Harmonious Labelings Via Cosets and Subcosets

Jared L. Painter
*University of North Alabama, jpainter@una.edu*

Holleigh C. Landers
*University of North Alabama, hlanders@una.edu*

Walker M. Mattox
*University of North Alabama, wmattox@una.edu*

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Harmonious Labelings Via Cosets and Subcosets

Cover Page Footnote
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Abstract

In [4], it is shown that the disjoint union of an odd cycle and certain paths is harmonious, and that certain starlike trees are harmonious using properties of cosets for a particular subgroup of $\mathbb{Z}_m$, where $m$ is the number of edges of the graph. We expand upon these results by first exploring the numerical properties when adding values from cosets and subcosets in $\mathbb{Z}_m$. We will then show that these properties may be used to harmoniously label graphs involving a more complex starlike tree, which we will call the snowflake graph.

1 Introduction

We adopt standard notation similar to [3] and [4]. Each graph $G = (V, E)$ will be a simple graph with $|V| = n$ and edge set $|E| = m$. A harmonious labeling of a graph $G$ is an injective function $f : V \rightarrow \mathbb{Z}_m$ where each edge $e \in E$ with endpoints $u, v \in V$ labeled $e = [f(u) + f(v)] \mod m$ is unique in $\mathbb{Z}_m$. Any graph with a harmonious labeling is said to be harmonious. In this paper, all graphs will consist of or contain trees, and we must allow for one vertex label to be repeated in our labeling since trees contain exactly one more vertex than they do edges. We still require that the edge labels are all distinct for a tree to be harmonious.

A comprehensive collection of harmonious graphs can be found in Gallian’s survey of graph labelings, [1]. In [4], the graph $T_{st+1}$ is defined to be the starlike tree consisting of $t$ paths, $P_s$, joined at an endpoint of each path by a central vertex called the root. Theorem 2 of [4] showed that the disjoint union $C_s \cup T_{st+1}$ is harmonious when $s \geq 3$ is odd and $t \geq 2$ is even. The harmonious labeling of $C_s \cup T_{st+1}$ is constructed by labeling the cycle and the paths of the starlike tree with elements from the cosets of $H = \langle t + 1 \rangle$ within the group $\mathbb{Z}_{s(t+1)}$. It is clear that when the cycle $C_s$ is labeled with elements from the zero coset, $0 + H$, that the edge labels of this cycle recover all elements from $0 + H$. The root of the starlike tree $T_{st+1}$ is labeled 0, which is the only repeated vertex label, and the individual paths are labeled with consecutive elements from each coset of $H$; $1 + H$, $2 + H$, $\ldots$, $(t-1) + H$. The vertex label connected to the root may be chosen to be the smallest element of the coset $i + H$, if $i$ is odd, where we label the remaining vertices of this path from the smallest to the largest element in $i + H$. If $i$ is even, we label the vertex of said path connected to the root with the largest element of $i + H$, where we proceed to label the remaining vertices of this path from the largest to the smallest elements of $i + H$. This is shown to return all unique edge labels in [4] using a number of cases.

The goal of this paper is to expand upon the use of the numerical properties of cosets of a cyclic subgroup $H$ of $\mathbb{Z}_m$, adopted in [4], to incorporate numerical properties of subcosets of a cyclic subgroup $K$ of $H$. In doing so, we will show that we can harmoniously label a more complex version of the starlike tree $T_{st+1}$ which we will refer to as a snowflake graph. We proceed in a slightly different manner than in [4], by proving several uniqueness properties of specific cosets and subcosets in a series of lemmas and theorems in the following section.
2 Sums of elements from Cosets and Subcosets

We begin this section by proving some general results on the location of sums of elements from cosets and subcosets of cyclic subgroups in $\mathbb{Z}_m$. We will show how to clearly describe which subcoset and coset the sum of two elements will lie, based on the subcoset and coset they are chosen from. We will go even further by indicating the exact ordered location such a sum lies, which we define in Definition 2.2 as the elemental position of an element in a given subcoset. In Section 3 we will use these results to prove that various graphs are harmonious.

The results in this section rely heavily on the use of properties of modular arithmetic. Many of the results in this section involve determining how to describe sums of elements from $\mathbb{Z}_m$. For example, we know that if $x, y \in \mathbb{Z}_m$, then $x + y$ is “equal” to $(x + y) \mod m \in \mathbb{Z}_m$. We will frequently write this as $x + y = (x + y) \mod m$ as opposed to using typical notation for congruence since we are primarily interested in the actual entry of $x + y$ as it appears in $\mathbb{Z}_m$, which will be $(x + y) \mod m$. Throughout we will assume that $m = ns$ with $n, s > 1$. The primary properties we use are summarized in the following lemma.

Lemma 2.1. Let $x, y \in \mathbb{Z}_m$ with $n, s > 1$, $x < n$, and $m = ns$. Then

(1) $yn = (y \mod s)n \in \mathbb{Z}_m$ and

(2) $x + yn = x + (y \mod s)n \in \mathbb{Z}_m$.

Proof. Using the division algorithm we know that $yn = yn(\mod ns) = r \in \mathbb{Z}_m$ where $yn = q(ns) + r$ with $r < ns$. This implies that $r$ is a multiple of $n$ which gives $r = \hat{r}n < ns \Rightarrow \hat{r} < s$. Dividing $yn = q(ns) + r$ by $n$ yields $y = qs + \hat{r}$ showing that $y \mod s = \hat{r}$. Thus

$$yn(\mod ns) = r = \hat{r}n = (y \mod s)n \in \mathbb{Z}_m.$$  

For (2) we observe that $yn(\mod ns) = (y \mod s)n \leq (s - 1)n$ by (1). Since $x < n$ by assumption then

$$x + (y \mod s)n < ns = m \Rightarrow x + yn = x + (y \mod s)n \in \mathbb{Z}_m.$$  

The properties of products and sums of elements in $\mathbb{Z}_m$ illustrated in Lemma 2.1 can be extended to the case when $s = pq$, that is, $\mathbb{Z}_m = \mathbb{Z}_{np}$. The specific property that we will use frequently is if $x, y, z \in \mathbb{Z}_m$ with $x < n$ and $y < p$, then

$$x + yn + znp = x + yn + (z \mod q)np \in \mathbb{Z}_m.$$  

(2.1.3)

We omit the full proof for property (2.1.3) here but notice that it follows in the same manner as property (2) in Lemma 2.1. We can easily see that since $x < n$ and $y < p$, then $x + yn < pn$ which implies that $x + yn + (z \mod q)np < npq$. What is not as clear is what happens to an element of the form $x + yn + zpn \in \mathbb{Z}_m$ when it is not assumed that both $x < n$ and $y < p$. These concepts are central to describing where sums of elements from specific cosets and subcosets of $\mathbb{Z}_m$ lie in terms of the coset location, subcoset location, and even the position within the subcoset. Lemma 2.4 will address this directly. First, we will describe the construction of the specific cosets and subcosets we are working with in $\mathbb{Z}_m$. 
Suppose that $H = \langle n \rangle$ in $\mathbb{Z}_m$ with $m = ns$ and $n, s > 1$ and both odd. Additionally, let $K = \langle np \rangle$, with $p > 1$ and odd. This implies that we may write $m = npq$ where $s = pq$ with both $p$ and $q$ odd. We note that $H$ is a subgroup of $\mathbb{Z}_m$ containing $s$ elements and having $n$ distinct cosets of the form

$$0 + H, 1 + H, \ldots, (n - 1) + H.$$  

Additionally, $K$ is a subgroup of $H$ with $q$ distinct elements and $p$ distinct subcosets for each coset of $H$. For example, we can see that the subcosets of $K$ in the zeroth coset of $H$ are

$$0 + K, n + K, 2n + K, \ldots, (p - 1)n + K.$$  

More generally, the subcosets of $K$ in the $i^{th}$ coset of $H$ are

$$i + (0 + K), i + (n + K), i + (2n + K), \ldots, i + [(p - 1)n + K].$$  

When referring to subcosets we adopt the convention of simply saying the $j^{th}$ subcoset of $K$ to reference the $jn + K$ subcoset. We should note that by construction there are exactly $np$ total subcosets of $K$ across all cosets of $H$ each containing $q$ distinct elements, therefore any two distinct subcosets are disjoint while the union of all subcosets is exactly $\mathbb{Z}_m$. We also notice that group properties of $\mathbb{Z}_m$ allow us to uniquely write each element of $\mathbb{Z}_m$ in terms of the coset, subcoset, and position within each subcoset that an element lies. We define the term “position” as the elemental position within a given subcoset of a particular coset. The following definition formalizes this terminology.

**Definition 2.2.** Let $0 \leq i \leq n - 1$, $0 \leq j \leq p - 1$, and $0 \leq r \leq q - 1$. If each of the $q$ elements in the $(jn) + K$ subcoset of the $i + H$ coset is ordered from smallest to largest, then we may write each such element uniquely as $i + (j + rp)n$, and it is said that this element is in the $r^{th}$ elemental position of the $j^{th}$ subcoset of the $i^{th}$ coset.

