On p-competition graphs of loopless Hamiltonian digraphs without symmetric arcs and graph operations

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Abstract

For a digraph $D$, the $p$-competition graph $C_p(D)$ of $D$ is the graph satisfying the following: $V(C_p(D)) = V(D)$, for $x, y \in V(C_p(D))$, $xy \in E(C_p(D))$ if and only if there exist distinct $p$ vertices $v_1, v_2, ..., v_p \in V(D)$ such that $x \rightarrow v_i$, $y \rightarrow v_i \in A(D)$ for each $i = 1, 2, ..., p$.

We show the $H_1 \cup H_2$ is a $p$-competition graph of a loopless digraph without symmetric arcs for $p \geq 2$, where $H_1$ and $H_2$ are $p$-competition graphs of loopless digraphs without symmetric arcs and $V(H_1) \cap V(H_2) = \{\alpha\}$. For $p$-competition graphs of loopless Hamiltonian digraphs without symmetric arcs, we obtain similar results. And we show that a star $K_{1,p}$ is a $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs if $n \geq 2p + 3$ and $p \geq 3$.

Based on these results, we obtain conditions such that spiders, caterpillars and cacti are $p$-competition graphs of loopless digraphs without symmetric arcs. We also obtain conditions such that these graphs are $p$-competition graphs of loopless Hamiltonian digraphs without symmetric arcs.

1 Introduction

In this paper we consider finite simple graphs and finite digraphs. For a graph $G$ and $S \subseteq V(G)$, $\langle S \rangle_G$ is the induced subgraph on $S$ of $G$. For a digraph $D$ and $v \in V(D)$, $\text{Out}_D(v) = \{u : v \rightarrow u \in A(D)\}$ and $\text{In}_D(v) = \{u : u \rightarrow v \in A(D)\}$. A pair of arcs $u \rightarrow v$ and $v \rightarrow u$ is called symmetric arcs.

Definition 1.1. For a digraph $D$, the $p$-competition graph $C_p(D)$ of $D$ is the graph satisfying the following:

1. $V(C_p(D)) = V(D)$,
2. For $x, y \in V(C_p(D))$, $xy \in E(C_p(D))$ if and only if there exist distinct $p$ vertices $v_1, v_2, ..., v_p \in V(D)$ such that $x \rightarrow v_i$, $y \rightarrow v_i \in A(D)$ for each $i = 1, 2, ..., p$.

A graph $G$ is called a $p$-competition graph if there exists a digraph $D$ such that $C_p(D) \cong G$.

The $p$-competition graphs have been extensively studied. For example, Kim et al. [4] gave a characterization of $p$-completion graphs of arbitrary digraphs. And Kim et al. [5] dealt with cycles in terms of $p$-competition graphs of arbitrary digraphs which are allowed to have loops and symmetric arcs. Kim et al. [4] also deal with $p$-competition graphs of acyclic graphs and loopless digraphs. Kidokoro et al. [2] studied $p$-competition graphs of loopless digraphs without symmetric arcs. Furthermore, they deal with $p$-competition graphs in term of the sum operation.

In the first part of this paper, we deal with properties of $p$-competition graphs with cut vertices and the union of two $p$-competition graphs. Furthermore, we give some family of $p$-competition graphs of loopless digraphs without symmetric arcs.

For a digraph $D$, a directed cycle, or dicycle, is a sequence of arcs of the form $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots \rightarrow v_i \rightarrow v_1$, where $v_1, v_2, ..., v_i$ are all distinct. A digraph $D$ is called Hamiltonian if there exists a dicycle, called a Hamiltonian dicycle, which includes every vertex of $D$. In

In the latter part of this paper, we deal with p-competition graphs of loopless Hamiltonian digraphs without symmetric arcs in terms of cut vertices and the union of two graphs.

Lundgren et al. [6] gave the following results on loopless Hamiltonian digraphs.

