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Prime labelings on planar grid graphs

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Prime labelings on planar grid graphs

Cover Page Footnote

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Abstract

It is known that for any prime p and any integer n such that $1 \leqslant n \leqslant p$, there exists a prime labeling on the $p \times n$ planar grid graph $P_p \times P_n$. We show that $P_p \times P_n$ has a prime labeling for any odd prime p and any integer n such that $p < n \leqslant p^2$.

1 Introduction

Graph labelings were formally introduced in the 1970s by Kotzig and Rosa [6]. Graph labelings have been applied to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should consult J. A. Gallian's comprehensive dynamic survey on graph labelings [3] for further investigation. We refer the reader to Chartrand, Lesniak, and Zhang [1] for concepts and notation not explicitly defined in this paper. All graphs in this paper are simple and connected.

Definition 1.1. *Let G be a graph on n vertices. We say that G is a prime graph if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ such that $f(u)$ and $f(v)$ are relatively prime whenever u is adjacent to v.*

Dean [2], and Ghorbani and Kamali [4] have independently shown that the ladder $P_2 \times P_n$ has a prime labeling. Kanetkar [5] found prime labelings for the grid graph $P_{n+1} \times P_{n+1}$ when n and $(n+1)^2+1$ are primes, and either $n = 5$ or $n \equiv 3$ or $9 \pmod{10}$, and a prime labeling on the grid graph $P_n \times P_{n+2}$ when $n \not\equiv 2 \pmod{7}$ is prime. Sundaram et al. [7] have shown that the grid graph $P_p \times P_n$ has a prime labeling when $p \geqslant 5$ is a prime and $n \leqslant p$. They conjecture that the planar grid graph $P_m \times P_n$ is prime for all positive integers m and n. In this paper we show that, for any odd prime p and any positive integer $p < n \leqslant p^2$, the $p \times n$ planar grid graph $P_p \times P_n$ has a prime labeling. Combining this result with that of Sundaram et al. [7], for any odd prime p and any positive integer $1 \leqslant n \leqslant p^2$, the $p \times n$ planar grid graph $P_p \times P_n$ has a prime labeling.

2 Preliminaries

Vilfred et al. [8] have shown the following result.

Theorem 2.1. *[8] Let p be a prime, and let n be an integer such that $n \leqslant 3$. Then there exists a prime labeling on $P_p \times P_n$.*

Sundaram et al. [7] have extended their result to the following theorem.

Theorem 2.2. *[7, Theorem 2.1] Let $p \geqslant 5$ be a prime, and let n be an integer such that $3 < n \leqslant p$. Then there exists a prime labeling on $P_p \times P_n$.*

Combining Theorems 2.1 and 2.2 yields the following result.

Theorem 2.3. *Let p be an odd prime, and let n be an integer such that $1 \leqslant n \leqslant p$. Then there exists a prime labeling on $P_p \times P_n$.*

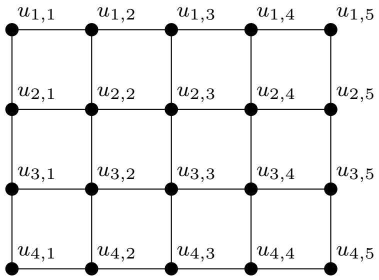

Figure 1: Grid graph $P_4 \times P_5$.

Remark 2.1. *We make use of the fact that* $\gcd(a, b) = \gcd(a, b + ha) = \gcd(a, ha - b)$ *for any integer* h.

Example 2.1. *We illustrate the grid graph* $P_4 \times P_5$ *in Figure 1. Due to the orientation of the vertices, we refer to the set of vertices* $\{u_{i,j} : 1 \leqslant j \leqslant n\}$ *as the* i*th row of* $P_m \times P_n$ *and* $\{u_{i,j} : 1 \leqslant i \leqslant m\}$ *as the* j*th column of* $P_m \times P_n$.

3 Prime labeling on $P_p \times P_n$

Remark 3.1. *In order to show that the* $p \times n$ *grid has a prime labeling, we begin with the labeling* $f(u_{i,j}) = (j-1)p+i$ *for all* $u_{i,j} \in V(P_p \times P_n)$. *We first show that the only labels that prevent this labeling from being prime are the labels on the vertices in the* p^{th} *row. We have* $\gcd((j-1)p+i, (j-1)p+i \pm 1) = 1$. *For* $1 \leqslant i < p$, *we have* $(j-1)p+i \equiv i \not\equiv 0 \pmod{p}$. *By Remark 2.1, we have* $\gcd((j-1)p+i, (j-1)p+i \pm p) = \gcd((j-1)p+i, p) = 1$.

We establish the main result of this paper.

Theorem 3.1. *Let* $p \geqslant 5$ *be a prime, and let* n *be an integer such that* $p < n \leqslant p^2$. *Then there exists a prime labeling on* $P_p \times P_n$.

Proof. The idea of the proof is to start with the labeling $f(u_{i,j}) = (j-1)p+i$ for all $u_{i,j} \in V(P_p \times P_n)$. By Remark 3.1, the labels in the p^{th} row prevent this labeling from being prime. So we swap the labels on some of the vertices in the p^{th} row with the labels on some of the vertices in the other rows to transform this labeling into a prime labeling. So in order to verify that the resulting labeling is prime, we need only check for primeness at the vertices $u_{i,j} \in V(P_p \times P_n)$ where $1 \leqslant i < p$ and $f(u_{i,j}) \neq (j-1)p+i$, and at the vertices in the p^{th} row.

We define the integers j_0, j_1, j_2, e_1, i_3, j_3, and k_j and ℓ_j for all $1 \leqslant j \leqslant j_1$ as follows. Let

$$j_0 = \left\lfloor \frac{n}{p} \right\rfloor \; j_1 = \left\lfloor \frac{n}{2p} \right\rfloor, \; j_2 = \left\lfloor \frac{n}{2} \right\rfloor,$$

$e_1 = \lfloor \log_2(p^2) \rfloor$, let $1 \leqslant i_3 \leqslant p$ and $1 \leqslant j_3 < p$ be the unique integers such that $(j_3-1)p+i_3 = n$, and for all $1 \leqslant j \leqslant j_1$, let k_j and ℓ_j be the unique integers such that $j\,2^{e_1+1} = (\ell_j-1)p+k_j$ and $1 \leqslant k_j \leqslant p$. Since $j \leqslant j_1 < p$, we have $1 \leqslant k_j < p$.

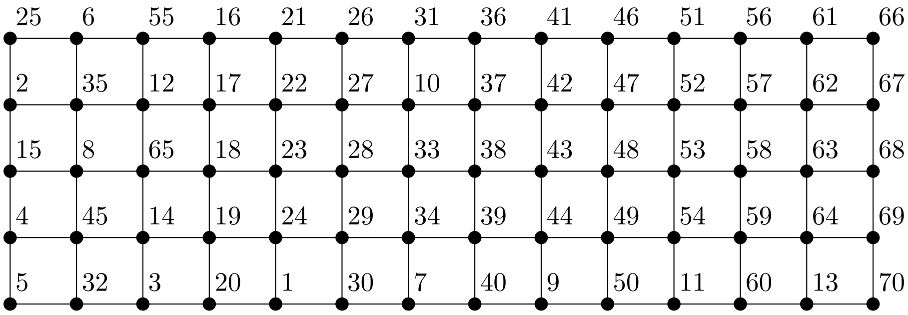

Figure 2: Prime labeling on $P_5 \times P_{14}$.

We need to consider the following four cases separately. These cases are $p < n \leqslant p^2 - p$ and j_0 is even, $p < n \leqslant p^2 - p$ and j_0 is odd, $p^2 - p < n < p^2$, and $n = p^2$.

Case 1. Suppose $p < n \leqslant p^2 - p$ and $j_0 = 2j_1$ is even. We define the labeling $f : V(P_p \times P_n) \to \{1, 2, \ldots, pn\}$ as follows.

