pseudometric $d_{w'}$ considered on the graph G' is one of the continuations of w' which is a metric. It is clear that $d_{w'}$ is also a continuation of the weight w. Hence, there exist at least two different continuations $d_{w'}$ and d_w of the weight w, since $r = d_{w'}(\{u,v\}) < d_w(\{u,v\})$. □

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Note that condition (2) of Theorem 2.1 can be reformulated as follows: for every two non-adjacent vertices $u, v \in V(G)$ the relation

$$\sup_{P \in \mathfrak{P}_{u,v}} \{2 \max_{e \in P} w(e) - w(P)\} = d_w(u, v) \tag{9}$$

holds.

Corollary 2.2. *Let (G, w) be a connected weighted graph with a pseudometrizable weight w. The weight w admits the unique continuation to a pseudometric which is not a metric if and only if the two following conditions hold simultaneously:*

(i) There exist $u, v \in V(G)$, $u \neq v$, such that $d_w(u, v) = 0$;

(ii) For every non-adjacent $u, v \in V(G)$ such that $d_w(u, v) \neq 0$ condition (9) holds.

Moreover, if ρ is the unique continuation of w, then $\rho(u, v) = d_w(u, v)$.

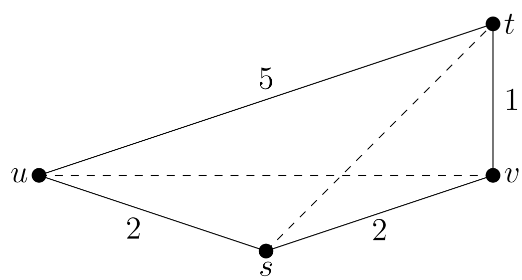

Figure 1: An example of a finite weighted graph (G, w) with the unique continuation.

Example 2.1. Let us consider a weighted graph (G, w) such that $V(G) = \{u, v, s, t\}$, $w(\{u, t\}) = 5$, $w(\{t, v\}) = 1$, $w(\{v, s\}) = w(\{s, u\}) = 2$. Consider infinite stationary sequences of paths $P_n = \{s, u, t\}$, $Q_n = \{u, t, v\}$ for every $n \in \mathbb{N}$, see Figure 1. Since

$$2 \max_{e \in P_n} w(e) - w(P_n) = 2 \cdot 5 - (5 + 2) = 3 = d_w(s, t)$$

and

$$2 \max_{e \in Q_n} w(e) - w(Q_n) = 2 \cdot 5 - (5 + 1) = 4 = d_w(u, v)$$

for every n, by Proposition 2.1 a continuation of the weight w to a metric is unique.

Example 2.2. Let us consider an infinite case. Let (G, w) be a weighted graph with $V(G) = \{u, v, x_1, x_2, ...\}$ and let $w(\{u, x_1\}) = w(\{x_1, v\}) = 2$ and $w(\{u, x_n\}) = 5$, $w(\{v, x_n\}) = 1 + 1/n$, $n = 2, 3, ...$, see Figure 2. By condition (iii) of Theorem 1.1 the weight w is pseudometrizable. Moreover, it is easy to see that it is also metrizable.

Consider a weighted graph $(\widetilde{G}, \widetilde{w})$ such that $V(\widetilde{G}) = V(G)$, and the set of edges $E(\widetilde{G})$ consists of all distinct unordered pairs of points from $V(G)$ except the pair $\{u, v\}$. For

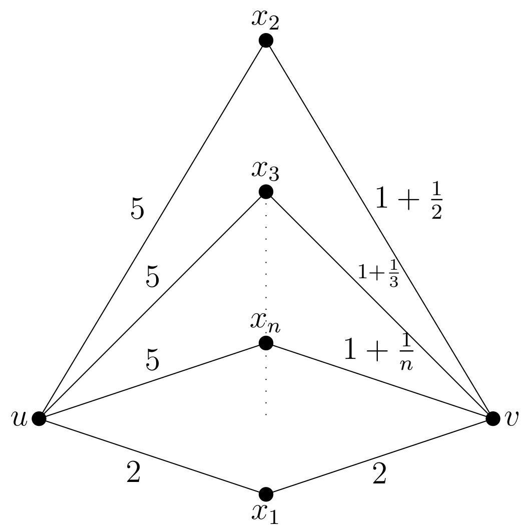

Figure 2: A weighted graph (G, w) with $\mathrm{card}(V(G)) = \aleph_0$.

every edge $\{x, y\} \in E(\widetilde{G})$ define its weight $\widetilde{w}(\{x, y\}) = d_w(u, v)$ where d_w is a shortest-path pseudometric of the graph (G, w), see (1).

Consider a sequence of paths $P_n = \{u, x_n, v\}$. Since the relation

$$2 \max_{e \in P_n} \widetilde{w}(e) - \widetilde{w}(P_n) = 2 \cdot 5 - (5 + 1 + \frac{1}{n}) \to 4 - 0 = d_{\widetilde{w}}(u, v) - 0, \quad \text{as} \quad n \to \infty,$$

holds, we have that the weight $\tilde{w}$ has a unique continuation to a metric ρ which must be established only for the edge $\{u, v\}$, i.e., $\rho(u, v) = d_{\tilde{w}}(u, v) = 4$.

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