**Theorem 1.1** (Lundgren et al. [6]).

1. For \( p \geq 2 \) and \( n \geq p + 3 \), \( C_n \) is the p-competition graph of a loopless Hamiltonian digraph.
2. For \( n \geq p + 2 \), \( K_n \) is the p-competition graph of a loopless Hamiltonian digraph.
3. For \( n \geq 2p + 1 \), \( K_n - e \) is the p-competition graph of a loopless Hamiltonian digraph.
4. For \( n \geq p + 3 \), \( P_n \) is the p-competition graph of a loopless Hamiltonian digraph.
5. For \( n \geq p + 3 \), a caterpillar with \( n \) vertices is the p-competition graph of a loopless Hamiltonian digraph.
6. For \( p \geq 2 \) and \( n \geq 2p \), a tree with \( n \) vertices is the p-competition graph of a loopless Hamiltonian digraph.
7. For \( p \geq 2 \) and two p-competition graphs of loopless Hamiltonian digraphs \( G_1 \) and \( G_2 \), \( G_1 \cup G_2 \) is the p-competition graph of a loopless Hamiltonian digraph.

Kidokoro et al. [3] gave some results on p-competition graphs of loopless Hamiltonian digraphs without symmetric arcs as follows.

**Theorem 1.2** (Kidokoro et al. [3]). Let \( p \) be a positive integer and \( n \geq 2p + 3 \). Then \( P_n \) is the p-competition graph of a loopless Hamiltonian digraph which has no symmetric arcs.

Lundgren et al. [6] dealt with loopless digraphs and in Theorem 1.1 (5) and (6) they combined p-competition graphs. Kidokoro et al. [3] and Nakada et al. [8] dealt with graph operations. In [2] Kidokoro et al. dealt with p-competition graphs in terms of sum operations. The sum \( G + I \) of two graphs \( G \) and \( I \) is the graph with the vertex set \( V(G + I) = V(G) \cup V(I) \) and the edge set \( E(G + I) = E(G) \cup E(I) \cup \{u, v\} : u \in V(G), v \in V(I) \}. \) They gave the next result.

**Theorem 1.3** (Kidokoro et al. [2]). Let \( G \) be a p-competition graph of a loopless digraph with no symmetric arcs, where \( G \) has no isolated vertices. Then \( G + K_n \) is the p-competition graph of a loopless digraph with no symmetric arcs.

Kidokoro et al. [2] showed the next result by Theorem 1.3 and Theorem 2.7

**Theorem 1.4** (Kidokoro et al. [2]). Let \( W_n \) be a wheel with order \( n \geq 4 \) and \( p \) be a positive integer with \( n \geq 2p + 4 \). Then \( W_n \) is the p-competition graph of a loopless digraph with no symmetric arcs.

Nakada et al. [8] dealt with full regular \( m \)-ary trees and another graph operation, that is, the union of graphs.

For graphs \( G \) and \( H \), the union of \( G \) and \( H \) is the graph \( G \cup H \) such that \( V(G \cup H) = V(G) \cup V(H) \) and \( E(G \cup H) = E(G) \cup E(H) \). For digraphs \( D \) and \( F \), the union of \( D \) and \( F \) is the digraph \( D \cup F \) such that \( V(D \cup F) = V(D) \cup V(F) \) and \( A(D \cup F) = A(D) \cup A(F) \).
Proof. Let $\overrightarrow{T_{m,n}}$ be a full regular $m$-ary rooted tree with height $n$, that is, every non-leaf has exactly $m$ children and the leaves, outdegree 0 vertices, being equidistance $n$ from the root. Let $T_{m,n}$ be the graph obtained from $\overrightarrow{T_{m,n}}$ without directions (see Figure 1). Then $T_{m,n}$ is a tree. The graph $K_{1,m}$ is a full regular $m$-ary tree with height 1, that is, $T_{m,1}$. Each subtree of $T_{m,n}$ induced by a non-leaf and its children is also a full regular $m$-ary tree with height 1. Kidokoro et al. [2] gave the following result.

**Theorem 1.5** (Kidokoro et al. [2]). Let $p$ be a positive integer and $p \leq \frac{m-1}{2}$. Then $K_{1,m}$ is the $p$-competition graph of a loopless digraph without symmetric arcs.

Since $T_{m,n}$ is the union of $K_{1,m}$, Nakada et al. [8] obtained the following result by Theorem 1.5.