1. For all $1 \leqslant i \leqslant j_0$ such that i is odd, let $f(u_{i,1}) = ip^2$.
2. For all i and j such that $i + j$ is even, and either
   - $j = 1$ and $j_0 < i < p$,
   - $1 < j < j_3$ and $1 \leqslant i < p$, or
   - $j = j_3$ and $1 \leqslant i \leqslant i_3$,

   let $f(u_{i,j}) = \big((j-1)p + i\big)p$.
3. For all $j_0 < j \leqslant n$ such that j is odd and $j \not\equiv 0 \pmod{p}$, let $f(u_{p,j}) = j$.
4. For all $1 \leqslant j \leqslant j_0$ and j is odd, let $f(u_{p,jp}) = j$.
5. For all $1 \leqslant j \leqslant j_1$, let $f(u_{p,2j}) = j\,2^{e_1+1}$. Then, for all $1 \leqslant j \leqslant j_0$ and j is even, $f(u_{p,j}) = j\,2^{e_1}$.
6. For all $1 \leqslant j \leqslant j_1$, let $f(u_{k_j,\ell_j}) = (2j)p$.
7. For all other vertices $u_{i,j}$, let $f(u_{i,j}) = (j-1)p + i$.

See Figure 2 for an example of this labeling on $P_5 \times P_{14}$. By Remark 3.1, in order to check that this labeling is prime, we need only check for primeness at all vertices $u_{i,j} \in V(P_p \times P_n)$ such that $1 \leqslant i < p$ and $f(u_{i,j}) \neq (j-1)p + i$, and at all vertices in the p^{th} row.

Subcase *(i)*. Suppose $j = 1$ and $1 \leqslant i \leqslant j_0 < p$ is odd. Then $f(u_{i,1}) = ip^2$, $f(u_{i\pm1,1}) = i \pm 1$, and $f(u_{i,2}) = p + i$. Since $\gcd(i, i\pm1) = \gcd(i\pm1, p) = 1$ and $\gcd(i, p+i) = \gcd(i,p) = \gcd(p, p+i) = 1$, we have $\gcd(ip^2, i\pm1) = 1$ and $\gcd(ip^2, p+i) = 1$.

Subcase *(ii)*. Suppose $i+j$ is even, and either $j = 1$ and $j_0 < i < p-1$, or $1 < j < j_3$ and $1 \leqslant i < p-1$, or $j = j_3$ and $1 \leqslant i \leqslant \min(i_3, p-2)$. Then $f(u_{i,j}) = \big((j-1)p+i\big)p$, $f(u_{i\pm1,j}) = (j-1)p+i\pm1$, and $f(u_{i,j\pm1}) = (j-1)p+i\pm p$. We have $\gcd\Big(\big((j-1)p+i\big)p, (j-1)p+i\pm1\Big) = 1$ since $\gcd\big((j-1)p+i, (j-1)p+i\pm1\big) = 1$ and $\gcd\big(p, (j-1)p+i\pm1\big) = 1$. Also, we have

$\gcd\big(((j-1)p+i)p, (j-1)p+i \pm p\big) = 1$ since $\gcd\big((j-1)p+i, (j-1)p+i \pm p\big) = 1$ and $\gcd\big(p, (j-1)p+i \pm p\big) = 1$. When $i = p-1$ and j is even, we have $f(u_{p,j}) = j2^{e_1}$. We will check that $\gcd(f(u_{p-1,j}), f(u_{p,j})) = 1$ in Subcase (iii).

Subcase (*iii*). Suppose $i = p-1$, $1 \leqslant j \leqslant j_3$, and j is even. If $i_3 = p-1$, then $j_3 = j_0 + 1$ is odd. Thus, $1 \leqslant j < j_3$, and j is even. (We will consider the situation $i_3 = p-1$ and $j_3 = j_0+1$ is even in Case 2.) Since $j_0 = j_3$ or $j_0 = j_3 - 1$, we have $1 \leqslant j \leqslant j_0 < p$, and j is even.

Thus, $f(u_{p-1,j}) = (jp-1)p$, $f(u_{p,j}) = j\,2^{e_1}$, $f(u_{p-1,j\pm1}) = jp-1\pm p$, and $f(u_{p-2,j}) = jp-2$. We observe that $\gcd\big((jp-1)p, j\big) = 1$ since $\gcd(jp-1, j) = 1$ and $\gcd(p, j) = 1$. Because $jp-1$ and p are odd, we have $\gcd\big((jp-1)p, j\,2^{e_1}\big) = \gcd\big((jp-1)p, j\big) = 1$. Since $f(u_{p-2,j}) = jp-2$ and $f(u_{p-1,j\pm1}) = jp-1\pm p$, an argument similar to that in Subcase (ii) shows that $\gcd\big((jp-1)p, jp-2\big) = \gcd\big((jp-1)p, jp-1\pm p\big) = 1$.

Subcase (*iv*). Suppose $i = p$, $1 \leqslant j < j_0$, and j is even. Then $f(u_{p,j}) = j\,2^{e_1}$, $f(u_{p-1,j}) = (jp-1)p$, and $f(u_{p,j\pm1}) = (j\pm1)p$. Since $j\pm1$ is odd, we have $\gcd(j\,2^{e_1}, j\pm1) = \gcd(j, j\pm1) = 1$. Since $1 \leqslant j < p$ and p is odd, we have $\gcd(j\,2^{e_1}, (j\pm1)p) = \gcd(j\,2^{e_1}, j\pm1) = 1$. It was shown in Subcase (iii) that $\gcd\big((jp-1)p, j\,2^{e_1}\big) = 1$.

Subcase (*v*). Suppose $i = p$ and $j = j_0$. Since j_0 is even, $f(u_{p,j_0}) = j_0\,2^{e_1}$, $f(u_{p-1,j_0}) = (j_0p-1)p$, $f(u_{p,j_0-1}) = (j_0-1)p$, and $f(u_{p,j_0+1}) = j_0+1$. Since j_0+1 is odd, we have $\gcd(j_0\,2^{e_1}, j_0+1) = \gcd(j_0, j_0+1) = 1$. It was shown in Subcase (iii) that $\gcd\big(j_0\,2^{e_1}, (j_0p-1)p\big) = 1$. It was shown in Subcase (iv) that $\gcd\big(j_0\,2^{e_1}, (j_0-1)p\big) = 1$.

Subcase (*vi*). Suppose $i = p$, $1 \leqslant j < j_0$, and j is odd. Then $f(u_{p,j}) = jp$, $f(u_{p-1,j}) = jp-1$, and $f(u_{p,j\pm1}) = (j\pm1)2^{e_1}$. By an argument similar to that in Subcase (iv), we have $\gcd\big(jp, (j\pm1)2^{e_1}\big) = 1$. Also, $\gcd(jp, jp-1) = 1$.

Subcase (*vii*). Consider the vertex $u_{p,jp}$, where $1 \leqslant j \leqslant j_0$ and j is odd. We have $f(u_{p,jp}) = j$, $f(u_{p,jp\pm1}) = (jp\pm1)p$, and $f(u_{p-1,jp}) = jp^2-1$. Since j is relatively prime to p, $jp\pm1$ and jp^2-1, we have $\gcd\big(j, (jp\pm1)p\big) = \gcd(j, jp^2-1) = 1$.

Subcase (*viii*). Suppose $i = p$, $j_0 < j \leqslant n$, j is odd, and $j \not\equiv 0 \pmod{p}$. If $j > j_0+1$, then $f(u_{p,j}) = j$, $f(u_{p,j\pm1}) = (j\pm1)p$, and $f(u_{p-1,j}) = jp-1$. We have $\gcd\big(j, (j\pm1)p\big) = \gcd(j, jp-1) = 1$ since j is relatively prime to p, $j\pm1$, and $jp-1$.