Our initial question of interest is when two elements are chosen in $\mathbb{Z}_m$ and we specify the coset, subcoset, and elemental position in which each of these elements are located, which coset and subcoset does the sum of these two elements lie? The answer is not as straightforward as it may seem as the location of the sum of these elements depends on whether or not the sum of the coset locations is subject to mod $n$. More precisely, the answer depends on whether or not the sum of the coset locations is greater than or equal to $n$. We will show that the subcoset where this sum lies is shifted up one unit mod $p$ when the sum of the coset locations is subject to mod $n$. A similar phenomenon occurs with the elemental position of the sum when the sum of the subcoset locations is subject to mod $p$. The following example will illustrate this “plus one shift” in the subcoset location and elemental location, as well as illustrate how we can apply (2.1.3) to determine where the sum of two elements lies in $\mathbb{Z}_m$.

**Example 2.3.** Suppose $H = \langle 3 \rangle$ and $K = \langle 15 \rangle$ in $\mathbb{Z}_{105}$, which implies $n = 3$, $p = 5$, and $q = 7$. The following table outlines the cosets and subcosets for $H$ and $K$ in $\mathbb{Z}_{105}$.

In the row labels where we write $0^{th}$, $1^{st}$, $2^{nd}$, we mean the zeroth and first subcosets etc. That is, $0(3) + K$, $1(3) + K$, etc. Applying Definition 2.2 we can write any given element uniquely in terms of the coset, subcoset, and elemental position within the given subcoset. For example, $40$, which is in the $2^{nd}$ elemental position of the $3^{rd}$ subcoset in the $1^{st}$ coset can be written as
Table 1: Subcosets for $K = \langle 15 \rangle$ and $H = \langle 3 \rangle$ in $\mathbb{Z}_{105}$

<table>
<thead>
<tr>
<th></th>
<th>$0 + H$</th>
<th>$1 + H$</th>
<th>$2 + H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0&lt;sup&gt;th&lt;/sup&gt;</td>
<td>{0,15,30,45,60,75,90}</td>
<td>{1,16,31,46,61,76,91}</td>
<td>{2,17,32,47,62,77,92}</td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>{3,18,33,48,63,78,93}</td>
<td>{4,19,34,49,64,79,94}</td>
<td>{5,20,35,50,65,80,95}</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>{6,21,36,51,66,81,96}</td>
<td>{7,22,37,52,67,82,97}</td>
<td>{8,23,38,53,68,83,98}</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>{9,24,39,54,69,84,99}</td>
<td>{10,25,40,55,70,85,100}</td>
<td>{11,26,41,56,71,86,101}</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>{12,27,42,57,72,87,102}</td>
<td>{13,28,43,58,73,88,103}</td>
<td>{14,29,44,59,74,89,104}</td>
</tr>
</tbody>
</table>

40 = 1 + 3 \cdot 3 + 2(3 \cdot 5). Choosing another element, say 93, which is in the 6<sup>th</sup> elemental position of the 1<sup>st</sup> subcoset in the zeroth coset, we get that 93 = 0 + 1 \cdot 3 + 6(3 \cdot 5). Adding 40 and 93 when in terms of their unique representations should provide insight to the resulting elemental position, subcoset, and coset in which the sum lies. Writing the sum out we can see that (2.1.3) applies and we get

$$(40 + 93) \mod 105 = \left[ [1 + 3 \cdot 3 + 2(3 \cdot 5)] + [0 + 1 \cdot 3 + 6(3 \cdot 5)] \right] \mod 105$$

$$= 1 + 4 \cdot 3 + (8 \mod 7)(3 \cdot 5) = 28.$$ 

This representation shows that 28 is in the $8 \mod 7$ or 1<sup>st</sup> elemental position of the 4<sup>th</sup> subcoset of the 1<sup>st</sup> coset, which we can verify is indeed the case. However, if we added $8 = 2 + 2 \cdot 3 + 0(3 \cdot 5)$ to 40 in this manner we get

$$40 + 8 = [1 + 3 \cdot 3 + 2(3 \cdot 5)] + [2 + 2 \cdot 3 + 0(3 \cdot 5)]$$

$$= 3 + 5 \cdot 3 + 2(2 \cdot 5) = 48.$$ 

Since the sum is less than 105 we omit the need to write $\mod 105$. We see that (2.1.3) does not immediately apply here because the sum of the coset locations is $x = 3$ and the sum of the subcoset locations is $y = 5$ which are not less than $n = 3$ and $p = 5$ respectively. We also notice that 48 is in the 3<sup>rd</sup> elemental position of the 1<sup>st</sup> subcoset of the zeroth coset per the table. This is because the sum of the coset locations being $x = 3$ is subject to $\mod 3$ which causes a shift of “plus one” in the sum of the subcoset locations, $y = 5$, to yield the actual subcoset location of $(5 + 1) \mod 5$. Similarly, we see that the sum of the elemental positions in the subcosets is also shifted by “plus one” since $5+1=6$ is subject to $\mod 5$.

We will classify the behavior on sums of elements generally in Lemma 2.4. First, for each of the following results we will make the blanket assumption that $H = \langle n \rangle$ and $K = \langle np \rangle$ are subgroups of $\mathbb{Z}_m$ with $m = npq$, $n, p, q \geq 3$, and all odd. We also note that if $x \in i + (jn + K)$, then we may write $x = i + (j + rp)n$ where $x$ is in the $r$<sup>th</sup> elemental position of $i + (jn + K)$ with $0 \leq i < n$, $0 \leq j < p$, and $0 \leq r < q$. It is a simple exercise to verify that these assumptions imply that $x = i + (j + rp)n < npq = m$. 

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Lemma 2.4. Let \( x \in \mathbb{Z}_{i_1+n+K} \) and \( y \in \mathbb{Z}_{i_2+(j_2+n)+K} \) with \( 0 \leq i_1, i_2 < n \) and \( 0 \leq j_1, j_2 < p \). Additionally, suppose that \( x \) is in the \( r_1 \) elemental position and \( y \) is in the \( r_2 \) elemental position with \( 0 \leq r_1, r_2 < q \).

(1) If \( i_1 + i_2 < n \) and \( j_1 + j_2 < p \), then \( x + y \in (i_1 + i_2) + ((j_1 + j_2)n + K) \) in the \((r_1 + r_2) \mod q\) elemental position.

(2) If \( i_1 + i_2 \geq n \) and \( j_1 + j_2 + 1 < p \), then \( x + y \in \hat{r} + [(j_1 + j_2+1)n + K] \) in the \((r_1 + r_2) \mod q\) elemental position where \( \hat{r} = (i_1 + i_2) \mod n \).

(3) If \( i_1 + i_2 < n \) and \( j_1 + j_2 \geq p \), then \( x + y \in (i_1 + i_2) + [\hat{s}n + K] \) in the \((r_1 + r_2 + 1) \mod q\) elemental position where \( \hat{s} = (j_1 + j_2) \mod p \).

(4) If \( i_1 + i_2 \geq n \) and \( j_1 + j_2 + 1 \geq p \), then \( x + y \in \hat{r} + [\hat{s}n + K] \) in the \((r_1 + r_2 + 1) \mod q\) elemental position where \( \hat{r} = (i_1 + i_2) \mod n \) and \( \hat{s} = (j_1 + j_2 + 1) \mod p \).

Proof. By definition of the elemental position we know that \( x = i_1 + (j_1 + r_1p)n \) and \( y = i_2 + (j_2 + r_2p)n \). Thus

\[
x + y = [(i_1 + i_2) + ((j_1 + j_2)n + K) + (r_1 + r_2p) \mod npq \in \mathbb{Z}_{npq}.
\]

If \( i_1 + i_2 < n \) and \( j_1 + j_2 < p \), then (2.1.3) applies and we may write

\[
x + y = (i_1 + i_2) + [(j_1 + j_2) + (r_1 + r_2) \mod q]p \mod npq,
\]

which proves (1).

If \( i_1 + i_2 \geq n \) and \( j_1 + j_2 + 1 < p \), then we have that \( n \leq i_1 + i_2 < 2n \), since \( 0 \leq i_1, i_2 < n \), and we may write that \( i_1 + i_2 = \hat{r} + n \Rightarrow (i_1 + i_2) \mod n = \hat{r} \). This gives that

\[
x + y = [\hat{r} + n + ((j_1 + j_2) + (r_1 + r_2p) \mod npq
\]
\[
= [\hat{r} + [(j_1 + j_2 + 1) + (r_1 + r_2p) \mod npq
\]
\[
= \hat{r} + [(j_1 + j_2 + 1) + [(r_1 + r_2 \mod q) \mod p]p \mod npq \in \mathbb{Z}_{npq} \quad \text{(by (2.1.3))}
\]

completing the proof of (2).

