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Black-Scholes Equation and Heat Equation

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BLACK-SCHOLES EQUATION AND HEAT EQUATION

An Honors Thesis submitted in partial fulfillment of the requirements for Honors in the Department of Mathematical Sciences

by

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Under the mentorship of Dr. Enkeleida Lakuriqi

ABSTRACT

First, we present and define the Black-Scholes equation which is used to model assets on the stock market. After that, we derive the heat equation that describes how the temperature increases through a homogeneous material. Finally, we detail how the two equations are related. We introduce and relate the Black-Scholes equation and Heat Equation.

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CHAPTER 1

INTRODUCTION

Finance is one of the fastest growing and continuously evolving areas of modern banking. The widespread reaches of finance touch and shape the world on a global as well as personal scale. Given the importance and ubiquitous nature of the area it is necessary and curious to understand such a field through the eyes of mathematics. With mathematics as the essential foundation, we can attempt to uncover the mechanics behind one of the modern world's largest driving forces.

Figure 1.1: Many assets on the Stock Market – as here in Frankfurt – are modelled using the Black-Sholes equation. Source [1]

In order to understand and analyze the construction of financial instruments we will discuss what is arguably the most impactful mathematical model in modern finance: the Black-Scholes Equation. The Black Scholes equation is a partial differential equation that was developed in the 1970's as a tool to value the price of a call or put option over time. Acclaimed for it simplicity and accessibility, the equation transformed markets and catalyzed advances in the field of financial mathematics. Moreover, the Black-Scholes equation has a very curious origin as it is also a two dimensional heat equation. From this connection stems a direct correlation between

the price fluctuations of an option and the movement of heat through a conducting medium. We will verify this connection, motivating future exploration towards the exact translation of financial phenomena into natural phenomena and vice versa [3].

CHAPTER 2

STOCHASTIC DIFFERENTIAL EQUATIONS

Stochastic differential equations serve as an essential and powerful tool for modeling the evolution and behavior of natural phenomena. In order to analyze and understand the models of such phenomena, particularly in finance, we must first introduce the foundations upon which such models are constructed. A **stochastic differential equation** is simply a differential equation where one or more of the terms is a stochastic process. A **stochastic process** can be defined as a set of random variables which express the evolution or development of a system over time. The **random variables** that define a stochastic process are variables that are subject to changes in value due to chance. Stochastic differential equations are effective in modeling real world systems largely due to the employment of random variables. These variables essentially account for just that, randomness. This allows for probabilistic analysis of patterns which yield important information about the behavior and trends of the system.

Given the defining characteristics of a stochastic differential equation it is not hard to imagine the far reaching applications in the field of Finance. Stochastic differential equations are particularly useful in considering the price fluctuations of an asset. An **asset** is an economic resource, tangible or intangible that be used to produce economic value. In considering the movement of such an asset, a reasonable assumption is that fluctuations in the value of an asset can be a consequence of a deterministic portion as well as a random portion. It is the modeling of the random portion of the asset price that lends itself nicely to stochastic differential equations. Another intrinsic property of assets which allows for the use of such differential equations is the type of stochastic process exhibited. In finance nearly all stochastic processes are considered a Markov process. A Markov property is simply a

stochastic differential equation that possesses the **Markov property**. In the realm of asset valuation, this means that the future value of a asset is completely independent of the past asset value. The Markov property is an interesting characteristic in financial models because it implies that all past information of an asset is encoded in the present value. This concept is known as market efficiency and is a fundamental principle in the field of Finance. This implication highlights a profound consequence which is that, based on the Markov property, it is impossible to beat or outperform the market as all relevant information is considered and incorporated by nature of the asset itself.

2.1   Stochastic Differential Equations and the Black-Scholes Equation

Now that we have laid the general framework for the purpose and application of stochastic differential equations, we will now view the the stochastic process in the light of the Black-Scholes equation. The Black-Scholes equation surfaced as a revolutionary tool used in the valuations of European call/put options. The equation derives it use from a simple construction and accessible variables, but would be meaningless if not for the stochastic process which is employed. The essence of the Black-Scholes equations stems from the stochastic dynamic of options, as well as other financial derivatives. The exact origins of the Black-Scholes equation will be presented through the derivation in the next chapter, but for now we will consider the stochastic differential equation at the core of the Black-Scholes equation.

$$dS = \sigma S dX + \mu S dt$$

This equation is a simple model describing the evolution of an asset price, S, over time. More precisely, the equation yields an infinitesimal change in S by an amount dS given a number of variables. We observe that dS is composed of two

portions, and as discussed earlier, one is random and the other predictable. It is the random portion $\sigma S dX$ that makes the equation stochastic. An equation is stochastic if at least one term is a stochastic process. Here σdX is the stochastic process in the equation where: σ is the volatility of the asset.

Figure 2.1: Evolution of the Dow Jones Index over the last twelve month. Apparent in the picture is the global variations together with the fluctuations due to the stochastic part. Source [2]

The **volatility** is a value that encodes the instability or risk of return associated with the asset. This makes sense if we consider an asset with 0 volatility, meaning absolutely no risk. Letting $\sigma = 0$ eliminates the random portion of the equation leaving an asset model that is wholly deterministic. However, it is the term dX which is most interesting in discussing the stochastic nature of the differential equation. The dX term is a sample of a random variable from a normal distribution. What distinguishes a random variable from a variable in the usual mathematical sense is that a random variable can take on many different values which each come with an associated probability. The values that can be associated to a random variable come from a distribution, which in this case is a normal distribution. In the context of this stochastic differential equation, dX is a time continuous stochastic process called a **Wiener process** which is defined by the following properties:

- dX is a random variable from a normal distribution.

- The mean of dX is 0.