**Theorem 1.6** (Nakada et al. [8]). Let $p$ be a positive integer and $p \leq \frac{m-1}{2}$. Then $T_{m,n}$ is the $p$-competition graph of a loopless digraph without symmetric arcs.

## 2 Cut-vertices and $p$-competition graphs

In this section we consider properties of $p$-competition graphs with cut-vertices and the union of two graphs. We obtain the following result.

**Theorem 2.1.** Let $p \geq 2$ be a positive integer. Let $H_1$ and $H_2$ be $p$-competition graphs of loopless digraphs without symmetric arcs and $V(H_1) \cap V(H_2) = \{\alpha\}$. Then $H_1 \cup H_2$ is the $p$-competition graph of a loopless digraph without symmetric arcs.

**Proof.** For $i = 1, 2$, let $D_i$ be a loopless digraph without symmetric arcs such that $C_p(D_i) = H_i$ and $V(D_1) \cap V(D_2) = \{\alpha\}$. The digraph $D_1 \cup D_2$ is a digraph with $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$. Furthermore for $v \in V(D_i) - \{\alpha\}$ ($i = 1, 2$), $Out_{D_1 \cup D_2}(v) = Out_{D_i}(v)$ and $Out_{D_1 \cup D_2}(\alpha) = Out_{D_1}(\alpha) \cup Out_{D_2}(\alpha)$. Since $Out_{D_1}(\alpha) \cap Out_{D_2}(\alpha) = \emptyset$, the digraph $D_1 \cup D_2$ has neither loops nor symmetric arcs.

For $u, v \in V(H_i)$, if $uv \in E(H_i)$, then $|Out_{D_1 \cup D_2}(u) \cap Out_{D_1 \cup D_2}(v)| = |Out_{D_1}(u) \cap Out_{D_1}(v)| \geq p$ and $uv \in E(C_p(D_1 \cup D_2)).$ For $u \in V(H_1) - \{\alpha\}$ and $v \in V(H_2) - \{\alpha\}$, then $|Out_{D_1 \cup D_2}(u) \cap Out_{D_1 \cup D_2}(v)| = |Out_{D_1}(u) \cap Out_{D_2}(v)| \leq 1$ and $uv \notin E(C_p(D_1 \cup D_2))$, because $p \geq 2$. Since for $u \in V(H_i)$ ($i = 1, 2$), $Out_{D_1 \cup D_2}(\alpha) \cap Out_{D_1 \cup D_2}(u) = Out_{D_1}(\alpha) \cap Out_{D_1}(u)$, $u\alpha \in E(H_i)$ if and only if $u\alpha \in E(C_p(D_1 \cup D_2))$.  

![Figure 1: A regular m-ary T_{m,1} and T_{m,2}](image-url)
Therefore \( C_p(D_1 \cup D_2) = H_1 \cup H_2 \).

By Theorem 2.1, we have the following result on cut-vertices.

**Corollary 2.2.** Let \( G \) be a connected graph with a cut-vertex \( \alpha \) and \( B_i \) \( (i = 1, 2, \ldots, l) \) be a connected component of \( G - \alpha \). Let \( p \geq 2 \) be an integer. If each induced subgraph \( \langle V(B_i) \cup \{\alpha\} \rangle_G \) is the \( p \)-competition graph of a loopless digraph without symmetric arcs, then \( G \) is the \( p \)-competition graph of a loopless digraph without symmetric arcs.

In [7] Nakada et al. dealt with spiders on \( p \)-completion graphs. The spider \( S_{m,n} \) is the following graph: \( V(S_{m,n}) = \{v_0\} \cup \{v_{i,j} : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n - 1\} \) and \( E(S_{m,n}) = \{\{v_0, v_{i,1}\} : i = 1, 2, \ldots, m\} \cup \{\{v_{i,j}, v_{i,j+1}\} : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n - 2\} \) (see Figure 2).

Note that \( S_{1,n} \sim P_n \), \( S_{2,n} \sim P_{2n-1} \) and \( S_{m,2} \sim K_{1,m} \).