If \( i_1 + i_2 < n \) and \( j_1 + j_2 \geq p \), then \( j_1 + j_2 \) is subject to \( \mod p \). In similar fashion to what we did above, we may write \( j_1 + j_2 = \hat{s} + p \) where \( \hat{s} = (j_1 + j_2) \mod p \). This gives that

\[
x + y = [(i_1 + i_2) + (\hat{s} + p) + (r_1 + r_2p) \mod npq
\]
\[
= [(i_1 + i_2) + [\hat{s} + (r_1 + r_2 + 1) \mod q]p \mod npq
\]
\[
= (i_1 + i_2) + [\hat{s} + [(r_1 + r_2 + 1) \mod q]p]n \quad \text{(by (2.1.3))}
\]

proving (3).
Finally, if \( i_1 + i_2 \geq n \) and \( j_1 + j_2 + 1 \geq p \), then it must be that \( n \leq i_1 + i_2 < 2n \) and \( p \leq j_1 + j_2 + 1 < 2p \). This means we may write \( i_1 + i_2 = \hat{r} + n \) and \( j_1 + j_2 + 1 = \hat{s} + p \) where \( \hat{r} = (i_1 + i_2) \mod n \) and \( \hat{s} = (j_1 + j_2 + 1) \mod p \). This gives that

\[
x + y = [(i_1 + i_2) + [(j_1 + j_2) + (r_1 + r_2)p] \mod npq
= [(\hat{r} + n) + [(j_1 + j_2) + (r_1 + r_2)p] \mod npq
= [\hat{r} + [(j_1 + j_2 + 1) + (r_1 + r_2)p] \mod npq
= [\hat{r} + [(\hat{s} + p) + [(r_1 + r_2)p] \mod npq
= \hat{r} + [\hat{s} + [(r_1 + r_2 + 1) \mod q] \mod p] \mod npq
\]

(by (2.1.3))

completing the proof. □

The significance of Lemma 2.4 is that adding two elements from \( i_1 \) and \( i_2 \) cosets of \( H \) where \( i_1 + i_2 \) is subject to \( \mod n \) yields a shift of “plus one” in the subcoset location for the sum of these two elements. If we assume that both \( x \) and \( y \) are in the same coset, say \( i + H \), then Lemma 2.4 says that \( x + y \) is in the \((j_1 + j_2) \mod p \) subcoset of \( K \) in the \( 2i \) coset of \( H \), if \( 2i < n \implies i \leq \frac{n-1}{2} \) since \( n \) is odd, or \( x + y \) is in the \((j_1 + j_2 + 1) \mod p \) subcoset of \( K \) in the \( (2i) \mod n \) coset of \( H \) if \( i \geq \frac{n+1}{2} \). We now wish to consider the set of all ordered sums of pairs of elements within a given coset. The following lemma will clearly classify all such sums.

**Lemma 2.5.** The sum of each pair of consecutive elements ordered from smallest to largest in \( i + H \), \( 0 \leq i \leq n - 1 \) yields distinct elements in \((2i) \mod n + H \). Moreover, if \( S \) is the set of all such sums, \( S = (2i) \mod n + H - \{x\} \) where \( x \) is the sum of the first and last element in \( i + H \).

**Proof.** By construction \( i + H = \{i, i + n, i + 2n, \ldots, i + (pq - 1)n\} \), and since each consecutive element increases by exactly \( n \) from left to right, the sum of each pair of consecutive elements will increase by \( 2n \). The ordered set \( S \) of consecutive sums of \( i + H \) will contain exactly \( pq - 1 \) elements, which is even, since \( pq \) is odd. We also note that if \( 2i < n \), then \((2i) \mod n = 2i\) but if \( 2i > n \), then \((2i) \mod n = \hat{r} \) where \( 2i = \hat{r} + n \), since \( 0 \leq i \leq n - 1 \). Thus the sum of consecutive elements will be in the \( 2i + H \) coset, if \( 2i < n \) and in the \( \hat{r} + H = (2i) \mod n + H \) coset, if \( 2i > n \). The sum of the \((j - 1)\text{th}\) and \(j\text{th}\) elements in \( i + H \) is

\[
[i + (j - 1)n] + [i + jn] = 2i + (2j - 1)n, \quad \text{for} \quad 0 < j \leq pq - 1.
\]

We observe that this sum will be recorded in our list \( S \) as exactly \( 2i + (2j - 1)n \) and is odd provided that \( 2i + (2j - 1)n < npq = m \). That is, if the element \( 2i + (2j - 1)n \) is not subject to \( \mod m \) in \( \mathbb{Z}_m \). Notice that \((2j - 1)n \) is not subject to \( \mod m \) whenever \( 2j - 1 < pq \implies j < \frac{pq + 1}{2} \) and is subject to \( \mod m \) if \( j \geq \frac{pq + 1}{2} \). We will show that \( 2i + (2j - 1)n \) is not subject to \( \mod m \) if \( j < \frac{pq + 1}{2} \) and is subject to \( \mod m \) if \( j \geq \frac{pq + 1}{2} \). To see this we will sum the largest two elements in \( i + H \) for which \( j < \frac{pq + 1}{2} \). That is, suppose that \( j = \frac{pq + 1}{2} - 1 \) and consider the sum of \( \frac{pq - 3}{2} \) and \( \frac{pq - 1}{2} \) in \( i + H \). This gives

\[
2i + (2j - 1)n = 2i + (pq - 2)n < npq \quad \text{since} \quad 0 \leq i \leq n - 1.
\]

Thus we can clearly see that the sum of all consecutive elements in \( i + H \) for \( 0 < j < \frac{pq + 1}{2} \) will yield exactly \( \frac{pq - 1}{2} \) odd elements in \((2i) \mod n + H \) of the form

\[
\{2i + n, 2i + 3n, 2i + 5n, \ldots, 2i + (pq - 2)n\}, \quad \text{if} \quad 2i < n \quad \text{(2.5.1)}
\]
\[ \{ \hat{r} + 2n, \hat{r} + 4n, \hat{r} + 6n, \ldots, \hat{r} + (pr - 1)n \}, \text{ if } 2i > n. \]  

(2.5.2)

We now show that if \( j > \frac{pq+1}{2} \), then the sum of consecutive elements is subject to mod \( m \) and that these sums yield even elements in \((2i) \mod n + H\). To see this let \( j = \frac{pq+1}{2} \), then the sum of the \((j - 1)^{th} \) and \(j^{th} \) elements is

\[
[2i + (2j - 1)n] \mod npq = [2i + (pq)n] \mod npq = 2i.
\]

If \( 2i < n \), then \( 2i \) is the first element of the coset \( 2i + H \). If \( 2i > n \), then \( 2i = n + \hat{r} \) and \( 2i \) is the second element of the coset \( \hat{r} + H \). Continuing to add consecutive elements where \( j > \frac{pq+1}{2} \) we know that each sum will increase by exactly \( 2n \) and the sum of the last two elements in \( i + H \) is

\[
[2i + (2pq - 3)n] \mod npq = 2i - (3n) \mod npq = 2i + (pq - 3)n.
\]

Thus we have exactly \( \frac{pq-1}{2} \) even elements in \((2i) \mod n + H\) of the form

\[
\{2i, 2i + 2n, 2i + 4n, \ldots, 2i + (pq - 3)n\} \text{ if } 2i < n, \text{ and }
\]

\[
\{\hat{r} + n, \hat{r} + 3n, \hat{r} + 5n, \ldots, \hat{r} + (pq - 2)n\} \text{ if } 2i > n.
\]

(2.5.3)

(2.5.4)

If \( 2i < n \), then we combine the odd elements from (2.5.1) with the even elements from (2.5.3) to get that

\[
S = \{2i, 2i + n, 2i + 2n, 2i + 3n, \ldots, 2i + (pq - 3)n, 2i + (pq - 2)n\}.
\]

In this case \( S \) consists of all elements in \( 2i + H \) except for the last element \( 2i + (pq - 1)n \), which is the sum of the first and last elements, \( i \) and \( i + (pq - 1)n \), of \( i + H \). On the other hand if \( 2i > n \), then we combine the odd elements from (2.5.2) with the even elements from (2.5.4) to get that

\[
S = \{\hat{r} + n, \hat{r} + 2n, \hat{r} + 3n, \hat{r} + 4n, \ldots, \hat{r} + (pq - 2)n, \hat{r} + (pq - 1)n\}.
\]

Here \( S \) consists of all elements in \( \hat{r} + H = (2i) \mod n + H \) except for the first element \( \hat{r} = (2i) \mod n \). In this case the sum of the first and last elements of \( i + H \) is

\[
[i + [i + (pq - 1)n]] \mod npq = [2i + (pq - 1)n] \mod npq = [\hat{r} + npq] \mod npq = \hat{r},
\]

completing the proof.

An important note from the proof of Lemma 2.5 is that we have no repeated elements in the \((2i) \mod n + H \) coset and that we will always get exactly \( \frac{pq-1}{2} \) odd elements and \( \frac{pq-1}{2} \) even elements. In total, this gives us exactly \( pq - 1 \) distinct elements of the \( pq \) total distinct elements from each coset. Moreover, we note that we are missing the last element of the \( 2i + H \) coset when \( 2i < n \) and are missing the first element of the \( \hat{r} + H \) coset, with \((2i) \mod n = \hat{r} \), when \( 2i > n \).