- The variance of dX is dt

The Wiener process, also called **Brownian motion**, is a type of stochastic process that exhibits the Markov property. Brownian motion, a particular type of Markov process, is considered a stochastic process given that is defined as a physical phenomenon. Brownion motion is the random movement of a particle suspended in fluid caused by collision with fast moving molecules on the fluid. Many financial models consider this movement as a defining characteristic of the asset being modeled. We define dX, the Brownian motion, explicitly,

$$dX = \phi \sqrt{dt}$$

Where ϕ is a random variable from a standardized normal distribution and $\sqrt{dt}$ is the standard deviation of the distribution. This is evident if we consider the properties of dX above and the fact the the standard deviation is simply the square root of the variance. In considering dX, the stochastic process, the standard deviation and variance give essential information on the distribution of values, however we must look at the sample space from which these value come as well as how probabilities are associated with each value. This is given by a probability density function which is a function that describes the likelihood of a random variables to take on a given value. The probability density function in this case is that of a standardized normal distribution.

This distribution is characterized by a few properties; the distribution has 0 mean, unit variance and has the probability density function:

$$\frac{1}{2\pi} e^{-\frac{1}{2}\phi^2}$$

Figure 2.2: A normal distribution [4]
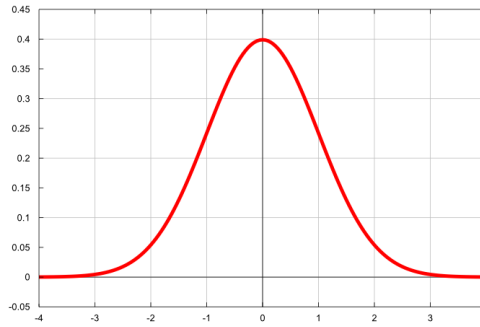
Given the probability density function, the probability of a random variable landing in a certain range is given by the integral of the variable's probability density function over the range.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\phi)e^{-\frac{1}{2}\phi^2}d\phi$$

This expression also defines the expectation operator of a random variable.

$$\mathbf{E}[F(\cdot)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\phi)e^{-\frac{1}{2}\phi^2}d\phi$$

The expected value of a random variable is the average long term value after a large number of repetitions. The expectation operator allows us to look at the average value that a random variable will take on over a period of time. Let us consider the expectation operator in the context of the random variable ϕ,

$$\mathbf{E}[\phi] = 0$$

and

$$\mathbf{E}[\phi^2] = 1$$

These properties are logical if we recognize that ϕ is drawn from a standardized normal distribution that has a zero mean and unit variance. The second property follows directly given a formal definition of the variance. The variance of a random

variable ϕ is defined as the expected value of of the squared deviation from the mean. Following this definition,

$$
\begin{aligned}
Var[\phi] &= \mathbf{E}[(\phi - \mathbf{E}[\phi])^2] \\
&= \mathbf{E}[\phi^2 - 2\phi\mathbf{E}[\phi] + \mathbf{E}[\phi]^2] \\
&= \mathbf{E}[\phi^2] - 2\mathbf{E}[\phi]\mathbf{E}[\phi] + \mathbf{E}[\phi]^2 \quad \text{(by linearity of the operator)} \\
&= \mathbf{E}[\phi^2] - 2\mathbf{E}[\phi]^2 + \mathbf{E}[\phi]^2 \\
&= \mathbf{E}[\phi^2] - \mathbf{E}[\phi]^2 \\
&= \mathbf{E}[\phi^2] \quad \text{(as } \mathbf{E}[\phi] = 0)
\end{aligned}
$$

Now we can consider the expected value in terms of the stochastic differential equation in order to gain insight into the long term behavior of the asset price.

$$
\mathbf{E}[dS] = \mathbf{E}[\sigma S dX + \mu S dt] = \mu S dt \quad \text{(since } \mathbf{E}[dX] = 0)
$$

This yields an important implication considering the fact the $\mu S dt$ is entirely deterministic. This implies that "on average" the next value for S in a sequence of infinitesimal time increments is higher than the previous value by an amount $\mu S dt$. In other words, by taking the expected value of the stochastic differential equation, we are able to calculate, on average, by just how much the asset in question will grow. This is an important consequence because it means that, given the necessary conditions, we can forecast the general behavior of an asset.

We are also able to compute another useful quantity defined by the expectation operator which is the variance of dS. The variance is useful because it describes the average distance of each value in a distribution from the mean. But. before considering the variance of dS, an important result must be defined: **Itô's Lemma** is a fundamental identity used in the treatment of random variables that relates a change in a function containing a random variable to a change in the random variable

itself. Itô's Lemma is essential because it defines how to differentiate functions of two variables that contain a Brownian motion. In other words, Itô's Lemma is to a random variable as the chain rule to a deterministic variable. Before continuing, we will present an informal, but revealing derivation contextualizing this important result in the case of our particular stochastic differential equation as well as the derivation of the Black-Scholes equation presented in the next chapter.

Consider $f(t, dX)$ where f is a function of time as well as a random variable dX, keeping consistent with the notation used. By Taylor expansion to degree 2,

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial dX}dX + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}dt + \frac{\partial^2 f}{\partial dX^2}dX\right)^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial dX}dX + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial t^2 \partial dX^2}dtdX + \frac{1}{2}\frac{\partial^2 f}{\partial dX^2}dX^2 \\
&+ \text{Higher Order Terms}
\end{aligned}
$$

Keeping only first degree dt terms and dX terms with degree ≤ 2.