If $j = j_0+1$, then $f(u_{p,j_0}) = j_0\,2^{e_1}$ and $f(u_{p,j_0+1}) = j_0+1$. Since $\gcd(j_0+1, j_0) = 1$ and j_0+1 is odd, we have $\gcd(j_0+1, j_02^{e_1}) = 1$.

Subcase (*ix*). Consider the vertex u_{k_j,ℓ_j} for some $1 \leqslant j \leqslant j_1$. We have $f(u_{k_j,\ell_j}) = (2j)p$. Recall that k_j and ℓ_j are the unique integers such that $j\,2^{e_1+1} = (\ell_j-1)p + k_j$ and $1 \leqslant k_j < p$. Since $e_1 = \lfloor \log_2(p^2) \rfloor$, we have $p^2 < 2^{e_1+1} < 2p^2$. Since $1 \leqslant j \leqslant j_1 \leqslant \frac{n}{2p}$, we have $n + p \leqslant p^2 < j\,2^{e_1+1} < np$.

Since $1 \leqslant k_j < p$, we have $f(u_{k_j,\ell_j\pm1}) = \big((\ell_j-1)\pm1\big)p + k_j = j\,2^{e_1+1} \pm p$. Since $j\,2^{e_1+1} \pm p = (\ell_j-1)p + k_j \pm p \equiv k_j \not\equiv 0 \pmod{p}$, we have $\gcd(p, j2^{e_1+1} \pm p) = 1$. By Remark 2.1, we have $\gcd(j, j2^{e_1+1} \pm p) = \gcd(j, p) = 1$. Thus, $\gcd(2jp, j2^{e_1+1} \pm p) = 1$.

If $1 < k_j < p$, we have $f(u_{k_j-1,\ell_j}) = j\,2^{e_1+1} - 1$. Also, if $1 \leqslant k_j < p-1$, we have $f(u_{k_j+1,\ell_j}) = j\,2^{e_1+1} + 1$. Since $j\,2^{e_1+1} \pm 1 = (\ell_j-1)p + k_j \pm 1 \equiv k_j \pm 1 \not\equiv 0 \pmod{p}$, $\gcd(p, j2^{e_1+1} \pm 1) = 1$. By Remark 2.1, we have $\gcd(j, j2^{e_1+1} \pm 1) = \gcd(j, 1) = 1$. Thus, $\gcd(2jp, j2^{e_1+1} \pm 1) = 1$.

Finally, we suppose $k_j = p-1$. Since $\ell_jp = j2^{e_1+1}+1$ is odd, ℓ_j is odd. Thus, $f(u_{k_j+1,\ell_j}) = f(u_{p,\ell_j}) = \ell_j$ if $\ell_j \not\equiv 0 \pmod{p}$ or $f(u_{k_j+1,\ell_j}) = f(u_{p,\ell_j}) = \frac{\ell_j}{p}$ if $\ell_j \equiv 0 \pmod{p}$. Since

$\ell_j p - j2^{e_1+1} = 1$, we have $\gcd(j2^{e_1+1}, \ell_j p) = 1$. Thus, $\gcd(2j, \ell_j) = 1$. If $\ell_j \not\equiv 0 \pmod{p}$, then $\gcd(2jp, \ell_j) = \gcd(p, \ell_j) = 1$. Since $\ell_j \leqslant p^2 - p$, we have $\frac{\ell_j}{p} \leqslant p - 1$. So, if $\ell_j \equiv 0 \pmod{p}$, then $\gcd(2jp, \frac{\ell_j}{p}) = \gcd(p, \frac{\ell_j}{p}) = 1$.

Case 2. Suppose $p < n \leqslant p^2 - p$ and $j_0 = 2j_1 + 1$ is odd. We observe that when we apply the labeling given in Case 1 to the present case, the labels $f(u_{p,j_0}) = j_0 p$ and $f(u_{p,j_0+1}) = (j_0+1)p$ (and also $f(u_{p-1,j_0+1}) = np$ if $n = j_0 p + p - 1$) are the only labels that prevent the labeling from being prime. We must find a vertex $u_{k',\ell'}$ in which to swap the value of $f(u_{p,j_0+1})$ with that of $f(u_{k',\ell'})$ so that the resulting labeling is prime.

Since j_0+1 is even, we write j_0+1 in the form of $j_0+1 = 2^\alpha \beta$, where α and β are the unique integers such that $\alpha \geqslant 1$ and β is odd. Let $e_0 = \lfloor \log_2(p) \rfloor$ and $e_1 = \lfloor \log_2(p^2) \rfloor$. Since $2^\alpha \beta < p$ and $\frac{1}{2}p < 2^{e_0} < p$, we have $\alpha \leqslant e_0$. We define the labeling $f : V(P_p \times P_n) \to \{1, 2, \ldots, pn\}$ by the labeling defined in Case 1 with the additional condition that we swap the labels on $f(u_{p,j_0+1})$ and $f(u_{k',\ell'})$ for some $u_{k',\ell'} \in V(P_p \times P_n)$ as follows.

1. If $\beta \geqslant 3$, we let k' and ℓ' be the unique integers such that $2^{e_1}\beta = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_1}\beta$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^\alpha \beta p$.
2. If $\beta = 1$ and $\alpha < e_0 - 1$, we let k' and ℓ' be the unique integers such that $2^{e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^\alpha p$.
3. If $\beta = 1$, $\alpha = e_0 - 1$, $e_1 = 2e_0$ is even, and p is not a Fermat prime, we let k' and ℓ' be the unique integers such that $2^{e_0+e_1-1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_0+e_1-1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0-1}p$.
4. If $\beta = 1$, $\alpha = e_0 - 1$, $e_1 = 2e_0$ is even, and p is a Fermat prime, we let k' and ℓ' be the unique integers such that $2^{e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0-1}p$.
5. If $\beta = 1$, $\alpha = e_0 - 1$, and $e_1 = 2e_0 + 1$ is odd, we let k' and ℓ' be the unique integers such that $2^{e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0-1}p$.
6. If $\beta = 1$, $\alpha = e_0$, and $e_1 = 2e_0$ is even, we let k' and ℓ' be the unique integers such that $2^{e_0+e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_0+e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0}p$.
7. If $\beta = 1$, $\alpha = e_0$, $e_1 = 2e_0 + 1$ is odd, and p is not a Mersenne prime, we let k' and ℓ' be the unique integers such that $2^{e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0}p$.
8. If $\beta = 1$, $\alpha = e_0$, $e_1 = 2e_0 + 1$ is odd, and p is a Mersenne prime, we let k' and ℓ' be the unique integers such that $2^{e_0+e_1} = (\ell'-1)p + k'$ and $1 \leqslant k' \leqslant p$. Then we let $f(u_{p,j_0+1}) = 2^{e_0+e_1}$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{e_0}p$.

We first observe that the values $2^{e_1}\beta$, 2^{e_1}, $2^{e_0+e_1-1}$, and $2^{e_0+e_1}$ are relatively prime to p. Thus, in all eight selections of the vertex $u_{k',\ell'}$, we have $1 \leqslant k' < p$.

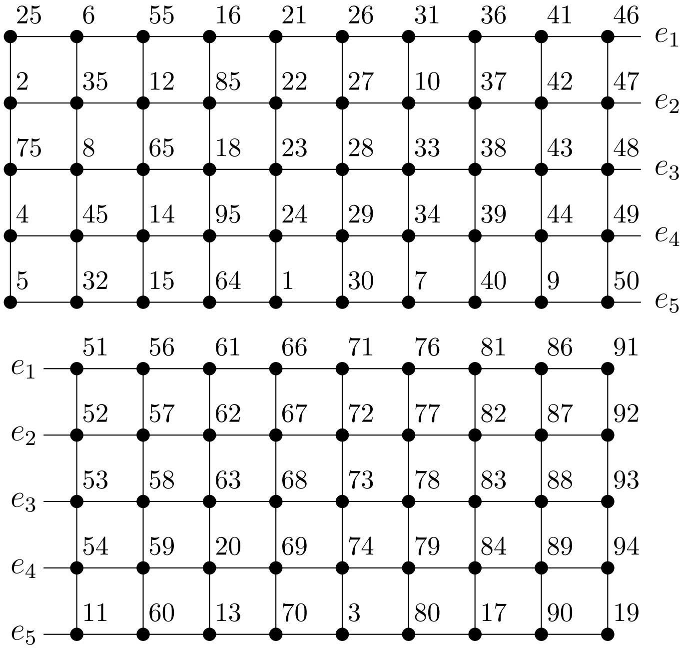

Figure 3: Prime labeling on $P_5 \times P_{19}$.