**Theorem 2.3** (Nakada et al. [7]). Let \( m \) be an even number and \( p \leq n - 2 \) be a positive integer. Then \( S_{m,n} \) \((m, n \geq 3)\) is the \( p \)-competition graph of a loopless digraph without symmetric arcs.

In [2] Kidokoro et al. gave the following result on paths in terms of \( p \)-competition graphs of loopless digraphs without symmetric arcs.

**Theorem 2.4** (Kidokoro et al. [2]). For a positive integer \( p \) and \( n \geq 2 \), \( P_n \) is the \( p \)-competition graph of a loopless digraph without symmetric arcs if and only if \( n \geq 2p + 3 \).

Using Theorem 2.1 and Theorem 2.4, we obtain the following result.

**Theorem 2.5.** Let \( p \geq 2 \) be a positive integer and \( n \geq 2p + 3 \). Then \( S_{m,n} \) \((m \geq 3, \text{ and } n \geq 7)\) is the \( p \)-competition graph of a loopless digraph without symmetric arcs.

**Proof.** Let \( F_i \) be a graph with \( V(F_i) = \{\alpha, v_{i,1}, v_{i,2}, \ldots, v_{i,n-1}\} \) and \( E(F_i) = \{\{\alpha, v_{i,1}\}\} \cup \{\{v_{i,j}, v_{i,j+1}\} : j = 1, 2, \ldots, n - 2\} \). Then \( F_i \) is a path \( P_n \) and \( \bigcup_{i=1}^{m} F_i \cong S_{m,n} \).

Since each \( F_i \) is the \( p \)-competition graph of a loopless digraph without symmetric arcs by Theorem 2.4, \( S_{m,n} \) is the \( p \)-competition graph of a loopless digraph without symmetric arcs.
A caterpillar is a tree $T$ with the property that the removal of degree one vertices of $T$ results in a path. Then degree one vertices are called leaves. By Theorem 1.5 and 2.1, we obtain the following result on caterpillars. For a caterpillar $G$, let $V_{NL}(G)$ be the vertex set of non-leaves of $G$, that is, the set of vertices whose degrees are greater than or equal to two.

**Theorem 2.6.** Let $p \geq 2$ be a positive integer and $p \leq \frac{m-1}{2}$. Then a caterpillar whose non-leaves have degrees at least $m + 1$ is the $p$-competition graph of a loopless digraph without symmetric arcs.

**Proof.** Let $G$ be a caterpillar and $V_{NL}(G) = \{u_1, u_2, ..., u_n\}$ be the vertex set of non-leaves of $G$. Then the induced subgraph $(V_{NL}(G))_G$ is a path. We assume that $E((V_{NL}(G))_G) = \{u_iu_{i+1} : i = 1, 2, ..., n - 1\}$ and $V_i = \{v \in V(G) : vu_i \in E(G), v \text{ is a leaf of } G\}$. Then $|V_i| \geq m - 1$ and $F_i = (\{u_i, u_{i+1}\} \cup V_i)_G \cong K_{1,|V_i|+1}$ for $i = 1, 2, ..., n - 1$. Furthermore $|V_n| \geq m$ and $F_n = (\{u_n\} \cup V_n)_G \cong K_{1,|V_n|}$. Since $F_i \cong K_{1,l_i}$ and $l_i \geq m$ for $i = 1, 2, ..., n$, $F_i$ is a $p$-competition graph of a loopless digraph without symmetric arcs by Theorem 1.5. Since $G \cong \bigcup_{i=1}^{n} F_i$, $G$ is a $p$-competition graph of a loopless digraph without symmetric arcs by Theorem 2.1. \qed

A nontrivial connected graph without cut-vertices is called a block. A block of a graph $G$ is a subgraph of $G$, which is itself a block and which is maximal with respect to that property. A cactus is a connected graph in which every block is an edge or a cycle. Thus, for a cactus $G$, we can obtain an edge partition $E(G) = E(C) \cup E(T)$, where $C$ is union of cycles and $T$ is a forest. Kidokoro et al. [2] also obtained the following result.

**Theorem 2.7** (Kidokoro et al. [2]). For a positive integer $p$ and $n \geq 3$, $C_n$ is the $p$-competition graph of a loopless digraph without symmetric arcs if and only if $n \geq 2p + 3$.