The remaining results of this section are devoted to showing that summing elements from the \( r^{th} \) position of each subcoset of \( K \) in \( i + H \) in a specific order, and summing the elements in the \( r^{th} \) elemental position of the \( j^{th} \) subcoset for each coset of \( H \) in a specific order will always yield distinct elements from \( \mathbb{Z}_m \). Before we prove our two main results of this section, we will introduce the two elemental orderings that will be integral in harmoniously labeling our graphs in the next section.
Definition 2.6. Let $H = \langle n \rangle$ and $K = \langle np \rangle$ be subgroups of $\mathbb{Z}_{npq}$ with $n, p, q \geq 3$ and all odd.

1. Fixing an elemental position $r$ for all subcosets of $K$ in $i + H$, $0 \leq i \leq n - 1$. We define the **split even/odd ordered subcoset list** with $r^\text{th}$ elemental position in $i + H$ to be

$$K_{ir} = \{i + (p - 1 + rp)n, \ldots, i + (4 + rp)n, i + (2 + rp)n, i + rpn,\
i + (p - 2 + rp)n, \ldots, i + (3 + rp)n, i + (1 + rp)n\}.$$ 

2. Fixing an elemental position $r$ for the $j^\text{th}$ subcoset of $K$, $0 \leq j \leq p - 1$, in each coset of $H$. We define the **split even/odd ordered coset list for a fixed subcoset** with $r^\text{th}$ elemental position to be

$$H_{jr} = \{(n - 1) + (j + rp)n, \ldots, 4 + (j + rp)n, 2 + (j + rp)n, (j + rp)n,\
(n - 2) + (j + rp)n, \ldots, 3 + (j + rp)n, 1 + (j + rp)n\}.$$ 

The **even/odd** logic behind the naming of these ordered lists comes from the fact that we have an odd number of subcosets and cosets, and thus have the same number of even and odd subcosets and cosets when we exclude the zeroth subcoset and coset. Thus we choose the middle element of $K_{ir}$ and $H_{jr}$ to be the zeroth subcoset and coset respectively. We then place all of the even subcoset/coset elements to the left of the middle element in an increasing order as we move away from the center, and place all of the odd subcoset/coset elements to the right of the middle element in a decreasing order as we move away from the center. These orderings provide interesting results when we sum consecutive elements of these ordered sets.

Theorem 2.7. Let $SK_{ir}$ be the set of consecutive sums of the ordered elements in $K_{ir}$, then

1. $SK_{ir_1} \cap SK_{ir_2} = \emptyset$, if either $i_1 \neq i_2$ or $r_1 \neq r_2$, that is, each pair of distinct sets $SK_{ir}$ are pairwise disjoint.

2. \[
\bigcup_{i=0}^{n-1} \left( \bigcup_{r=0}^{q-1} SK_{ir} \right) = \mathbb{Z}_{npq} - \left[ \left( \bigcup_{l=0}^{n+1} (2l + K) \right) \bigcup \left( \bigcup_{l=0}^{n+1} [(2l + 1) + (n + K)] \right) \right].
\]

Proof. The sums of consecutive elements in $K_{ir}$ up to the middle element (sums for the even subcoset elements) will be

$$\{2i + 2(p - 2 + rp)n, 2i + 2(p - 4 + rp)n, \ldots, 2i + 2(3 + rp)n, 2i + 2(1 + rp)n\}$$

and the sums from the middle element to the last element will be

$$\{2i + (p - 2 + 2rp)n, 2i + 2(p - 3 + rp)n, \ldots, 2i + 2(4 + rp)n, 2i + 2(2 + rp)n\}.$$ 

Combining these elements and reordering for clarity, we get that

$$SK_{ir} = \{2i + 2(p - j + rp)n\}_{j=2}^{p-1} \cup \{2i + (p - 2 + 2rp)n\}.$$ (2.7.3)
To complete the proof of (1) we will show that if general elements of $SK_{i_1r_1}$ and $SK_{i_2r_2}$ are equal, then $i_1 = i_2$ and $r_1 = r_2$. We will need to consider several cases, including when elements are subject to $\mod npq$, and we must also show that the element $2i + (p - 2 + 2rp)n$ cannot be equal to another general element.

**Case 1:** We begin by assuming that $2i_1 + 2(p - j + r_1p)n \in SK_{i_1r_1}$ and $2i_2 + 2(p - k + r_2p)n \in SK_{i_2r_2}$ where neither element is subject to $\mod npq$, that is the elements are identically the same and we may write

$$2i_1 + 2(p - j + r_1p)n = 2i_2 + 2(p - k + r_2p)n$$

$$\implies i_1 - i_2 = [j - k + (r_2 - r_1)p]n.$$ 

However, this would imply that $i_1 - i_2$ is a multiple of $n$ and thus $i_1 - i_2 = 0 \Rightarrow i_1 = i_2$ since $0 \leq i_1, i_2 \leq n - 1$ by definition. This would also imply that $j - k + (r_2 - r_1)p = 0$ and thus $k - j = (r_2 - r_1)p$. Since $2 \leq j, k \leq p - 1$ we must also have that $k = j$ and in turn $r_2 = r_1$ completing case 1.

**Case 2:** Next we will assume that these two general elements are equal subject to $\mod npq$. We can assume that only one of the elements is subject to $\mod npq$ since assuming both were subject to $\mod npq$ would yield the same argument as in Case 1. Without loss of generality, suppose

$$[2i_1 + 2(p - j + r_1p)n] \mod npq = 2i_2 + 2(p - k + r_2p)n,$$

that is, $npq \leq 2i_1 + 2(p - j + r_1p)n < 2npq$ but $0 \leq 2i_2 + 2(p - k + r_2p)n < npq$. Thus we may write

$$2i_1 + 2(p - j + r_1p)n = npq + 2i_2 + 2(p - k + r_2p)n$$

which is impossible since $2i_1 + 2(p - j + r_1p)n$ is even and $npq + 2i_2 + 2(p - k + r_2p)n$ is odd since $n, p, q$ are all odd.

**Case 3:** Finally, we will show that the elements $2i_1 + (p - 2 + 2r_1p)n$ and $2i_2 + (p - 2 + 2r_2p)n$, can only be equal if $i_1 = i_2$ and $r_1 = r_2$. Additionally, we need to show that $2i + (p - 2 + 2rp)n \neq 2k + 2(p - j + sp)n$. If $2i_1 + (p - 2 + 2r_1p)n = 2i_2 + (p - 2 + 2r_2p)n$ where neither are subject to $\mod npq$, then $i_1 - i_2 = (r_2 - r_1)pn$ and we must have that $i_1 = i_2$ and $r_2 = r_1$. If $[2i_1 + (p - 2 + 2r_1p)n] \mod npq = 2i_2 + (p - 2 + 2r_2p)n$ (i.e. $2i_1 + (p - 2 + 2r_1p)n$ is subject to $\mod npq$), then we may write

$$2i_1 + (p - 2 + 2r_1p)n = npq + 2i_2 + (p - 2 + 2r_2p)n$$

which is impossible since $2i_1 + (p - 2 + 2r_1p)n$ is odd and $npq + 2i_2 + (p - 2 + 2r_2p)n$ must be even. Finally, since $2i + (p - 2 + 2rp)n$ is odd and $2k + 2(p - j + sp)n$ is even, these cannot be equal unless one of these values is subject to $\mod npq$. Without loss of generality, suppose that $2k + 2(p - j + sp)n$ is subject to $\mod npq$ and $2i + (p - 2 + 2rp)n$ is not. As was done in the previous case we may write

$$2k + 2(p - j + sp)n = npq + 2i + (p - 2 + 2rp)n$$

$$\implies 2(k - i) = npq + [2j - p - 2 + 2(r - s)p]n$$

$$\implies 2(k - i) = [pq + 2j - p - 2 + 2(r - s)p]n$$
and thus \( k = i \) and \( pq + 2j - p - 2 + 2(r - s)p = 0 \) since \( 0 \leq i, k \leq n - 1 \). Now by reordering the second equation we have \( 2(j - 1) = [1 - q + 2(s - r)]p \) which implies that \( j - 1 = 0 \) or equivalently \( j = 1 \). Which is a contradiction by (2.7.3) since \( 2 \leq j \leq p - 1 \). Therefore, \( 2i + (p - 2 + 2rp)n \neq 2k + 2(p - j + sp)n \), which proves that \( SK_{i, r_1} \cap SK_{i, r_2} = \emptyset \) whenever \( i_1 \neq i_2 \) or \( r_1 \neq r_2 \).