See Figure 3 for an example of this labeling on $P_5 \times P_{19}$. We verify that each choice of $f(u_{p,j_0})$ and $f(u_{k',\ell'})$ results in a prime labeling. In order to check for primeness at vertex $u_{k',\ell'}$, we need to determine the labels at each of its neighboring vertices. In particular, we want to show that $f(u_{k',\ell'\pm 1}) = (\ell'-1)p + k' \pm p$, $f(u_{k'-1,\ell'}) = (\ell'-1)p + k' - 1$ if $1 < k' < p$, and either

- $f(u_{k'+1,\ell'}) = (\ell'-1)p + k' + 1$ if $1 \leqslant k' < p-1$,
- $f(u_{k'+1,\ell'}) = \ell'$ if $k' = p-1$ and $\ell' \not\equiv 0 \pmod{p}$, or
- $f(u_{k'+1,\ell'}) = \frac{\ell'}{p}$ if $k' = p-1$ and $\ell' \equiv 0 \pmod{p}$.

In addition, we need to choose a vertex $u_{k',\ell'}$ whose label has not been previously swapped with another value. Thus, we need to choose a vertex $u_{k',\ell'}$ whose label is $f(u_{k',\ell'}) = (\ell'-1)p + k'$. Hence, we need to show $(\ell'-1)p + k' \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$ and $n + p < (\ell'-1)p + k' \leqslant np$. When $j_0 > 1$, we have $f(u_{p,2}) = 2^{e_1+1}$ and $f(u_{k_1,\ell_1}) = 2p$. In Case 1, we verified that $n + p < 2^{e_1+1} \leqslant np$. So if $(\ell'-1)p + k' < 2^{e_1+1}$, we will only need to verify that $n + p < (\ell'-1)p + k'$. Otherwise, if $2^{e_1+1} < (\ell'-1)p + k'$, we will only need to verify that $(\ell'-1)p + k' \leqslant np$. However, in the case when $j_0 = 1$, we will need to verify $n + p < (\ell'-1)p + k' \leqslant np$.

Subcase (*i*). Assume $\beta \geqslant 3$. Then k' and ℓ' are the unique integers such that $2^{e_1}\beta = (\ell'-1)p + k'$ and $1 \leqslant k' < p$. We have $f(u_{p,j_0+1}) = 2^{e_1}\beta$ and $f(u_{k',\ell'}) = (j_0+1)p = 2^{\alpha}\beta p$.

We first show that $n + p < 2^{e_1}\beta \leqslant np$ and $2^{e_1}\beta \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. We know that $n+p \leqslant p^2 < 2^{e_1+1}$. Since $\beta \geqslant 3$, we have $n+p < 3 \cdot 2^{e_1} \leqslant 2^{e_1}\beta$. Since $2j_1+2 = j_0+1 = 2^{\alpha}\beta \geqslant 6$, we have $2 \leqslant j_1 \leqslant \frac{n}{2p}$. Thus, $n \geqslant 4p$. We observe that $\beta = \frac{j_0+1}{2^{\alpha}} = \frac{2j_1+2}{2^{\alpha}} \leqslant j_1 + 1 \leqslant \frac{n}{2p} + 1$. Since $2^{e_1} < p^2$ and $n > 2p$, we have $2^{e_1}\beta < p^2(\frac{n}{2p} + 1) < np$. Also, since every value in $\{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$ is divisible by 2^{e_1+1} and $2^{e_1}\beta$ is not divisible by 2^{e_1+1}, we have $2^{e_1}\beta \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$.

We consider vertex u_{p,j_0+1}. We have $f(u_{p,j_0+1}) = 2^{e_1}\beta$, $f(u_{p,j_0}) = j_0 p$, $f(u_{p,j_0+2}) = j_0 + 2$, and either

- $f(u_{p-1,j_0+1}) = (j_0+1)p - 1 = 2^\alpha \beta p - 1$ if $i_3 \neq p-1$ or
- $f(u_{p-1,j_0+1}) = \big((j_0+1)p - 1\big)p = \big(2^\alpha \beta p - 1\big)p$ if $i_3 = p-1$.

Since p is odd and $1 \leqslant \beta < p$, we have $\gcd(2\beta, p) = 1$. Since j_0 and j_0+2 are relatively prime to $j_0 + 1 = 2^\alpha\beta$, we have $\gcd(j_0 p, 2^{e_1}\beta) = 1$ and $\gcd(j_0+2, 2^{e_1}\beta) = 1$. Since $(j_0+1)p - 1 = 2^\alpha\beta - 1$ is relatively prime to 2 and β, we have $\gcd((j_0+1)p-1, 2^{e_1}\beta) = 1$. Also, since $\gcd(p, 2\beta) = 1$, we have $\gcd\big(\big((j_0+1)p-1\big)p, 2^{e_1}\beta\big) = 1$.

Next consider vertex $u_{k',\ell'}$. If $1 < k' < p$, then $f(u_{k'-1,\ell'}) = 2^{e_1}\beta - 1$. Since $2^{e_1}\beta - 1 \equiv k' - 1 \not\equiv 0 \pmod{p}$ and $\gcd(2\beta, 2^{e_1}\beta - 1) = 1$, we have $\gcd(2^\alpha\beta p, 2^{e_1}\beta - 1) = 1$. If $1 \leqslant k' < p-1$, then $f(u_{k'+1,\ell'}) = 2^{e_1}\beta + 1$. Since $2^{e_1}\beta + 1 \equiv k' + 1 \not\equiv 0 \pmod{p}$ and $\gcd(2\beta, 2^{e_1}\beta + 1) = 1$, we have $\gcd(2^\alpha\beta p, 2^{e_1}\beta + 1) = 1$. Suppose $k' = p-1$. Since $\ell' p = 2^{e_1}\beta + 1$ is odd, ℓ' is odd. Thus, $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \ell'$ if $\ell' \not\equiv 0 \pmod{p}$ and $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \frac{\ell'}{p}$ if $\ell' \equiv 0 \pmod{p}$. Since $\ell' p - 2^{e_1}\beta = 1$, we have $\gcd(2^{e_1}\beta, \ell' p) = 1$. Thus, $\gcd(2^\alpha\beta p, \ell') = 1$ if $\ell' \not\equiv 0 \pmod{p}$. We observe that $\frac{\ell'}{p} \leqslant \frac{n}{p} \leqslant p - 1$. Thus, $\gcd(2^\alpha\beta p, \frac{\ell'}{p}) = 1$ if $\ell' \equiv 0 \pmod{p}$. Since $1 \leqslant k' < p$, we have $f(u_{k',\ell'\pm 1}) = 2^{e_1}\beta \pm p$. By Remark 2.1, we have $\gcd(2^{e_1}\beta \pm p, 2^\alpha\beta) = \gcd(p, 2^\alpha\beta) = 1$. We observe that $2^{e_1}\beta \pm p \equiv k' \not\equiv 0 \pmod{p}$. Thus, $\gcd(2^{e_1}\beta \pm p, 2^\alpha\beta p) = 1$.