By Theorem 2.7, Theorem 1.5, Theorem 2.4 and Theorem 2.1 we obtain the following result on cacti.

**Theorem 2.8.** Let $p \geq 2$ be a positive integer. Then a cactus $G$ is the $p$-competition graph of a loopless digraph without symmetric arcs if there exists an edge partition $E(G) = E(C) \cup E(T)$ such that

1. $E(C) = \bigcup E(C_{\alpha_i})$, where $C_{\alpha_i}$ is a cycle with $\alpha_i \geq 2p + 3$,

2. $E(T) = (\bigcup E(K_{1,\beta_j})) \cup (\bigcup E(P_{\gamma_k}))$ such that

   2.1 each $K_{1,\beta_j}$ is a maximal subgraph isomorphic to a star graph and $\frac{\beta_j - 1}{2} \geq p$, that is, $\beta_j \geq 2p + 1$,

   2.2 for all $P_{\gamma_k}, \gamma_k \geq 2p + 3$.

**Proof.** By assumption, $G = (\bigcup C_{\alpha_i}) \cup (\bigcup K_{1,\beta_j}) \cup (\bigcup P_{\gamma_k})$. By Theorem 2.7, Theorem 1.5 and Theorem 2.4, all $C_{\alpha_i}, K_{1,\beta_j}$ and $P_{\gamma_k}$ are $p$-competition graphs of loopless digraphs without symmetric arcs.

Since $G$ is a cactus, for any two of $C_{\alpha_i}, K_{1,\beta_j}$ and $P_{\gamma_k}$, the intersection of these two has at most one vertex. Therefore, a cactus satisfying the conditions of the theorem is the $p$-competition graph of a loopless digraph without symmetric arcs by Theorem 2.1. \qed
3 Cut-vertices and $p$-competition graphs of Hamiltonian digraphs

In this section we consider properties of $p$-competition graphs of loopless Hamiltonian digraphs without symmetric arcs. We obtain the following result.

**Theorem 3.1.** Let $p \geq 3$ be a positive integer. Let $H_1$ and $H_2$ be $p$-competition graphs of loopless Hamiltonian digraphs without symmetric arcs and $V(H_1) \cap V(H_2) = \{\alpha\}$. Then $H_1 \cup H_2$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.

**Proof.** For $i = 1, 2$, let $D_i$ be a loopless Hamiltonian digraph without symmetric arcs such that $C_p(D_i) = H_i$ and $V(D_1) \cap V(D_2) = \{\alpha\}$. Then $C_p(D_1 \cup D_2) = H_1 \cup H_2$ by the similar way of the proof of Theorem 2.1.

Let $\alpha \rightarrow w_{1,1} \rightarrow w_{1,2} \rightarrow \cdots \rightarrow w_{1,k_1} \rightarrow \alpha$ be a Hamiltonian dicycle of $D_1$ and $\alpha \rightarrow w_{2,1} \rightarrow w_{2,2} \rightarrow \cdots \rightarrow w_{2,k_2} \rightarrow \alpha$ be a Hamiltonian dicycle of $D_2$.

Adding an arc $w_{1,k_1} \rightarrow w_{2,1}$ to the digraph $D_1 \cup D_2$, we make a new digraph $D^*$ with $V(D^*) = V(D_1) \cup V(D_2)$ and $A(D^*) = A(D_1) \cup A(D_2) \cup \{w_{1,k_1} \rightarrow w_{2,1}\}$.

Then $Out_{D^*}(v) = Out_{D_1 \cup D_2}(v)$ for $v \in V(D^*)-\{w_{1,k_1}\}$ and $Out_{D^*}(w_{1,k_1}) = Out_{D_1 \cup D_2}(w_{1,k_1}) \cup \{w_{2,1}\}$. For $v \in V(H_2)$, $|Out_{D^*}(w_{1,k_1}) \cap Out_{D^*}(v)| = |(Out_{D_1 \cup D_2}(w_{1,k_1}) \cup \{w_{2,1}\}) \cap Out_{D_1 \cup D_2}(v)| \leq 2$. Thus $C_p(D^*) = H_1 \cup H_2$, because $p \geq 3$.