For the proof of (2) we will show that none of the elements in \( SK_{i, r} \) lie in \( 2l + K \) with \( 0 \leq l \leq \frac{n - 1}{2} \) nor can they lie in \((2l + 1) + (n + K)\) with \( 0 \leq l \leq \frac{n - 3}{2} \). Additionally, we will show that combining all elements from both

\[
\bigcup_{i=0}^{n-1} \left( \bigcup_{r=0}^{q-1} SK_{i, r} \right) \quad \text{and} \quad \left( \bigcup_{i=0}^{n-1} (2l + K) \right) \bigcup \left( \bigcup_{l=0}^{\frac{n-3}{2}} [(2l + 1) + (n + K)] \right),
\]

we get exactly \( npq \) elements by (1) and the fact that elements in distinct subcosets are distinct. First, we observe that if \( 2i + 2(p - j + rp)n \) with \( 2 \leq j \leq p - 1 \), is to be an element of the \((2l + 1) + (n + K)\) subcoset, then we must have that \( 2i \) is subject to \( \text{mod} \ n \) and \([2(p - j) + 1] \text{mod} \ p = 1 \) by Lemma 2.4. But this would mean that either \( p - j = 0 \) or \( 2(p - j) = p \), both of which are impossible since \( 0 \leq j < p \) and \( p \) is odd. Similarly, we can see that \( 2i + (p - 2 + rp)n \) cannot be in \((2l + 1) + (n + K)\) since we would have to have that \( p - 2 + 1 = p - 1 = 1 \implies p = 0 \). Now, if either \( 2i + 2(p - j + rp)n \) or \( 2i + (p - 2 + 2rp)n \) was to be an element of \( 2l + K \), we would have to have that \( 2(p - j) = 0 \) or \( p - 2 = 0 \) respectively. Of course, these equations imply that either \( p = j \) or \( p = 2 \) both of which are impossible. To tally all elements when taking the union of these sets, we notice that since each set \( SK_{i, r} \) contains exactly \( p - 1 \) elements, \( \bigcup_{i=0}^{n-1} \left( \bigcup_{r=0}^{q-1} SK_{i, r} \right) \) contains exactly \((p - 1)npq\) distinct elements by (1). Also we have that each subcoset contains exactly \( q \) elements which gives that

\[
\left( \bigcup_{l=0}^{\frac{n-1}{2}} (2l + K) \right) \bigcup \left( \bigcup_{l=0}^{\frac{n-3}{2}} [(2l + 1) + (n + K)] \right) \quad \text{contains} \quad \frac{n + 1}{2}q + \frac{n - 1}{2}q = nq
\]
distinct element since distinct subcosets contain distinct elements. Thus we have that

\[
\bigcup_{i=0}^{n-1} \left( \bigcup_{r=0}^{q-1} SK_{i, r} \right) \bigcup \left[ \left( \bigcup_{l=0}^{\frac{n-1}{2}} (2l + K) \right) \bigcup \left( \bigcup_{l=0}^{\frac{n-3}{2}} [(2l + 1) + (n + K)] \right) \right]
\]
contains exactly \((p - 1)npq + nq = npq\) distinct elements and therefore must be equal to \( \mathbb{Z}_{npq} \).

**Theorem 2.8.** Let \( SH_{j, r} \) be the set of consecutive sums of the ordered elements in \( H_{j, r} \), then

1. \( SH_{j_1, r_1} \cap SH_{j_2, r_2} = \emptyset \), if either \( j_1 \neq j_2 \) or \( r_1 \neq r_2 \), that is, each pair of distinct sets \( SH_{j, r} \) are pairwise disjoint.
2. \( \bigcup_{j=0}^{p-1} \left( \bigcup_{r=0}^{q-1} SH_{j, r} \right) = \mathbb{Z}_{npq} - H \).
Proof. For the sake of simplifying notation, define $\beta = 2j + 2rp$. The sums of consecutive elements in $H_{jr}$ up to the middle element (sums for the even coset elements) will be

$$\{2(n-2) + \beta n, 2(n-4) + \beta n, \ldots, 6 + \beta n, 2 + \beta n\}$$

and the sums from the middle element to the last element will be

$$\{(n-2) + \beta n, 2(n-3) + \beta n, \ldots, 8 + \beta n, 4 + \beta n\}.$$

Combining these elements and reordering for clarity, we get that

$$SH_{jr} = \{2(n-i) + \beta n\}_{i=2}^{n-1} \cup \{(n-2) + \beta n\}.$$ 

To complete the proof of (1), we will show that if general elements of $SH_{j_1r_1}$ and $SH_{j_2r_2}$ are equal, then $j_1 = j_2$ and $r_1 = r_2$. As was the case in Theorem 2.7, we will need to consider several cases, including when elements are subject to $\text{mod } npq$ and showing that the element $(n-2) + \beta n$ cannot be equal to another general element.

Case 1. We begin by assuming that $2(n-i) + \beta_1 n \in SH_{j_1r_1}$ and $2(n-k) + \beta_2 n \in SH_{j_2r_2}$ where neither element is subject to $\text{mod } npq$, that is the elements are identically the same and we may write

$$2(n-i) + \beta_1 n = 2(n-k) + \beta_2 n$$

$$\implies 2(k-i) = (\beta_2 - \beta_1)n.$$ 

However, since $n$ is odd this would imply that $k-i = 0$ since $2 \leq i, k \leq n-1$ and $\beta_2 - \beta_1 = 0$. Both imply that $k = i$ and $\beta_2 = \beta_1$ which yields

$$2j_2 + 2r_2p = 2j_1 + 2r_1p \implies (j_2 - j_1) = (r_1 - r_2)p.$$ 

Since $0 \leq j_1, j_2 \leq p - 1$, we must have the $j_2 - j_1 = 0 \implies j_2 = j_1$ and in turn must also have that $r_2 = r_1$.

Case 2. Next we will assume that these two general elements are equal subject to $\text{mod } npq$. We can assume that only one of the elements is subject to $\text{mod } npq$ since assuming both were subject to $\text{mod } npq$ would yield the same argument as in Case 1. Without loss of generality, suppose

$$[2(n-i) + \beta_1 n] \text{ mod } npq = 2(n-k) + \beta_2 n,$$

that is, $npq < 2(n-i) + \beta_1 n < 2npq$ but $0 \leq 2(n-k) + \beta_2 n < npq$. This would mean that we could write

$$2(n-i) + \beta_1 n = npq + 2(n-k) + \beta_2 n$$

$$\implies 2(k-i) = (\beta_2 - \beta_1 + pq)n.$$ 

Again this would imply that $k-i = 0 \implies k = i$ and also that $\beta_2 - \beta_1 + pq = 0$. Expanding $\beta_1$ and $\beta_2$, we get

$$2(j_2 + 2r_2p) - (2j_1 + 2r_1p) + pq = 0$$

$$\implies 2(j_2 - j_1) = (2r_1 - 2r_2 - q)p.$$
Thus \( j_2 - j_1 = 0 \implies j_2 = j_1 \) and \( 2r_1 - 2r_2 - q = 0 \implies 2(r_1 - r_2) = q \) which is impossible since \( q \) is odd. This contradicts the assumption that two elements \( 2(n - i) + \beta_1n \) and \( 2(n - k) + \beta_2n \) can be equal where one is subject to \( \text{mod } npr \) and the other is not.

**Case 3.** Finally, we will show that the elements \( (n - 2) + \beta_1n \) and \( (n - 2) + \beta_2n \), can only be equal if \( j_1 = j_2 \) and \( r_1 = r_2 \). Additionally, we will show that \( (n - 2) + \beta_1n \neq 2(n - i) + \beta_2n \). First we will note that \( (n - 2) + \beta_1n \neq (n - 2) + \beta_2n \) when one of these elements is subject to \( \text{mod } npq \), since \( (n - 2) + \beta n \) is odd and \( [(n - 2) + \beta n] \text{mod } npq \) would be even. When we do assume that \( (n - 2) + \beta_1n = (n - 2) + \beta_2n \), we may assume, without loss of generality, that neither are subject to \( \text{mod } npq \) and it is immediate that

\[
(2j_1 + 2r_1p)n = (2j_2 + 2r_2p)n \implies j_1 = j_2 \text{ and } r_1 = r_2.
\]

We should also note that \( (n - 2) + \beta_1n \) can never equal \( 2(n - i) + \beta_2n \) unless one of these elements is subject to \( \text{mod } npr \) since \( (n - 2) + \beta_1n \) is odd and \( 2(n - i) + \beta_2n \) is even when modular arithmetic is not considered. Without loss of generality, suppose that \( 2(n - i) + \beta_2n \) is subject to \( \text{mod } npr \) and that \( (n - 2) + \beta_1n \) is not. Then we may write

\[
2(n - i) + \beta_2n = npr + (n - 2) + \beta_1n
\]

implying that \( 2 - 2i \equiv 0 \mod n \) and thus \( 2 - 2i = 0 \) implies that \( i = 1 \) since \( n \) is odd. However, this is impossible since \( 2 \leq i \leq n - 1 \), completing the proof of (1).