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial dX}dX + \frac{1}{2}\frac{\partial^2 f}{\partial dX^2}dX^2 \\
&= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial dX^2}\right)dt + \frac{\partial f}{\partial dX}dX
\end{aligned}
$$

Here it becomes clear that through the Taylor expansion Itô's Lemma creates a link between the function and the random variable allowing manipulations with random variables to be a possibility. Now that the process is clear, we can understand Itô's Lemma as a tool in considering the variance of dS,

$$
Var[dS] = \mathbf{E}[dS^2] - \mathbf{E}[dS]^2
$$

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If we expand dS^2 and apply Itô's Lemma,

$$dS^2 = \sigma^2 S^2 dX^2$$

$$= \sigma^2 S^2 dt \qquad (\text{given that } dX^2 \longrightarrow dt \text{ as } dt \longrightarrow 0)$$

This implies,

$$Var[dS] = \mathbf{E}[\sigma^2 S^2 dt] - \mathbf{E}[\mu^2 S^2 dt^2]$$

$$= \mathbf{E}[\sigma^2 S^2 dt] \qquad (\text{as } dt \longrightarrow 0)$$

$$= \mathbf{E}[dS^2]$$

In other words, if we consider the distribution of small changes in the asset S the distance of these random variables away from the mean will be $\mathbf{E}[dS^2]$.

Faced with the question of how to most accurately model natural phenomena, a universal challenge presents itself across all applications. The problem is how to address the ever present element of randomness that occurs in real life. In the context of financial assets, while there are some elements of an asset price that can be easily accounted for, it is known that random fluctuations exist and cannot be ignored. This is evident in simply looking at plots of stock prices; how can these sharp, jagged movements that characterize sudden changes in the value of a derivative be accounted for in a mathematical model. The answer is through the use of a stochastic process. We have discussed the applications and importance of such processes given the construction of random variables. Random variables are the tools that are available to

model the element of randomness that exists. These variables are subject to probabilistic analysis which allows for a wealth of information to be extracted. While such analysis does not imply predictability, it does shed light on the long term behavior and quantitative trends through statistical tools.

CHAPTER 3

THE BLACK-SCHOLES EQUATION

3.1 History of the Black-Scholes Equation

In this chapter we will examine the Black-Scholes equation which is a powerful partial differential equation that evaluates the price of a financial derivative. The impact of the Black-Scholes equation cannot be over stressed as it forever changed the face of modern Finance and financial markets. Here we will consider where the Black-Scholes originated from and the consequences. As the name suggests, the Black-Scholes equation was first published by two economists Fischer Black and Myron Scholes in their 1973 paper "The Pricing of Options and Corporate Liabilities". Black, a financial consultant, and Scholes, a young assistant professor of Finance, conceived the idea for the model at MIT in the early 1970's. In the paper Black and Scholes derive a partial differential equation which serves as a model for the valuation of an option over a period of time. In addition to Black and Scholes, Robert Merton is also credited with the success of the model as he was the first to publish a paper deepening the mathematical understanding of the model. In 1997, both Merton and Scholes received the Nobel Prize in Economics for their work on the Black-Scholes model. Black had passed away two years prior, but was recognized for his contributions.

Before the Black-Scholes equation invaded financial markets, few people traded in financial derivatives as traders could only speculate as to their value. Having no way to value these derivatives created risk that made the practice of trading with these options nothing more than a gamble. The reason that Black-Scholes had such a profound impact on financial markets was because the model was able to value these financial derivatives. Options were no longer bets on an asset, but could be considered assets themselves. People were able to trade on these options now knowing what their value was. This innovation spawned the creation of derivatives markets and options

exchanges that are worth $710 trillion today. In the remaining portions of the section we will present and explain the pieces of the Black-Scholes equation and its origins to better understand just how this machinery transformed modern financial markets.

3.2 The Evolution of an Asset

Before delving into the behavior of an asset we will first expound on the meaning and implications of a financial asset and its properties. As mentioned earlier, an asset is an economic means that maintains a value which motivates market behavior. Assets come in many different shapes and sizes and can be used in many different ways. People buy, trade, and sell assets all under the premise that doing so will produce positive economic value. Assets can be thought of as a kind of economic currency, and similar to paper currency the value changes with time, sometimes drastically. Consider a common type of asset, stock. A stock is a type of asset that represents partial ownership of a corporation. If I buy $20 worth of stock I possess an asset that can be used to make a profit because my stock has value. However the question of how much profit can be made exists because the value of the shares fluctuate. I can maximize my profit if I know when and how the value of my stock will change. Price optimization is the aim in all financial markets and would not be possible without mathematical models. An elementary example is presented to solidify such a relevant circumstance. If the company for which I own stock creates a new successful product the value of my shares of stock will go up. Conversely if the company reports low quarterly earnings the share price will go down. We can see from the image on the next page a graphical representation depicting changes in the value of a particular asset, which in this case is Coca-Cola stock. Notice how sporadic and random portions of the graph are. This behavior is precisely the characteristic that makes mathematical

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models so difficult. Financial mathematical models attempt to model the evolution of an asset in order to forecast future values and identify important trends.