Since $\alpha \rightarrow w_{1,1} \rightarrow w_{1,2} \rightarrow \cdots \rightarrow w_{1,k_1} \rightarrow w_{2,1} \rightarrow w_{2,2} \rightarrow \cdots \rightarrow w_{2,k_2} \rightarrow \alpha$ is a Hamiltonian dicycle of $D^*$, $H_1 \cup H_2$ is a $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.

By Theorem 3.1 we have the following result on cut-vertices.

**Corollary 3.2.** Let $G$ be a connected graph with a cut-vertex $\alpha$ and $B_i$ $(i = 1, 2, \ldots, l)$ be a connected component of $G - \alpha$. Let $p \geq 3$ be an integer. If each induced subgraph $(V(B_i) \cup \{\alpha\})_G$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs, then $G$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.

Using Theorem 3.1 and Theorem 1.2, we obtain the following result.

**Theorem 3.3.** Let $p \geq 3$ be a positive integer and $n \geq 2p + 3$. Then $S_{m,n}$ $(m \geq 3, n \geq 9)$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.

In the proof of Theorem 2.7, Kidokoro et al. [2] constructed the loopless digraph $D$ without symmetric arcs whose $p$-competition graph is a cycle $C_n$ as follows: $V(D) = \{v_0, v_1, \ldots, v_{n-1}\}$, and $A(D) = \{v_i \rightarrow v_{i+j} : i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, p + 1\}$, where all subscript arithmetic is taken modulo $n$.

We examine the construction of this digraph $D$ and observe that $D$ has a Hamiltonian dicycle $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_0$. So, we obtain the following result.

**Theorem 3.4.** Let $p$ be a positive integer and $n \geq 2p + 3$. Then $C_n$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.
Since $p \leq \frac{m-1}{2}$ means $m \geq 2p+1$, for a positive integer $p$ and $n \geq 2p+1$, $K_{1,n}$ is the $p$-competition graph of a loopless digraph without symmetric arcs. By Theorem 1.5.

In the proof of Theorem 1.5, Kidokoro et al. constructed a loopless digraph $D_1$ without symmetric arcs whose $p$-competition graph is $K_{1,n}$, where $V(D_1) = \{v_0, v_1, ..., v_{n-1}\} \cup \{u\}$, and $A(D_1) = \{v_i \rightarrow v_{i+j} : i = 0, 1, ..., n-1, j = 1, 2, ..., p\} \cup \{u \rightarrow v_i : i = 0, 1, ..., n-1\}$, where all subscript arithmetic is taken modulo $n$.

We observe that we can modify $D$ so that it is a Hamiltonian digraph, which we will refer to as $D_1$. So, we obtain the following result.

**Theorem 3.5.** Let $p \geq 3$ be a positive integer and $n \geq 2p+3$. Then $K_{1,n}$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs.

**Proof.** Let $K_{1,n}$ be a star with $V(K_{1,n}) = \{v_0, v_1, ..., v_{n-1}\} \cup \{u\}$ and $E(K_{1,n}) = \{\{u, v_0\}, \{u, v_1\}, ..., \{u, v_{n-1}\}\}$. All subscript arithmetic is taken modulo $n$. We construct a digraph $H$ which is modified the digraph $D_1$ as follows

1. $V(H) = \{v_0, v_1, ..., v_{n-1}\} \cup \{u\}$, and
2. $A(H) = \{v_i \rightarrow v_{i+k} : i = 0, 1, ..., n-1, k = 1, 2, ..., p\} \cup \{u \rightarrow v_i : i = 1, 2, ..., n-1\} \cup \{v_i \rightarrow v_{i+2k} : i = n-1, n-2, ..., n-p\} \cup \{v_0 \rightarrow u\}$.