For the proof of (2), we first note that since each \( SH_{jr} \) contains exactly \( n - 1 \) distinct elements, then we must have that \( \bigcup_{j=0}^{p-1} \bigcup_{r=0}^{q-1} SH_{jr} \) contains exactly \((n - 1)pq\) distinct elements by (1). This is precisely the same number of elements that are in \( \mathbb{Z}_{npq} - H \) since \( H \) contains exactly \( pq \) distinct elements. Thus to complete the proof we only need to argue that we never get elements from \( H \) in \( SH_{jr} \). Recall that by construction

\[
SH_{jr} = \{2(n - i) + \beta n\}_{i=2}^{n-1} \cup \{(n - 2) + \beta n\}.
\]

The coset location for an element in \( SH_{jr} \) is computed by

\[
2(n - i) \mod n, \text{ for } 2 \leq i \leq n - 1 \text{ and } (n - 2) \mod n = n - 2.
\]

Notice that

\[
2(n - i) \mod n = -2i \mod n = \begin{cases} n - 2i & 2i < n \\ 2(n - i) & 2i > n \end{cases}.
\]

Since \( n \) is odd and \( 2 \leq i \leq n - 1 \), neither \( n - 2i \) nor \( 2(n - i) \) can equal 0. Therefore none of the elements in \( SH_{jr} \) will be in \( H \) completing the proof. \( \square \)
3 Harmoniously Labeling the Snowflake Graph

We recall that in [4], it is shown that $C_s \cup T_{st+1}$, with odd $s \geq 3$ and even $t \geq 2$, is harmonious. Though it is not mentioned in [4], we would like to observe an immediate corollary to Theorem 2 in [4]. Namely, we could break the graph $C_s$ at the vertex labeled zero and connect the vertex labeled $(s-1)(t+1)$ to the root of $T_{st+1}$ which is also labeled zero. This preserves all of the edge and vertex labels in the original harmonious labeling of $C_s \cup T_{st+1}$ and thus provides a harmonious labeling of $T_{s(t+1)+1}$.

**Corollary 3.1.** Let $s, t \geq 3$ be odd. Then $T_{st+1}$ is harmonious.

**Example 3.2.** The following graph illustrates Corollary 3.1 providing a harmonious labeling of $T_{(5)(5)+1}$.

![Figure 1: Harmonious Labeling for $T_{(5)(5)+1}$](image)

Our motivation for the following results was an attempt to expand what was done in [4] to include the use of subcosets to harmoniously label extensions of the graph $T_{st+1}$. The primary hurdle we faced in doing this was the shift that occurred in both the subcoset position and in the elemental position which we have addressed in Lemma 2.4. The extension of $T_{st+1}$ we are considering we call a “snowflake” graph and is defined as follows.

**Definition 3.3.** The graph $SF_{str+1}$, with $s, t, r \geq 1$ and all odd but not all 1, is the graph formed by overlaying the central vertex of $P_r$ on top of every vertex of $T_{st+1}$ except for the root. We refer to this as a snowflake graph with $t$ branches of length $s$ and subbranches of length $r$. 
If \( s = t = r = 1 \), then \( SF_{str+1} \) is \( P_2 \) which is harmonious. If \( t = r = 1 \) and \( s \geq 3 \), then \( SF_{str+1} \) is \( P_s+1 \) which is also known to be harmonious. If \( r = 1 \) and \( s, t \geq 3 \), then \( SF_{str+1} \) is exactly \( T_{st+1} \) which is harmonious from Corollary 3.1. If \( s = r = 1 \) and \( t \geq 3 \), it is easy to harmoniously label \( SF_{str+1} \) by labeling the root zero and each of the single vertices on each branch by the elements in \( \mathbb{Z}_t \). Finally, if \( s = t = 1 \), then \( SF_{str+1} \) consists of 2 adjacent vertices in which the central vertex of \( P_r \) is placed on top of one of these vertices. This graph will have exactly \( r \) edges which implies we are working in \( \mathbb{Z}_r \). To harmoniously label \( SF_{(1)(1)r+1} \), we consider the ordered set

\[ H = \{ r - 1, r - 3, \ldots, 2, 0, r - 2, r - 4, \ldots, 3, 1 \} = \mathbb{Z}_r. \]

The set \( H \) is analogous to the split even/odd ordered coset list from Definition 2.6, where the nonzero even elements from \( \mathbb{Z}_r \) are to the left of zero and the odd elements from \( \mathbb{Z}_r \) are to the right of zero. Both groups of even and odd nonzero elements are ordered from largest to smallest moving from left to right. Taking consecutive sums of elements in \( H \), similarly to what we did in Theorem 2.8, we get

\[ S H = \{ 2r - 4, 2r - 8, \ldots, 6, 2, r - 2, 2r - 6, 2r - 10, \ldots, 8, 4 \}. \]

Reordering these elements for clarity we get that

\[ S H = \{ 2(r - i) \}_{i=2}^{r-2} \cup \{ r - 2 \}. \]

Applying the same techniques used in the proof of Theorem 2.8 we can see that

\[ S H = \{ 1, 2, 3, \ldots, r - 2, r - 1 \} \text{ after applying } \text{mod} \ r. \]

Thus labeling the vertices of \( P_r \) with the ordered elements from \( H \), we recover every element in \( \mathbb{Z}_r \) except for zero. However, since the central vertex of \( P_r \) will be labeled zero and this vertex is also adjacent to the root of the snowflake graph, which we will also label zero, the edge label of zero will will be recovered proving that \( SF_{(1)(1)r+1} \) is harmonious. We visually see the procedure required to harmoniously label \( SF_{(1)(1)r+1} \) with the following graph where \( r = 9 \).

The graph in Figure 2 does not accurately demonstrate the rational behind referring to \( SF_{str+1} \) as the “snowflake graph.” To better illustrate this terminology we consider the following example.
Example 3.4. The snowflake graph $SF_{(3)(5)(5)+1}$ can be visualized as follows,

![Image of the Snowflake Graph $SF_{(3)(5)(5)+1}$](image)

Following Definition 3.3, we have $t = 5$ branches coming out of the central vertex (or the root) each with $s = 3$ vertices on each branch, this part of the graph is exactly $T_{(3)(5)+1}$. Then we overlay the central vertex of a path $P_r$ with $r = 5$ on top of every vertex except for the root to obtain our snowflake graph. We will frequently refer to the central path of length $s$ from each branch as the spine of the branch. We should also note that the graph has $(3)(5)(5) + 1 = 76$ vertices with exactly $(3)(5)(5) = 75$ edges. In general the graph $SF_{str+1}$ has $str + 1$ vertices and $str$ edges.

Our first attempt at harmoniously labeling the snowflake graph involved a direct extension of what was done in [4], which was ultimately unsuccessful due to the shift in subcoset location and elemental position. However, we did find that if we labeled each branch of the snowflake graph with elements from the $i + H$ coset with $2i$ is not subject to mod the generator of $H$, then we would return unique edge labels when the subbranches of the snowflake graph were labeled with the ordered elements from $K_{sr}$ in the appropriate manner. From there we found we could simply connect path graphs with the same number of vertices as there are elements in each coset of $H$ to the root of the snowflake graph in a manner that does yield a harmonious labeling. The following theorem shows the harmonious labeling of the non-disjoint union of the snowflake graph with appropriate path graphs.

**Theorem 3.5.** Let $s, t, r \geq 1$, $str \geq 3$ and odd. Then the graph $G$, which consists of the snowflake graph $SF_{str+1}$ whose root is adjacent to a single vertex in each of the $(t - 1)$ copies of $P_{sr}$, is harmonious.

**Proof.** We begin the proof by describing the labeling process for this graph. By construction $G$ has $(t - 1)(sr - 1) + str + (t - 1) = sr(2t - 1)$ edges when we connect one vertex from each
path $P_{st}$ to the root of $SF_{str+1}$. Now we define $H = \langle 2t - 1 \rangle$ and $K = \langle r(2t - 1) \rangle$, in $\mathbb{Z}_{sr(2t-1)}$. This means that $H$ has $2t - 1$ cosets each containing $sr$ elements and $K$ has exactly $r$ subcosets for each coset of $H$, each containing $s$ elements. We label each branch of $SF_{str+1}$ with elements from the coset $i + H$ for $0 \leq i \leq t - 1$ such that each subbranch on the $i$th branch is labeled with elements from the split even/odd ordered subcoset list, $K_{ij}$, with the largest elemental position $j$ being closest to the root if $i$ is even and then ordered from the largest to the smallest elemental position moving away from the root. If $i$ is odd, then we label the subbranch on the $i$th branch with the elements of $K_{ij}$, such that the smallest elemental position $j$ is closest to the root and then order from the smallest to the largest elemental position as we move away from the root. We should recall that the set $K_{ij}$ is by definition an ordered set, and therefore each subbranch is labeled with these elements in order which implies that the middle element in $K_{ij}$, which is the element in the zeroth subcoset of $i + H$ in the $j$th elemental position, lies on the spine of the $i$th branch of $SF_{str+1}$.

We now proceed by labeling the $t - 1$ paths of the graph with the coset elements $i + H$ ordered from smallest to largest, for $t \leq i \leq 2(t - 1)$. We will refer to each path according to the coset it is labeled with, that is the path labeled with elements from $i + H$ will be called the $i$th-path. We will describe how we connect each path graph to the root of $SF_{str+1}$ at the end of the proof.