Figure 3.1: Plot of KO stock price with respect to time [5]

Consider an asset price S whose evolution we will try to model. In order to look at this phenomenon we will associate a return, or the relative change in price. Lets say at time t the value of the asset is S and on a subsequent time interval, $t + dt$, we have $S + dS$. This represents a change in S by and infinitesimal amount dS over an infinitesimal interval of time, dt. In modeling our return on the asset we consider dS relative to S, as this scaling preserves proportion, giving $\frac{dS}{S}$. The most basic financial models break down the return into two portions: the random portion and the deterministic portion, where the latter is analogous to a risk free return on investment. This portion we denote μdt and is typically a constant in basic financial models, however μ can be a function in more complex models. The second portion contains the randomness of the asset price denoted σdX. Here σ is the volatility of the asset which is formally defined as a measure of the standard deviation of the returns of the asset. More informally, the volatility of an asset can be thought of as the amount of risk associated with the value of an asset. Asset prices are not static and some fluctuate more so than others which creates an element of risk when

investing; σ accounts for this risk. The standard deviation, which defines volatility, is defined as the dispersion of data points, which represent historical returns, from the mean. In other words, given the historical returns of an asset over a period of time and considering their distribution we can see the volatility. The volatility will be expressed through the dispersion of data points from the mean. So the greater the dispersion the higher the volatility. Asset volatility is an important and curious topic for which we will devote more time later in the chapter. Note that σ is then scaled by dX which is a random sample from a normal distribution. The choice of a sample from a normal distribution results from the modeling of a random and volatile phenomenon. It is obvious that seemingly random fluctuations exist in the market, but the distributions cannot be predicted. However, it is known, by the Central Limit Theorem, that given sufficiently many random variables the average will converge to a normal distribution. It is for this reason that dX is defined as a random variable from a normal distribution. Bringing the portions together to model our return we have the stochastic differential equation,

$$\frac{dS}{S} = \sigma dX + \mu dt$$

This equation is an example of a random walk. A **random walk** is defined as a sequence of discrete random steps. Random walks are used in many different fields, but are omnipresent in Finance. Figure 3.1 is an example of a random walk as the graph is composed of a connected series of discrete points. Random walks are useful because they are strictly a mathematical construction that tracks real life phenomena. This makes sense given the model above as we observe the relation between our variables. Given S over a time interval dt, we consider dS as a step in the sequence. A known fact it that prices in financial markets follow a random walk. However, it is important to note precisely what this means. Random walks do not forecast asset prices deterministically, but they are able to reveal interesting information about
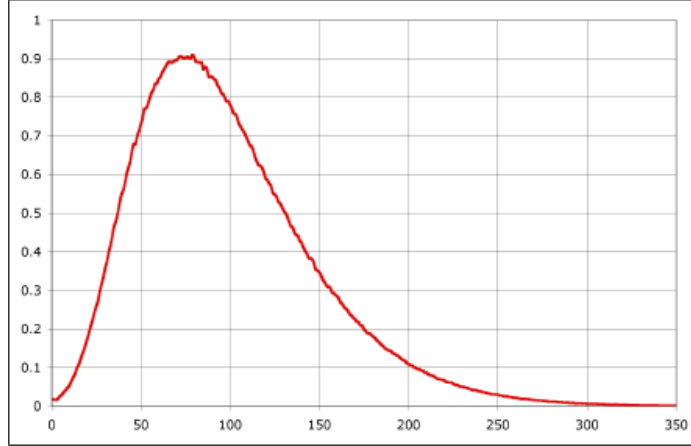
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Figure 3.2: A lognormal distribution [6]

asset behavior in a probabilistic sense. The fact that random walks are "memoryless"
characterizes their importance in such models. Random walks are indeed a Markov
process which exhibit the necessary properties. We can now consider the random walk
of the stochastic differential equation discussed. If an asset S follows the random walk
given by this equation then the random variable's probability density function is that
of a slightly skewed normal distribution called a lognormal distribution. A **lognormal**
distribution of a random variable is continuous distribution whose natural log is
normally distributed. A lognormal distribution is assumed in the case of the Black-
Scholes particularly because it does not allow for negative values. This is rather
intuitive for the reason that we cannot have a negative asset price.

3.3   Black-Scholes Derivation

In deriving and analyzing the Black- Scholes equation we will first look at the stochas-
tic differential equation from which it stems,

$$\frac{dS}{S} = \underbrace{\sigma dX}_{\text{Random}} + \underbrace{\mu dt}_{\text{Deterministic}} \tag{3.1}$$

Here, μdt is the average rate of growth of the asset price, which is predictable.

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This portion of the equation is also known as "the drift". Conversely, σdX measures the volatility of the underlying asset and is entirely random.

In order to better understand the mechanics of (3.1) we will take a function of S, which represents the value of the underlying asset. We can take $f(S)$ and expand the function, $df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2f}{dS^2}dS^2 + ...$. In doing this we see that varying S by a small amount dS is the same as varying $f(S)$. This concept is crucial in not only deriving, but also understanding the derivation of the Black Scholes equation. Furthermore, this idea captures the significance of Ito's Lemma, which serves a an essential tool in the derivation.

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2f}{dS^2}dS^2 + ... \tag{3.2}$$

In addition to this expansion, we will introduce one more result that is key in the derivation. We claim that dX^2 converges to dt as dt tends to 0.

In other words, for smaller dt, (as the time interval decreases), the more dX^2 behaves like dt. Now looking at 3.1:

$$
\begin{aligned}
dS^2 &= (S\sigma dX + S\mu dt)^2 \\
&= \sigma^2 S^2 dX^2 + 2\mu\sigma S^2 dXdt + \mu^2 S^2 dt^2
\end{aligned}
$$

Given that $dX = \sqrt{dt}$,

$$
= \sigma^2 S^2 dt + 2\mu \ \sigma S^2 dt^{\frac{3}{2}} + \mu^2 \sigma^2 dt^2
$$

as $dX = \sqrt{dt}$

We see that $\sigma^2 S^2 dX^2$ is the "leading" term as we are interested in a small dt. This results from the idea that analyzing the behavior of dS over the smallest time interval will result in the most accurate evaluation of the asset. So,

$$dS^2 = \sigma^2 S^2 dX^2 + \underbrace{\cdots}_{\text{Insignificant higher order terms}}$$

$$\Rightarrow dS^2 = \sigma^2 S^2 dX^2$$

Here the dots indicate the higher order terms not considered given that the leading term yields the smallest dt. Now substituting this into 3.2 we have,

$$
\begin{aligned}
df &= \frac{df}{dS}(S\sigma dX + S\mu dt) + \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2 dt \\
&= \sigma S dX \frac{df}{dS} + \mu S \frac{df}{dS}dt + \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2 dt \\
&= \sigma S dX \frac{df}{dS} + \left(\mu S \frac{df}{dS} + \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2\right)dt
\end{aligned}
$$