Since $n \geq 2p+3, v_{i-(p+1)} \neq v_{i+k}$ for $i = n-1, n-2, ..., n-p$ and $k = 1, 2, ..., p$. Then $Out_H(v_i) = \{v_{i+k} : 1 \leq k \leq p\} \cup \{v_{i-(p+1)}\}$ for $i = n-1, n-2, ..., n-p$, $Out_H(v_i) = \{v_{i+k} : 1 \leq k \leq p\}$ for $i = n-p-1, n-p-2, ..., 1$, $Out_H(v_0) = \{v_i : 1 \leq i \leq p\} \cup \{u\}$ and $Out_H(u) = \{v_1, v_2, ..., v_{n-1}\}$.

And for $i = 0, 1, ..., n-1$, max\{ $l : v_l \in Out_H(v_i)$ $\} = i+p$. For $i = n-p-1, n-p-2, ..., 0$, $\min\{ l : v_l \in Out_H(v_i) \} = i$ and for $i = n-1, n-2, ..., n-p$, $\min\{ l : v_l \in Out_H(v_i) \} = i - (p+1)$.

Since $n \geq 2p+3, v_{i+(p+1)+p} \notin In_H(v_i)$, and there exist no symmetric arcs between $v_i$ and $v_{i+k}$ for $i = 0, 1, ..., n-1$ and $k = 1, 2, ..., p$. For $i = n-1, n-2, ..., n-p$, max\{ $l : v_l \in Out_H(v_{i+(p+1)}) \} = i - (p+1) + p = i - 1$. Thus $v_{i-(p+1)} \notin In_H(v_i)$ and there exist no symmetric arcs between $v_i$ and $v_{i-(p+1)}$. Therefore $H$ has no symmetric arcs and no loops.

For $i = n-1, n-2, ..., n-p$, $Out_H(u) \cap Out_H(v_i) = \{v_{i+1}, v_{i+2}, ..., v_{i+p}, v_{i-(p+1)}\} - \{v_0\}$ and $|Out_H(u) \cap Out_H(v_i)| = p$ and $\{u, v_i\} \in E(C_p(H))$.

For $i = n-p-1, n-p-2, ..., 0$, $Out_H(u) \cap Out_H(v_i) = \{v_{i+1}, v_{i+2}, ..., v_{i+p}\}$ and $|Out_H(u) \cap Out_H(v_i)| = p$ and $\{u, v_i\} \in E(C_p(H))$.

We consider $v_i$ and $v_j$ for $i, j = n-1, n-2, ..., n-p$. Since $p \geq 3$, we consider that for $i < j$ and $j-i = k$, $Out_H(v_i) \cap Out_H(v_{i+k}) = \{v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}, v_{i-(p+1)}, v_{i+k-(p+1)}\}$ and $k \leq p-1$. If $v_{i+k-(p+1)} \in Out_H(v_i) - \{v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}\}$, then $i+1 \leq i+k-(p+1) \leq i+k$ and thus $p+2 \leq k$, which is a contradiction. So $Out_H(v_i) \cap Out_H(v_{i+k}) \subseteq \{v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}, v_{i-(p+1)}\}$. And $\{|v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}, v_{i-(p+1)}\| = (i+p+1) - (i+k) = p+1 - k \leq p-1$ without the case $k = 1$. So, we consider the case $Out_H(v_i) \cap Out_H(v_{i+k})$. If $i-(p+1) \equiv i+1+p \pmod{n}$, then $i-(p+1) = i+1+p-n$ because $i-(p+1) \leq i+1+p$. Since $n \geq 2p+3, v_{i-(p+1)} \notin In_H(v_i)$.

Next we consider $v_i$ and $v_j$ for $i = n-p-1, n-p-2, ..., 0$ and $j = n-1, n-2, ..., 0$.
For $i < j$, $j - i = k$ and $i,j = n - p - 1,n - p - 2,...,0$, $\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+k}) = \{v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}\}$ and $|\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+k})| = i + p - (i + k) = p - k < p$.