Subcase (*ii*). Assume $\beta = 1$, $\alpha < e_0 - 1$, and $j_0 > 1$. Then k' and ℓ' are the unique integers such that $2^{e_1} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1}$, and $f(u_{k',\ell'}) = (j_0+1)p = 2^\alpha p$. Since $p > 4$, we have

$$n + p < (j_0+1)p + p = 2^\alpha p + p \leqslant 2^{e_0-2}p + p < \tfrac{1}{4}p^2 + p < \tfrac{1}{2}p^2 < 2^{e_1}.$$

Also, $2^{e_1} < 2^{e_1+1} \leqslant np$. Since $2^{e_1} < 2^{e_1+1}$ and 2^{e_1+1} is the minimum value in $\{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$, we have $2^{e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$.

We first consider vertex u_{p,j_0+1}. We have $f(u_{p,j_0+1}) = 2^{e_1}$, $f(u_{p,j_0}) = j_0 p$, $f(u_{p,j_0+2}) = j_0 + 2$, and either

- $f(u_{p-1,j_0+1}) = (j_0+1)p - 1 = 2^\alpha p - 1$ if $i_3 \neq p-1$ or
- $f(u_{p-1,j_0+1}) = \big((j_0+1)p - 1\big)p = \big(2^\alpha p - 1\big)p$ if $i_3 = p-1$.

Since p, j_0, j_0+2, and $(j_0+1)p - 1$ are odd, we have $\gcd(j_0 p, 2^{e_1}) = \gcd(j_0+2, 2^{e_1}) = \gcd((j_0+1)p-1, 2^{e_1}) = \gcd\big(\big((j_0+1)p-1\big)p, 2^{e_1}\big) = 1$.

Next consider vertex $u_{k',\ell'}$. If $1 < k' < p$, then $f(u_{k'-1,\ell'}) = 2^{e_1} - 1$. Since $2^{e_1} - 1 \equiv k' - 1 \not\equiv 0 \pmod{p}$ and $2^{e_1} - 1$ is odd, we have $\gcd(2^\alpha p, 2^{e_1} - 1) = 1$. If $1 \leqslant k' < p-1$, then $f(u_{k'+1,\ell'}) = 2^{e_1} + 1$. Since $2^{e_1} + 1 \equiv k' + 1 \not\equiv 0 \pmod{p}$ and $2^{e_1} + 1$ is odd, we have $\gcd(2^\alpha p, 2^{e_1} + 1) = 1$. Suppose $k' = p-1$. Since $\ell' p = 2^{e_0} + 1$ is odd, ℓ' is odd. Thus, $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \ell'$ if $\ell' \not\equiv 0 \pmod{p}$ and $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \frac{\ell'}{p}$ if $\ell' \equiv 0 \pmod{p}$. Since ℓ' is odd, we have $\gcd(2^\alpha p, \ell') = 1$ if $\ell' \not\equiv 0 \pmod{p}$. We observe that $\frac{\ell'}{p} \leqslant \frac{n}{p} \leqslant p - 1$. Thus, $\gcd(2^\alpha p, \frac{\ell'}{p}) = 1$ if $\ell' \equiv 0 \pmod{p}$. We next consider $f(u_{k',\ell'\pm 1}) = 2^{e_1} \pm p$. By Remark

2.1, we have $\gcd(2^{e_1} \pm p, 2^\alpha) = \gcd(p, 2^\alpha) = 1$. We observe that $2^{e_1} \pm p \equiv k' \not\equiv 0 \pmod{p}$. Thus, $\gcd(2^{e_1} \pm p, 2^\alpha p) = 1$.

Subcase *(iii)*. Assume $\beta = 1$, $\alpha = e_0 - 1$, $j_0 > 1$, $e_1 = 2e_0$, and p is not a Fermat prime. Then k' and ℓ' are the unique integers such that $2^{e_1+e_0-1} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1+e_0-1}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0-1}p$.

We need to show that $n + p < 2^{e_1+e_0-1} \leqslant np$. Let $r_1 = p - 2^{e_0}$. Since r_1 is odd and p is not a Fermat prime, we have $r_1 \geqslant 3$. Thus, $p \geqslant 2^{e_0} + 3$. Since $2^{e_0} \geqslant 4$, we have

$$2^{e_1+e_0-1} = 2^{3e_0-1} < \tfrac{1}{2}(2^{e_0})^3 + 2(2^{e_0})^2 - \tfrac{3}{2}(2^{e_0}) - 9 = (2^{e_0-1} - 1)(2^{e_0} + 3)^2 \leqslant j_0 p^2 \leqslant np.$$

Also, $n + p < 2^{e_1+1} \leqslant 2^{e_0+e_1-1}$.

Since $j_0 + 1 = 2^{e_0-1}$, the largest power of 2 in $\{j : 1 \leqslant j \leqslant j_1\}$ is 2^{e_0-3}. Thus, the largest power of 2 in $\{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$ is $2^{e_0+e_1-2}$. Hence, $2^{e_0+e_1-1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Subcase *(iv)*. Assume $\beta = 1$, $\alpha = e_0 - 1$, $e_1 = 2e_0$, $j_0 > 1$, and p is a Fermat prime. Then k' and ℓ' are the unique integers such that $2^{e_1} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0-1}p$.

We need to show that $n + p < 2^{e_1} \leqslant np$. We have $j_0 + 2 = 2^{e_0-1} + 1$. Since p is a Fermat prime and $\frac{1}{2}p < 2^{e_0} < p$, we have $p = 2^{e_0} + 1$. Since $2^{e_0} \geqslant 4$, we have

$$n + p < (j_0 + 2)p = (2^{e_0-1} + 1)(2^{e_0} + 1) = \tfrac{1}{2}(2^{e_0})^2 + \tfrac{3}{2}(2^{e_0}) + 1 < (2^{e_0})^2 = 2^{e_1}.$$

Also, $2^{e_1} < 2^{e_1+1} \leqslant np$. The argument given in Subcase (ii) shows that $2^{e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Subcase *(v)*. Assume $\beta = 1$, $\alpha = e_0 - 1$, $j_0 > 1$, and $e_1 = 2e_0 + 1$ is odd. Then k' and ℓ' are the unique integers such that $2^{e_1} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0-1}p$.

We have $j_0 + 2 = 2^{e_0-1} + 1$. Because $\frac{1}{2}p < 2^{e_0} < p$, we have $p < 2^{e_0+1}$. Since $2^{e_0} \geqslant 4$,

$$n + p < (j_0 + 2)p = (2^{e_0-1} + 1)(2^{e_0+1}) = (2^{e_0})^2 + 2(2^{e_0}) < 2(2^{e_0})^2 = 2^{e_1}.$$

Also, $2^{e_1} < 2^{e_1+1} \leqslant np$. The argument given in Subcase (ii) shows that $2^{e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Subcase *(vi)*. Assume $j_0 = 1$. Since $j_0 + 1 = 2 = 2^\alpha$, we have $\alpha = 1$. For the primes 5 and 7, we have $\alpha = e_0 - 1$ since $e_0 = 2$. For all other primes $p \geqslant 11$, we have $e_0 \geqslant 3$ which implies that $\alpha < e_0 - 1$.

We first consider $p = 5$. Since $p = 5$ is a Fermat prime, Subcase (iv) applies and thus $2^{e_1} = (\ell' - 1)p + k'$. Then $f(u_{p,2}) = 2^{e_1} = 16$ and $f(u_{k',\ell'}) = 2p = 10$. A calculation shows that $n + p < (j_0 + 2)p = 15 < 16 = 2^{e_1}$ and $2^{e_1} = 16 < 25 = j_0 p^2 \leqslant np$.

We next consider $p = 7$. Since $e_1 = 2e_0 + 1$ for $p = 7$, Subcase (v) applies and thus $2^{e_1} = (\ell' - 1)p + k'$. Then $f(u_{p,2}) = 2^{e_1} = 32$ and $f(u_{k',\ell'}) = 2p = 14$. A calculation shows that $n + p < (j_0 + 2)p = 21 < 32 = 2^{e_1}$ and $2^{e_1} = 32 < 49 = j_0 p^2 \leqslant np$.