Again, in using Itô's lemma what we have done is related a small change in a the function of a variable to a small change in the variable itself. It is worth mentioning that this exact process can be done with a function of two variables say $f(S,t)$ for example. Where the independent variables are S, the value of the underlying asset, and t, time. The following is the above result for $V(S,t)$, where V is the value of a portfolio,

$$dV = \sigma S \frac{\partial V}{\partial S}dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt$$

From here we will derive the Black-Scholes equation. To do this let us construct a portfolio:

$$\Pi = V - \Delta S$$

This portfolio consists of one option V and we assign to Δ a number which scales the underlying, S. This scaling value will consequently adjust the value of the option

and this the portfolio itself. We will discuss a value for Δ later on, however a natural choice will emerge which will play an important role in the final steps of the derivation. Given our definition of the above portfolio we can say that the jump in the value of this portfolio in one time step is:

$$d\Pi = dV - \Delta dS$$

Substituting dV and dS above yields,

$$
\begin{aligned}
d\Pi &= \sigma S \frac{\partial V}{\partial S} dX + \mu S \frac{\partial V}{\partial S} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + \frac{\partial V}{\partial t} dt - \Delta S \sigma dX - \Delta S \mu dt \\
&= (\sigma S \frac{\partial V}{\partial S} - \Delta S \sigma) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + \Delta S \mu \right) dt \\
&= \underbrace{\sigma S (\frac{\partial V}{\partial S} - \Delta) dX}_{\text{random portion}} + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + \Delta S \mu \right) dt
\end{aligned}
$$

We can now see how a natural choice for Δ emerges which will eliminate the random component present in the natural walk. Clearly, this leaves us will solely the deterministic component allowing us to better forecast asset behavior.

$$\Delta = \frac{\partial V}{\partial S}$$

$$\Rightarrow d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

Here we consider a more subtle concept of supply and demand called arbitrage. Arbitrage is the process of continuously buying and selling assets in order to profit from differences in price. Because arbitrage exists in the marketplace we can equate the return of a portfolio Π with a riskless, return from an investment, denoted $r\Pi dt$.

$$\Rightarrow r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

Substituting our portfolio and our Δ into the above equation yields,

$$r(V - \Delta S)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

$$\Leftrightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Finally, we have derived the Black-Scholes equation presented above.

3.4 Implied Volatility and the 'Smile'

Apart from the Black-Schole's new and innovative approach for valuation of options, the equation has many other auxiliary effects. Many of these consequences are worth noting, however here we will focus on the concept of implied volatility and an unusual phenomenon known graphically as the 'smile'.
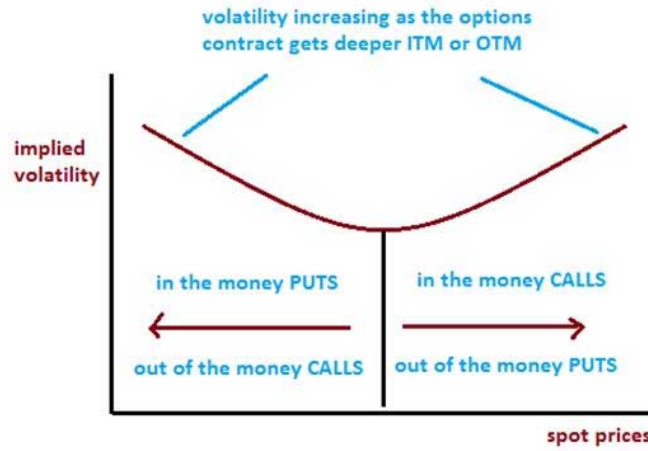
The Black-Scholes became such a powerful tool largely in part as a result of its accessible and simple construction. To illustrate this we will look at the variables present in the equation. The factors that are considered in the model of such an option are the stock price, the exercise price, the interest rate, the time to expiration, and the volatility. Most of these variables are self explanatory, however we require one definition. The **exercise price** of an option, also known as the strike price, is the value at which the owner of an option is entitled to buy or sell the asset. These are all reasonable and relevant variables to utilize due particularly to their availability. The stock price, exercise price, interest rate, and time to expiration are all current quantities that are known, making them easily employable. However the volatility is

an important factor that is not so easily employed. A direct measurement of volatility is inherently difficult to obtain given the nature of the factor. The volatility is not a constant value. This highlights a major flaw in the Black-Scholes model because in the equation σ is a constant. Regardless, the past volatility is of no use in considering the present volatility over the life of the option. Yet, it is true that options are being quoted and traded in the market on the basis of this volatility. This means that even though we may not know the volatility the market does. In considering this fact we can again employ the Black-Scholes equation only this time from a different angle. We need the stock price, exercise price, interest rate, and time to expiration, which are easy enough to locate as they are constantly quoted or prescribed in the option contract. Next we find the option price that is quoted in the market, and work backwards. Instead of solving for the option price, we are solving for the market implied volatility. This is possible as there exists a one to one correspondence between implied volatility and market price. This directly proportional relation between the two factors serves as the ground from which we are able to draw the market implied volatility. This result is powerful in that it allows us to quantify and analyze an elusive, but essential variable in the valuation of options.