For $i < j$, $j - i = k$, $i = n - p - 1,n - p - 2,...,0$ and $j = n - 1,n - 2,...,n - p$, $\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+k}) \subseteq \{v_{i+k+1}, v_{i+k+2}, ..., v_{i+p}, v_{i+k-(p+1)}\}$ and $|\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+k})| \leq p - k + 1$. So $|\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+k})| \leq p - 1$ without the case $k = 1$. Then $\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+1}) \subseteq \{v_{i+2}, v_{i+3}, ..., v_{i+p}, v_{i-1}\}$ and $\{v_{i+1}\} = \text{Out}_H(v_i) - \{v_{i+2}, v_{i+3}, ..., v_{i+p}\}$. If $v_{i-1} \in \text{Out}_H(v_i) \cap \text{Out}_H(v_{i+1})$, then $v_{i-1} = v_{i+1}$ and $i + 1 = i - p$, which is a contradiction. So $\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+1}) = \{v_{i+2}, v_{i+3}, ..., v_{i+p}\}$ and $|\text{Out}_H(v_i) \cap \text{Out}_H(v_{i+1})| = p - 1 < p$.

Thus $\{v_i, v_j\} \notin E(C_p'(H))$ for $i, j = n - 1, n - 2, ..., 0$.

Therefore $C_p'(H) \cong K_{1,n}$.

Then $v_0 \rightarrow u \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_0$ is a Hamiltonian dicycle. Thus $H$ is the loopless Hamiltonian digraph with no symmetric arcs whose $p$-competition graphs is $K_{1,n}$.

By Theorem 3.1 and 3.5, we obtain the following result on caterpillars.

**Theorem 3.6.** Let $p \geq 3$ be a positive integer and $m \geq 2p + 3$. Then a caterpillar whose non-leaves have degrees at least $m + 1$ is the $p$-competition graph of a loopless digraph without symmetric arcs.

**Proof.** Let $G$ be a caterpillar, $V_{NL}(G) = \{u_1, u_2, ..., u_n\}$ be a vertex set of non-leaves of $G$ and $V_i = \{v \in V(G) : vu_i \in E(G), v \text{ is a leaf of } G\}$. By a proof similar to that of Theorem 2.6, we obtain $G \cong \bigcup_{i=1}^{n} F_i$, where $F_i = \{\{u_i, u_{i+1}\} \cup V_i\}_G \cong K_{1,|V_i|+1}$ for $i = 1, 2, ..., n - 1$ and $F_n = \{\{u_n\} \cup V_n\}_G \cong K_{1,|V_n|}$. Since $|V_i| \geq m - 1$ for $i = 1, 2, ..., n - 1$ and $|V_n| \geq m$, each $F_i$ is a $p$-competition graph of a loopless digraph without symmetric arcs by Theorem 3.5. So, $G$ is the $p$-competition graph of a loopless digraph without symmetric arcs by Theorem 3.1.

By Theorem 3.4, Theorem 3.5, Theorem 1.2 and Theorem 3.1, we obtain the following result on cacti.

**Theorem 3.7.** Let $p \geq 3$ be a positive integer. Then a cactus $G$ is the $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs if there exists an edge partition $E(G) = E(C) \cup E(T)$ such that

1. $E(C) = \bigcup E(C_{\alpha_i})$, where $C_{\alpha_i}$ is a cycle with $\alpha_i \geq 2p + 3$,

2. $E(T) = (\bigcup E(K_{1,\beta_j})) \cup (\bigcup E(P_{\gamma_k}))$ such that

   - (2-1) each $K_{1,\beta_j}$ is a maximal subgraph isomorphic to a star graph and $\beta_j \geq 2p + 3$,
   - (2-2) for all $P_{\gamma_k}$, $\gamma_k \geq 2p + 3$.

**Proof.** By assumption, $G = (\bigcup C_{\alpha_i}) \cup (\bigcup K_{1,\beta_j}) \cup (\bigcup P_{\gamma_k})$. By Theorem 3.4, Theorem 3.5 and Theorem 1.2, all $C_{\alpha_i}$, $K_{1,\beta_j}$ and $P_{\gamma_k}$ are $p$-competition graphs of loopless Hamiltonian digraphs without symmetric arcs.

Since $G$ is a cactus, for any two of $C_{\alpha_i}$, $K_{1,\beta_j}$ and $P_{\gamma_k}$, the intersection has at most one vertex. Therefore, a cactus satisfying the conditions of theorem is a $p$-competition graph of a loopless Hamiltonian digraph without symmetric arcs by Theorem 3.1.
References