We first note that by Lemma 2.5 the $i$th-path with $t \leq i \leq 2(t-1)$ returns edge labels $\hat{r} + H - \{x\}$ where $\hat{r} = (2i) \mod (2t - 1)$ is odd, and $x$ is the first element of $\hat{r} + H$. This gives all elements back from all odd cosets except for the first element.

Now by Theorem 2.7, the subbranches of the $i$th branch of $SF_{str+1}$, with $0 \leq i \leq t - 1$ gives all elements back from the even cosets except for the elements in the $0$th subcoset of $K$, that is we are missing the edge labels from $\bigcup_{i=0}^{t-1} (2i + K)$. Thus the subbranches of the snowflake graph accounts for exactly $str - st = st(r-1)$ distinct elements as edge labels of the $str$ total elements from the $t$ even cosets, while the edge labels on the paths account for $(sr-1)(t-1)$ distinct elements of the $sr(t-1)$ total elements from the $t-1$ odd cosets. The remainder of the proof will be devoted to accounting for the $st$ missing elements in $\bigcup_{i=0}^{t-1} (2i + K)$ and the $t - 1$ first elements missing in $\hat{r} + H$ where $\hat{r} = (2i) \mod (2t - 1)$ for $t \leq i \leq 2(t-1)$.

For the missing elements in each $2i + K$ we recall that the middle elements of each $K_{ij}$ are the elements in the $0$th subcoset of $K$ (which we will simply refer to as $K$) and are connected on the spine of each branch. These vertex labels on the spine of the $i$th branch from $K$ are labeled from largest to smallest beginning at the root when $i$ is even and smallest to largest beginning at the root when $i$ is odd. More precisely, the vertex labels on the spine of $i$th branch are the ordered elements

$$\{i, i + r(2t - 1), i + 2r(2t - 1), \ldots, i + (s - 1)r(2t - 1)\}$$

ordered from largest to smallest if $i$ is odd and smallest to largest if $i$ is even. By definition $0 \leq i \leq t - 1$ and since these elements are in $K$, we know from Lemma 2.4 that we do not incur a shift in the elemental position when summing consecutive elements of this list. Moreover, summing these consecutive elements yields the ordered elements

$$\{2i + (2l + 1)r(2t - 1)\}_{l=0}^{s-3} \cup \{2i + (2l)r(2t - 1)\}_{l=0}^{s-3}$$

in $K$ from $2i + H$. This consists of exactly $s - 1$ distinct elements from $K$ in $2i + H$ where the only element missing is the sum of the first element $i$ and the last element $i + (s - 1)r(2t - 1), \ldots, i + (s - 1)r(2t - 1)$. 


which is \(2i + (s - 1)r(2t - 1)\), the last element of \(K\) in \(2i + H\). Now recall that on the graph of \(SF_{str+1}\) we connected the smallest element, \(i\), of \(K\) in the \(i^{th}\) coset to the root when \(i\) is odd and connected the largest element, \(i + (s - 1)r(2t - 1)\), of \(K\) in the \(i^{th}\) coset to the root when \(i\) is even. We should also recall from the beginning of the proof we observed that we were missing the first elements from \(\hat{r} + H\) where \(\hat{r} = (2i) \mod (2t - 1)\) and \(t \leq i \leq 2(t - 1)\). Summarizing these statements we must still recover edge labels from the last elements of \(K\) in for each even coset and the first elements from each odd coset. By construction when \(0 \leq i \leq t - 1\) we have that \(i\) is connected to the root when \(i\) is odd and \(i + (s - 1)r(2t - 1)\) is connected to the root when \(i\) is even, thus these elements are recovered as edge labels. When \(t \leq i \leq 2(t - 1)\) we have labeled each \(i^{th}\)-path with the ordered elements from \(i + H\). Therefore we may choose to connect the element \(i\) to the root from each path when \(i\) is odd and the element \(i + (s - 1)r(2t - 1)\) from each path to the root when \(i\) is even for \(t \leq i \leq 2(t - 1)\). This recovers all missing edge labels from \(\mathbb{Z}_{sr(2t-1)}\) completing the proof.

Figure 4: Harmonious Labeling of \(4P_{(3)(5)} \cup SF_{(3)(5)(5)+1}\)

**Example 3.6.** We consider the graph graph \(G = 4P_{(3)(5)} \cup SF_{(3)(5)(5)+1}\), shown in Figure 4, where we connect exactly one vertex from each path \(P_{15}\) to the root of \(SF_{(3)(5)(5)+1}\). For this graph we set \(H = \langle 9 \rangle\) and \(K = \langle 45 \rangle\) in \(\mathbb{Z}_{135}\). We have two paths that correspond to odd cosets, \(5 + H\) and \(7 + H\), and connect the first element of each, 5 and 7, to the root of our snowflake. We also have two paths that correspond to even cosets, \(6 + H\) and \(8 + H\), and connect the last element of \(0 + K\),
which is $6 + (3-1) \cdot 5(2(5)-1) = 96$ from $6 + H$ and $8 + (3-1) \cdot 5(2(5)-1) = 98$ from $8 + H$ to the root of our snowflake. We can see doing this and using the labeling algorithm described in the proof of Theorem 3.5 provides a harmonious labeling of $G$.

Unfortunately the method described in Theorem 3.5 did not provide any insight on how to harmoniously label the snowflake graph by itself. We also didn’t see any easy method to find such a labeling. In fact, we discovered the solution to harmoniously labeling the snowflake by itself on accident while checking for harmonious labelings on small snowflake graphs using Mathematica. We used brute force to determine how many harmonious labelings there would be, if any, on $SF_{(1)(3)(5)+1}$ and it turned out there were in fact many such labelings. We randomly chose one such harmonious labeling and happened to notice that each branch of the snowflake was labeled by the subcoset location instead of the coset location and the vertices on each subbranch were ordered by coset location as described in the *split even/odd ordered coset list* from Definition 2.6.

We then attempted to extend this notion to larger snowflake graphs and observed that we could harmoniously label the snowflake graph alone by assigning each branch a subcoset position and ordering each subbranch with the ordered elements in $H_{str}$ where we choose the smallest to largest elemental position moving away from the root on a branch when the subcoset is odd and largest to smallest elemental position as we move away from the root when the subcoset is even. Notice this is exactly how we choose the elemental position placement on each subbranch from Theorem 3.5 but we have switched the roles of the cosets and subcosets. We should also note that this requires us to generate $H$ and $K$ differently to obtain the proper number of cosets and subcosets. Specifically, we need $H = \langle r \rangle$ and $K = \langle tr \rangle$. We detail the method we use to harmoniously label the snowflake graph in the proof of the following theorem.

**Theorem 3.7.** The graph $SF_{str+1}$ with $s, t, r \geq 1$, $str \geq 3$ and odd is harmonious.

**Proof.** We begin the proof by noting that per the discussion immediately following Definition 3.3 we know that $SF_{str+1}$ is harmonious if $s, t, r$ is 1 and all odd with $str > 1$. Thus for the remainder of the proof we will assume that $s, t$, and $r$ are all odd and each greater than 1.

As we have done before we label the root of $SF_{str+1}$ with zero. Now recall that our graph has $str$ edges so we are working in $\mathbb{Z}_{str}$. Let $H = \langle r \rangle$ and $K = \langle tr \rangle$. This gives that $H$ has $r$ cosets and $K$ has $t$ subcosets for each coset of $H$, each of which contain exactly $s$ elements. This implies that each subcoset has $0, 1, 2, \ldots, s - 1$ elemental positions. We begin by fixing a subcoset for each branch, say the $j$th subcoset, and we will refer to this branch as the $j$th branch. We will label each $P_r$ subbranch on the $j$th branch with the ordered elements from the *split even/odd ordered coset list*, $H_{ji}$, where $0 \leq i \leq s - 1$. If $j$ is odd, then we will label each subbranch with the ordered elements from $H_{ji}$ where the elemental position $i$ is ordered from smallest to largest as our subbranches move away from the root. If $j$ is even, then we will label each subbranch with the ordered elements from $H_{ji}$ where the elemental position is ordered from largest to smallest as our subbranches move away from the root. Our claim is that this labeling is harmonious. By Theorem 2.8 this produces edge labels from the set $SH_{ji}$, and labeling every $P_r$ subbranch of each branch in this manner produces exactly $st(r - 1)$ distinct edge labels in $\mathbb{Z}_{str} - H$ when only the paths on the subbranches are considered.