While this technique does shed light on the markets view of volatility, an interesting phenomenon can be observed that highlights a flaw in the model. To best see the discrepancy between the Black-Scholes model and the real world phenomenon we introduce the 'smile'. The 'smile' is the resulting curve when implied volatility and and strike price are plotted accordingly. According to the Black-Scholes model, given the same strike price and maturity date, volatility should be held constant. In other words according to the model, implied volatility should be static across all strike prices, all else held constant. This is not the case. In reality if we plot strike price and volatility we see an upward sloping curve that looks like a smile, hence the

Figure 3.3: A description of the volatility smile. [7]

volatility increasing as the options contract gets deeper ITM or OTM

implied volatility

in the money PUTS

in the money CALLS

out of the money CALLS

out of the money PUTS

spot prices

name. This suggest that the deeper we are in or out of the money yields in increase in volatility, but this can never happen in the model, which again stresses an important flaw in the model. The 'smile' is a popular topic of current research.

CHAPTER 4

THE HEAT EQUATION

The significance of partial differential equations in financial modeling is enormous given that PDE's function as the machinery which makes these models possible. The importance of PDE's is logical considering that most often there are many dynamic factors which constitute the value of an continuously evolving asset. In order to contextualize PDE's in the light of Finance we begin by examining the well known heat equation. A basic but enlightening example of a partial differential equation, the heat equation is to PDE's what the logistics equation is to ODE's. The heat equation is a partial differential equation that models heat flow through a medium over time.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Solutions to this equation, $u(x, t)$, describe the temperature at a certain point of the medium as a function of space and time.

Before we present a derivation we must first introduce several necessary assumptions and laws essential in the derivation. The assumption that is made in considering the one dimensional heat equation is that we have long, thin uniform metal bar that is perfectly insulated on its sides such that the only way for heat to escape is at the ends. Furthermore, we assume that there is no heat source within the metal bar itself. Given the nature of the equation we can see that the flow of heat through the metal bar depends only on the distance x and the time t.

In figure 4.1; A metal bar of length l rests on the x-axis, denoting the spacial component, with time t on the vertical axis. Given our assumption of perfect insulation, heat can only be lost or gained at the boundaries 0 and l. In this instance we will assume that heat flows from left to right. Furthermore, given non-uniform temperature in the rod heat will flow from areas of higher temperature to areas of
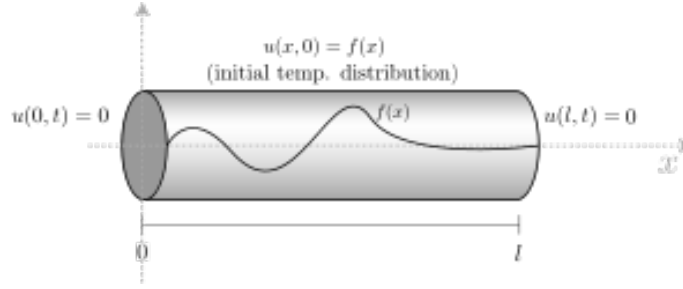
Figure 4.1: Heat traveling along a rod. [8]

lower temperature. We are able to exploit these characteristics to derive the heat equation.

First, we introduce the formula for the **Transfer of Heat Energy**.

$$dE = c \ m \ du$$

Where c is the specific heat of the substance. The specific heat is defined as the amount of energy needed to raise the temperature by one unit. Here m is the mass of the object and du is the change in temperature of the object. This formula relates the heat energy lost or gained in an object to the temperature change and consequently highlights the subtle but important distinction between heat energy and temperature. Heat energy is a measure of total energy due to molecular motion in an object. Temperature can be thought of as a measure of heat energy which is defined as the measure of the average energy of molecular motion. Next, we will introduce **Fourier's Law of Heat Transfer**.

$$\frac{dE}{dt} \propto k \ A \ \frac{du}{dx}$$

This law states the fact the the change in energy with respect to time is proportional to the change in temperature with respect to space. Furthermore, Fourier's Law can be expanded stating an equality.

$$dE = k \ A \ \frac{du(x,t)}{dx} \ dt$$

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Here, we clearly see that now we have another expression for dE defined through the change in temperature as a function of two variables.

Lastly we must define the **Law of Conservation of Energy**. Conservation of Energy states that the total energy of an isolated system remains constant or is conserved; that is, energy can neither be created nor destroyed in such a system, only transferred. The law of conservation of energy combined with our initial assumptions create the first step in deriving the heat equation.

Consider an arbitrary, thin cross-sectional slice of the metal rod with length dx. We define the temperature at the left end of the slice, $u(x,t)$, and the right end $u(x + dx, t)$. Through conservation of energy we are able to equate expressions for the change in heat energy in the rod.

$$\left(kA\frac{du(x,t)}{dx} - kA\frac{du(x+dx,t)}{dx} \right) dt = c\, m\, du(x,t)$$

On the left, we now have the net heat in the rod for a time increment dt by Fourier's Law. On the right we will rewrite the formula for the transfer of heat energy given that $m = \rho\, V$, where ρ is the density of the metal and V is the volume. A step further, we can write $V = A\, dx$. This gives an equivalent expression $m = \rho\, A\, dx$. Substituting into the above equation.

$$\left(kA\frac{du(x,t)}{dx} - kA\frac{du(x+dx,t)}{dx} \right) dt = c\ \rho\ A\ dx\ du(x,t)$$

$$\frac{d}{dx}k\left(\frac{du(x,t) - du(x+dx,t)}{dx} \right) = c\ m\ \frac{du(x,t)}{dt}$$

$$\frac{d}{dx}\left(\frac{du(x,t) - du(x+dx,t)}{dx} \right) = \frac{c\ m}{k}\ \frac{du(x,t)}{dt}$$