Finally, we consider primes $p \geqslant 11$. Since $\alpha < e_0 - 1$, Subcase (ii) applies and thus $2^{e_1} = (\ell' - 1)p + k'$. Then $f(u_{p,2}) = 2^{e_1}$ and $f(u_{k',\ell'}) = 2p$. The argument given in Subcase (ii) shows that $n + p < 2^{e_1} < np$.

Also, the argument given in Subcase (ii) shows that $2^{e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that each of these labelings is prime.

Subcase (vii). Assume $\beta = 1$, $\alpha = e_0$, and $e_1 = 2e_0$. Then k' and ℓ' are the unique integers such that $2^{e_1+e_0} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1+e_0}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0}p$.

We need to show that $n + p < 2^{e_1+e_0} \leqslant np$. We have $j_0 = 2^{e_0-1} - 1$. Since p is odd and $\frac{1}{2}p < 2^{e_0} < p$, we have $p \geqslant 2^{e_0} + 1$. Since $2^{e_0} \geqslant 4$, we have

$$2^{e_1+e_0} = 2^{3e_0} < (2^{e_0})^3 + (2^{e_0})^2 - (2^{e_0}) - 1 = (2^{e_0} - 1)(2^{e_0} + 1)^2 \leqslant j_0p^2 \leqslant np.$$

Also, $n + p < 2^{e_1+1} < 2^{e_1+e_0}$.

Since $j_0 + 1 = 2^{e_0}$, the largest power of 2 in $\{j : 1 \leqslant j \leqslant j_1\}$ is 2^{e_0-2}. Thus, the largest power of 2 in $\{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$ is $2^{e_0+e_1-1}$. Hence, $2^{e_0+e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Subcase $(viii)$. Assume $\beta = 1$, $\alpha = e_0$, $e_1 = 2e_0 + 1$, and p is not a Mersenne prime. Then k' and ℓ' are the unique integers such that $2^{e_1} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0}p$.

We need to show that $n + p < 2^{e_1} \leqslant np$. We have $j_0 + 2 = 2^{e_0} + 1$. Let $r_2 = 2^{e_0+1} - p$. Since r_2 is odd and p is not a Mersenne prime, we have $r_2 \geqslant 3$. Thus, $p \leqslant 2^{e_0+1} - 3$. Hence,

$$n + p < (j_0 + 2)p \leqslant (2^{e_0} + 1)(2^{e_0+1} - 3) = 2^{2e_0+1} - (2^{e_0}) - 3 < 2^{e_1}.$$

Also, $2^{e_1} < 2^{e_1+1} \leqslant np$. The argument given in Subcase (ii) shows that $2^{e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Subcase (ix). Assume $\beta = 1$, $\alpha = e_0$, $e_1 = 2e_0 + 1$, and p is a Mersenne prime. Then k' and ℓ' are the unique integers such that $2^{e_1+e_0} = (\ell' - 1)p + k'$ and $1 \leqslant k' < p$, $f(u_{p,j_0+1}) = 2^{e_1+e_0}$, and $f(u_{k',\ell'}) = (j_0 + 1)p = 2^{e_0}p$.

We need to show that $n + p < 2^{e_1+e_0} \leqslant np$. We have $j_0 = 2^{e_0} - 1$. Since $p < 2^{e_0+1} < 2p$ and p is a Mersenne prime, we have $p = 2^{e_0+1} - 1$. Since $2^{e_0} \geqslant 4$, we have

$$2^{e_1+e_0} = 2^{3e_0+1} < 4(2^{e_0})^3 - 8(2^{e_0})^2 + 5(2^{e_0}) - 1 = (2^{e_0} - 1)(2^{e_0+1} - 1)^2 = j_0p^2 \leqslant np.$$

Also, $n + p < 2^{e_1+1} < 2^{e_1+e_0}$. The argument given in Subcase (vii) shows that $2^{e_0+e_1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant j_1\}$. An argument similar to the one given in Subcase (ii) demonstrates that this is a prime labeling.

Case 3. Suppose $p^2 - p < n < p^2$. Then $j_0 = p - 1$ is even and $j_1 = \frac{p-1}{2}$. We let $r = n - (p^2 - p)$. Then $0 < r < p$ and $n + p = p^2 + r$. Let $t_r = \lfloor \log_2(p^2 + r) \rfloor$. Then $n + p < 2^{t_r+1} < 2(n + p)$. Let $1 \leqslant j \leqslant \frac{p-1}{2}$. Then $n + p < j2^{t_r+1} < (p - 1)(n + p) < np$. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let $k_{r,j}$ and $\ell_{r,j}$ be the unique integers such that $j\,2^{t_r+1} = (\ell_{r,j} - 1)p + k_{r,j}$ and $1 \leqslant k_{r,j} < p$. We define the labeling $f : V(P_p \times P_n) \to \{1, 2, \ldots, pn\}$ as follows.

1. For all $1 \leqslant i < p$ such that i is odd, let $f(u_{i,1}) = ip^2$.
2. For all i and j such that $i + j$ is even, and either

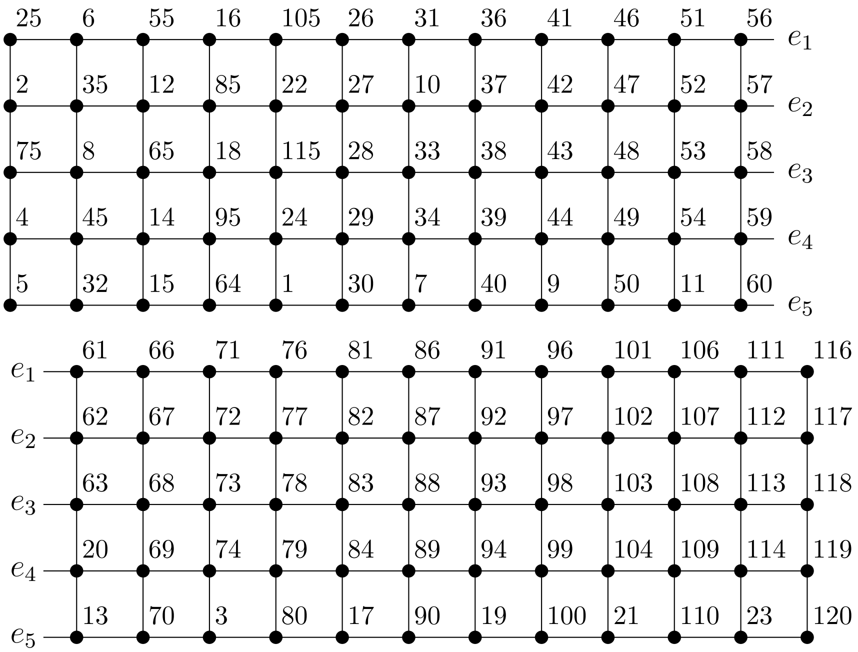

Figure 4: Prime labeling on $P_5 \times P_{24}$.

- $1 < j < p$ and $1 \leqslant i < p$, or
- $j = p$ and $1 \leqslant i \leqslant r$,

let $f(u_{i,j}) = \big((j-1)p+i\big)p$.

3. For all $p < j \leqslant n$ such that j is odd and $j \not\equiv 0 \pmod{p}$, let $f(u_{p,j}) = j$.
4. For all $1 \leqslant j \leqslant p-1$ and j is odd, let $f(u_{p,jp}) = j$.
5. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let $f(u_{p,2j}) = j\,2^{t_r+1}$. Then, for all $1 \leqslant j \leqslant p-1$ and j is even, we have $f(u_{p,j}) = j\,2^{t_r}$.
6. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let $f(u_{k_{r,j},\ell_{r,j}}) = (2j)p$.
7. For all other vertices $u_{i,j}$, let $f(u_{i,j}) = (j-1)p+i$.

See Figure 4 for an example of this labeling on $P_5 \times P_{24}$. An argument similar to the one given in Case 1 demonstrates that this is a prime labeling.