To complete the proof we only need to show that we recover all $st$ elements from $H$ on the edges from the spine of each branch. First notice that by construction the central vertex of each $P_r$ subbranch is labeled with an element from the $0 + H$ coset and these central vertices are also on the spine of each branch and thus have adjacent edges. It is clear that the sum of any two vertex
labels from $0 + H$ will yield an edge label from $0 + H$ by Lemma 2.4. Thus for the $j^{th}$ branch we have consecutive elements from the $j^{th}$ subcoset in $0 + H$ adjacent to each other. Ignoring the edges coming directly out of the root on each spine our construction returns sums of the ordered elements from the $j^{th}$ subcoset in $0 + H$. That is, we are summing elements in order from the list
\[
\{[j + 0(t)]r, [j + 1(t)]r, \ldots, [j + (s - 1)t]r\}.
\]
Since each element from this list is increasing by $tr$, sums of consecutive elements will be increasing by $2tr$ in $\mathbb{Z}_{str}$. Summing the elements in the $k - 1$ and $k$ elemental positions, $1 \leq k \leq s - 1$ we get
\[
[j + (k - 1)t]r + [j + (k)t]r = [(2j + (2k - 1)t)r] \mod str.
\]
As we did in the proof of Lemma 2.5, we observe that $[2j + [(2k - 1)t]]r$ is not subject to mod $str$ if $k < \frac{s + 1}{2}$ and is subject to mod $str$ if $k \geq \frac{s + 1}{2}$. To see this consider the two cases,

\[
k = \frac{s - 1}{2} \implies [2j + (2k - 1)t]r = [2j + (s - 2)t]r < str, \text{ since } 0 \leq j \leq t - 1
\]

\[
k = \frac{s + 1}{2} \implies [2j + (2k - 1)t]r = 2jr + (s)tr > str
\]

\[
\implies [2j + [(2k - 1)t]r] \mod str = (2jr + str) \mod str = 2jr.
\]

Thus we can see that the sum of all consecutive elements in $j^{th}$ subcoset of $0 + H$ for $0 < k < \frac{s + 1}{2}$ will yield exactly $\frac{s - 1}{2}$ odd elements in $[(2j) \mod t]r + K$ of the form

\[
\{(2j + t)r, (2j + 3t)r, (2j + 5t)r, \ldots, [2j + (s - 2)t]r\} \text{ if } 2j < t \text{ and}
\]

\[
\{(\hat{s} + 2t)r, (\hat{s} + 4t)r, (\hat{s} + 6t)r, \ldots, [\hat{s} + (s - 1)]tr\} \text{ if } 2j > t,
\]

where $\hat{s} = (2j) \mod t$. We also note that $\hat{s}$ is odd since $2j$ is even and $t$ is odd, which shows that the elements above are indeed all odd.

To get the even elements of $[2j \mod t]r + K$ we simply let $\frac{s + 1}{2} \leq k \leq s - 1$ which yields exactly $\frac{s - 1}{2}$ even elements of the form

\[
\{(2j)r, (2j + 2t)r, (2j + 4t)r, \ldots, [2j + (s - 3)t]r\}, \text{ if } 2j < t \text{ and}
\]

\[
\{(\hat{s} + t)r, (\hat{s} + 3t)r, (\hat{s} + 5t)r, \ldots, [\hat{s} + (s - 2)t]r\}.
\]

Combining the even and odd elements together we have that the set $R$ of all sums is

\[
R = \{(2j)r, (2j + t)r, (2j + 2t)r, (2j + 3t)r, \ldots, [2j + (s - 2)t]r\} \text{ if } 2j < t \text{ and}
\]

\[
R = \{(\hat{s} + t)r, (\hat{s} + 2t)r, (\hat{s} + 3t)r, (\hat{s} + 4t)r, \ldots, [\hat{s} + (s - 1)t]r\} \text{ if } 2j > t.
\]

This shows that we get the first $s - 1$ elements of the $2j$ subcoset of $0 + H$ when $2j < t$ and get the last $s - 1$ elements of $\hat{s}$ subcoset of $0 + H$ when $2j > t$. Of course, we have $\hat{s} = 2j \mod t \in \{1, 3, \ldots, t - 2\}$ yields odd subcosets for

\[
j \in \left\{\frac{t + 1}{2}, \frac{t + 3}{2}, \ldots, t - 1\right\} \implies 2j \in \{t + 1, t + 3, \ldots, 2(t - 1)\},
\]
and all even subcosets are obtained from
\[
j \in \left\{ 1, 2, \ldots, \frac{t - 1}{2} \right\} \implies 2j \in \{2, 4, \ldots, t-1\}.
\]

Finally, we recall that the odd subcoset branches are labeled with the smallest elemental position closest to the root and the even subcoset branches have the largest elemental position closest to the root. That gives that the smallest element in each odd subcoset of \(0 + H\) is connected to the root and the largest element of each even subcoset of \(0 + H\) is connected to the root. However, we also notice that for our odd subcosets, when \(2j > t\), we are missing the first element of the \(s\) subcoset in \(R\) which is then recovered since each of the smallest elements of the odd subcosets are connected to the root which is labeled zero. Similarly, since we are missing the largest element from each even subcoset in \(R\) this is also recovered since the largest element of each even subcoset of \(0 + H\) is connected to the root. Thus we have recovered every element of \(H\) as an edge label which gives that the set of all edge labels is \((\mathbb{Z}_{str} - H) \cup H = \mathbb{Z}_{str}\). Given that there are exactly \(str\) edges and we have recovered every element in \(\mathbb{Z}_{str}\) as an edge label, we have shown that \(SF_{str+1}\) is harmonious.

Example 3.8. We now apply Theorem 3.7 to \(SF_{(3)(5)(5)+1}\). In contrast to our harmonious labeling in Example 3.6 we will label each branch of the graph dependent on a particular subcoset. The setup for our subgroups are as follows, \(H = \langle r \rangle = \langle 5 \rangle\) and \(K = \langle rt \rangle = \langle 25 \rangle\). This yields five cosets of \(H\) and five subcosets of \(K\) for each coset of \(H\) each containing exactly \(s = 3\) elements.
that is, each subcoset has three elemental positions 0, 1, and 2. The paths $P_5$ on each branch are labeled using the split even/odd ordered coset list, $H_{jr}$, for the $j$th subcoset chosen for the given branch. If $j$ is even, we label $P_5$ closest to the root with the elements from $H_{j2}$ in order since 2 is the largest elemental position, and order from largest elemental position to smallest for subsequent $P_5$ subbranches as we move away from the origin. If $j$ is odd, we label each $P_5$ subbranch with ordered elements from $H_{jr}$ from smallest to largest elemental position as we move away from the root. This procedure gives a harmonious labeling for $SF_{(3)(5)(5)+1}$ as shown in Figure 5.

4 Further Examples and Questions

It is noted in [4] that it is not known if $C_s \cup P_{st+1}$ or $C_s \cup T_{st+1}$ are harmonious when $s$ and $t$ are both odd. Similarly it is not known if $T_{st+1}$ is harmonious when either $s$ or $t$ are even. We have similar questions for the snowflake graph $SF_{str+1}$ as having $s$, $t$, and $r$ all odd is what allows us to show that the snowflake graph is harmonious.

**Question.** Is $SF_{str+1}$ harmonious when at least one of the parameters are even, that is, if $s$ or $t$ is even?

We must maintain that $r$ is odd by the definition of the snowflake graph, since $r$ must be odd for the path $P_r$ to have a central vertex. Though it is not clear how to prove this in general, we did check for harmonious labelings using brute force in Mathematica. Since the number of permutations of labels is high, even for a small graph, we did not invest time into brute force testing for larger examples, nor was attempting to do so part of the scope of this paper. However, we did find several harmonious labelings for $SF_{(2)(2)(3)+1}$ which we will discuss in the following example.

**Example 4.1.** The following is a harmonious labeling of the graph $SF_{(2)(2)(3)+1}$.

![Figure 6: Harmonious Labeling of $SF_{(2)(2)(3)+1}$](image)

The labeling chosen here is just one of many harmonious labelings that we observed through our brute force calculation. Though many harmonious labelings were observed, only 1 in 16,632 labelings were harmonious when all vertex permutations were considered with the root fixed at zero. That being said, we did not seem to observe a distinct pattern in the labelings that were harmonious but further study of these graphs when $s$ or $t$ are even may provide a general construction. Moreover, if a method to harmoniously label $T_{st+1}$ is found with either $s$ or $t$ even, then it may be
possible to extend this method to harmoniously label $SF_{str+1}$ with either $s$ or $t$ even, in a similar manner.

We should note that the methods employed in [4] and in this paper fail quickly if $m$ is even. This is because if we are working in $\mathbb{Z}_m$ with $m$ even, then adding consecutive elements, say in the $i + H$ coset as we did in Lemma 2.5, $a$ would always yield elements in the $(2i) \mod m + H$ coset but $(2i) \mod m$ would always be even since $m$ is even. This will always produce duplicate edge labels when any of our methods from this paper or from [4] are used.

The final question that we ask is can we extend the concept of the snowflake graph and such a harmonious labeling further and harmoniously label such graphs using subcosets of subcosets. We admit that the pure notion of this seems quite daunting. However, if proper notation was used, we feel it may be possible to inductively define and extension of the snowflake graph and harmoniously label it using similar techniques. More work is needed to explore the feasibility of extending these methods.

References