(Consolidating all constants as γ and taking the limit as $dx \to 0$ we get by linearity:)

$$\lim_{dx\to 0}\ \frac{d}{dx}\left(\frac{du(x,t) - du(x+dx,t)}{dx} \right) = \gamma\ \frac{du(x,t)}{dt}$$

$$\frac{d}{dx}\ \lim_{dx\to 0}\ \left(\frac{du(x,t) - du(x+dx,t)}{dx} \right) = \gamma\ \frac{du(x,t)}{dt}$$

$$\frac{d^2u(x,t)}{dx^2} = \gamma\frac{du(x,t)}{dt}$$

CHAPTER 5

CORRESPONDENCE

Given what we know about the Black-Scholes equation and the heat equation it is now possible to analyze the correlation between the two equations. The well known heat equation has many parallels with the Black-Scholes equation. This similarity is the reason why such a profound correspondence exists. We are able to gain perspective into the Black-Scholes equation by exploring it in the form of the heat equation and ultimately deriving a solution. In order to do this we will first convert the Black-Scholes equation into the heat equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, 0 \leq S, 0 \leq t \leq T$$

Given S as the value of the underlying, t representing time, and E as the strike price of the call option, the equation has a terminal boundary condition giving the price of the call option.

$$V(S,T) = f(S) = \max(S - E, 0)$$

Also, $V(0,t) = 0$ considering that the option becomes worthless if the value of the underlying drops to 0. Without any manipulations we can compare the Black-Scholes equation and the heat equation noticing the biggest similarity is that both equations contain a first order derivative with respect to time and up to a second order derivative with respect to space, considering S as the spacial equivalent. An initial and well motivated transformation of variables will highlight the likeness.

$$S = e^x, \ t = T - \frac{\tau}{\sigma^2}$$

$$V(S,t) = v(x, \tau) = v\left(ln(S), \frac{\sigma^2}{2}(T - t)\right)$$

Calculating the respective derivatives present in the original equation.

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}$$

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial s}\left(\frac{\partial v}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) \\
&= -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}\frac{\partial x}{\partial S}\frac{\partial v}{\partial x} \\
&= -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial}{\partial x}\frac{\partial v}{\partial x} \\
&= -\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}
\end{aligned}$$

Substituting the new derivatives back into the Black-Scholes equation and grouping like terms.

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v$$

Since we have combined parameters we can now let $\kappa = \frac{2r}{\sigma^2}$ and replace τ with t. Also, we redefine the boundary conditions and the domain of the new variables.

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (\kappa - 1)\frac{\partial v}{\partial x} - \kappa v$$

We now have an equation with constant coeffecients that is defined on the interval $-\infty < x < \infty$. Note that the old interval $0 < S < \infty$ is still defined on the new

interval. The time interval is rescaled with a new upper bound $0 \le t \le \frac{\sigma^2}{2}T$. An added consequence of the substitution is the transformation of the Black-Scholes terminal boundary condition to an initial condition relevant to the heat equation. From our substitution of S.

$$v(x,0) = V(e^x, T) = f(e^x) = \max(e^x - E, 0)$$

At this point it is noticeable that the new equation is quite similar to the heat equation, the biggest exception being the two terms on the right hand side of the equation. We eliminate these terms through one more substitution of variables.

$$v(x,t) = \underbrace{e^{\alpha x + \beta t}}_{\phi} \underbrace{(x,t)}_{u} = \phi u$$

We will again compute the corresponding derivatives.

$$\frac{\partial v}{\partial t} = \beta \phi u + \phi \frac{\partial u}{\partial t}$$

$$\frac{\partial v}{\partial x} = \alpha \phi u + \phi \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\alpha \phi u + \phi \frac{\partial u}{\partial t}\right)$$
$$= \alpha^2 \phi u + 2\alpha \phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}$$

Substituting the derivatives into the recently transformed PDE we gather like terms and can cancel the common $e^{\alpha x + \beta t}$ as it is an exponential function which is always positive.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + [2\alpha + (k-1)]\frac{\partial u}{\partial x} + [\alpha^2 + (k-1)\alpha - k - \beta]u$$

Since α and β are arbitrary constants we choose them appropriately in order to eliminate the $\frac{\partial u}{\partial x}$ and u terms.

$$
\begin{aligned}
\alpha &= -\frac{k-1}{2} \\
\beta &= \alpha^2 + (k-1)\alpha - k = -\frac{(k+1)^2}{4}
\end{aligned}
$$

Subsequently α and β create coeffecients of 0 which eliminate the $\frac{\partial u}{\partial x}$ and u terms respectively. The equation is then reduced to the one dimensional heat equation.

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,\ 0 \le t \le \frac{\sigma^2}{2}T
$$

We now need to transform the initial condition.

$$
u(x,0) = e^{-\alpha x}v(x,0) = e^{-\alpha x}f(e^x)
$$

For a call option with strike price E, $f(x) = \max(x - E, 0)$, so

$$
u(x,0) = e^{-\alpha x}f(e^x) = e^{-\alpha x}\max(e^x - E, 0)
$$

We can now employ the well known heat equation solution representation formula for the Black-Scholes equation.

$$
u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-\frac{(x-s)^2}{4\tau}}\,ds
$$

In order to integrate we perform a change of variables. Let $z = \frac{(s-x)}{\sqrt{2\tau}} \Rightarrow dz = -\frac{1}{\sqrt{2\tau}}dx$

$$
u(x,\tau) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} u_0(z\sqrt{2\tau} + x)e^{-\frac{z^2}{2}}\,dz
$$

Note that $u_0(s)$ is positive only if the argument $s > 0$, that is $z\sqrt{2\tau} + x > 0$ or $z > -\frac{x}{\sqrt{2\tau}}$. On this domain $u_0 > 0$. Given that $u(x, 0) = e^{-\alpha x}\max(e^x - 1, 0)$.