Case 4. Suppose $n = p^2$. Then $j_0 = p$, $j_1 = \frac{p-1}{2}$ and $n+p = p^2+p$. Let $e_3 = \lfloor \log_2(p^2+p) \rfloor$. Then $n+p < 2^{e_3+1} < 2(n+p)$. Let $1 \leqslant j \leqslant \frac{p-1}{2}$. Then $n+p < j2^{e_3+1} < (p-1)(n+p) < np$. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let k_j and ℓ_j be the unique integers such that $j\,2^{e_3+1} = (\ell_j-1)p+k_j$ and $1 \leqslant k_j < p$. Let k' and ℓ' be the unique integers such that $(p^2-1)2^{e_0-1} = (\ell'-1)p+k'$ and $1 \leqslant k' \leqslant p$. We define the labeling $f : V(P_p \times P_n) \to \{1, 2, \ldots, pn\}$ as follows.

1. For all $1 \leqslant i < p$ such that i is odd, let $f(u_{i,1}) = ip^2$.
2. For all i and j such that $i+j$ is even, $1 < j \leqslant p$, and $1 \leqslant i < p$, let $f(u_{i,j}) = \big((j-1)p+i\big)p$.

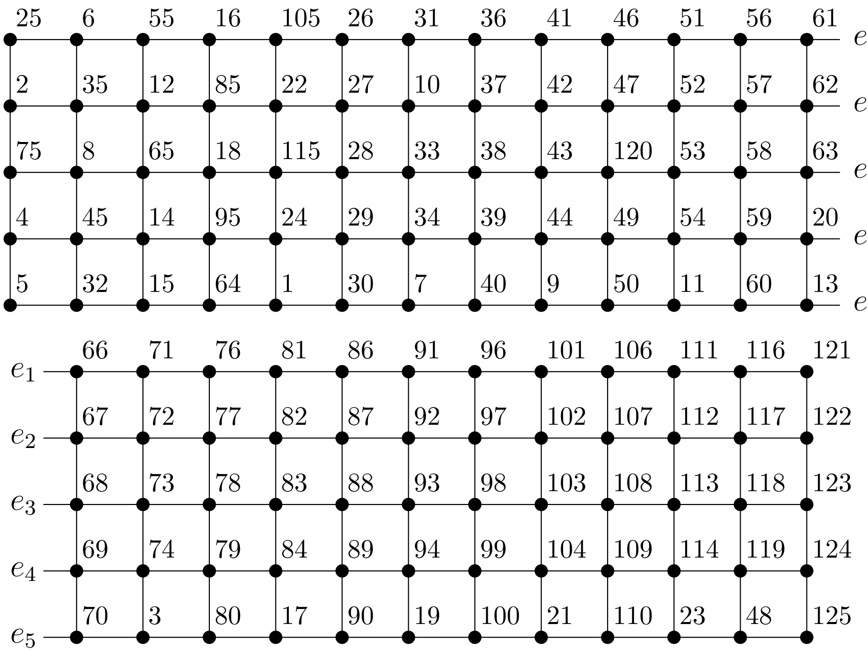

Figure 5: Prime labeling on $P_5 \times P_{25}$.

3. For all $p < j < p^2$ such that j is odd and $j \not\equiv 0 \pmod{p}$, let $f(u_{p,j}) = j$.

4. For all $1 \leqslant j < p$ and j is odd, let $f(u_{p,jp}) = j$.

5. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let $f(u_{p,2j}) = j\,2^{e_3+1}$. Then, for all $1 \leqslant j \leqslant p-1$ and j is even, we have $f(u_{p,j}) = j\,2^{e_3}$.

6. For all $1 \leqslant j \leqslant \frac{p-1}{2}$, let $f(u_{k_j,\ell_j}) = (2j)p$.

7. Let $f(u_{p,p^2-1}) = (p^2-1)2^{e_0-1}$ and $f(u_{k',\ell'}) = (p^2-1)p$.

8. For all other vertices $u_{i,j}$, let $f(u_{i,j}) = (j-1)p + i$.

See Figure 5 for an example of this labeling on $P_5 \times P_{25}$.

This labeling is similar to the labeling in Case 1 except for the labels given by $f(u_{p,p^2-1}) = (p^2-1)2^{e_0-1}$ and $f(u_{k',\ell'}) = (p^2-1)p$. Thus, we only need to check for primeness at these vertices.

We first show that $p^2 + p < (p^2-1)2^{e_0-1} < p^3$. Suppose $p > 5$. Since $\frac{1}{4}p < 2^{e_0-1} < \frac{1}{2}p$, we have $p^2 + p < \frac{1}{4}p(p^2-1) < 2^{e_0-1}(p^2-1) < \frac{1}{2}p(p^2-1) < p^3$. When $p = 5$, we have $e_0 = 2$. Thus, $p^2 + p = 30 < 48 = 2^{e_0-1}(p^2-1)$ and $2^{e_0-1}(p^2-1) = 48 < 125 = p^3$.

We next show that $(p^2-1)2^{e_0-1} \notin \{j2^{e_1+1} : 1 \leqslant j \leqslant \frac{p-1}{2}\}$. Every element of $\{j2^{e_1+1} : 1 \leqslant j \leqslant \frac{p-1}{2}\}$ is divisible by 2^{e_1+1}. We show that $2^{e_0-1}(p^2-1)$ is not divisible by 2^{e_1+1}.

Suppose $p \equiv 1 \pmod{4}$. Since $p+1 \equiv 2 \pmod{4}$, $p+1$ is divisible by 2 but not 4. Since $p-1 \equiv 0 \pmod{4}$ and $\frac{1}{2}p < 2^{e_0} < p$, $p-1$ is divisible by $2^{\alpha'}$ for some $2 \leqslant \alpha' \leqslant e_0$. Thus, $(p^2-1)2^{e_0-1}$ is divisible by $2^{e_0+\alpha'}$ for some $2 \leqslant \alpha' \leqslant e_0$. Hence, $(p^2-1)2^{e_0-1}$ is not divisible by 2^{2e_0+1}. Since $2e_0 \leqslant e_1$, $2^{e_0-1}(p^2-1)$ is not divisible by 2^{e_1+1}.

Suppose $p \equiv 3 \pmod 4$. Since $p-1 \equiv 2 \pmod 4$, $p-1$ is divisible by 2 but not 4. Since $p+1 \equiv 0 \pmod 4$ and $p < 2^{e_0+1} < 2p$, $p+1$ is divisible by $2^{\alpha'}$ for some $2 \leqslant \alpha' \leqslant e_0+1$. Thus, $(p^2-1)2^{e_0-1}$ is divisible by $2^{e_0+\alpha'}$ for some $2 \leqslant \alpha' \leqslant e_0+1$. We consider the cases $\alpha' < e_0+1$ and $\alpha' = e_0+1$ separately. Suppose $\alpha' < e_0+1$. Thus, $(p^2-1)2^{e_0-1}$ is divisible by $2^{e_0+\alpha'}$ for some $2 \leqslant \alpha' \leqslant e_0$. Then $(p^2-1)2^{e_0-1}$ is not divisible by 2^{2e_0+1}. Since $2e_0 \leqslant e_1$, $2^{e_0-1}(p^2-1)$ is not divisible by 2^{e_1+1}. Suppose $\alpha' = e_0+1$. Because 2^{e_0+1} divides $p+1$ and $p < 2^{e_0+1} < 2p$, $p = 2^{e_0+1}-1$ is a Mersenne prime. We observe that $\frac{1}{2}p < 2^{e_0} < \frac{1}{\sqrt{2}}p$. Thus, $e_1 = 2e_0+1$. Hence, $2^{e_0-1}(p^2-1)$ is divisible by $2^{2e_0+1} = 2^{e_1}$, but it is not divisible by 2^{e_1+1}.