$$u_0 = e^{-\frac{(k-1)}{2}}\left(e^x - 1\right) = e^{\frac{(k+1)}{2}(x+z\sqrt{2\tau})} - e^{\frac{(k-1)}{2}(x+z\sqrt{2\tau})}$$

Given the new construction of u_0 we consider two separate integrals.

$$\underbrace{\frac{1}{2\sqrt{\pi}}\int_{-\frac{x}{2\tau}}^{\infty} e^{\frac{(k+1)}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}}\,dz}_{I_1} - \underbrace{\frac{1}{2\sqrt{\pi}}\int_{-\frac{x}{2\tau}}^{\infty} e^{\frac{(k-1)}{2}(x+z\sqrt{2\tau})}e^{-\frac{z^2}{2}}\,dz}_{I_2}$$

We now have two separate integrals I_1 and I_2 which we can evaluate. We will evaluate I_1 first. Working in the exponent of I_1 we complete the square to simplify the process.

$$\frac{(k+1)}{2}(x + z\sqrt{2\tau}) - \frac{z^2}{2}$$

$$= -\frac{z^2}{2} + \frac{(k+1)\sqrt{2\tau}z}{2} + \frac{(k+1)x}{2}$$

$$= \left(-\frac{1}{2}\right)\left(z^2 - (k+1)\sqrt{2\tau}z\right) + \frac{(k+1)x}{2}$$

$$= \left(-\frac{1}{2}\right)\left(z^2 - (k+1)\sqrt{2\tau}z + \frac{(k+1)^2\tau}{2}\right) + \frac{(k+1)^2\tau}{4} + \frac{(k+1)x}{2}$$

$$= \left(-\frac{1}{2}\right)\left(z - (k+1)\sqrt{\frac{\tau}{2}}\right)^2 + \frac{(k+1)^2\tau}{4} +$$

Putting the exponent back into I_1 we can pull out the last two terms as they are constants with respect to z.

$$\frac{e^{\frac{(k+1)^2\tau}{4}} + e^{\frac{(k+1)x}{2}}}{\sqrt{2\pi}}\int_{-\frac{x}{2\tau}}^{\infty} e^{-\frac{1}{2}(z-(k+1)\sqrt{\frac{\tau}{2}})^2}\,dz$$

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One more change of variables is required to complete the integral.

$$y = z - \sqrt{\frac{\tau}{2}}(k+1) \quad \Longrightarrow \quad dy = dz$$

$$\frac{e^{\frac{(k+1)^2\tau}{4}} + e^{\frac{(k+1)x}{2}}}{\sqrt{2\pi}} \int_{-\frac{x}{2\tau} - \sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dz$$

Drawing from chapter 2, notice that the integrand is just the probability density function of a normal distribution. The integral can be represented as the cumulative distribution function of a normal random variable, d. The **cumulative distribution function**, ϕ, of a normal random variable, d, is the probability that ϕ will take a value less than or equal to d. For our particular instance we consider the cumulative distribution function in a continuous case which is just the area under the probability density function from $-\infty$ to x.

$$\phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Therefore,

$$I_1 = e^{\frac{(k+1)^2\tau}{4}} + e^{\frac{(k+1)x}{2}} \phi(d_1), \text{ where } d_1 = \frac{x}{2\tau} + \sqrt{\frac{\tau}{2}}(k+1\phi(d_2))$$

I_2 is computed the exact same way except that $(k+1)$ is replaced with $(k-1)$. Therefore, the solution to this heat equation initial value problem.

$$u(x,\tau) = e^{\frac{(k+1)^2\tau}{4}} + e^{\frac{(k+1)x}{2}} \phi(d_1) - e^{\frac{(k-1)^2\tau}{4}} + e^{\frac{(k-1)x}{2}} \phi(d_2)$$

We now must retrace our steps and undo each change of variables, starting with $v(x,\tau) = e^{-\frac{(k+1)^2\tau}{4} - \frac{(k+1)x}{2}} u(x,\tau)$. Combined and simplifying the left hand side.

$$v(x,\tau) = e^x \phi(d_1) - e^{-\tau k} \phi(d_2)$$

Replacing $x = ln(S)$ and $\tau = \frac{\sigma^2}{2}(T-t)$ and $V(S,t) = v(x, \tau)$. Also, note $k = \frac{2r}{\phi^2}$

$$V(S,t) = S\phi(d_1) - e^{-r(T-t)}\phi(d_2)$$

The last modification needed is the transformation of d_1 and d_2 back in terms of initial variables. We do this the exact same was as their coefficients.

$$
\begin{aligned}
d_1 &= \frac{ln(S) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{ln(S) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

The end result is the Black-Scholes formula for the price of European call option with strike price K at time T, if the current time, underlying price, interest rate, and volatility are t, S, r, σ respectively. Most of the time the formula is not presented in the full closed form solution, but rather piecemeal or:

$$V_c(S,t) = S \cdot \phi(d_1) - e^{-r(T-t)} \cdot \phi(d_2)$$

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CHAPTER 6

CONCLUSION

To conclude, we have presented the powerful applications of such mathematical models as well as their tremendous implications in finance. Using mathematical tools the substance behind such models can be deconstructed, reconstructed, and improved allowing for a better understanding and utility. In deconstructing the Black-Scholes equation we rebuild the structure with a new face and name which is that of the familiar heat equation. This connection alone highlights intriguing properties shared by the Black-Scholes equation and the heat equation, both as parabolic partial differential equations. Much more room is left for future exploration in the area motivated by the relation between mathematical finance and natural phenomena.

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