We consider vertex u_{p,p^2-1}. We have $f(u_{p,p^2-1}) = 2^{e_0-1}(p^2-1)$, $f(u_{p,p^2-2}) = p^2-2$, $f(u_{p,p^2}) = p^3$, and $f(u_{p-1,p^2-1}) = (p^2-1)p-1$. We observe that p^2-1 is relatively prime to p, p^2-2, and $(p^2-1)p-1$. Since p, p^2-2, and $(p^2-1)p-1$ are odd, we have $\gcd\bigl(2^{e_0-1}(p^2-1), p^3\bigr) = \gcd\bigl(2^{e_0-1}(p^2-1), p^2-2\bigr) = \gcd\bigl(2^{e_0-1}(p^2-1), (p^2-1)p-1\bigr) = 1$.

We next consider vertex $u_{k',\ell'}$. Since p is odd and p is relatively prime to p^2-1, $\gcd\bigl(2^{e_0-1}(p^2-1), p\bigr) = 1$. Thus, $k' \equiv 2^{e_0-1}(p^2-1) \not\equiv 0 \pmod p$. Hence, $1 \leqslant k' < p$. Since $\ell' p = 2^{e_0-1}(p^2-1)+1$ is odd, ℓ' is odd. We have $f(u_{k',\ell'}) = p(p^2-1)$, $f(u_{k',\ell'\pm 1}) = 2^{e_0-1}(p^2-1) \pm p$, $f(u_{k'-1,\ell'}) = 2^{e_0-1}(p^2-1)-1$ if $1 < k' < p$, and either

- $f(u_{k'+1,\ell'}) = 2^{e_0-1}(p^2-1)+1$ if $1 \leqslant k' < p-1$,
- $f(u_{k'+1,\ell'}) = \ell'$ if $k' = p-1$ and $\ell' \not\equiv 0 \pmod p$, or
- $f(u_{k'+1,\ell'}) = \frac{\ell'}{p}$ if $k' = p-1$ and $\ell' \equiv 0 \pmod p$.

By Remark 2.1, we have $\gcd(2^{e_0-1}(p^2-1) \pm p, p^2-1) = \gcd(p, p^2-1) = 1$. Since $2^{e_0-1}(p^2-1) \pm p \equiv k' \not\equiv 0 \pmod p$, we have $\gcd(2^{e_0-1}(p^2-1) \pm p, p) = 1$. Thus, $\gcd\bigl(2^{e_0-1}(p^2-1) \pm p, (p^2-1)p\bigr) = 1$.

For $1 < k' < p$, we have $f(u_{k'-1,\ell'}) = 2^{e_0-1}(p^2-1)-1$, and for $1 \leqslant k' < p-1$, we have $f(u_{k'+1,\ell'}) = 2^{e_0-1}(p^2-1)+1$. By Remark 2.1, we have $\gcd\bigl(2^{e_0-1}(p^2-1) \pm 1, p^2-1\bigr) = \gcd(1, p^2-1) = 1$. Since $2^{e_0-1}(p^2-1) \pm 1 \equiv k' \pm 1 \not\equiv 0 \pmod p$, we have $\gcd(2^{e_0-1}(p^2-1) \pm 1, p) = 1$. Thus, $\gcd\bigl(2^{e_0-1}(p^2-1) \pm 1, (p^2-1)p\bigr) = 1$.

Suppose $k' = p-1$. Thus, $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \ell'$ if $\ell' \not\equiv 0 \pmod p$ and $f(u_{k'+1,\ell'}) = f(u_{p,\ell'}) = \frac{\ell'}{p}$ if $\ell' \equiv 0 \pmod p$. Since $\ell' p - 2^{e_0-1}(p^2-1) = 1$, we have $\gcd\bigl(\ell' p, 2^{e_0-1}(p^2-1)\bigr) = 1$. Thus, $\gcd(\ell', p^2-1) = 1$. Hence, $\gcd\bigl(\ell', (p^2-1)p\bigr) = 1$ if $\ell' \not\equiv 0 \pmod p$. Since $2^{e_0-1} < \frac{1}{2}p$, we have $(\ell'-1)p < (\ell'-1)p + k' = 2^{e_0-1}(p^2-1) < \frac{1}{2}p(p^2-1)$. Thus, $\ell' < \frac{1}{2}p^2 + \frac{1}{2}$, which, in turn, implies that $\frac{\ell'}{p} < \frac{1}{2}p + \frac{1}{2p} < p$. Hence, $\gcd\bigl(\frac{\ell'}{p}, (p^2-1)p\bigr) = 1$ if $\ell' \equiv 0 \pmod p$. □

Remark 3.2. *Suppose* $p < n < p^2$. *We see from the proof of Theorem 3.1 that we can swap the labels* $f(u_{2i-1,1}) = (2i-1)p^2$ *and* $f(u_{p,2i-1}) = (2i-1)p$, *for all integers* $1 \leqslant i \leqslant \lfloor \frac{n+p}{2p} \rfloor$, *so that the resulting labeling is prime. Thus, a lower bound on the number of distinct prime labelings on the* $p \times n$ *grid is* $2^{\lfloor \frac{n+p}{2p} \rfloor}$ *if* $p < n < p^2$.

Suppose $n = p^2$. *Then we can swap the labels* $f(u_{2i-1,1}) = (2i-1)p^2$ *and* $f(u_{p,2i-1}) = (2i-1)p$, *for all integers* $1 \leqslant i \leqslant \frac{p-1}{2}$, *so that the resulting labeling is prime. Thus, a lower bound on the number of distinct prime labelings on the* $p \times p^2$ *grid is* $2^{(p-1)/2}$.

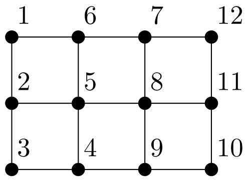

Figure 6: Prime labeling on $P_3 \times P_4$.

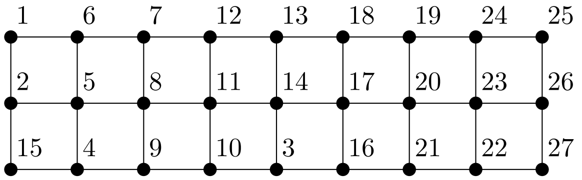

Figure 7: Prime labeling on $P_3 \times P_9$.

4 Prime labeling on $P_3 \times P_n$

We consider prime labelings on the $3 \times n$ grid.

Theorem 4.1. *Let n be a positive integer such that $n \leqslant 9$, then $P_3 \times P_n$ has a prime labeling.*

Proof. **Case 1.** Suppose $1 \leqslant n \leqslant 4$. Consider the labeling on $P_3 \times P_n$ given by $f(u_{i,j}) = 3(j-1)+i$ if j is odd, and $f(u_{i,j}) = 3(j-1)+4-i$ if j is even. The reader can observe that this is a prime labeling on $P_3 \times P_n$. See Figure 6.

Case 2. Suppose $5 \leqslant n \leqslant 9$. Consider the labeling on $P_3 \times P_n$ given by $f(u_{i,j}) = 3(j-1)+i$ if j is odd and $(i,j) \notin \{(3,1),(3,5)\}$, $f(u_{i,j}) = 3(j-1)+4-i$ if j is even, $f(u_{3,1}) = 15$, and $f(u_{3,5}) = 3$. The reader can observe that this is a prime labeling on $P_3 \times P_n$. See Figure 7.

$\square$

Combining Theorems 2.3, 3.1, and 4.1 yields the following result.

Theorem 4.2. *Let p be an odd prime, and let n be an integer such that $1 \leqslant n \leqslant p^2$. Then there exists a prime labeling on $P_p \times P_n$.*

5 Further problems for investigation

Sundaram et al. [7] conjecture that $P_m \times P_n$ has a prime labeling for all positive integer m and n. We propose two more modest versions of this conjecture.

Conjecture 5.1. *Let p be a prime, and let n be a positive integer. Then there exists a prime labeling on $P_p \times P_n$.*

Conjecture 5.2. *Let p and q be primes, and let n be a positive integer. Then there exists a prime labeling on $P_{pq} \times P_n$.*

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